Uniform exponential decay for viscous damped systems

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Dedicated to Ferruccio Colombini with friendship

Abstract. We consider a class of viscous damped vibrating systems. We prove that, under the assumption that the damping term ensures the exponential decay for the corresponding inviscid system, then the exponential decay rate is uniform for the viscous one, regardless what the value of the viscosity parameter is. Our method is mainly based on a decoupling argument of low and high frequencies. Low frequencies can be dealt with because of the effectiveness of the damping term in the inviscid case while the dissipativity of the viscous term guarantees the decay of the high frequency components. This method is inspired in previous work by the authors on time-discretization schemes for damped systems in which a numerical viscosity term needs to be added to ensure the uniform exponential decay with respect to the time-step parameter.

1. Introduction

Let X and Y be Hilbert spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Let $A : \mathcal{D}(A) \subset X \to X$ be a skew-adjoint operator with compact resolvent and $B \in \mathfrak{L}(X, Y)$.

We consider the system described by

$$\dot{z} = Az + \varepsilon A^2 z - B^* B z, \quad t \geq 0, \qquad z(0) = z_0 \in X. \tag{1.1}$$

Here and henceforth, a dot ($\dot{}$) denotes differentiation with respect to time t. The element $z_0 \in X$ is the initial state, and $z(t)$ is the state of the system. Most of the linear equations modeling the damped viscous vibrations of elastic structures (strings, beams, plates,...) can be written in the form (1.1) or some variants that we shall also discuss, in which the viscosity term has a more general form, namely,

$$\dot{z} = Az + \varepsilon \mathcal{V}_\varepsilon z - B^* B z, \quad t \geq 0, \qquad z(0) = z_0 \in X, \tag{1.2}$$

for a suitable viscosity operator $\mathcal{V}_\varepsilon$, which might depend on ε.

We define the energy of the solutions of system (1.1) by

$$E(t) = \frac{1}{2} \|z(t)\|_X^2 , \quad t \geq 0, \tag{1.3}$$

which satisfies

$$\frac{dE}{dt}(t) = - \|Bz(t)\|_Y^2 - \varepsilon ||Az||_X^2, \quad t \geq 0. \tag{1.4}$$

In this paper, we assume that system (1.1) is exponentially stable when $\varepsilon = 0$. For the sake of completeness and clarity we distinguish the case in which the viscosity parameter vanishes

$$\dot{z} = Az - B^*Bz, \quad t \geq 0, \qquad z(0) = z_0 \in X. \tag{1.5}$$

This model corresponds to a conservative system in which a bounded damping term has been added. The damped wave and Schrödinger equations enter in this class, for instance.

Thus, we assume that there exist positive constants μ and ν such that any solution of (1.5) satisfies

$$E(t) \leq \mu \, E(0) \exp(-\nu t), \quad t \geq 0. \tag{1.6}$$

Our goal is to prove that the exponential decay property (1.6) for (1.5) implies the uniform exponential decay of solutions of (1.1) with respect to the viscosity parameter $\varepsilon > 0$.

This result might seem immediate a priori since the viscous term that (1.1) adds to (1.5) should in principle increase the decay rate of the solutions of the later. But, this is far from being trivial because of the possible presence of overdamping phenomena. Indeed, in the context of the damped wave equation, for instance, it is well known that the decay rate does not necessarily behave monotonically with respect to the size of the damping operator (see, for instance, [6, 7, 15]). In our case, however, the viscous damping operator is such that the decay rate is kept uniformly on ε. This is so because it adds dissipativity to the high frequency components, while it does not deteriorate the low frequency damping that the bounded feedback operator $-B^*B$ introduces.

The main result of this paper is that system (1.1) enjoys a uniform stabilization property. It reads as follows:

Theorem 1.1. *Assume that system* (1.5) *is exponentially stable and satisfies* (1.6) *for some positive constants μ and ν, and that $B \in \mathfrak{L}(X, Y)$.*

Then there exist two positive constants μ_0 and ν_0 depending only on $\|B\|_{\mathfrak{L}(X,Y)}$, ν and μ such that any solution of (1.1) *satisfies* (1.6) *with constants μ_0 and ν_0 uniformly with respect to the viscosity parameter $\varepsilon > 0$.*

Our strategy is based on the fact that the uniform exponential decay properties of the energy for systems (1.5) and (1.1), respectively, are equivalent to observability properties for the conservative system

$$\dot{y} = Ay, \quad t \in \mathbb{R}, \qquad y(0) = y_0 \in X, \tag{1.7}$$

and its viscous counterpart

$$\dot{u} = Au + \varepsilon A^2 u, \quad t \in \mathbb{R}, \qquad u(0) = u_0 \in X. \tag{1.8}$$

For (1.7) the observability property consists in the existence of a time $T^* > 0$ and a positive constant $k_* > 0$ such that

$$k_* \|y_0\|_X^2 \leq \int_0^{T^*} \|By(t)\|_Y^2 \ dt, \tag{1.9}$$

for every solution of (1.7) (see [11]).

A similar argument can be applied to the viscous system (1.8). In this case the relevant inequality is the following: There exist a time $T > 0$ and a constant $k_T > 0$ such that any solution of (1.8) satisfies

$$k_T \|u_0\|_X^2 \leq \int_0^T \|Bu(t)\|_Y^2 \ dt + \varepsilon \int_0^T \|Au(t)\|_X^2 \ dt. \tag{1.10}$$

Note however that, for the uniform exponential decay property of the solutions of (1.1) to be independent of ε, we also need the time T and the observability constant k_T in (1.10) to be uniform. Actually we will prove the observability property (1.10) for the time $T = T^*$ given in (1.9).

