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Manuscrit

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par

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Contribution en contrôlabilité et problèmes inverses pour quelques équations aux dérivées partielles

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Je saisis enfin cette occasion pour remercier mes amis pour leur patience et leur soutien sans faille durant toutes ces années. Je ne les cite pas ici, mais je sais qu'ils sauront se reconnaître. Merci pour tout, votre amitié m'est très précieuse. À ceux qui, de près ou de loin, ont été à mes cotés pendant ces quelques années, qui ont supporté mes doutes et m'ont encouragé à persévérer, à tout ceux là, je n'ai qu'un mot à dire : merci.

Présentation générale du manuscrit

Ce mémoire d'habilitation rend compte d'une large partie des recherches que j'ai effectuées après ma thèse. J'ai organisé la lecture de mes travaux en quatre parties correspondant à des thématiques distinctes et pouvant être lues indépendamment. Chacune présente un point de vue sur un sujet que j'ai exploré à travers plusieurs travaux, incluant des commentaires et des questions ouvertes mentionnés au fil de la lecture.

Dans une première partie, je présenterai les travaux [EZ10, EZ11c, EZ13] concernant la contrôlabilité des systèmes conservatifs et l'approximation numérique des contrôles. Je m'intéresserai dans un premier temps à un résultat de régularité de l'opérateur de contrôle donné par la méthode d'unicité de Hilbert développée par J.-L. Lions [Lio88a], dans l'esprit du travail récent de B. Dehman et G. Lebeau [DL09]. Il s'agit en particulier de montrer que le contrôle de norme L^2 minimale (éventuellement modifié convenablement) prend en compte la régularité des données à contrôler, et construit automatiquement une trajectoire contrôlée avec la même régularité que les données. Je montrerai ensuite comment ce résultat permet de construire des approximations numériques des contrôles avec des ordres de convergence explicites. Je me concentrerai notamment sur deux cas, selon que l'on se base uniquement sur l'observabilité des modèles continus ou que l'on suppose en plus l'observabilité des systèmes discrétisés, uniformément par rapport au(x) paramètre(s) de discrétisation.

Dans une deuxième partie, je m'intéresserai à un problème inverse pour les ondes, étudié dans [BDBE13, BE13, BEO]. La question posée est de retrouver un potentiel dans une équation des ondes à partir de la connaissance du flux de la solution sur le bord ou une partie de bord. Bien qu'il s'agisse d'une question reliée à l'observabilité de l'équation des ondes, le problème est non-linéaire en le potentiel, et cela amène un certain nombre de difficultés. Dans un premier temps, je rappellerai comment on peut établir la stabilité de ce problème inverse via des inégalités de Carleman pour les ondes en suivant l'approche de [IY01]. J'expliquerai que l'on peut alors déduire des inégalités de Carleman et du mécanisme de la preuve de la stabilité un algorithme convergeant vers le potentiel à retrouver, sans nécessairement en être proche initialement. Par la suite, je m'intéresserai à la convergence des problèmes inverses correspondants pour l'équation des ondes semi-discrète vers le problème inverse pour l'équation des ondes continue. Pour cela, j'établirai en particulier des inégalités de Carleman pour l'équation des ondes semi-discrétisée en espace.

Dans une troisième partie, je mentionnerai plusieurs transformations intégrales, étudiées dans [EZ11a, EZ11b, EZ], et leur intérêt pour les problèmes de contrôlabilité. En particulier, à l'inverse de la transformation proposée par L. Miller dans [Mil04], je présenterai une transformation permettant d'associer aux solutions de l'équations de la chaleur une solution de l'équation des ondes. J'en déduirai des résultats nouveaux sur l'espace atteignable pour l'équation de la chaleur sous des conditions de contrôle géométrique garantissant l'observabilité de l'équation des ondes. J'appliquerai également une idée similaire pour obtenir des résultats d'observabilité pour les systèmes conservatifs discrétisés en temps. Je présenterai notamment deux transformations intégrales permettant de relier les solutions des équations discrétisées en temps aux solutions de l'équation continue. Je montrerai ainsi un résultat d'observabilité uniforme par rapport au pas de discrétisation pour les approximations semi-discrètes en temps de systèmes conservatifs, avec un temps optimal, contrairement au précédent résultat obtenu dans [EZZ08].

