Observability properties of a semi-discrete 1d wave equation derived from a mixed finite element method on nonuniform meshes

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Abstract

The goal of this article is to analyze the observability properties for a space semi-discrete approximation scheme derived from a mixeed finite element method of the 1d wave equation on nonuniform meshes. More precisely, we prove that observability properties hold uniformly with respect to the mesh-size under some assumptions, which, roughly, measures the lack of uniformity of the meshes, thus extending the work [4] to nonuniform meshes. Our results are based on a precise description of the spectrum of the discrete approximation schemes on nonuniform meshes, and the use of Ingham's inequality. We also mention applications to the boundary null controllability of the 1d wave equation, and to stabilization properties for the 1d wave equation.

1 Introduction

The goal of this article is to address the observability properties for a semi-discrete 1d wave equation.

We consider the following 1d wave equation

$$\begin{cases} \partial_{tt}^2 u - \partial_{xx}^2 u = 0, & (x,t) \in (0,1)\times \mathbb{R}, \\ u(0,t) = u(1,t) = 0, & t \in \mathbb{R}, \\ u(x,0) = u^0(x),\ \partial_t u(x,0) = u^1(x), & x \in (0,1), \end{cases} \tag{1.1}$$

where $u^0 \in H_0^1(0,1)$ and $u^1(x) \in L^2(0,1)$. The energy of solutions of (1.1), given by

$$E(t) = \frac{1}{2}\int_0^1 |\partial_t u(t,x)|^2 + |\partial_x u(t,x)|^2 \, dx, \tag{1.2}$$

is constant.

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It is well-known (see [15]) that for all $T > 0$, there exists a constant K_T such that the admissibility inequality

$$\int_0^T |\partial_x u(0,t)|^2 \, dt \leq K_T E(0) \tag{1.3}$$

holds for any solution of (1.1) with $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$.

Besides, for any time $T > 2$, there exists a positive constant k_T such that the boundary observability inequality

$$k_T E(0) \leq \int_0^T |\partial_x u(0,t)|^2 \, dt \tag{1.4}$$

holds for any solution of (1.1) with $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$.

Inequalities (1.3)-(1.4) arise naturally when dealing with boundary controllability properties of the 1d wave equation, see [15]. Indeed, the observability and controllability properties are dual notions. We will make these links clearer in Section 3.

Let us also present another relevant observability inequality, which is useful when dealing with distributed controls or stabilization properties of damped wave equations (see [11, 15]). If (a,b) denotes a non empty subinterval of $(0,1)$, the following distributed observability property holds: for any time $T > 2\max\{a, 1-b\}$, there exists a constant C_1 such that any solution of (1.1) with $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ satisfies

$$E(0) \leq C_1 \int_0^T \int_a^b |\partial_t u(x,t)|^2 \, dx \, dt. \tag{1.5}$$

In the sequel, we will consider observability properties for the 1d space semi-discrete wave equation derived from a mixed finite element method on a nonuniform mesh.

For any integer $n \in \mathbb{N}^*$, let us consider a mesh $\mathcal{S}_n$ given by $n+2$ points

$$\left\{ \begin{array}{l} 0 = x_{0,n} < x_{1,n} < \cdots < x_{n,n} < x_{n+1,n} = 1, \\ h_{j+1/2,n} = x_{j+1,n} - x_{j,n}, \quad j \in \{0, \cdots, n\}. \end{array} \right. \tag{1.6}$$

On $\mathcal{S}_n$, the mixed finite element approximation scheme for system (1.1) reads as (see [5], [10] or [4])

$$\left\{ \begin{array}{l} \dfrac{h_{j-1/2,n}}{4}(u''_{j-1,n} + u''_{j,n}) + \dfrac{h_{j+1/2,n}}{4}(u''_{j,n} + u''_{j+1,n}) \\ \qquad\qquad = \dfrac{u_{j+1,n} - u_{j,n}}{h_{j+1/2,n}} - \dfrac{u_{j,n} - u_{j-1,n}}{h_{j-1/2,n}}, \quad j = 1, \cdots n, \; t \in \mathbb{R}, \\ u_{0,n}(t) = u_{n+1,n}(t) = 0, \quad t \in \mathbb{R}, \\ u_j(0) = u^0_{j,n}, \qquad u'_j(0) = u^1_{j,n}, \qquad j = 1, \cdots, n. \end{array} \right. \tag{1.7}$$

The notations we use are the standard ones: A prime denotes differentiation with respect to time, and $u_{j,n}(t)$ is an approximation of the solution u of (1.1) at the point $x_{j,n}$ at time t.

System (1.7) is conservative. The energy of solutions u of (1.7), given by

$$E_n(t) = \frac{1}{2}\sum_{j=0}^{n} h_{j+1/2,n}\Big(\frac{u_{j+1,n}(t)-u_{j,n}(t)}{h_{j+1/2,n}}\Big)^2 + \frac{1}{2}\sum_{j=0}^{n} h_{j+1/2,n}\Big(\frac{u'_{j+1,n}(t)+u'_{j,n}(t)}{2}\Big)^2, \quad t\in\mathbb{R}, \quad (1.8)$$

is constant.

In this semi-discrete setting, we will investigate the observability properties corresponding to (1.4) and (1.5), and especially under which assumptions on the meshes $\mathcal{S}_n$ we can guarantee discrete observability inequalities to be uniform with respect to n.

For this purpose, we introduce the notion of regularity of a mesh:

Definition 1.1. For a mesh $\mathcal{S}_n$ given by $n+2$ points as in (1.6), we define the regularity of the mesh $\mathcal{S}_n$ by

$$\mathrm{Reg}(\mathcal{S}_n) = \frac{\max_j\{h_{j+1/2,n}\}}{\min_j\{h_{j+1/2,n}\}}. \quad (1.9)$$

Given $M \geq 1$, we say that a mesh $\mathcal{S}_n$ given by $n+2$ points as in (1.6) is M-regular if

$$\mathrm{Reg}(\mathcal{S}_n) = \frac{\max_j\{h_{j+1/2,n}\}}{\min_j\{h_{j+1/2,n}\}} \leq M. \quad (1.10)$$

Obviously, a 1-regular mesh is uniform. In other words, the regularity of the mesh $\mathrm{Reg}(\mathcal{S}_n)$ measures the lack of uniformity of the mesh.

Within this class, we will prove the following observability properties:

Theorem 1.2. *Let M be a real number greater than one, and consider a sequence $(\mathcal{S}_n)_n$ of M-regular meshes.*

Then for any time $T>2$, there exist positive constants k_T and K_T such that for all integer n, any solution u_n of (1.7) satisfies

$$k_T E_n(0) \leq \int_0^T \Big(\Big|\frac{u_{1,n}(t)}{h_{1/2,n}}\Big|^2 + |u'_{1,n}(t)|^2\Big)dt \leq K_T E_n(0). \quad (1.11)$$

Besides, if $J=(a,b)\subset(0,1)$ denotes a subinterval of $(0,1)$, then for any time $T>2\max\{a,1-b\}$, there exists a constant C_1 such that for all integer n, any solution u_n of (1.7) satisfies

$$E_n(0) \leq C_1 \int_0^T \sum_{x_{j,n}\in J} h_{j+1/2,n}\Big(\frac{u'_{j,n}(t)+u'_{j+1,n}(t)}{2}\Big)^2 dt. \quad (1.12)$$

Obviously, these properties are discrete versions of inequalities (1.3)-(1.4) and (1.5). Also note that the right hand-side inequality in (1.11) holds, as (1.3), for all time $T>0$, taking $K_T = K_3$ for $T\leq 2$.

Theorem 1.2 is based on an explicit spectral analysis of (1.7) ăin the discrete setting, that allows us to prove the existence of a gap between the eigenvalues of the space discrete operator in (1.7). Thanks to Ingham's inequality [13], this reduces the analysis to the study of the observability properties of the eigenvectors of (1.7), which will again be deduced from the explicit form of the spectrum of (1.7).

Besides, we emphasize that Theorem 1.2 provides uniform (with respect to n) observability results. Therefore, using precisely the same duality as in the continuous setting, Theorem 1.2 has several applications to controllability and stabilization properties for the space semi-discrete 1d wave equations (1.7). In Section 3, similarly as in [4], we present an application to the boundary null controllability of the space semi-discrete approximation scheme of the 1d wave equation. Later, in Section 4, following [1], we study the decay properties of the energy for semi-discrete approximation schemes of 1d damped wave equations.

Let us briefly comment some relative works. Similar problems have been extensively studied in the last decade for various space semi-discrete approximation schemes of the 1d wave equation, see for instance the review article [26]. The numerical schemes on uniform meshes provided by finite difference and finite element approximation schemes do not have uniform observability properties, whatever the time T is ([12]). This is due to high frequency waves that do not propagate, see [23, 16]. In other words, these numerical schemes create some spurious high-frequency wave solutions that are localized.

However some remedies exist. The most natural one consists in filtering the initial data and removing these spurious waves, as in [12, 25]. Another way to filter is to use the bi-grid method as introduced and developed in [9] and analyzed in [19]. A new approach was proposed recently in [18] based on wavelet filtering. Let us also mention the results [22, 21, 20, 8] that amounts to adding an extra term in (1.12) which is non-negligible only for the high frequencies. A last possible cure was proposed in [1, 10] and later analyzed in [4]: a 1d semi-discrete scheme derived from a mixed finite element method was proposed, which has the property that the group velocity of the waves is bounded from below.

To the best of our knowledge there is no result at all for the space semi-discrete wave equation on nonuniform meshes, although most of the domains used in practice are recovered by non periodic triangulations. A first step in this direction can be found in [20], in which a study of a non homogeneous string equation on a uniform mesh was proposed. This can indeed be seen, up to a change of variable, almost as a discretization of a wave equation with constant velocity on a nonuniform mesh.

Let us also mention that some results are available in the context of the heat equation for space semi-discrete approximation schemes on nonuniform meshes in [14], even in dimension greater than 1.

The outline of this paper is as follows. In Section 2, we precisely describe the spectrum of the space semi-discrete operator and prove Theorem 1.2. Section 3 and 4 respectively aim at presenting precise applications of Theorem 1.2 to controllability and stabilization properties.