The observability inequality (1.10) can not be obtained directly from (1.9) since the viscosity operator εA^2 is an unbounded perturbation of the dynamics associated to the conservative system (1.7). Therefore, we decompose the solution u of (1.8) into its low and high frequency parts, that we handle separately. We first use the observability of (1.7) to prove (1.10), uniformly on ε, for the low frequency components. Second, we use the dissipativity of (1.8) to obtain a similar estimate for the high-frequency components.

In this way, we derive observability properties of the low and high frequency components separately, that, together, yield the needed observability property (1.10) leading to the uniform exponential decay result.

Our arguments do not apply when the damping operator B is not bounded, as it happens when the damping is concentrated on the boundary for the wave equation, see for instance [7]. Dealing with unbounded damping operators B needs further work.

As we mentioned above, the results in this paper are related with the literature on the uniform stabilization of numerical approximation schemes for damped equations of the form (1.5) and in particular with [21, 20, 18, 19, 9]. Similar techniques have also been employed to obtain uniform dispersive estimates for numerical approximation schemes to Schrödinger equations in [12].

The recent work [8] is also worth mentioning. There, observability issues were discussed for time and fully discrete approximation schemes of (1.7) and was one of the sources of motivation for this work.

The outline of this paper is as follows.
In Section 2, we recall the results of [8] and prove Theorem 1.1. In Section 3, we present a generalization of Theorem 1.1 to other viscosity operators. We also

specify an application of our technique for viscous second order in time evolution equations which fit (1.2). In Section 4, we present some applications to viscous approximations of damped Schrödinger and wave equations. Finally, some further comments and open problems are collected in Section 5.

2. Proof of Theorem 1.1

We first need to introduce some notations.

Since A is a skew-adjoint operator with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j \ : \ j \in \mathbb{N}\}$, where $(\mu_j)_{j\in\mathbb{N}}$ is a sequence of real numbers such that $|\mu_j| \to \infty$ when $j \to \infty$. Set $(\Phi_j)_{j\in\mathbb{N}}$ an orthonormal basis of eigenvectors of A associated to the eigenvalues $(i\mu_j)_{j\in\mathbb{N}}$, that is

$$A\Phi_j = i\mu_j\Phi_j. \tag{2.1}$$

Moreover, define

$$\mathcal{C}_s = \text{span}\,\{\Phi_j : \text{ the corresponding } i\mu_j \text{ satisfies } |\mu_j| \le s\}. \tag{2.2}$$

In the sequel, we assume that system (1.5) is exponentially stable and that $B \in \mathfrak{L}(X,Y)$, i.e. there exists a constant K_B such that

$$\|Bz\|_Y \le K_B \|z\|_X, \qquad \forall z \in X. \tag{2.3}$$

The proof is divided into several steps.

First, we write carefully the energy identity for z solution of (1.1).

Consider z a solution of (1.1). Its energy $\|z(t)\|_X^2$ satisfies

$$\|z(T)\|_X^2 + 2\int_0^T \|Bz(t)\|_Y^2\ dt + 2\int_0^T \varepsilon \|Az(t)\|_Y^2\ dt = \|z(0)\|_X^2. \tag{2.4}$$

Therefore our goal is to prove that, with T^* as in (1.9), there exists a constant $c > 0$ such that any solution of (1.1) satisfies

$$c\,\|z(0)\|_X^2 \le \int_0^{T^*} \|Bz(t)\|_Y^2\ dt + \varepsilon \int_0^{T^*} \|Az(t)\|_X^2\ dt. \tag{2.5}$$

It is easy to see that, combining (2.4) and (2.5), the semigroup S_ε generated by (1.1) satisfies

$$\|S_\varepsilon(T^*)\| \le \gamma = 1 - c, \tag{2.6}$$

for a constant $0 < \gamma < 1$ independent of $\varepsilon > 0$. This, by the semigroup property, yields the uniform exponential decay result.

We also claim that, for (2.5) to hold for the solutions of (1.1), it is sufficient to show (1.10) for solutions of (1.8). To do that, it is sufficient to follow the argument in [11] developed in the context of system (1.5).

We decompose z as $z = u + w$ where u is the solution of the system (1.8) with initial data $u(0) = z_0$ and w satisfies

$$\dot{w} = Aw + \varepsilon A^2 w - B^*Bz, \quad t \ge 0, \qquad w(0) = 0. \tag{2.7}$$

Indeed, multiplying (2.7) by w and integrating in time, we get

$$\|w(t)\|_X^2 + 2\varepsilon \int_0^t \|Aw(s)\|_X^2 \ ds + 2 \int_0^t < Bz(s), Bw(s) >_Y \ ds = 0.$$

Using that B is bounded, this gives

$$\|w(t)\|_X^2 + 2\varepsilon \int_0^t \|Aw(s)\|_X^2 \ ds \leq \int_0^t \|Bz(s)\|_Y^2 + K_B^2 \int_0^t \|w(s)\|_X^2 \ ds. \quad (2.8)$$

Grönwall's inequality then gives a constant G, that depends only on K_B and T^*, such that

$$\sup_{t\in[0,T^*]} \left\{ \|w(t)\|_X^2 \right\} + \varepsilon \int_0^{T^*} \|Aw(s)\|_X^2 \ ds \leq G \int_0^{T^*} \|Bz(s)\|_Y^2 \ ds. \quad (2.9)$$

Therefore in the sequel we deal with solutions u of (1.8), for which we prove (1.10) for $T = T^*$.