Dans une quatrième et dernière partie, je me concentrerai sur la contrôlabilité des fluides visqueux non-homogènes dans les cas compressibles [EGGP12] et incompressibles [BEG]. La difficulté de ces modèles vient du couplage entre l'équation de transport satisfaite par la densité du fluide et l'équation parabolique satisfaite par la vitesse du fluide. Il s'agit alors de développer un outil capable de gérer simultanément les propriétés des deux équations. Pour cela, nous développerons dans un premier temps une inégalité de Carleman parabolique proche de celle présentée par A. Fursikov et O. Imanuvilov [FI96], mais avec un poids dépendant des variables d'espace et de temps. Cela permettra en particulier de considérer des poids constants le long des caractéristiques de la trajectoire cible. Nous pourrons ainsi obtenir des espaces à poids appropriés pour étudier la contrôlabilité des équations de Navier-Stokes avec densité non-homogène. Dans le cas des fluides compressibles, il faudra également veiller à une bonne compréhension du couplage de la vitesse et de la densité, ce qui nous amènera à introduire la vitesse effective [BD03, BDL03] et le flux visqueux effectif [Lio98]. Dans le cas des fluides incompressibles, nous devrons prendre soin d'obtenir une estimée d'observabilité suffisamment précise pour l'équation de Stokes pour permettre de prendre en compte le couplage avec la densité.

Certaines des questions que j'ai développées depuis ma thèse, un peu plus marginales dans mon travail de recherche, ne seront pas abordées dans ce mémoire, ou bien seulement indirectement, afin de maintenir une certaine homogénéité dans mon propos. Ainsi, les travaux [DE] et [EdG11], respectivement en collaboration avec B. Dehman et F. de Gournay, ne seront mentionnés et discutés que brièvement dans les Sections 5.3.1 et 6.5.3. D'autres travaux ne seront pas évoqués dans le texte, en particulier :

- Le travail [CEG09] en collaboration avec J.-M. Coron et O. Glass prouvant des propriétés d'observabilité uniforme de l'équation des ondes discrétisée à l'aide du schéma d'approximation de Glimm ;
- Le travail [EV10] en collaboration avec J. Valein prouvant des propriétés d'observabilité uniforme pour des équations paraboliques discrétisées en temps ;
- Le travail [EV14] en collaboration avec M. Vanninathan, qui étudie les propriétés de contrôlabilité d'un modèle simplifié d'interaction fluide-structure où le fluide est modélisé par une équation des ondes et la structure par un oscillateur ;
- Le travail [EHL14] en collaboration avec C. Lacave et M. Hillairet sur le comportement en temps long d'une boule rigide dans une fluide visqueux incompressible en dimension 2, en particulier dans les espaces L^p.

Mes articles sont disponibles sur internet à l'adresse
http ://www.math.univ-toulouse.fr/∼ervedoza/publis.html.

General presentation

In this dissertation, I will account for a large part of the research I did after my PhD. This report is composed of four parts which correspond to distinct thematics and can be read independently. Each one presents a point of view on a topic I explored through several works, and includes comments and open questions mentioned along the text.

In Part I, I will present the works [EZ10, EZ11c, EZ13] on the controllability of conservative systems and the numerical approximation of controls. First, I focus on a regularity result of the control operator given by the Hilbert uniqueness method developed by J.-L. Lions in [Lio88a], in the spirit of the recent work [DL09] by B. Dehman and G. Lebeau. In particular, we show that the standard control minimizing the L^2 norm (or a suitably modified version of it) takes into account the regularity of the data to be controlled and automatically builds controlled trajectories with the same regularity as the data. I will then explain how this result allows us to design numerical approximations of the controls with explicit rates of convergence. I will focus on two cases, depending on whether we only assume the observability of the continuous models or if we also assume that the discrete systems are uniformly observable with respect to the discretization parameter(s).

In Part II, I will focus on an inverse problem for the wave equation, studied in [BDBE13, BE13, BEO]. The question is to find a potential in a wave equation from the knowledge of the flux of the solution on the boundary of the domain, or a part of it. Although it is a matter related to the observability of the wave equation, the problem is nonlinear in the potential, and this brings a number of challenges. At first, I recall how one can establish the stability of the inverse problem via Carleman estimates for waves following the approach of [IY01]. I will then explain that from Carleman estimates and from the mechanism of the proof of the stability, we can actually deduce an algorithm converging to the potential to be estimated, without necessarily starting from an initial guess close to it. Subsequently, I will focus on the convergence of inverse problems for the space semi-discrete wave equation to the inverse problem for the continuous wave equation. This will require developing Carleman estimates for the space semi-discrete wave equation.