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2 Spectral Theory

In this Section, we first study the spectrum of the space semi-discrete operator in (1.7) on a general mesh $\mathcal{S}_n$ given by $n+2$ points as in (1.6). Second, we derive more precise estimates on the spectrum when $\mathcal{S}_n$ is an M-regular mesh. Third, we derive Theorem 1.2 from our analysis. Finally, we discuss the assumption of M-regularity of the meshes, and show that, in some sense, this assumption is sharp with respect to the observability properties given in Theorem 1.2.

Given a mesh $\mathcal{S}_n$ of $n+2$ points as in (1.6), since the system (1.7) is conservative, the spectral problem for (1.7) reads as: Find $\lambda_n \in \mathbb{R}$ and a non-trivial solution ϕ_n such that

$$\left\{ \begin{aligned} & -\frac{\lambda_n^2}{4}(h_{j-1/2,n}(\phi_{j,n}+\phi_{j-1,n}) + h_{j+1/2,n}(\phi_{j,n}+\phi_{j+1,n})) \\ & \qquad = \frac{\phi_{j+1,n}-\phi_{j,n}}{h_{j+1/2,n}} - \frac{\phi_{j,n}-\phi_{j-1,n}}{h_{j-1/2,n}}, \quad j = 1, \cdots, n, \\ & \phi_{0,n} = \phi_{n+1,n} = 0. \end{aligned} \right. \tag{2.1}$$

2.1 Computations of the eigenvalues for a general mesh

In this Subsection, we consider a general mesh $\mathcal{S}^n$ given by $n+2$ points as in (1.6).

Theorem 2.1. *The spectrum of system* (1.7) *is precisely the set of* $\pm\lambda_n^k$ *with* $k \in \{1, \cdots, n\}$*, where* λ_n^k *is defined by the implicit formula*

$$\sum_{j=0}^{n} \arctan\Big(\frac{\lambda_n^k h_{j+1/2,n}}{2}\Big) = k\frac{\pi}{2}. \tag{2.2}$$

The gap between two eigenvalues is bounded from below:

$$\min_{k\in\{1,\cdots,n-1\}} \{\lambda_n^{k+1} - \lambda_n^k\} \geq \pi. \tag{2.3}$$

Besides, for each $k \in \{1, \cdots, n\}$*, the following estimate holds:*

$$\lambda_n^k \geq \lambda_{*n}^k = 2(n+1)\tan\Big(\frac{k}{n+1}\frac{\pi}{2}\Big) \geq k\pi. \tag{2.4}$$

Remark 2.2. Note that λ_{*n}^k coincides with the k-th eigenvalue of system (1.7) for a uniform mesh constituted by $n+2$ points. Also note that $k\pi$ is the k-th eigenvalue of system (1.1). In other words, inequality (2.4) implies that the dispersion diagrams corresponding to the spectrum of (1.7) for a general nonuniform mesh, for a uniform mesh, and for the continuous system (1.1) are sorted.

Proof. To simplify notation, we drop the unnecessary subscript n.

Let us introduce functions p and q corresponding to $\partial_x \phi$ and $i\lambda\phi$ in the continuous case:

$$\begin{cases} p_{j+1/2} = \dfrac{\phi_{j+1} - \phi_j}{h_{j+1/2}}., \\ q_{j+1/2} = \dfrac{i\lambda}{2}(\phi_j + \phi_{j+1}). \end{cases} \tag{2.5}$$

The spectral system (2.1) then becomes :

$$\begin{cases} \dfrac{i\lambda}{2}(h_{j-1/2,n} q_{j-1/2} + h_{j+1/2,n} q_{j+1/2}) = p_{j+1/2} - p_{j-1/2}, \ j = 1, \cdots, n, \\ \dfrac{i\lambda}{2}(h_{j-1/2,n} p_{j-1/2} + h_{j+1/2,n} p_{j+1/2}) = q_{j+1/2} - q_{j-1/2}, \ j = 1, \cdots, n, \end{cases} \tag{2.6}$$

with boundary conditions

$$\frac{i\lambda h_{n+1/2}}{2} p_{n+1/2} + q_{n+1/2} = 0, \qquad \frac{i\lambda h_{1/2}}{2} p_{1/2} - q_{1/2} = 0.$$

Equations (2.6) rewrites, for $j \in \{1, \cdots, n\}$, as:

$$\begin{cases} \Big(\dfrac{i\lambda h_{j-1/2}}{2} q_{j-1/2} + p_{j-1/2}\Big) + \Big(\dfrac{i\lambda h_{j+1/2}}{2} q_{j+1/2} - p_{j+1/2}\Big) = 0, \\ \Big(\dfrac{i\lambda h_{j-1/2}}{2} p_{j-1/2} + q_{j-1/2}\Big) + \Big(\dfrac{i\lambda h_{j+1/2}}{2} p_{j+1/2} - q_{j+1/2}\Big) = 0, \end{cases} \tag{2.7}$$

For $j \in \{1, \cdots, n\}$, this leads to:

$$\begin{aligned} \Big(1 + \frac{i\lambda h_{j-1/2}}{2}\Big)(p_{j-1/2} + q_{j-1/2}) &= \Big(1 - \frac{i\lambda h_{j+1/2}}{2}\Big)(p_{j+1/2} + q_{j+1/2}) \\ \Big(1 - \frac{i\lambda h_{j-1/2}}{2}\Big)(p_{j-1/2} - q_{j-1/2}) &= \Big(1 + \frac{i\lambda h_{j+1/2}}{2}\Big)(p_{j+1/2} - q_{j+1/2}). \end{aligned}$$

These two equations can be seen as propagation formulas, each term corresponding to $\partial_t w \pm \partial_x w$. Especially, they imply:

$$p_{j+1/2} + q_{j+1/2} = (p_{1/2} + q_{1/2})\Big(\frac{2 + i\lambda h_{1/2}}{2 - i\lambda h_{j+1/2}}\Big) \prod_{k=1}^{j-1} \frac{2 + i\lambda h_{k+1/2}}{2 - i\lambda h_{k+1/2}}, \tag{2.8}$$

$$p_{j+1/2} - q_{j+1/2} = (p_{1/2} - q_{1/2})\Big(\frac{2 - i\lambda h_{1/2}}{2 + i\lambda h_{j+1/2}}\Big) \prod_{k=1}^{j-1} \frac{2 - i\lambda h_{k+1/2}}{2 + i\lambda h_{k+1/2}}. \tag{2.9}$$

We remark that each term in the product has modulus 1, and therefore there exists $\alpha_{j+1/2} \in (-\pi, \pi]$, given by $\tan(\alpha_{j+1/2}/2) = \lambda h_{j+1/2}/2$, such that :

$$\frac{2 + i\lambda h_{j+1/2}}{2 - i\lambda h_{j+1/2}} = \exp(i\alpha_{j+1/2}).$$

We also denote by β_j the coefficient

$$\beta_j = \frac{2 + i\lambda h_{1/2}}{2 - i\lambda h_{j+1/2}},$$

which satisfies

$$\frac{\beta_j}{\bar{\beta}_j} = \exp(i\alpha_{j+1/2}) \exp(i\alpha_{1/2}).$$

Combined with the boundary conditions, identities (2.8)-(2.9) give:

$$\begin{aligned} p_{n+1/2}\Big(1 - \frac{i\lambda h_{n+1/2}}{2}\Big) &= \beta_n \exp\Big(i\sum_{k=1}^{n-1} \alpha_{k+1/2}\Big) p_{1/2}\Big(1 + \frac{i\lambda h_{1/2}}{2}\Big) \\ p_{n+1/2}\Big(1 + \frac{i\lambda h_{n+1/2}}{2}\Big) &= \bar{\beta}_n \exp\Big(-i\sum_{k=1}^{n-1} \alpha_{k+1/2}\Big) p_{1/2}\Big(1 - \frac{i\lambda h_{1/2}}{2}\Big). \end{aligned}$$

Then, if λ is an eigenvalue, λ satisfies:

$$\Big(\frac{\beta_n}{\bar{\beta}_n}\Big)^2 \exp\Big(2i\sum_{k=1}^{n-1} \alpha_{k+1/2}\Big) = \exp\Big(2i\sum_{k=0}^{n} \alpha_{k+1/2}\Big) = 1.$$

To simplify notation, we define:

$$f(\lambda) = 4\sum_{k=0}^{n} \arctan\Big(\frac{\lambda h_{k+1/2}}{2}\Big).$$

Then, if λ is an eigenvalue, there exists an integer k such that:

$$f(\lambda) = 2k\pi.$$

The image of f is exactly $(-2(n+1)\pi, 2(n+1)\pi)$, and therefore k must belong to $\{-n, \cdots, n\}$.

Conversely, if λ is a solution of $f(\lambda) = 2k\pi$ for an integer $k \in \{-n, \cdots, n\}$, then λ is an eigenvalue, except if $k = 0$, which corresponds to $p_{j+1/2} = q_{j+1/2} = 0$ for all $j \in \{0, \cdots, n\}$. This gives us exactly $2n$ eigenvalues $\lambda^{\pm k}$, $k \in \{1, \cdots, n\}$. Since f is odd, we easily get $\lambda^k = -\lambda^{-k}$.

Moreover, the derivative of f is explicit:

$$f'(\lambda) = 8\sum_{k=0}^{n} \frac{1}{4 + (\lambda h_{k+1/2})^2} h_{k+1/2}.$$

It follows that

$$0 \le f'(\lambda) \le 2\sum_{k=0}^{n} h_{k+1/2,n} = 2.$$

Since all the eigenvalues are simple and $f(\lambda^{k+1}) - f(\lambda^k) = 2\pi$ for all $k \in \{1, \cdots, n\}$, this implies that the gap between the eigenvalues is bounded from below by π, and therefore (2.3) holds.

Using the concavity of arctan gives the following estimate:

$$\arctan\Big(\frac{\lambda_n^k}{2(n+1)}\Big) = \arctan\Big(\frac{1}{2(n+1)}\sum_{j=0}^{n}\lambda_n^k h_{j+1/2}\Big)$$
$$\geq \frac{1}{n+1}\sum_{j=0}^{n}\arctan\Big(\frac{\lambda_n^k h_{j+1/2}}{2}\Big) = \frac{k}{n+1}\frac{\pi}{2}.$$

In other words,

$$\lambda_n^k \geq 2(n+1)\tan\Big(\frac{k}{n+1}\frac{\pi}{2}\Big),$$

and (2.4) follows. □

We illustrate this result on Figures 1-2 by computing dispersion diagrams for various nonuniform meshes $\mathcal{S}_n$, that we characterize by their regularity $\mathrm{Reg}(\mathcal{S}_n)$, as defined in (1.9).