As said in the introduction, we decompose the solution u of (1.8) into its low and high frequency parts. To be more precise, we consider

$$u_l = \pi_{1/\sqrt{\varepsilon}} u, \quad u_h = (I - \pi_{1/\sqrt{\varepsilon}})u, \quad (2.10)$$

where $\pi_{1/\sqrt{\varepsilon}}$ is the orthogonal projection on $\mathcal{C}_{1/\sqrt{\varepsilon}}$ defined in (2.2). Here the notation u_l and u_h stands for the low and high frequency components, respectively.

Note that both u_l and u_h are solutions of (1.8) since the projection $\pi_{1/\sqrt{\varepsilon}}$ and the viscosity operator A^2 commute.

Besides, u_h lies in the space $\mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$, in which the following property holds:

$$\sqrt{\varepsilon} \|Ay\|_X \geq \|y\|_X, \quad \forall y \in \mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}. \quad (2.11)$$

In a first step, we compare u_l with y_l solution of (1.7) with initial data $y_l(0) = u_l(0)$. Now, set $w_l = u_l - y_l$. From (1.9), which is valid for solutions of (1.7), we get

$$k_* \|u_l(0)\|_X^2 = k_* \|y_l(0)\|_X^2 \leq 2 \int_0^{T^*} \|Bu_l(t)\|_Y^2 \ dt + 2 \int_0^{T^*} \|Bw_l(t)\|_Y^2 \ dt. \quad (2.12)$$

In the sequel, to simplify the notation, $c > 0$ will denote a positive constant that may change from line to line, but which does not depend on ε.

Let us therefore estimate the last term in the right hand side of (2.12). To this end, we write the equation satisfied by w_l, which can be deduced from (1.7) and (1.8):

$$\dot{w}_l = Aw_l + \varepsilon A^2 u_l, \quad t \geq 0, \qquad w_l(0) = 0.$$

Note that $w_l \in \mathcal{C}_{1/\sqrt{\varepsilon}}$, since u_l and y_l both belong to $\mathcal{C}_{1/\sqrt{\varepsilon}}$. Therefore, the energy estimate for w_l leads, for $t \geq 0$, to

$$\|w_l(t)\|_X^2 = -2\varepsilon \int_0^t < Au_l(s), Aw_l(s) >_X ds$$
$$\leq \varepsilon \int_0^t \|Au_l(s)\|_X^2 \, ds + \int_0^t \|w_l(s)\|_X^2 \, ds.$$

Grönwall's Lemma applies and allows to deduce from (2.12) and the fact that the operator B is bounded, the existence of a positive c independent of ε, such that

$$c\,\|u_l(0)\|_X^2 \leq \int_0^{T^*} \|Bu_l(t)\|_Y^2 \; dt + \varepsilon \int_0^{T^*} \|Au_l(s)\|_X^2 \, ds.$$

Besides,

$$\int_0^{T^*} \|Bu_l(t)\|_Y^2 \; dt \leq 2 \int_0^{T^*} \|Bu(t)\|_Y^2 \; dt + 2 \int_0^{T^*} \|Bu_h(t)\|_Y^2 \; dt$$

and, since $u_h(t) \in \mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$ for all t,

$$\int_0^{T^*} \|Bu_h(t)\|_Y^2 \; dt \leq K_B^2 \int_0^{T^*} \|u_h(t)\|_X^2 \; dt \leq K_B \varepsilon \int_0^{T^*} \|Au_h(t)\|_X^2 \; dt.$$

It follows that there exists $c > 0$ independent of ε such that

$$c\,\|u_l(0)\|_X^2 \leq \int_0^{T^*} \|Bu(t)\|_Y^2 \; dt + \varepsilon \int_0^{T^*} \|Au(s)\|_X^2 \, ds. \tag{2.13}$$

Let us now consider the high frequency component u_h. Since $u_h(t)$ is a solution of (1.8) and belongs to $\mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$ for all time $t \geq 0$, the energy dissipation law for u_h solution of (1.8) reads

$$\|u_h(t)\|_X^2 + 2\varepsilon \int_0^t \|Au_h(s)\|_X^2 \; ds = \|u_h(0)\|_X^2\,, \quad t \geq 0, \tag{2.14}$$

and

$$\|u_h(t)\|_X^2 \leq \exp(-2t)\, \|u_h(0)\|_X^2\,, \quad \forall t \geq 0.$$

In particular, these two last inequalities imply the existence of a constant $c > 0$ independent of ε such that any solution u_h of (1.8) with initial data $u_h(0) \in \mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$ satisfies

$$c\,\|u_h(0)\|_X^2 \leq \varepsilon \int_0^{T^*} \|Au_h(s)\|_X^2 \; ds. \tag{2.15}$$

Combining (2.13) and (2.15) leads to the observability inequality (1.10). This, combined with the arguments of [11] and (2.9), allows to prove that any solution z of (1.1) satisfies (2.5), and proves (2.6), from which Theorem 1.1 follows.