In Part III, I will mention several integral transforms, studied in [EZ11a, EZ11b, EZ], and their interest in controllability theory. In particular, unlike the integral transform proposed by L. Miller [Mil04], I will present a transform associating a solution of the wave equation to any solution of the heat equation. That way, we deduce new results on the reachable set for the heat equation under geometric conditions ensuring the observability of the wave equation. We will also apply a similar idea to get observability results for time semi-discrete approximations of conservative systems. We will in particular present two specific integral transforms to link the solutions of the time-discrete approximate equations to the solutions of the continuous equation. I thus show a result of uniform observability with respect to the time discretization parameter for time semi-discrete approximations of conservative systems, with an optimal time, in contrast to the previous results obtained in [EZZ08].

In Part IV, I will focus on the controllability of viscous non-homogeneous fluids in the compressible [EGGP12] and incompressible [BEG] cases. In these models, the difficulty comes from the coupling between the transport equation satisfied by the fluid density and the parabolic equation satisfied by the fluid velocity. It is then needed to develop a tool to handle the properties of the two equations simultaneously. We therefore start by developing a parabolic Carleman inequality similar to the one produced by A. Fursikov and O. Imanuvilov [FI96], but with a weight function depending on the space and time variables. This will in particular allow us to choose weight functions constant along the characteristics of the target velocity. That way, we can construct suitable weighted spaces to study the controllability of the Navier-Stokes equations with non-homogeneous density. In the case of compressible fluids, we must also ensure a good understanding of the coupling of the velocity and of the density, which leads us to introduce the effective velocity [BD03, BDL03] and the effective viscous flux [Lio98]. In the case of incompressible fluids, we must be cautious and obtain a sufficiently precise observability estimate for the Stokes equation for handling the coupling with the density.

Some of the questions I have developed since my PhD will not be addressed in this text, or only indirectly, to maintain some homogeneity in the discussion. This is the case of the works [DE] and [EdG11], done with B. Dehman and F. de Gournay respectively, which will be mentioned and discussed only briefly in Section 5.3.1 and 6.5.3, respectively. Some other works will not be referred to in the text, in particular:

- The work [CEG09] with J.-M. Coron and O. Glass proving uniform observability properties for the wave equation discretized using Glimm's scheme;
- The work [EV10] with J. Valein proving uniform observability properties for time-discrete parabolic equations;
- The work [EV14] with M. Vanninathan studying the controllability of a simplified model of fluid-structure interaction in which the fluid is modeled by a wave equation and the structure by an oscillator;
- The work [EHL14] with C. Lacave and M. Hillairet on the large time behavior of a rigid ball in a 2d viscous incompressible fluid, in particular in L^p spaces.

My articles are available on the web at
http://www.math.univ-toulouse.fr/∼ervedoza/publis.html.

List of presented publications

Here is the list of the publications presented in this document.

[BDBE13] L. Baudouin, M. De Buhan, and S. Ervedoza. Global Carleman estimates for waves and applications. *Comm. Partial Differential Equations*, 38(5):823–859, 2013.

[BE13] L. Baudouin and S. Ervedoza. Convergence of an inverse problem for a 1-D discrete wave equation. *SIAM J. Control Optim.*, 51(1):556–598, 2013.

[BEG] M. Badra, S. Ervedoza, and S. Guerrero. Local controllability to trajectories for non-homogeneous 2-d incompressible Navier-Stokes equations. *Preprint, in revision*, 2014.

[BEO] L. Baudouin, S. Ervedoza, and A. Osses. Stability of an inverse problem for the discrete wave equation and convergence results. *J. Math. Pures Appl., to appear.*

[EGGP12] S. Ervedoza, O. Glass, S. Guerrero, and J.-P. Puel. Local exact controllability for the one-dimensional compressible Navier-Stokes equation. *Arch. Ration. Mech. Anal.*, 206(1):189–238, 2012.

[EZ] S. Ervedoza and E. Zuazua. Transmutation techniques and observability for time-discrete approximation schemes of conservative systems. *Numer. Math., to appear.*

[EZ10] S. Ervedoza and E. Zuazua. A systematic method for building smooth controls for smooth data. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1375–1401, 2010.

[EZ11a] S. Ervedoza and E. Zuazua. Observability of heat processes by transmutation without geometric restrictions. *Math. Control Relat. Fields*, 1(2):177–187, 2011.