Let us briefly explain the two ways we have chosen for generating them.

- **Method** 1. In Figure 1, we create a random vector h of length $n+1$ whose values are chosen according to a uniform law on $(0,1)$. This vector is then normalized such that the sum of its components is one, so that h corresponds to the vector $(h_{1/2,n},\cdots,h_{n+1/2,n})$, which describes the mesh in a unique way.
- **Method** 2. In Figure 2, we create a random vector x of length n whose components are chosen according to a uniform law on $(0,1)$. Then we sort its components in an increasing way to obtain a vector $(x_{1,n},\cdots,x_{n,n})$, which represents the mesh points.

In both cases, the diagrams look the same. It is particularly striking that the shape of the dispersion diagrams does not seem to depend significantly on the meshes.

2.2 Spectral properties on M-regular meshes

This subsection is devoted to prove more properties for the spectrum of (1.7) when the mesh $\mathcal{S}_n$ is M-regular for some $M \geq 1$.

Theorem 2.3. *Let $M \geq 1$.*

Then, for any M-regular mesh $\mathcal{S}_n$, the eigenvalue λ_n^n of (2.1) on $\mathcal{S}^n$ satisfies

$$\lambda_n^n \leq \frac{4M}{\pi}(n+1)^2. \tag{2.10}$$

Besides, for any M-regular mesh $\mathcal{S}_n$, if ϕ_n^k denotes the eigenvector corresponding to λ_n^k in (2.1), then its energy

$$E_n^k = \frac{1}{2}\sum_{j=0}^{n} h_{j+1/2,n}\Big(\Big|\frac{\phi_{j+1,n}^k - \phi_{j,n}^k}{h_{j+1/2,n}}\Big|^2 + |\lambda_n^k|^2\Big|\frac{\phi_{j,n}^k + \phi_{j+1,n}^k}{2}\Big|^2\Big) \tag{2.11}$$

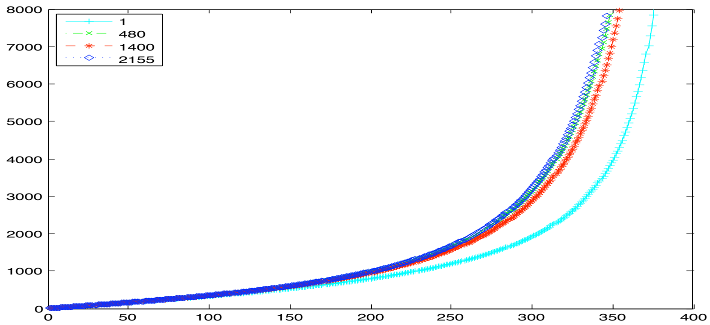

Figure 1: Dispersion diagrams for various meshes constituted by 400 points generated by Method 1 for different values of Reg.

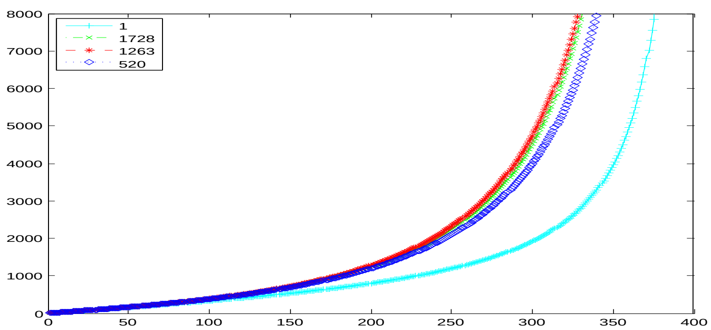

Figure 2: Dispersion diagrams for various meshes constituted by 400 points generated by Method 2 for different values of Reg.

satisfies

$$\frac{1}{1+M^2}\Big(\Big|\frac{\phi^k_{1,n}}{h_{1/2,n}}\Big|^2+\frac{h^2_{1/2,n}}{4}\Big|\frac{\lambda^k_n\phi^k_{1,n}}{h_{1/2,n}}\Big|^2\Big)\leq E^k_n$$
$$\leq(1+M^2)\Big(\Big|\frac{\phi^k_{1,n}}{h_{1/2,n}}\Big|^2+\frac{h^2_{1/2,n}}{4}\Big|\frac{\lambda^k_n\phi^k_{1,n}}{h_{1/2,n}}\Big|^2\Big), \quad (2.12)$$

Moreover, if $\omega=(a,b)$ is some subinterval of $(0,1)$, then the energy of the k-th eigenvector ϕ^k_n in ω, defined by

$$E^k_{\omega,n}=\frac{1}{2}\sum_{x_{j,n}\in\omega}h_{j+1/2,n}\Big(\Big|\frac{\phi^k_{j+1,n}-\phi^k_{j,n}}{h_{j+1/2,n}}\Big|^2+|\lambda^k_n|^2\Big|\frac{\phi^k_{j,n}+\phi^k_{j+1,n}}{2}\Big|^2\Big), \quad (2.13)$$

satisfies

$$E_n^k \leq \frac{M^2}{|\omega|} E_{J,n}^k. \tag{2.14}$$

Remark 2.4. These inequalities roughly says that the eigenvectors cannot concentrate in some part of an M-regular mesh. These properties are indeed the one needed for control and stabilization purposes, as we will see in next Sections.

Remark 2.5. Note that Theorem 2.1 gives the estimate

$$\lambda_n^n \geq 2(n+1)\tan\Big(\Big(1-\frac{1}{n+1}\Big)\frac{\pi}{2}\Big) \underset{n\to\infty}{\simeq} \frac{4}{\pi}(n+1)^2.$$

and therefore estimate (2.10) implies that λ_n^n really grows as n^2 when $n \to \infty$.

Proof. Along the proof, we fix an integer n, a real number $M \geq 1$ and an M-regular mesh $\mathcal{S}_n$, so that we can remove the index n without confusion.

Inequality (2.10) is a consequence of (2.2). Indeed, if we set $h = \min\{h_{j+1/2}\}$ and $H = \max\{h_{j+1/2}\}$, then we have

$$1 \leq (n+1)H \leq (n+1)Mh. \tag{2.15}$$

Using (2.2), we get

$$\sum_{j=0}^{n} \arctan\Big(\frac{\lambda^n h_{j+1/2}}{2}\Big) = \frac{n\pi}{2} \geq (n+1)\arctan\Big(\frac{\lambda^n h}{2}\Big),$$

which provides

$$\frac{\lambda^n}{(n+1)^2} \leq \frac{2}{h(n+1)^2}\tan\Big(\frac{\pi}{2}\Big(1-\frac{1}{n+1}\Big)\Big) \leq M \sup_{\eta\in[0,1]} \Big\{2\eta\tan\Big(\frac{\pi}{2}(1-\eta)\Big)\Big\},$$

from which (2.12) can be easily deduced.

To derive the properties (2.12) and (2.14) of the eigenvectors, we use the computations and notations (2.5) introduced in the proof of Theorem 2.1. Namely, we introduce

$$\begin{cases} q_{j+1/2}^k = \dfrac{i\lambda^k}{2}(\phi_j^k + \phi_{j+1}^k), \\ p_{j+1/2}^k = \dfrac{\phi_{j+1}^k - \phi_j^k}{h_{j+1/2}}. \end{cases}$$

Then the previous computations, and in particular identities (2.8)-(2.9), give

$$\begin{aligned}
E^k &= \frac{1}{2}\sum_{j=0}^{n} h_{j+1/2}\Big(|p^k_{j+1/2}|^2 + |q^k_{j+1/2}|^2\Big)\\
&= \frac{1}{4}\sum_{j=0}^{n} h_{j+1/2}\Big(|p^k_{j+1/2} - q^k_{j+1/2}|^2 + |p^k_{j+1/2} + q^k_{j+1/2}|^2\Big)\\
&= \frac{1}{4}\sum_{j=0}^{n} h_{j+1/2}\Big(|\bar{\beta}_j|^2|p^k_{1/2} - q^k_{1/2}|^2 + |\beta_j|^2|p^k_{1/2} + q^k_{1/2}|^2\Big)\\
&= \frac{1}{4}\sum_{j=0}^{n} h_{j+1/2}\frac{4+(\lambda h_{1/2})^2}{4+(\lambda h_{j+1/2})^2}\Big(|p^k_{1/2} - q^k_{1/2}|^2 + |p^k_{1/2} + q^k_{1/2}|^2\Big).
\end{aligned}$$

Using the definition (2.5) of $(p^k_{1/2}, q^k_{1/2})$, this leads to

$$E^k = \frac{1}{2}\left(\sum_{j=0}^{n}\frac{h_{j+1/2}}{4+(\lambda^k h_{j+1/2})^2}\right)\Big(4+(\lambda^k h_{1/2})^2\Big)\Big(\Big|\frac{\phi^k_1}{h_{1/2}}\Big|^2 + \frac{h^2_{1/2}}{4}\Big|\frac{\lambda^k\phi^k_1}{h_{1/2}}\Big|^2\Big). \tag{2.16}$$

Given an interval ω, the same computations give for E^k_ω :

$$E^k_\omega = \frac{1}{2}\left(\sum_{x_j\in\omega}\frac{h_{j+1/2}}{4+(\lambda^k h_{j+1/2})^2}\right)\Big(4+(\lambda^k h_{1/2})^2\Big)\Big(\Big|\frac{\phi^k_1}{h_{1/2}}\Big|^2 + \frac{h^2_{1/2}}{4}\Big|\frac{\lambda^k\phi^k_1}{h_{1/2}}\Big|^2\Big). \tag{2.17}$$

Inequalities (2.12) and (2.14) easily follow from (2.16)-(2.17) and the M-regularity assumption. □

2.3 Proof of Theorem 1.2

Our strategy is based on Ingham's Lemma on non-harmonic Fourier series, which we recall hereafter (see [13, 24]):

Lemma 2.6. *Ingham's Lemma [13] Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of real numbers and $\gamma > 0$ be such that*

$$\lambda_{k+1} - \lambda_k \geq \gamma > 0, \ \forall k \in \mathbb{N}. \tag{2.18}$$

For any $T > 2\pi/\gamma$, there exist two positive constants $c = c(T,\gamma) > 0$ and $C = C(T,\gamma) > 0$ such that, for any sequence $(a_k)_{k\in\mathbb{N}}$,

$$c\sum_{k\in\mathbb{N}}|a_k|^2 \leq \int_0^T \Big|\sum_{k\in\mathbb{N}} a_k e^{i\lambda_k t}\Big|^2\, dt \leq C\sum_{k\in\mathbb{N}}|a_k|^2. \tag{2.19}$$

Proof of Theorem 1.2. Let us consider a sequence $(\mathcal{S}_n)_n$ of M-regular meshes.