3. Variants of Theorem 1.1

3.1. General viscosity operators

Other viscosity operators could have been chosen. In our approach, we used the viscosity operator εA^2, which is unbounded, but we could have considered the viscosity operator

$$\varepsilon \mathcal{V}_\varepsilon = \frac{\varepsilon A^2}{I - \varepsilon A^2}, \tag{3.1}$$

which is well defined, since A^2 is a definite negative operator, and commutes with A. This choice presents the advantage that the viscosity operator now is bounded, keeping the properties of being small at frequencies of order less than $1/\sqrt{\varepsilon}$ and of order 1 on frequencies of order $1/\sqrt{\varepsilon}$ and more. Again, the same proof as the one presented above works.

The following result constitutes a generalization of Theorem 1.1, which applies to a wide range of viscosity operators, and, in particular, to (3.1).

Theorem 3.1. *Assume that system* (1.5) *is exponentially stable and satisfies* (1.6), *and that* $B \in \mathfrak{L}(X, Y)$.

Consider a viscosity operator $\mathcal{V}_\varepsilon$ *such that*

1. $\mathcal{V}_\varepsilon$ *defines a self-adjoint definite negative operator.*
2. *The projection* $\pi_{1/\sqrt{\varepsilon}}$ *and the viscosity operator* $\mathcal{V}_\varepsilon$ *commute.*
3. *There exist positive constants* c *and* C *such that for all* $\varepsilon > 0$,

$$\begin{cases} \sqrt{\varepsilon} \left\| \left(\sqrt{-\mathcal{V}_\varepsilon}\right) z \right\|_X \leq C \left\| z \right\|_X, \ \forall z \in \mathcal{C}_{1/\sqrt{\varepsilon}}, \\ \sqrt{\varepsilon} \left\| \left(\sqrt{-\mathcal{V}_\varepsilon}\right) z \right\|_X \geq c \left\| z \right\|_X, \ \forall z \in \mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}. \end{cases}$$

Then the solutions of (1.2) *are exponentially decaying in the sense of* (1.6), *uniformly with respect to the viscosity parameter* $\varepsilon \geq 0$.

The proof of Theorem 3.1 can be easily deduced from the one of Theorem 1.1 and is left to the reader.

Especially, note that the second item implies that both spaces $\mathcal{C}_{1/\sqrt{\varepsilon}}$ and $\mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$ are left globally invariant by the viscosity operator $\mathcal{V}_\varepsilon$. Therefore, if $u_l \in \mathcal{C}_{1/\sqrt{\varepsilon}}$ and $u_h \in \mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$, we have

$$< \mathcal{V}_\varepsilon(u_l + u_h), (u_l + u_h) >_X = < \mathcal{V}_\varepsilon u_l, u_l >_X + < \mathcal{V}_\varepsilon u_h, u_h >_X .$$

Also remark that the second item is always satisfied when the operators $\mathcal{V}_\varepsilon$ and A commute.

3.2. Wave type systems

In this subsection we investigate the exponential decay properties for viscous approximations of second order in time evolution equation.

Let H be a Hilbert space endowed with the norm $\|\cdot\|_H$. Let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint positive operator with compact resolvent and $C \in \mathfrak{L}(H, Y)$.

We then consider the initial value problem

$$\begin{cases} \ddot{v} + A_0 v + \varepsilon A_0 \dot{v} + C^* C \dot{v} = 0, \quad t \geq 0, \\ v(0) = v_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{v}(0) = v_1 \in H. \end{cases} \tag{3.2}$$

System (3.2) can be seen as a particular instance of (1.2) modeling wave and beams equations.

The energy of solutions of (3.2) is given by

$$E(t) = \frac{1}{2} \|\dot{v}(t)\|_H^2 + \frac{1}{2} \left\| A_0^{1/2} v(t) \right\|_H^2, \tag{3.3}$$

and satisfies

$$\frac{dE}{dt}(t) = - \|C\dot{v}(t)\|_Y^2 - \varepsilon \left\| A_0^{1/2} \dot{v}(t) \right\|_H^2. \tag{3.4}$$

As before, we assume that, for $\varepsilon = 0$, the system

$$\ddot{v} + A_0 v + C^* C \dot{v} = 0, \quad t \geq 0, \qquad v(0) = v_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{v}(0) = v_1 \in H, \tag{3.5}$$

is exponentially stable, i.e. (1.6) holds.

We are indeed in the setting of (1.2), since (3.2) can be written as

$$\dot{Z} = AZ + \varepsilon \mathcal{V}_\varepsilon Z - B^* B Z, \tag{3.6}$$

with

$$Z = \begin{pmatrix} v \\ \dot{v} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \mathcal{V}_\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & -A_0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & C \end{pmatrix}. \tag{3.7}$$

Note that the viscosity operator $\mathcal{V}_\varepsilon$ introduced in (3.7) does not satisfy Condition 1 in Theorem 3.1. Though, we can prove the following theorem:

Theorem 3.2. *Assume that system* (3.5) *is exponentially stable and satisfies* (1.6) *for some positive constants μ and ν, and that $C \in \mathfrak{L}(H, Y)$. Set $K < \infty$.*

Then there exist two positive constants μ_K and ν_K depending only on $\|C\|_{\mathfrak{L}(H,Y)}$, K, ν and μ such that any solution of (3.2) *satisfies* (1.6) *with constants μ_0 and ν_0 uniformly with respect to the viscosity parameter $\varepsilon \in [0, K]$.*

Before going into the proof, we introduce the spectrum of A_0. Since A_0 is self-adjoint positive definite with compact resolvent, its spectrum is discrete and $\sigma(A_0) = \{\lambda_j^2 : j \in \mathbb{N}\}$, where λ_j is an increasing sequence of real positive numbers such that $\lambda_j \to \infty$ when $j \to \infty$. Set $(\Psi_j)_{j \in \mathbb{N}}$ an orthonormal basis of eigenvectors of A_0 associated to the eigenvalues $(\lambda_j^2)_{j \in \mathbb{N}}$.