[EZ11b] S. Ervedoza and E. Zuazua. Sharp observability estimates for heat equations. *Arch. Ration. Mech. Anal.*, 202(3):975–1017, 2011.

[EZ11c] S. Ervedoza and E. Zuazua. The wave equation: Control and numerics. In P. M. Cannarsa and J. M. Coron, editors, *Control of Partial Differential Equations*, Lecture Notes in Mathematics, CIME Subseries. Springer Verlag, 2011.

[EZ13] S. Ervedoza and E. Zuazua. *Numerical approximation of exact controls for waves.* Springer Briefs in Mathematics. Springer, New York, 2013.

Contents

Part I

Computation of controls for conservative models

Chapter 1

Introduction

This part concerns the controllability of conservative models, which encompass some classical models as for instance the wave and Schrödinger equations.

In the following, we consider the following abstract control problem given by

$$y' = Ay + Bv, \quad t \geq 0, \qquad y(0) = y_0. \tag{1.1}$$

In (1.1), A is a skew-adjoint operator defined on some Hilbert space X, with dense domain $\mathscr{D}(A)$ and with compact resolvent, $y(t)$ is the state of the system at time t, the prime $'$ denotes differentiation with respect to time, B is the control operator, assumed to belong to $\mathscr{L}(U, \mathscr{D}(A)')$ where U is another Hilbert space, and v is the control function, that we look for in the space $L^2(0,T;U)$, where $T > 0$ is some finite time horizon.

Under this setting, the operator A drives the dynamics of the state y. The control operator B describes the way one can act on the system, and we choose the control function v in order to steer the system from some initial state to some desired state, for instance a stationary state.

In the sequel, we assume that system (1.1) is well-posed for control functions $v \in L^2(0,T;U)$, in the sense that, given any $y_0 \in X$ and $v \in L^2(0,T;U)$, there exists a unique solution y of (1.1) in the class $C([0,T];X)$.

We will focus on the question of exact controllability for system (1.1) at time $T > 0$. System (1.1) is said to be exactly controllable if, given any $y_0 \in X$ and $y_1 \in X$, there exists a control $v \in L^2(0,T;U)$ such that the solution y of (1.1) satisfies

$$y(T) = y_1. \tag{1.2}$$

Thanks to the property $A^* = -A$, system (1.1) is time reversible. As it is also linear, one easily checks that the exact controllability property for system (1.1) at time $T > 0$ is equivalent to the following null-controllability property at time $T > 0$: given any $y_0 \in X$, there exists a control $v \in L^2(0,T;U)$ such that the solution y of (1.1) satisfies

$$y(T) = 0. \tag{1.3}$$

Following the Hilbert Uniqueness Method [Lio88b, Lio88a], using duality, the controllability of (1.1) is equivalent to the observability problem for the adjoint equation:

$$z' = Az, \quad t \geq 0, \qquad z(0) = z_0. \tag{1.4}$$

Here, we used the assumption $A^* = -A$, and this explains why the adjoint equation looks the same as (1.1) (In general, one should rather consider the dynamics $-z' = A^* z$ with $z(T) = z_T$ as initial data, the problem then being posed backward in time.)

Namely, system (1.4) is said to be observable through B^* in time $T > 0$ if there exists a constant C_{obs} such that for all $z_0 \in \mathscr{D}(A)$, the solution z of (1.4) satisfies

$$\|z_0\|_X \leq C_{obs} \|B^* z\|_{L^2(0,T;U)} . \tag{1.5}$$

In (1.5), we assumed $z_0 \in \mathscr{D}(A)$ so that the solution z of (1.4) belongs to the space $C([0,T]; \mathscr{D}(A))$ and the right hand-side of (1.5) makes sense. But it is convenient to extend this inequality to any trajectory z solution of (1.4) with initial data $z_0 \in X$. This is equivalent to the hidden regularity or admissibility property for B^*, namely the existence, for all $T > 0$, of a constant C_T such that any solution z of (1.4) with initial data $z_0 \in \mathscr{D}(A)$ satisfies

$$\|B^* z\|_{L^2(0,T;U)} \leq C_T \|z_0\|_X . \tag{1.6}$$

Under this admissibility property, both estimates (1.5) and (1.6), if satisfied when $z_0 \in \mathscr{D}(A)$, can be extended to any trajectory z solution of (1.4) with initial data $z_0 \in X$. Actually, the hidden-regularity condition (1.6) for B^* is equivalent to the well-posedness in $C([0,T]; X)$ of system (1.1) with controls in $L^2(0,T;U)$, see [TW09, Chapter 4]. Of course, when B^* is bounded from X to U, property (1.6) is obvious.