According to Lemma 2.6 and inequalities (2.3)-(2.4), we only need to prove the observability inequalities (1.11)-(1.12) for the stationnary solutions

$$u^k_n(t) = \exp(i\lambda^k_n t)\phi^k_n$$

of (1.7) corresponding to the eigenvectors ϕ_n^k of system (2.1) on $\mathcal{S}_n$.

Since each mesh $\mathcal{S}^n$ is M-regular, we can apply Theorem 2.3. Especially, inequality (2.12) holds, and therefore Ingham's inequality (2.19) implies (1.11).

To prove (1.12), we fix $J = (a, b) \subset (0, 1)$ a subinterval of $(0, 1)$. According to Ingham's Lemma and (2.3), it is sufficient to prove that there exists a constant C independent of n such that for any eigenvector ϕ_n^k solution of (2.1) on $\mathcal{S}^n$ corresponding to the eigenvalue λ_n^k, the quantity

$$I_{J,n}^k = \sum_{x_{j,n} \in J} h_{j+1/2,n} |\lambda_n^k|^2 \Big(\frac{\phi_{j,n}^k + \phi_{j+1,n}^k}{2} \Big)^2 \tag{2.20}$$

satisfies

$$E_n^k \leq C I_{J,n}^k. \tag{2.21}$$

We thus investigate inequality (2.21) on a mesh $\mathcal{S}_n$ by using a multiplier technique.

Let ω be a strict subinterval of J and let us denote by η a function of $x \in [0, 1]$ such that:

$$\begin{cases} \eta(x) = 0, \quad \forall x \in (0,1) \backslash J, \\ \eta(x) = 1, \quad \forall x \in \omega, \\ \|\eta\|_\infty \leq 1, \\ \|\eta'\|_\infty \leq C_{J,\omega}. \end{cases} \tag{2.22}$$

To simplify notation, we drop the exponent k and the index n hereafter. Below, we denote by η_j the value of η in the mesh point x_j.

We consider system (2.1) and multiply each equation by $\eta_j^2 \phi_j$. Discrete integration by parts leads:

$$\lambda^2 \sum_{j=0}^n h_{j+1/2} \Big(\frac{\phi_j + \phi_{j+1}}{2} \Big) \Big(\frac{\eta_j^2 \phi_j + \eta_{j+1}^2 \phi_{j+1}}{2} \Big) = \sum_{j=0}^n h_{j+1/2} \Big(\frac{\phi_{j+1} - \phi_j}{h_{j+1/2}} \Big) \Big(\frac{\eta_{j+1}^2 \phi_{j+1} - \eta_j^2 \phi_j}{h_{j+1/2}} \Big).$$

Then,

$$\lambda^2 \sum_{j=0}^n h_{j+1/2} \Big(\frac{\eta_j^2 + \eta_{j+1}^2}{2} \Big) \Big(\frac{\phi_j + \phi_{j+1}}{2} \Big)^2 - \sum_{j=0}^n h_{j+1/2} \Big(\frac{\eta_j^2 + \eta_{j+1}^2}{2} \Big) \Big(\frac{\phi_{j+1} - \phi_j}{h_{j+1/2}} \Big)^2 = A_1 + A_2, \tag{2.23}$$

where A_1 and A_2 are defined by

$$
\begin{aligned}
A_1 &= -\frac{\lambda^2}{2}\sum_{j=0}^{n} h_{j+1/2}^3\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)\Big(\frac{\phi_{j+1}-\phi_j}{h_{j+1/2}}\Big)\Big(\frac{\eta_{j+1}-\eta_j}{h_{j+1/2}}\Big)\Big(\frac{\eta_j+\eta_{j+1}}{2}\Big),\\
A_2 &= 2\sum_{j=0}^{n} h_{j+1/2}\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)\Big(\frac{\phi_{j+1}-\phi_j}{h_{j+1/2}}\Big)\Big(\frac{\eta_{j+1}-\eta_j}{h_{j+1/2}}\Big)\Big(\frac{\eta_j+\eta_{j+1}}{2}\Big).
\end{aligned}
$$

Then, for any choices of positive parameters δ_1 and δ_2, we get:

$$
|A_1| \le \frac{1}{4\delta_1}\sum_{j=0}^{n} h_{j+1/2}\lambda^2\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)^2\Big(\frac{\eta_{j+1}-\eta_j}{h_{j+1/2}}\Big)^2 + \frac{\delta_1}{4}\sum_{j=0}^{n} h_{j+1/2}(\lambda^2 h_{j+1/2}^4)\Big(\frac{\phi_{j+1}-\phi_j}{h_{j+1/2}}\Big)^2\Big(\frac{\eta_j+\eta_{j+1}}{2}\Big)^2
$$

$$
|A_2| \le \frac{1}{\delta_2}\sum_{j=0}^{n} h_{j+1/2}\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)^2\Big(\frac{\eta_{j+1}-\eta_j}{h_{j+1/2}}\Big)^2 + \delta_2\sum_{j=0}^{n} h_{j+1/2}\Big(\frac{\phi_{j+1}-\phi_j}{h_{j+1/2}}\Big)^2\Big(\frac{\eta_j+\eta_{j+1}}{2}\Big)^2.
$$

From estimates (2.10) and (2.15), we get

$$
\lambda^2 h_{j+1/2}^4 \le \Big(\frac{4M}{\pi}(n+1)^2\Big)^2\Big(\frac{M}{(n+1)}\Big)^4 \le \Big(\frac{4}{\pi}\Big)^2 M^4.
$$

Therefore, if we set

$$
\delta_1 = \frac{\pi^2}{16M^4} \quad ; \quad \delta_2 = \frac{1}{4},
$$

using the classical inequality

$$
\Big(\frac{\eta_j+\eta_{j+1}}{2}\Big)^2 \le \frac{\eta_j^2+\eta_{j+1}^2}{2}.
$$

we deduce from (2.23) the existence of two constants independent of k and n such that

$$
\begin{aligned}
\frac{1}{2}\sum_{j=0}^{n} h_{j+1/2}\Big(\frac{\eta_j^2+\eta_{j+1}^2}{2}\Big)\Big(\frac{\phi_{j+1}-\phi_j}{h_{j+1/2}}\Big)^2 &\le \lambda^2\sum_{j=0}^{n} h_{j+1/2}\Big(\frac{\eta_j^2+\eta_{j+1}^2}{2}\Big)\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)^2\\
&+ C_1\sum_{j=0}^{n} h_{j+1/2}\lambda^2\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)^2\Big(\frac{\eta_{j+1}-\eta_j}{h_{j+1/2}}\Big)^2\\
&+ C_2\sum_{j=0}^{n} h_{j+1/2}\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)^2\Big(\frac{\eta_{j+1}-\eta_j}{h_{j+1/2}}\Big)^2.
\end{aligned}
$$

But $|\lambda|$ is also uniformly bounded from below (see (2.4)), which implies that

$$\sum_{j=0}^{n} h_{j+1/2}\Big(\frac{\eta_j^2+\eta_{j+1}^2}{2}\Big)\Big(\frac{\phi_{j+1}-\phi_j}{h_{j+1/2}}\Big)^2 \leq \lambda^2 \sum_{j=0}^{n} h_{j+1/2}\Big(\frac{\eta_j^2+\eta_{j+1}^2}{2}\Big)\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)^2$$
$$+ C\sum_{j=0}^{n} h_{j+1/2}\lambda^2\Big(\frac{\phi_j+\phi_{j+1}}{2}\Big)^2\Big(\frac{\eta_{j+1}-\eta_j}{h_{j+1/2}}\Big)^2.$$

Using the properties (2.22) of the function η leads us to the following result:

$$E_{\omega,n}^k \leq C I_{J,n}^k.$$

Therefore inequality (2.21) can be deduced from inequality (2.14) applied to ω. □

2.4 The regularity assumption

Let us discuss the assumption on the regularity on the meshes.

2.4.1 Concentration effects without the M-regularity assumption

Here, we construct a sequence of meshes $\mathcal{S}_n$ such that:

- The sequence $\mathrm{Reg}(\mathcal{S}_n)$ goes to infinity arbitrarily slowly when $n \to \infty$.
- There exists an interval $J = [a, b]$ for which there is no constant C such that for all n, for all eigenvectors ϕ_n^k of (2.1) on $\mathcal{S}_n$,

$$E_n^k \leq C E_{J,n}^k, \tag{2.24}$$

  where E_n^k and $E_{J,n}^k$ are, respectively, as in (2.11) and (2.13).

Choose a strict non-empty closed subinterval J of $(0,1)$, and a sequence K_n going to infinity when $n \to \infty$.

Introduce a sequence of meshes $(\mathcal{S}_n)$, each one constituted by $n+2$ points such that

$$x_{0,n} = 0, \quad x_{n+1,n} = 1, \quad \begin{cases} x_{j+1,n} - x_{j,n} = H_n, & \text{if } [x_{j,n}, x_{j+1,n}] \subset J, \\ x_{j+1,n} - x_{j,n} = h_n, & \text{if } [x_{j,n}, x_{j+1,n}] \subset [0,1]\backslash J, \end{cases}$$

where $H_n = K_n h_n$. Remark that the mesh $\mathcal{S}_n$ is then totally described by the quantity K_n. Thus, from identities (2.16)-(2.17), we get:

$$\frac{E_n^k}{E_{J,n}^k} = 1 + \frac{E_{(0,1)\backslash J,n}^k}{E_{J,n}^k} = 1 + \frac{1-|J|}{|J|}\ \frac{4+(\lambda_n^k H_n)^2}{4+(\lambda_n^k h_n)^2}.$$

But

$$\frac{|J|}{H_n} + \frac{1-|J|}{h_n} = n+1,$$

and so $(n+1)h_n = (1-|J|)+|J|/K_n$ converges to $1-|J|$. But inequality (2.4) gives

$$\frac{\lambda_n^n h_n}{2} \geq (n+1)h_n \ \tan\Big(\frac{n}{n+1}\ \frac{\pi}{2}\Big),$$

and then $(\lambda_n^n h_n)_n$ goes to infinity when $n \to \infty$. Especially, this implies that

$$\frac{E_n^n}{E_{J,n}^n} \simeq \frac{1-|J|}{|J|}\ \frac{H_n^2}{h_n^2} = \frac{1-|J|}{|J|}\ K_n^2 \to \infty,$$

and therefore there is no constant such that (2.24) holds uniformly with respect to $n \in \mathbb{N}$ and $k \in \{1, \cdots, n\}$.