These notations are consistent with the ones introduced in Section 2, by setting A as in (3.7), and

$$\mu_{\pm j} = \pm \lambda_j, \qquad \Phi_j = \begin{pmatrix} \dfrac{1}{i\mu_j} \Psi_j \\ \Psi_j \end{pmatrix}.$$

For convenience, similarly as in (2.2), we define

$$\mathfrak{C}_s = \operatorname{span} \{\Psi_j : \text{ the corresponding } \lambda_j \text{ satisfies } |\lambda_j| \leq s\}, \tag{3.8}$$

which satisfies $\mathcal{C}_s = (\mathfrak{C}_s)^2$.

Sketch of the proof. The proof of Theorem 3.2 closely follows the one of Theorem 1.1.

As before, we read the exponential stability of (3.5) into the following observability inequality: There exist a time T^* and a positive constant k_* such that any solution of

$$\ddot{y} + A_0 y = 0, \quad t \geq 0, \qquad y(0) = y_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{y}(0) = y_1 \in H, \tag{3.9}$$

satisfies

$$k_* \Big(\|y_1\|_H^2 + \left\| A_0^{1/2} y_0 \right\|_H^2 \Big) \leq \int_0^{T^*} \|C\dot{y}(t)\|_Y^2 \ dt. \tag{3.10}$$

Due to (3.4), as in (2.5), the exponential decay of the energy for solutions of (3.2) is equivalent to the following observability inequality: There exist a time $\tilde{T}$ and a positive constant c such that for any $\varepsilon \in [0, K]$,

$$c \Big(\|v_1\|_H^2 + \left\| A_0^{1/2} v_0 \right\|_H^2 \Big) \leq \int_0^{\tilde{T}} \|C\dot{v}(t)\|_Y^2 \ dt + \varepsilon \int_0^{\tilde{T}} \left\| A_0^{1/2} \dot{v}(t) \right\|_H^2 \ dt \tag{3.11}$$

holds for any solution v of (3.2).

Using the same perturbative arguments as in [11] or (2.7)-(2.9), the observability inequality (3.11) holds if and only if there exist a time T and a positive constant $k_T > 0$ such that, for any $\varepsilon \in [0, K]$, the observability inequality

$$k_T \Big(\|u_1\|_H^2 + \left\| A_0^{1/2} u_0 \right\|_H^2 \Big) \leq \int_0^{T} \|C\dot{u}(t)\|_Y^2 \ dt + \varepsilon \int_0^{T} \left\| A_0^{1/2} \dot{u}(t) \right\|_H^2 \ dt \tag{3.12}$$

holds for any solution u of

$$\ddot{u} + A_0 u + \varepsilon A_0 \dot{u} = 0, \quad t \geq 0, \qquad u(0) = u_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{u}(0) = u_1 \in H. \tag{3.13}$$

As before, we then focus on the observability inequality (3.12) for solutions of (3.13). As in the proof of Theorem 1.1, we now decompose the solution of (3.13) into its low and high frequency parts, that we handle separately. To be more precise, we consider

$$u_l = P_{1/\sqrt{\varepsilon}} u, \qquad u_h = (I - P_{1/\sqrt{\varepsilon}}) u.,$$

where $P_{1/\sqrt{\varepsilon}}$ is the orthogonal projection in H on $\mathfrak{C}_{1/\sqrt{\varepsilon}}$ as defined in (3.8). Again, both u_l and u_h are solutions of (3.13) since $P_{1/\sqrt{\varepsilon}}$ commute with A_0.

Arguing as before, the low frequency component u_l can be compared to y_l solution of (3.9) with initial data $(y_0, y_1) = (P_{1/\sqrt{\varepsilon}} u_0, P_{1/\sqrt{\varepsilon}} u_1)$, and using (3.10) for solutions of (3.9), we obtain the existence of a positive constant c_1 such that

$$\begin{aligned} c_1 \Big(\left\|P_{1/\sqrt{\varepsilon}} u_1\right\|_H^2 + \left\| A_0^{1/2} P_{1/\sqrt{\varepsilon}} u_0 \right\|_H^2 \Big) \\ \leq \int_0^{T^*} \|C\dot{u}(t)\|_Y^2 \ dt + \varepsilon \int_0^{T^*} \left\| A_0^{1/2} \dot{u}(t) \right\|_H^2 \ dt. \end{aligned} \tag{3.14}$$

For the high frequency component u_h, the situation is slightly more intricate than in Theorem 1.1. The energy of the solution u_h satisfies the dissipation law

$$\frac{1}{2}\frac{d}{dt}\Big(\|\dot u_h(t)\|_H^2 + \left\|A_0^{1/2}u_h(t)\right\|_H^2 \Big) = -\varepsilon \left\|A_0^{1/2}\dot u_h\right\|_H^2 \leq -\|\dot u_h\|_H^2 , \tag{3.15}$$

where the last inequality comes from $\dot u_h \in \mathfrak{C}^{\perp}_{1/\sqrt{\varepsilon}}$.