In the following, we will always assume that system (1.4) observed through B^* satisfies the admissibility condition (1.6) and the observability condition (1.5) for some time $T > 0$. Under these two conditions, system (1.1) is controllable in time T, see [Lio88a]. Besides, [Lio88a] exhibits a constructive manner to derive the control.

Namely, given $y_0 \in X$, the control function v can be computed as follows. Let us introduce the functional[1]

$$J(z_0) = \frac{1}{2} \int_0^T \|B^* z(t)\|_U^2 \, dt + \langle z_0, y_0 \rangle_X, \tag{1.7}$$

defined for $z_0 \in X$, where z is the solution of (1.4) with initial condition z_0. From condition (1.6), this functional is well-defined for $z_0 \in X$. From condition (1.5), the quantity $\|B^* z\|_{L^2(0,T;U)}$ defines a norm on X, and the functional J is therefore coercive and strictly convex. Following, J admits a unique minimizer $Z_0 \in X$ corresponding to a trajectory Z of (1.4). Setting

$$v = B^* Z, \quad t \geq 0, \tag{1.8}$$

the function v is a suitable control function for (1.1).

Another way to understand this control process is to introduce the Gramian operator defined by the bilinear form

$$\langle \Lambda_T z_{0,a}, z_{0,b} \rangle_X = \int_0^T \langle B^* z_a(t), B^* z_b(t) \rangle_U \, dt, \tag{1.9}$$

for all $(z_{0,a}, z_{0,b}) \in X^2$, where z_a, z_b denote the corresponding solutions of (1.4) with initial data $z_{0,a}$, $z_{0,b}$ respectively. Using this representation, one immediately sees that the functional J in (1.7) simply is

$$J(z_0) = \frac{1}{2} \langle \Lambda_T z_0, z_0 \rangle_X + \langle z_0, y_0 \rangle_X. \tag{1.10}$$

As Λ_T is obviously symmetric due to its definition, the minimizer Z_0 is simply given by the equation

$$\Lambda_T Z_0 = -y_0, \tag{1.11}$$

[1] For sake a simplicity, we restrict ourselves from now to the case of real Hilbert spaces X and U, but our strategy can be applied with minor changes to complex Hilbert spaces as well.

It thus provides a way to rewrite the minimization problem above as the computation of the inverse of a matrix, which exists thanks to (1.5).

The goal of the next sections is to present two results.

First, in Chapter 2, we focus on the following question. If y_0 is smoother, for instance in $\mathscr{D}(A)$, does Z_0 belong to $\mathscr{D}(A)$? More generally, the question is to check if the following above construction maintains the regularity of the initial data to be controlled, and this can be seen at different levels: on Z_0 defined by (1.11), on the control function v given by (1.8), and on the corresponding controlled trajectory y solution of (1.1).

It turns out that this question has a negative answer in the case of unbounded control operators, see Section 2.1.2, and we shall therefore design an alternate technique to compute the controls. Our technique will simply consist in introducing a smooth weight in time within the Gramian operator vanishing at $t = 0$ and $t = T$. This harmless modification will avoid the creation of singularities in the control process.

Before going further, let us emphasize that the question here is to design one process which gives a positive answer to the above question for all levels of regularity higher than X. In particular, we do not want to design a process adapted to the degree of smoothness of the initial data.

Second, in Chapter 3, we shall explain how our understanding of this question helps in designing numerical methods for the computation of the controls. We shall develop these results in two directions, namely in the case in which the discrete versions of the equation (1.4) are observable uniformly with respect to the discretization parameters, and in the case in which this result is not available - and possibly wrong. In both cases, it will be of primary importance to consider a control process which maintains the degree of regularity of the initial data.

Chapter 2

Control process and smoothness issues

2.1 Study of the classical control process

Our first aim is to discuss the classical case and to recall some previous results.