2.4.2 Partial regularity assumptions

Without the M-regularity assumption, one can derive partial results, due to the explicit form (2.16) of the energy.

For instance, identity (2.16) on the energy of the k-th eigenvector ϕ_n^k on $\mathcal{S}_n$ gives:

$$E_n^k \leq \frac{4+(\lambda_n^k h_{1/2,n})^2}{4+\inf\limits_j(\lambda_n^k h_{j+1/2,n})^2}\Big(\Big|\frac{\phi_{1,n}^k}{h_{1/2,n}}\Big|^2 + \frac{h_{1/2,n}^2}{4}\Big|\frac{\lambda_n^k\phi_{1,n}^k}{h_{1/2,n}}\Big|^2\Big).$$

In particular, if there exists a constant $M_1 > 0$ such that for all n,

$$h_{1/2,n} \leq M_1 \inf_j h_{j+1/2,n}, \tag{2.25}$$

then for all n and k,

$$E_n^k \leq (1+M_1^2)\Big(\Big|\frac{\phi_{1,n}^k}{h_{1/2,n}}\Big|^2 + \frac{h_{1/2,n}^2}{4}\Big|\frac{\lambda_n^k\phi_{1,n}^k}{h_{1/2,n}}\Big|^2\Big).$$

Now, consider the reverse equality. From (2.16), we get

$$E_n^k \geq \frac{4+(\lambda_n^k h_{1/2,n})^2}{4+\sup\limits_j(\lambda_n^k h_{j+1/2,n})^2}\Big(\Big|\frac{\phi_{1,n}^k}{h_{1/2,n}}\Big|^2 + \frac{h_{1/2,n}^2}{4}\Big|\frac{\lambda_n^k\phi_{1,n}^k}{h_{1/2,n}}\Big|^2\Big).$$

In particular, if there exists a constant $M_2 > 0$ such that for all n,

$$\sup_j h_{j+1/2,n} \leq M_2 h_{1/2,n}, \tag{2.26}$$

then, for all n and k, we get

$$E_n^k \geq \frac{1}{1+M_2^2}\Big(\Big|\frac{\phi_{1,n}^k}{h_{1/2,n}}\Big|^2 + \frac{h_{1/2,n}^2}{4}\Big|\frac{\lambda_n^k\phi_{1,n}^k}{h_{1/2,n}}\Big|^2\Big).$$

Besides, as in Subsubsection 2.4.1, for each integer n, we can consider sequences of meshes $\mathcal{S}_n n$ given as in (1.6) defined by

$$x_{1,n} - x_{0,n} = h_{1/2,n}, \qquad x_{j+1,n} - x_{j,n} = h_n, \quad \forall j \in \{1, \cdots, n\},$$

where $h_{1/2,n}$ and h_n are two sequences going to zero. It is then easy to check that if condition (2.26) is not satisfied, that is if $h_n/h_{1/2,n} \to \infty$ when $n \to \infty$, then there is no positive constant c such that

$$E_n^k \geq c\Big(\Big|\frac{\phi_{1,n}^k}{h_{1/2,n}}\Big|^2 + \frac{h_{1/2,n}^2}{4}\Big|\frac{\lambda_n^k \phi_{1,n}^k}{h_{1/2,n}}\Big|^2\Big)$$

uniformly in k and n.

On the contrary, if $h_n/h_{1/2,n} \to 0$ when $n \to \infty$, then there is no constant C such that

$$E_n^k \leq C\Big(\Big|\frac{\phi_{1,n}^k}{h_{1/2,n}}\Big|^2 + \frac{h_{1/2,n}^2}{4}\Big|\frac{\lambda_n^k \phi_{1,n}^k}{h_{1/2,n}}\Big|^2\Big)$$

uniformly in k and n.

Therefore, if we consider a sequence of meshes $\mathcal{S}_n$ such that $\mathrm{Reg}(\mathcal{S}_n)$ is unbounded, we cannot have both observability and admissibility properties as in (1.11) uniformly with respect to n.

Remark 2.7. If we are interested in the observability inequality (1.12) for a particular subinterval $(a,b) \subset (0,1)$, the situation is more intricate. As above, due to the explicit description of the energies (2.16) and (2.17), one easily check that if there exists a constant M_3 such that for all $n \in \mathbb{N}$,

$$\sup_{x_{j,n} \in (a,b)} \{h_{j+1/2,n}\} \leq M_3 \inf_{x_{j,n} \notin (a,b)} \{h_{j+1/2,n}\}, \tag{2.27}$$

then for all $n \in \mathbb{N}$ and for all $k \in \{1, \cdots, n\}$,

$$E_{(a,b),n}^k \leq \frac{M_3^2}{(b-a)} E_n^k.$$

However, under the only condition (2.27), the estimates (2.10) on the eigenvalues might be false, and therefore the proof presented above of inequality (2.21) (with $J = (a,b)$) fails. We do not know if assumption (2.27) suffices to guarantee (2.21) to hold uniformly with respect to $n \in \mathbb{N}$ and $k \in \{1, \cdots, n\}$.

Also remark that if assumption (2.27) holds for a sequence of meshes $\mathcal{S}^n$ for any subinterval $(a,b) \subset (0,1)$, then there exists a real number M such that all the meshes $\mathcal{S}^n$ are M-regular.

3 Application to the null controllability of the wave equation

3.1 The continuous setting

Let us first present the problem. It is well-known that for any time $T > 2$, given any initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$, we can find a control function $v(t) \in L^2(0,T)$ such that the solution of

$$\begin{cases} \partial_{tt}^2 y - \partial_{xx}^2 y = 0, & (x,t) \in (0,1) \times (0,T), \\ y(0,t) = v(t),\ y(1,t) = 0, & t \in (0,T), \\ y(x,0) = y^0(x),\ \partial_t y(x,0) = y^1(x), & x \in (0,1), \end{cases} \tag{3.1}$$

satisfies

$$y(T) = 0, \quad \partial_t y(T) = 0. \tag{3.2}$$

By duality (namely the Hilbert Uniqueness Method, or HUM in short), this property is equivalent to the observability inequality (1.4), see [15].

Note that there might be several controls $v \in L^2(0,T)$ such that (3.2) holds for solutions of (3.1). In the sequel, we will say that such a v is an admissible control for (3.1).

Besides, there is an explicit method to compute the so-called HUM control v_{HUM}, which is the one of minimal $L^2(0,T)$ norm among all admissible controls for (3.1). Indeed, consider the functional

$$\begin{aligned} &\mathcal{J} : H_0^1(0,1) \times L^2(0,1) \to \mathbb{R} \\ &\mathcal{J}(z^0, z^1) = \frac{1}{2}\int_0^T (\partial_x z)^2(0,t)\ dt - \int_0^1 y^0(x)\partial_t z(x,0)\ dx \\ &\qquad\qquad + < y^1, z(.,0) >_{H^{-1}\times H_0^1}, \end{aligned} \tag{3.3}$$

where z is the solution of the backward conservative wave equation

$$\begin{cases} \partial_{tt}^2 z - \partial_{xx}^2 z = 0, & (x,t) \in (0,1) \times (0,T), \\ z(0,t) = z(1,t) = 0, & t \in (0,T), \\ z(x,T) = z^0(x),\ \partial_t z(x,T) = z^1(x), & x \in (0,1). \end{cases} \tag{3.4}$$

Then $\mathcal{J}$ is strictly convex, coercive (see (1.4)), and therefore has a minimizer $(Z^0, Z^1) \in H_0^1(0,1) \times L^2(0,1)$. The HUM control is then given by $v_{HUM}(t) = \partial_x Z(0,t)$, where Z is the solution of (3.4) with initial data (Z^0, Z^1).

Note also that the HUM control is the only admissible control v for (3.1) that can be written as $v(t) = \partial_x z(0,t)$ for some z solution of (3.4) with initial data in $H_0^1(0,1) \times L^2(0,1)$.

It is then natural to try to construct this control numerically. This will be investigate in the sequel.

3.2 The semi-discrete setting

This part is inspired by [4] where similar results were derived for uniform meshes.

We consider a mesh $\mathcal{S}_n$ as in (1.6) and derive an approximation scheme for (3.1) from a mixed finite element method. The problem reads as follows: given y_n^0 and y_n^1 defined on $\mathcal{S}_n$, find a discrete control $v_n \in L^2(0,T)$ such that the solution y_n of

$$\begin{cases} \dfrac{h_{j-1/2,n}}{4}(y''_{j-1,n}+y''_{j,n})+\dfrac{h_{j+1/2,n}}{4}(y''_{j,n}+y''_{j+1,n}) \\ \qquad\qquad = \dfrac{y_{j+1,n}-y_{j,n}}{h_{j+1/2,n}}-\dfrac{y_{j,n}-y_{j-1,n}}{h_{j-1/2,n}}, \quad j=1,\cdots,n,\ t\in[0,T], \\ y_{0,n}(t)=v_n(t), \quad y_{n+1,n}(t)=0, \quad t\in(0,T), \\ y_{j,n}(0)=y^0_{j,n} \quad y'_{j,n}(0)=y^1_{j,n},\ j=1,\cdots,n, \end{cases} \tag{3.5}$$

satisfies

$$y_{j,n}(T)=0, \quad y'_{j,n}(T)=0, \quad j=1,\cdots,n. \tag{3.6}$$

Again, the study of this problem is based on a duality principle. Given any $T>2$, we choose $\epsilon>0$ such that $T-4\epsilon>2$ and a smooth function ρ satisfying

$$\begin{cases} \rho(t)=1, \quad \text{if } t\in[2\epsilon,T-2\epsilon], \\ \rho(t)=0, \quad \text{if } t\in[0,\epsilon]\cup[T-\epsilon,T], \\ 0\le\rho(t)\le 1, \quad \forall t. \end{cases} \tag{3.7}$$