Setting

$$E_h(t) = \frac{1}{2}\|\dot u_h(t)\|_H^2 + \frac{1}{2}\left\|A_0^{1/2}u_h(t)\right\|_H^2 ,$$

we thus obtain that

$$E_h(t) + \int_0^t \|\dot u_h(s)\|_H^2 \ ds \leq E_h(0). \tag{3.16}$$

We now prove the so-called equirepartition of the energy for the solutions u of (3.13). Multiplying (3.13) by u and integrating by parts between 0 and t, we obtain

$$< \dot u(t), u(t) >_H - < \dot u(0), u(0) >_H - \int_0^t \|\dot u(s)\|_H^2 \ ds + \int_0^t \left\|A_0^{1/2}u(s)\right\|_H^2 \ ds \\ + \varepsilon \int_0^t < A_0^{1/2}\dot u(s), A_0^{1/2}u(s) >_H \ ds = 0.$$

In particular,

$$\int_0^t \|\dot u(s)\|_H^2 \ ds = \int_0^t \left\|A_0^{1/2}u(s)\right\|_H^2 \ ds + \frac{\varepsilon}{2}\Big(\left\|A_0^{1/2}u(t)\right\|_H^2 - \left\|A_0^{1/2}u_0\right\|_H^2 \Big) \\ + < \dot u(t), u(t) >_H - < \dot u(0), u(0) >_H . \tag{3.17}$$

Now, for u_h, which is a solution of (3.13), for all $t \geq 0$, $u_h(t) \in \mathfrak{C}^{\perp}_{1/\sqrt{\varepsilon}}$. In particular, for all $t \geq 0$, we have

$$\Big| < \dot u_h(t), u_h(t) >_H \Big| \leq \frac{\sqrt{\varepsilon}}{2}\|\dot u_h\|_H^2 + \frac{1}{2\sqrt{\varepsilon}}\|u_h(t)\|_H^2 \leq \sqrt{\varepsilon}E_h(t), \tag{3.18}$$

where we used that for $\phi \in \mathfrak{C}^{\perp}_{1/\sqrt{\varepsilon}}$,

$$\|\phi\|_H^2 \leq \varepsilon \left\|A_0^{1/2}\phi\right\|_H^2 .$$

Combining (3.18) with identity (3.17) for u_h, we obtain

$$\int_0^t \|\dot u_h(s)\|_H^2 \ ds \geq \int_0^t \left\|A_0^{1/2}u_h(s)\right\|_H^2 \ ds - \Big(\sqrt{\varepsilon}+\varepsilon\Big)(E_h(t)+E_h(0)). \tag{3.19}$$

This yields

$$\int_0^t \|\dot u_h(s)\|_H^2 \ ds \geq \int_0^t E_h(s)\ ds - \frac{1}{2}\Big(\sqrt{\varepsilon}+\varepsilon\Big)(E_h(t)+E_h(0)). \tag{3.20}$$

Combined with (3.16), we obtain

$$\Big(1-\frac{1}{2}(\sqrt{\varepsilon}+\varepsilon)\Big)E_h(t)+\int_0^t E_h(s)\ ds \leq E_h(0)\Big(1+\frac{1}{2}(\sqrt{\varepsilon}+\varepsilon)\Big) \tag{3.21}$$

Assuming that $K \geq 1$, which can always be assumed, for $\varepsilon \in [0, K]$, we thus have

$$(1-K)E_h(t)+\int_0^t E_h(s)\ ds \leq (1+K)E_h(0).$$

The decay of $E_h(t)$, guaranteed by the dissipation law (3.15), then proves that

$$(t+1-K)E_h(t) \leq (1+K)E_h(0).$$

For $t = 1+3K$, we thus have $E_h(1+3K) \leq E_h(0)/2$. We then deduce from the dissipation law (3.15) the existence of a positive constant c_K such that

$$c_K E_h(0) \leq \varepsilon \int_0^{1+3K} \left\| A_0^{1/2}\dot{u}_h(s)\right\|_H^2\ ds. \tag{3.22}$$

We finally conclude Theorem 3.2 by combining (3.14) and (3.22) as before. □

Remark 3.3. One cannot expect the results of Theorem 3.2 to hold uniformly with respect to $\varepsilon \in [0, \infty]$. Indeed, an overdamping phenomenon appears when $\varepsilon \to \infty$. This can indeed be deduced from the existence of the following solutions of (3.13):

$$u_j(t) = \exp(t\tau_j^\varepsilon)\Psi_j, \quad t \geq 0, \quad \text{where } \tau_j^\varepsilon = \frac{\varepsilon\lambda_j^2}{2}\left(\sqrt{1-\frac{4}{(\varepsilon\lambda_j)^2}}-1\right) \underset{\varepsilon\lambda_j\to\infty}{\sim} -\frac{1}{\varepsilon}.$$

Plugging these solutions in (3.12), one can check that the observability inequality (3.12) cannot hold uniformly with respect to $\varepsilon \in [0, \infty)$. Finally, using the equivalence between the observability inequality (3.12) for solutions of (3.13) and the observability inequality (3.11) for solutions of (3.2), this proves that the results of Theorem 3.2 do not hold uniformly with respect to $\varepsilon \in [0, \infty]$.

Remark 3.4. To avoid the overdamping phenomenon when $\varepsilon \to \infty$, one can for instance add a dispersive term in (3.2), and consider the initial value problem

$$\begin{cases} \ddot{v} + A_0 v + \varepsilon A_0 \dot{v} + \varepsilon A_0 v + C^*C\dot{v} = 0, \quad t \geq 0, \\ v(0) = v_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{v}(0) = v_1 \in H. \end{cases} \tag{3.23}$$

The energy of solutions of (3.23) is now given by

$$E_\varepsilon(t) = \frac{1}{2}\left\|\dot{v}(t)\right\|_H^2 + \Big(\frac{1+\varepsilon}{2}\Big)\left\|A_0^{1/2}v(t)\right\|_H^2. \tag{3.24}$$

One can then prove that, if system (3.5) is exponentially stable, then the energy E_ε of solutions of systems (3.23) is exponentially stable, uniformly with respect to the viscosity parameter $\varepsilon \in [0, \infty)$. The proof can be done similarly as the one of Theorem 3.2 and is left to the reader. The main difference that the dispersive term introduces is that the high frequency solutions u_h of

$$\ddot{u}_h + A_0 u_h + \varepsilon A_0 \dot{u}_h + \varepsilon A_0 u_h = 0, \quad t \geq 0, \tag{3.25}$$

with initial data $(u_h(0), \dot{u}_h(0)) \in (\mathfrak{C}^{\perp}_{1/\sqrt{\varepsilon}})^2 \cap (\mathcal{D}(A_0^{1/2}) \times H)$ now satisfy, instead of (3.19), which deteriorates when $\varepsilon \to \infty$, the following property of equirepartition of the energy

$$\left| \int_0^t \|\dot{u}_h\|_H^2 \, ds - (1+\varepsilon) \int_0^t \left\| A_0^{1/2} u(s) \right\|_H^2 \, ds \right| \leq 2E_{h,\varepsilon}(t) + 2E_{h,\varepsilon}(0), \tag{3.26}$$

where $E_{h,\varepsilon}$ is the energy of the solutions u_h of (3.25).

4. Applications

This section is devoted to present some precise examples.

4.1. The viscous Schrödinger equation

Let Ω be a smooth bounded domain of $\mathbb{R}^N$.

Let us now consider the following damped Schrödinger equation:

$$\begin{cases} i\dot{z} + \Delta_x z + i a(x)\, z = 0, & \text{in } \Omega \times (0,\infty), \\ z = 0, & \text{on } \partial\Omega \times (0,\infty), \\ z(0) = z_0, & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $a = a(x)$ is a nonnegative damping function in $L^\infty(\Omega)$, that we assume to be positive in some open subdomain ω of Ω, that is there exists $a_0 > 0$ such that

$$a(x) \geq a_0, \quad \forall x \in \omega. \tag{4.2}$$

The energy of solutions of (4.1), given by

$$E(t) = \frac{1}{2} \|z(t)\|^2_{L^2(\Omega)}, \tag{4.3}$$

satisfies

$$\frac{dE}{dt}(t) = -\int_\Omega a(x) |z(t,x)|^2 \, dx. \tag{4.4}$$

The stabilization problem for (4.1) has already been studied in the recent years. Let us briefly present some known results. Some of them concern the problem of exact controllability but, as explained for instance in [16], it is equivalent to the observability and the stabilization ones addressed in this article in the case where the damping operator B is bounded.

For instance, in [14], it is proved that the Geometric Control Condition (GCC) is sufficient to guarantee the stabilization property (1.6) for the damped Schrödinger equation (4.1). The GCC can be, roughly, formulated as follows (see [2] for the precise setting): The subdomain ω of Ω is said to satisfy the GCC if there exists a time $T > 0$ such that all rays of Geometric Optics that propagate inside the domain Ω at velocity one reach the set ω in time less than T. This condition is necessary and sufficient for the stabilization property to hold for the wave equation.

But, in fact, the Schrödinger equation behaves slightly better than a wave equation from the stabilization point of view because of the infinite velocity of propagation and, in this case, the GCC is sufficient but not always necessary. For instance, in [13], it has been proved that when the domain Ω is a square, for any non-empty bounded open subset ω, the stabilization property (1.6) holds for system (4.1). Other geometries have been also dealt with: We refer to the articles [4, 1].

Now, we assume that ω satisfies the GCC and, consequently, that we are in a situation where the stabilization property (1.6) for (4.1) holds, and we consider the viscous approximations

$$\begin{cases} i\dot{z} + \Delta_x z + ia(x)\, z - i\sqrt{\varepsilon}\Delta_x z = 0, & \text{in } \Omega \times (0,\infty), \\ z = 0, & \text{on } \partial\Omega \times (0,\infty), \\ z(0) = z_0, & \text{in } \Omega, \end{cases} \tag{4.5}$$

where $\varepsilon \geq 0$.

System (4.1) can be seen as a Ginzburg-Landau type approximation. More precisely, system (4.1) is the inviscid limit of (4.5). We refer to the works [17, 3] where inviscid limits were analyzed in a nonlinear context.