We then introduce the following functional $\mathcal{J}_n$ as:

$$\begin{aligned} \mathcal{J}_n(z^0_n,z^1_n) &= \frac{1}{8}\int_0^T \rho(t)|z'_{1,n}|^2(t)\ dt+\frac{1}{2}\int_0^T \Big(\frac{z_{1,n}(t)}{h_{1/2,n}}\Big)^2\ dt \\ &+\left(\frac{h_{1/2,n}}{4}y^1_{1,n}z_{1,n}(0)+\sum_{j=1}^n\frac{h_{j+1/2,n}}{4}(y^1_{j,n}+y^1_{j+1,n})(z_{j,n}(0)+z_{j+1,n}(0))\right) \\ &-\left(\frac{h_{1/2,n}}{4}y^0_{1,n}z'_{1,n}(0)+\sum_{j=1}^n\frac{h_{j+1/2,n}}{4}(y^0_{j,n}+y^0_{j+1,n})(z'_{j,n}(0)+z'_{j+1,n}(0))\right), \end{aligned} \tag{3.8}$$

where z_n is the solution of

$$\begin{cases} \dfrac{h_{j-1/2,n}}{4}(z''_{j-1,n}+z''_{j,n})+\dfrac{h_{j+1/2,n}}{4}(z''_{j,n}+z''_{j+1,n}) \\ \qquad\qquad = \dfrac{z_{j+1,n}-z_{j,n}}{h_{j+1/2,n}}-\dfrac{z_{j,n}-z_{j-1,n}}{h_{j-1/2,n}},\ j=1,\cdots,n,\ t\in[0,T], \\ z_{0,n}(t)=z_{n+1,n}(t)=0, \quad t\in(0,T), \\ z_{j,n}(T)=z^0_{j,n}, \quad z'_{j,n}(T)=z^1_{j,n},\ j=1,\cdots,n. \end{cases} \tag{3.9}$$

Then the following Lemma holds:

Lemma 3.1. *For any integer n, the functional $\mathcal{J}_n$ is strictly convex and coercive, and then has a unique minimizer (Z^0_n,Z^1_n). Besides, for all n, if v_n is the solution of*

$$\begin{cases} -\dfrac{h_{1/2,n}}{4}v''_n+\dfrac{1}{h_{1/2,n}}v_n=-\dfrac{1}{4}(\rho Z'_{1,n})'+\dfrac{1}{h^2_{1/2,n}}Z_{1,n}, \quad t\in[0,T], \\ v'_n(0)=v'_n(T)=0, \end{cases} \tag{3.10}$$

where Z_n is the solution of (3.9) *with initial data* (Z_n^0, Z_n^1)*, then* $v_n(t)$ *is a control of* (3.5) *in time* T*.*

The proof of Lemma 3.1 is the same as in [4]. For completeness, we will give a sketch of the proof hereafter.

For convenience, we introduce the operators $\mathbb{P}_{\mathcal{S}_n}$, $\mathbb{Q}_{\mathcal{S}_n}$ and $\mathbb{R}_{\mathcal{S}_n}$ which map discrete data $a = (a_j)_{j\in\{1,\cdots,n\}}$ given on a mesh $\mathcal{S}_n$ as in (1.6) to functions defined on $(0,1)$ by :

$$\left.\begin{aligned}\mathbb{P}_{\mathcal{S}_n} a(x) &= a_j + (a_{j+1} - a_j)\Big(\frac{x - x_{j,n}}{h_{j+1/2,n}}\Big),\\ \mathbb{Q}_{\mathcal{S}_n} a(x) &= \frac{a_j + a_{j+1}}{2},\\ \mathbb{R}_{\mathcal{S}_n} a(x) &= \frac{h_{j+1/2,n}}{4}(a_j + a_{j+1}) + \sum_{k=j+1}^{n} h_{k+1/2,n}\Big(\frac{a_k + a_{k+1}}{2}\Big),\end{aligned}\right\} \text{ on } [x_{j,n}, x_{j+1,n}],$$

with the convention $a_0 = a_{n+1} = 0$. With these definitions, $\mathbb{P}_{\mathcal{S}_n}$ and $\mathbb{Q}_{\mathcal{S}_n}$ are extension operators and $\mathbb{R}_{\mathcal{S}_n}$ can be seen as an approximation of the discrete integral $x \mapsto \int_x^1 a(s)\, ds$.

Let us rewrite all discrete computations in terms of the operators $\mathbb{P}_{\mathcal{S}_n}, \mathbb{Q}_{\mathcal{S}_n}, \mathbb{R}_{\mathcal{S}_n}$. First, for any solution z_n of (3.9), the energy (1.8) writes

$$E_n(t) = \frac{1}{2}\left\|\mathbb{Q}_{\mathcal{S}_n} z_n(t)\right\|^2_{L^2(0,1)} + \frac{1}{2}\left\|\partial_x(\mathbb{P}_{\mathcal{S}_n} z_n(t))\right\|^2_{L^2(0,1)}. \tag{3.11}$$

Second, the functional $\mathcal{J}_n$ reads as

$$\begin{aligned}\mathcal{J}_n(z_n^0, z_n^1) = \frac{1}{8}\int_0^T \rho(t)|z'_{1,n}|^2(t)\, dt + \frac{1}{2}\int_0^T \Big(\frac{z_{1,n}(t)}{h_{1/2,n}}\Big)^2\, dt \\ + \int_0^1 (\mathbb{R}_{\mathcal{S}_n} y_n^1)(\partial_x \mathbb{P}_{\mathcal{S}_n} z_n(0))\, dx - \int_0^1 (\mathbb{Q}_{\mathcal{S}_n} y_n^0)(\mathbb{Q}_{\mathcal{S}_n} z'_n(0))\, dx.\end{aligned} \tag{3.12}$$

We are now in position to sketch the proof of Lemma 3.1.

Sketch of the proof of Lemma 3.1. Fix an integer $n \in \mathbb{N}$. The functional $\mathcal{J}_n$ is strictly convex, and its coercivity is obvious since we are working in a finite dimensional setting. It follows that $\mathcal{J}_n$ has a unique minimizer (Z_n^0, Z_n^1).

Let us compute the Fréchet derivative of $\mathcal{J}_n$ in the minimizer (Z_n^0, Z_n^1): For any (z_n^0, z_n^1), the solution z_n of (3.9) on $\mathcal{S}^n$ satisfies

$$\begin{aligned}0 = \int_0^T \Big(-\frac{1}{4}(\rho(t)Z'_{1,n}(t))' + \frac{1}{h^2_{1/2,n}} Z_{1,n}(t)\Big) z_{1,n}(t)\, dt \\ + \int_0^1 (\mathbb{R}_{\mathcal{S}_n} y_n^1)(\partial_x \mathbb{P}_{\mathcal{S}_n} z_n(0))\, dx - \int_0^1 (\mathbb{Q}_{\mathcal{S}_n} y_n^0)(\mathbb{Q}_{\mathcal{S}_n} z'_n(0))\, dx,\end{aligned}$$

which rewrites in terms of v_n defined in (3.10)ăas

$$0 = \frac{1}{4}\int_0^T h_{1/2,n} v_n' z_{1,n}' \, dt + \int_0^T v_n \frac{z_{1,n}}{h_{1/2,n}} \, dt + \int_0^1 (\mathbb{R}_{\mathcal{S}_n} y_n^1)(\partial_x \mathbb{P}_{\mathcal{S}_n} z_n(0)) \, dx - \int_0^1 (\mathbb{Q}_{\mathcal{S}_n} y_n^0)(\mathbb{Q}_{\mathcal{S}_n} z_n'(0)) \, dx. \quad (3.13)$$

Now, consider y_n the solution of (3.5) with boundary control v_n. Multiplying (3.5) by z_n solution of (3.9) with initial data (z_n^0, z_n^1), we get, after tedious computations that are left to the reader, that

$$0 = \frac{1}{4}\int_0^T h_{1/2,n} v_n' z_{1,n}' \, dt + \int_0^T v_n \frac{z_{1,n}}{h_{1/2,n}} \, dt + \int_0^1 (\mathbb{R}_{\mathcal{S}_n} y_n^1)(\partial_x \mathbb{P}_{\mathcal{S}_n} z_n(0)) \, dx - \int_0^1 (\mathbb{Q}_{\mathcal{S}_n} y_n^0)(\mathbb{Q}_{\mathcal{S}_n} z_n'(0)) \, dx - \int_0^1 (\mathbb{R}_{\mathcal{S}_n} y_n'(T))(\partial_x \mathbb{P}_{\mathcal{S}_n} z_n^0) \, dx + \int_0^1 (\mathbb{Q}_{\mathcal{S}_n} y_n(T))(\mathbb{Q}_{\mathcal{S}_n} z_n^1) \, dx. \quad (3.14)$$

Combined with (3.13), this yields that the solution y_n of (3.5) satisfies the following property: For any (z_n^0, z_n^1),

$$-\int_0^1 (\mathbb{R}_{\mathcal{S}_n} y_n'(T))(\partial_x \mathbb{P}_{\mathcal{S}_n} z_n^0) \, dx + \int_0^1 (\mathbb{Q}_{\mathcal{S}_n} y_n(T))(\mathbb{Q}_{\mathcal{S}_n} z_n^1) \, dx = 0.$$

This obviously implies (3.6). □

It is natural to ask if the discrete controls v_n constructed in Lemma 3.1 converge to an admissible control for (3.1) under some assumptions on the convergence of (y_n^0, y_n^1). We will prove that this is indeed the case.

Given a sequence of meshes $(\mathcal{S}_n)_n$, we say that the sequence of discrete data $(a_n, b_n)_n$ defined on the meshes $\mathcal{S}_n$ strongly converges to (a, b) in $L^2(0,1) \times H^{-1}(0,1)$ if:

$$\begin{aligned} &\mathbb{Q}_{\mathcal{S}_n} a_n \to a \quad \text{in } L^2(0,1), \\ &\mathbb{R}_{\mathcal{S}_n} b_n \to (x \mapsto \textstyle\int_x^1 b(s) \, ds) \quad \text{in } L^2(0,1). \end{aligned} \quad (3.15)$$

Theorem 3.2. *Let $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ and $T > 2$.*

Given $M \geq 1$, we consider a sequence $(\mathcal{S}_n)$ of M-regular meshes, and a sequence of initial data (y_n^0, y_n^1) which strongly converges to (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$ in the sense of (3.15).

Then the sequence of discrete controls $(v_n)_n$ given by Lemma 3.1 strongly converges in $L^2(0,T)$ to the HUM control v_{HUM} for (3.1) *with initial data (y^0, y^1).*

First of all, let us mention that given $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$, it is possible to find a sequence of initial data (y_n^0, y_n^1) which strongly converges to (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$ in the sense of (3.15).

The proof of Theorem 3.2 is mainly based on inequality (1.11), that implies that the discrete controls v_n are bounded in $L^2(0,T)$. Once this is proved, the result can be deduced from classical convergence properties of the scheme.

Proof. The proof is divided into several steps. First, we prove uniform bounds on the sequence v_n. Second, we prove that any weak limit of v_n is a control of (3.1). Third, we prove that there is only one weak limit, which coincides with the HUM-control v_{HUM} of (3.1). We finally prove the strong convergence of the controls v_n in $L^2(0,T)$.

Uniform bounds. Since $\mathcal{J}_n(Z_n^0, Z_n^1) \leq \mathcal{J}_n(0,0) = 0$, we have that

$$\frac{1}{8}\int_0^T \rho(t)|Z'_{1,n}|^2(t)\ dt + \frac{1}{2}\int_0^T \Big(\frac{Z_{1,n}(t)}{h_{1/2,n}}\Big)^2\ dt \\ \leq \sqrt{2E_*^n(0)}\sqrt{\|\mathbb{R}_{\mathcal{S}_n} y_n^1\|^2_{L^2(0,1)} + \|\mathbb{Q}_{\mathcal{S}_n} y_n^0\|^2_{L^2(0,1)}},$$

where $E_*^n(t)$ denotes the energy of $Z_n(t)$, which is constant. In view of the definition of ρ, since we assume that the meshes $\mathcal{S}_n$ are M-regular, inequality (1.11) holds. This, combined with the fact that $(\mathbb{Q}_{\mathcal{S}_n} y_n^0)$ and $(\mathbb{R}_{\mathcal{S}_n} y_n^1)$ are convergent in $L^2(0,1)$ and therefore bounded, leads us to

$$k_T E_*^n(T) \leq \frac{1}{8}\int_0^T \rho(t)|Z'_{1,n}|^2(t)\ dt + \frac{1}{2}\int_0^T \Big(\frac{Z_{1,n}(t)}{h_{1/2,n}}\Big)^2\ dt \leq C. \tag{3.16}$$

Besides, multiplying (3.10) by $h_{1/2,n} v_n$ and integrating in time gives

$$\int_0^T \frac{h_{1/2,n}^2}{4}|v_n'(t)|^2 + |v_n(t)|^2\ dt = \int_0^T \Big(\frac{h_{1/2,n}}{4}\rho(t) Z'_{1,n}(t) v_n'(t) + \frac{Z_{1,n}(t)}{h_{1/2,n}} v_n(t)\Big)\ dt \\ \leq \Big(\int_0^T \frac{h_{1/2,n}^2}{4}|v_n'(t)|^2 + |v_n(t)|^2\ dt\Big)^{1/2} \\ \Big(\int_0^T \frac{\rho(t)}{8}|Z'_{1,n}|^2(t)\ dt + \frac{1}{2}\int_0^T \Big(\frac{Z_{1,n}(t)}{h_{1/2,n}}\Big)^2\ dt\Big)^{1/2}, \tag{3.17}$$

and therefore we obtain

$$\int_0^T \frac{h_{1/2,n}^2}{4}|v_n'(t)|^2 + |v_n(t)|^2\ dt \leq C. \tag{3.18}$$

We have thus proved, using the M-regularity assumption, that the sequence of discrete controls v_n is bounded in $L^2(0,T)$, and therefore there exists a function $v \in L^2(0,T)$ such that

$$v_n \rightharpoonup v, \quad \text{in } L^2(0,T)\ \textit{weak}, \quad h_{1/2,n} v_n' \rightharpoonup 0, \quad \text{in } L^2(0,T)\ \textit{weak}. \tag{3.19}$$

The second statement in (3.19) comes from the continuity of the derivation in the sense of distributions.

The function v is an admissible control for (3.1). We need the following classical Lemma on the convergence of the numerical schemes (which can be found for instance in [5]):

Lemma 3.3. *Consider two smooth functions (u^0, u^1) on $(0,1)$ such that $u^0(0) = u^0(1) = 0$ and $u(x,t)$ the solution of the conservative system (1.1) with initial data (u^0, u^1).*

Given a sequence $(\mathcal{S}_n)_n$ of M-regular meshes, for all $n \in \mathbb{N}$, we denote by $u_{j,n}(t)$ the solution of the conservative semi-discrete scheme (1.7) with initial data

$$u^0_{j,n} = u^0(x_{j,n}), \quad u^1_{j,n} = u^1(x_{j,n})), \quad j \in \{1, \cdots, n\}.$$

Then $(\mathbb{P}_{\mathcal{S}_n} u_{j,n}, \mathbb{Q}_{\mathcal{S}_n} u'_{j,n})$ strongly converges to (u, u') in $C([0,T]; H^1_0(0,1) \times L^2(0,1))$ and

$$\frac{u_{1,n}(t)}{h_{1/2,n}} \to \partial_x u(0,t) \text{ in } L^2(0,T), \qquad u'_{1,n}(t) \to 0 \text{ in } L^2(0,T). \tag{3.20}$$

This result is of course still true for the backward system (3.4) and its semi-discrete approximations (3.9).

Now, if we consider two smooth functions (z^0, z^1) as in Lemma 3.3 for the backward wave equation (3.4) and its semi-discrete approximations (3.9), using (3.19), we can pass to the limit in (3.13) and obtain that the solution z of (3.4) satisfies:

$$0 = \int_0^T v(t) \partial_x z(0,t)\, dt + < y^1, z(.,0) >_{H^{-1}(0,1) \times H^1_0(0,1)} \\ - \int_0^1 y^0(x) \partial_t z(x,0)\, dx. \tag{3.21}$$

By a density argument, this equation can be extended to any $(z^0, z^1) \in H^1_0(0,1) \times L^2(0,T)$.

Besides, for any $(z^0, z^1) \in H^1_0(0,1) \times L^2(0,1)$, as in (3.14), multiplying the solution of (3.1) with boundary condition $y(0,t) = v(t)$ and initial data (y^0, y^1) by z solution of (3.4) with initial data (z_0, z_1), we obtain that

$$0 = \int_0^T v(t) \partial_x z(0,t)\, dt + + < y^1, z(.,0) >_{H^{-1}(0,1) \times H^1_0(0,1)} \\ - \int_0^1 y^0(x) \partial_t z(x,0)\, dx - < \partial_t y(T), z_0 >_{H^{-1}(0,1) \times H^1_0(0,1)} + \int_0^1 y(T,x) z_1(x)\, dx.$$

Hence we deduce from (3.21) that

$$< \partial_t y(T), z_0 >_{H^{-1}(0,1) \times H^1_0(0,1)} - \int_0^1 y(T,x) z_1(x)\, dx = 0.$$

Therefore y satisfies (3.2). This precisely means that v is an admissible control for (3.1).

The limit v is the HUM control v_{HUM}. It is sufficient to prove that $v(t)$ coincides with some $\partial_x z(t,0)$, where z is the solution of (3.4) for some initial data $(z^0, z^1) \in H_0^1(0,1) \times L^2(0,1)$, see for instance [15].

From (3.16), there exist two functions $Z^0 \in H_0^1(0,1)$ and $Z^1 \in L^2(0,1)$ such that

$$\mathbb{P}_{\mathcal{S}_n} Z_n^0 \rightharpoonup Z^0,\ H_0^1(0,1)\ weak, \quad \mathbb{Q}_{\mathcal{S}_n} Z_n^1 \rightharpoonup Z^1,\ L^2(0,1)\ weak.$$

Using the weak formulations of (3.9) and the conservation of the energy, we can prove (the proof can be adapted in a standard way from the arguments in [5]) that, for all $t \in [0,T]$,

$$\begin{aligned} &\forall t,\ (\mathbb{P}_{\mathcal{S}_n} Z_n(t), \mathbb{Q}_{\mathcal{S}_n} Z_n(t)) \rightharpoonup (Z(t), Z'(t)) \text{ in } H_0^1(0,1) \times L^2(0,1)\ weak, \\ &(\mathbb{P}_{\mathcal{S}_n} Z_n, \mathbb{Q}_{\mathcal{S}_n} Z_n) \rightharpoonup (Z, Z') \text{ in } L^\infty(0,T; H_0^1(0,1) \times L^2(0,1))\ *weak, \end{aligned} \tag{3.22}$$

where Z is the solution of (3.4) with initial data (Z^0, Z^1). Besides, one easily shows that

$$\frac{Z_{1,n}}{h_{1/2,n}} - \frac{h_{1/2,n}}{4} Z_{1,n}'' \rightharpoonup \partial_x Z(0,t), \quad \text{in} \quad \mathcal{D}'(0,T). \tag{3.23}$$

But $Z_{1,n}/h_{1/2,n}$ is bounded in $L^2(0,T)$ from (3.16), and therefore $h_{1/2,n} Z_{1,n}'' \rightharpoonup 0$ in $\mathcal{D}'(0,T)$. This also gives that

$$\left\{ \begin{array}{l} \dfrac{Z_{1,n}}{h_{1/2,n}} \rightharpoonup \partial_x Z, \\ Z_{1,n} \rightharpoonup 0, \\ h_{1/2,n} (\rho Z_{1,n}')' \rightharpoonup 0, \end{array} \right. \quad \text{in } \mathcal{D}'(0,T). \tag{3.24}$$

Combined with the definition of v_n in Lemma 3.1, it follows that

$$-\frac{h_{1/2,n}^2}{4} v_n'' + v_n \rightharpoonup \partial_x Z(0,t), \text{ in } \mathcal{D}'(0,T).$$

But, since v_n is bounded in $L^2(0,T)$ by (3.18),

$$h_{1/2,n}^2 v_n'' \rightharpoonup 0 \text{ in } \mathcal{D}'(0,T),$$

and therefore $v(t) = \partial_x Z(0,t)$ in $\mathcal{D}'(0,T)$.