For the stabilization problem, Theorem 3.1 applies and provides the following result:

Theorem 4.1. *Assume that system* (4.1) *is exponentially stable, i.e. it satisfies* (1.6)*.*

Then the solutions of (4.5) *are exponentially decaying in the sense of* (1.6)*, uniformly with respect to the viscosity parameter* $\varepsilon \geq 0$*.*

Proof. Let us check the hypothesis of Theorem 3.1.

This example enters in the abstract setting given in the introduction: The operator $A = i\Delta_x$ with the Dirichlet boundary conditions is indeed skew-adjoint in $L^2(\Omega)$ with compact resolvent and domain $\mathcal{D}(A) = H^2 \cap H_0^1(\Omega) \subset L^2(\Omega)$. Since a is a nonnegative function, the damping term in (4.1) takes the form B^*Bz where B is defined as the multiplication by $\sqrt{a(x)}$, which is obviously bounded from $L^2(\Omega)$ to $L^2(\Omega)$.

The viscosity operator is

$$\varepsilon\mathcal{V}_\varepsilon = \sqrt{\varepsilon}\Delta_x = -i\sqrt{\varepsilon}A = -\sqrt{\varepsilon}|A|.$$

Obviously, this viscosity operator $\mathcal{V}_\varepsilon$ satisfies the assumptions 1, 2 and 3, and therefore Theorem 3.1 applies. □

4.2. The viscous damped wave equation

Again, let Ω be a smooth bounded domain of $\mathbb{R}^N$.

We now consider the damped wave equation

$$\begin{cases} \ddot{v} - \Delta_x v + a(x)\,\dot{v} = 0, & \text{in } \Omega \times (0,\infty), \\ v = 0, & \text{on } \partial\Omega \times (0,\infty), \\ v(0) = v_0, \quad \dot{v}(0) = v_1 & \text{in } \Omega, \end{cases} \tag{4.6}$$

where a is a nonnegative function as before, and satisfies (4.2) for some non-empty open subset ω of Ω.

The energy of solutions of (4.6), given by

$$E(t) = \frac{1}{2}\,\|\dot{v}\|^2_{L^2(\Omega)} + \frac{1}{2}\,\|\nabla v\|^2_{L^2(\Omega)}\,, \tag{4.7}$$

satisfies the dissipation law

$$\frac{dE}{dt}(t) = -\int_\Omega a(x)|\dot{v}|^2\,dx. \tag{4.8}$$

We assume that system (4.6) is exponentially stable. From the works [2, 5], this is the case if and only if ω satisfies the Geometric Control Condition given above.

We now consider viscous approximations of (4.6) given, for $\varepsilon > 0$, by

$$\begin{cases} \ddot{v} - \Delta_x v + a(x)\dot{v} - \varepsilon\Delta_x \dot{v} = 0, & \text{in } \Omega \times (0,\infty), \\ v = 0, & \text{on } \partial\Omega \times (0,\infty), \\ v(0) = v_0 \in H^1_0(\Omega), \quad \dot{v}(0) = v_1 \in L^2(\Omega). \end{cases} \tag{4.9}$$

Setting $A_0 = -\Delta_x$ with Dirichlet boundary conditions and $C = \sqrt{a(x)}$, Theorem 3.2 applies:

Theorem 4.2. *Assume that ω satisfies the Geometric Control Condition.*

Then the solutions of (4.9) *decay exponentially, i.e. satisfy* (1.6) *uniformly with respect to the viscosity parameter $\varepsilon \in [0,1]$. To be more precise, there exist positive constants μ_0 and ν_0 such that for all $\varepsilon \in [0,1]$, for any initial data in $H^1_0(\Omega) \times L^2(\Omega)$, the solution of* (4.9) *satisfies*

$$E(t) \le \mu_0\,E(0)\exp(-\nu_0 t), \quad t \ge 0. \tag{4.10}$$

5. Further comments

1. In this article, we have identified a class of damped systems, with added viscosity term, in which overdamping does not occur. This is to be compared with the existing literature on the overdamping phenomenon for the damped wave equation ([6, 7]).

2. As we mentioned in the introduction, our methods and results require the assumption that the damping operator B is bounded. This is due to the method we employ, which is based on the equivalence between the exponential decay of the energy and the observability properties of the conservative system, that requires the damping operator to be bounded. However, in several relevant applications, as for instance when dealing with the problem of boundary stabilization of the

wave equation (see [16]), the feedback law is unbounded, and our method does not apply. This issue requires further work.

3. The same methods allow obtaining numerical approximation schemes with uniform decay properties.

The discrete analogue of the viscosity term added above for the stabilization of the wave equation has already been discussed in the works [21, 20, 18, 9] for space semi-discrete approximation schemes of damped wave equations. In those articles, though, the viscosity term is needed due to the presence of high-frequency spurious solutions that do not propagate and therefore are not efficiently damped by the damping operator B^*B when it is localized in space as in the examples considered above.

Following the same ideas as in [21, 20, 18, 9], if observability properties such as (1.9) hold for fully discrete approximation schemes of the conservative linear system (1.7) in a filtered space (see [8]), then adding a suitable viscosity term to the corresponding fully discrete version of the dissipative system (1.5) suffices to obtain uniform (with respect to space time discretization parameters) stabilization properties. This issue is currently investigated by the authors and will be published in [10].

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