Since we have already proved that v is an admissible control for (1.1), this proves that v is the HUM control v_{HUM}.

Strong convergence Since the weak convergence is already proven, it is sufficient to prove the convergence of the $L^2(0,T)$ norms.
Since $v(t) = \partial_x Z(0,t)$ where Z is the solution of (3.4) with initial data (Z^0, Z^1),

we get from (3.21) that :

$$0=\int_0^T (\partial_x Z(0,t))^2\ dt- < y^1, Z(.,0) >_{H^{-1}(0,1)\times H_0^1(0,1)} \\ - \int_0^1 y^0(x)\partial_t Z(x,0)\ dx. \quad (3.25)$$

But (3.13) gives:

$$0=\frac{1}{4}\int_0^T \rho(t)|Z_{1,n}'(t)|^2\ dt+\int_0^T \Big|\frac{Z_{1,n}(t)}{h_{1/2,n}^2}\Big|^2\ dt \\ +\int_0^1 (\mathbb{R}_{\mathcal{S}_n} y_n^1)(x)\partial_x(\mathbb{P}_{\mathcal{S}_n} Z_n)(x,0)\ dx-\int_0^1 (\mathbb{Q}_{\mathcal{S}_n} y_n^0)(x)(\mathbb{Q}_{\mathcal{S}_n} z_{*,n}')(x,0)\ dx.$$

Convergences (3.22) and (3.15) imply that we can pass to the limit in the linear term, and therefore, by (3.25), we get:

$$\frac{1}{4}\int_0^T \rho(t)|Z_{1,n}'(t)|^2\ dt+\int_0^T \Big|\frac{Z_{1,n}(t)}{h_{1/2,n}^2}\Big|^2\ dt \to \int_0^T |\partial_x Z(0,t)|^2\ dt.$$

Combined with the weak convergences (3.24), this proves the following strong convergences:

$$\left\{\begin{array}{l} \sqrt{\rho}\, Z_{1,n}' \to 0, \\ \dfrac{Z_{1,n}}{h_{1/2,n}}(t) \to \partial_x Z(0,t), \end{array}\right. \qquad \text{in } L^2(0,T).$$

But, from the definition (3.10) of v_n, the convergence (3.19) implies that:

$$\int_0^T \frac{h_{1/2,n}^2}{4}|v_n'(t)|^2+|v_n(t)|^2\ dt=\int_0^T \frac{h_{1/2,n}}{4}\rho(t)Z_{1,n}'(t)v_n'(t)+\frac{Z_{1,n}(t)}{h_{1/2,n}}v_n(t)\ dt \\ \to \int_0^T \partial_x Z(0,t)v(t)\ dt=\int_0^T v(t)^2\ dt.$$

Hence we deduce from (3.19) that:

$$\left\{\begin{array}{l} h_{1/2,n}v_n' \to 0 \\ v_n \to v=v_{HUM} \end{array}\right. \qquad \text{in } L^2(0,T),$$

which concludes the proof of Theorem 3.2. □

Remark 3.4. The proof of Theorem 3.2 slightly differs from the one in [4], which presented an approach based on the spectral decomposition of the solutions. This technique, in our context, seems more technically involved than the one presented above, since the spectrum is not as explicit as in the case of a uniform mesh.

4 Application to the damped wave equation

4.1 The continuous setting

We consider the continuous damped wave equation on the interval $(0,1)$:

$$\begin{cases} \partial_{tt}^2 w - \partial_{xx}^2 w + 2\sigma\partial_t w = 0, & (x,t) \in (0,1)\times(0,\infty), \\ w(0,t) = w(1,t) = 0, & t \in (0,\infty), \\ w(x,0) = w^0(x), \quad \partial_t w(x,0) = w^1(x), & x \in (0,1), \end{cases} \tag{4.1}$$

with $w^0 \in H_0^1(0,1)$ and $w^1 \in L^2(0,1)$.

We assume that the damping function $\sigma = \sigma(x)$ is bounded, non-negative and bounded from below by a positive number on a subinterval J, that is there exists $\alpha > 0$, such that

$$\sigma(x) \geq \alpha, \quad \forall x \in J, \qquad \|\sigma\|_\infty = K. \tag{4.2}$$

Then the energy, defined by (1.2), satisfies the dissipation law

$$\frac{dE}{dt}(t) = -2\int_0^1 \sigma(x)|\partial_t w(t,x)|^2 \, dx. \tag{4.3}$$

It is well-known that, under the assumption (4.2), the energy is exponentially decaying: There exist positive constants C and μ such that

$$E(t) \leq C\, E(0)\exp(-\mu t), \quad t \in \mathbb{R}. \tag{4.4}$$

Using classical arguments in stabilization theory (see [11]), the energy of (4.1) is exponentially decaying if and only if the observability inequality (1.5) holds for solutions of the conservative system (1.1).

4.2 The semi-discrete setting

We consider a mesh $\mathcal{S}_n$ as in (1.6), and discretize equation (4.1) according to the mixed finite element method:

$$\begin{cases} \dfrac{h_{j-1/2,n}}{4}(w''_{j-1,n} + w''_{j,n}) + \dfrac{h_{j+1/2,n}}{4}(w''_{j,n} + w''_{j+1,n}) = \\ \qquad -\dfrac{h_{j-1/2,n}\sigma_{j-1/2,n}}{2}(w'_{j-1,n} + w'_{j,n}) - \dfrac{h_{j+1/2,n}\sigma_{j+1/2,n}}{2}(w'_{j,n} + w'_{j+1,n}) \\ \qquad\qquad + \dfrac{w_{j+1,n} - w_{j,n}}{h_{j+1/2,n}} - \dfrac{w_{j,n} - w_{j-1,n}}{h_{j-1/2,n}}, \quad j = 1,\cdots,n, \; t \in [0,\infty), \\ w_0(t) = w_{n+1}(t) = 0, \quad t \in [0,\infty), \\ w_j(0) = w^0_{j,n}, \quad w'_j(0) = w^1_{j,n}, \quad j = 1,\cdots,n, \end{cases} \tag{4.5}$$

where $\sigma_{j+1/2,n}$ is an approximation on $[x_{j,n}, x_{j+1,n}]$ of the damping function σ in (4.1) which is bounded, non-negative and satisfies

$$\sigma_{j+1/2,n} \geq \alpha, \quad \forall [x_{j,n}, x_{j+1,n}] \subset J, \qquad \sigma_{j+1/2,n} \leq K, \; j = 0,\cdots,n, \tag{4.6}$$

where α and K are as in (4.2).

The energy (1.8) of solutions of (4.5) satisfies

$$\frac{dE-n}{dt}(t) = -2\sum_{j=0}^{n} h_{j+1/2,n}\sigma_{j+1/2,n}\left(\frac{w'_{j,n}(t)+w'_{j+1,n}(t)}{2}\right)^2. \tag{4.7}$$

Obviously, this dissipation law corresponds to a discrete version of (4.3).

The question we investigate is the following: Given a sequence $(\mathcal{S}_n)_n$ of meshes, can we find positive constants C and μ independent of n such that

$$E_n(t) \le C\, E_n(0)\exp(-\mu t), \quad t \in (0,\infty), \tag{4.8}$$

for any solution of (4.5) on $\mathcal{S}_n$?

Similarly as in the continuous setting, this property is equivalent to the uniform observability inequality (1.12) for solutions of the conservative system (1.7) (see for instance [22]). Therefore Theorem 1.2 leads to the following result:

Theorem 4.1. *Let $M \ge 1$, and consider a sequence $(\mathcal{S}_n)_n$ of M-regular meshes.*

Then there exist positive constants C and μ such that for all n, inequality (4.8) *holds for any solution of* (4.5) *on $\mathcal{S}_n$.*

The proof of Theorem 4.1, which can be adapted in a standard way from [11] or [22], is left to the reader.

Remark 4.2. Note that this method yields an estimate on the decay rate μ appearing in (4.8), which is far from being optimal in general. This is a drawback of the method, which is based on a perturbation argument of the conservative system. Even in the continuous setting, the decay rate parameter obtained through this method is not in general the sharp one, which is known to coincide (at least in the one dimensional case) with the spectral abscissa (see [6]).

5 Further comments

In this paper, we have given a space semi-discrete scheme derived from a mixed finite element method for a 1d wave equation, which has a good behavior with respect to both stabilization and controllability properties on a large class of nonuniform meshes.

1. The key point of our analysis is the description of the spectrum of the discrete operator given in Theorem 2.1 on any mesh in a surprisingly explicit formulation. This description does not seem available for other classical schemes as the ones provided by finite difference or finite element methods. To our knowledge, in these cases, only asymptotic distributions of the eigenvalues are available, see for instance [3] and the literature therein.

2. It would be interesting to estimate the (asymptotic) decay rate for the semi-discrete damped equation as in [6]. To our knowledge, this is still an open problem even in the case of uniform meshes. Some partial results in this

direction are given in [8] in the context of the Perfectly Matched Layer (PML in short) equations (see [2]), which is a kind of damped wave equation.

3. There exist different methods to prove observability inequalities: Another one might be given via multiplier techniques. However, it seems difficult to find a good multiplier on a nonuniform mesh.

4. It would be particularly challenging to understand the behavior of the discrete waves in higher dimension on nonuniform meshes. To our knowledge, this question has not been addressed so far. We expect this question to be difficult to address with the tools used until now, which require either a good knowledge on the eigenvalues (see [12, 19, 17, 20, 18, 25] and our own approach) or the existence of multipliers that behave well (see [22, 21, 8]) on the discrete systems.

5. Let us mention the recent work [7], which studied observability properties for time-discrete approximation schemes of linear conservative systems in a very general abstract setting. The approach developed in [7] allows to derive uniform observability inequalities for time-discrete approximation schemes in a systematic way. One of the interesting features of this technique is that it can be applied to fully discrete schemes as soon as the space semi-discrete approximation schemes satisfy uniform observability properties (see [7, Section 5]). Note that the study presented here fits in this abstract setting. Therefore, combining Theorem 1.2 and the results in [7], one can derive uniform (with respect to the time and space discretization parameters) observability properties for time-discrete approximation schemes of the space semi-discrete approximation scheme (1.7).

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