2.1.1 State of the art

We focus on the case of the wave equation set in a smooth bounded domain $\Omega \subset \mathbb{R}^d$ and controlled from an open subset $\omega \subset \Omega$:

$$\begin{cases} \partial_{tt} y - \Delta y = v\chi, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y_0, y_1), & x \in \Omega. \end{cases} \tag{2.1}$$

Here, χ denotes a (possibly smooth) indicator function of the set ω (i.e. $\omega = \{\chi > 0\}$). The natural choice of functional setting for this problem is to choose control functions v in $L^2((0,T) \times \omega)$ and initial data in $H_0^1(\Omega) \times L^2(\Omega)$.

In this setting, it is well-known that system (2.1) is controllable in time $T > 0$ if and only if (ω, Ω, T) satisfies the celebrated Geometric Control Condition (GCC) of C. Bardos, G. Lebeau and J. Rauch [BLR88, BLR92]. In short, (ω, Ω, T) satisfies the GCC if and only if all geodesic rays - or rays of geometric optics - traveling at velocity 1 in the domain Ω and bouncing on the boundary $\partial\Omega$ according to Descartes Snell's laws meet ω in time less than T.

In the following, for $s \geq 0$, we introduce the functional spaces $H^s_{(0)}(\Omega) = \mathscr{D}((-\Delta_D)^{s/2})$, where $-\Delta_D$ is the Dirichlet Laplace operator on Ω defined on $L^2(\Omega)$ and with domain $\mathscr{D}(-\Delta_D) = H^2 \cap H_0^1(\Omega)$. Note that these spaces include in their definition some boundary conditions compatible with the ones required by equation (2.1).

We should first mention the works [BLR88, BLR92] which prove that, under the GCC, if $(y_0, y_1) \in H^{s+1}_{(0)}(\Omega) \times H^s_{(0)}(\Omega)$ for some $s \geq 0$, then there exists a control function v such that the controlled trajectory y satisfies $y \in C^0([0,T]; H^{s+1}_{(0)}(\Omega)) \cap C^1([0,T]; H^s_{(0)}(\Omega))$. Note however that the controls constructed in [BLR88, BLR92] depend on the degree of regularity s.

Later on in [DL09], it was proved that if χ has the form

$$\chi(t,x) = \eta(t)\chi_0(x), \qquad (t,x) \in (0,T) \times \Omega, \tag{2.2}$$

where $\eta = \eta(t)$ is a nonnegative smooth cut-off function of time flat at $t = 0$ and $t = T$ with $\{\eta > 0\} = (0, T)$, and $\chi_0 = \chi_0(x)$ is a nonnegative smooth function of space such that $(\{\chi_0 > 0\}, \Omega, T)$ satisfies the GCC, then the control of minimal L^2-norm preserves the regularity of the initial data.

To be more precise, here, the control is computed as follows. Introduce the functional J^χ defined by

$$J^\chi(z_0, z_1) = \frac{1}{2}\int_0^T \int_\omega \chi^2(t,x)|z(t,x)|^2\,dt\,dx + \langle (z_0, z_1), (y_0, y_1)\rangle_{L^2\times H^{-1}, H_0^1\times L^2}, \tag{2.3}$$

over the set $(z_0, z_1) \in L^2(\Omega)\times H^{-1}(\Omega)$, where z denotes the solution of the adjoint wave equation

$$\begin{cases} \partial_{tt} z - \Delta z = 0, & (t,x) \in (0,T)\times\Omega, \\ z(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega, \\ (z(0,\cdot), \partial_t z(0,\cdot)) = (z_0, z_1), & x\in\Omega, \end{cases} \tag{2.4}$$

and the duality product $\langle (z_0, z_1), (y_0, y_1)\rangle_{L^2\times H^{-1}, H_0^1\times L^2}$ in (2.3) means

$$\langle (z_0, z_1), (y_0, y_1)\rangle_{L^2\times H^{-1}, H_0^1\times L^2} = \int_\Omega z_0 y_1 - \int_\Omega \nabla((-\Delta_D)^{-1} z_1)\nabla y_0. \tag{2.5}$$

Under the GCC, the functional J^χ has a unique minimizer $(Z_0, Z_1) \in L^2(\Omega)\times H^{-1}(\Omega)$. If we denote by Z the corresponding solution of (2.4), the function

$$v(t,x) = \chi(t,x)Z(t,x), \qquad (t,x)\in(0,T)\times\Omega, \tag{2.6}$$

is a control function for (2.1), and it is the one of minimal $L^2((0,T)\times\omega)$-norm.

Note that here, though the operator driving the dynamics of y is skew-adjoint on $H_0^1(\Omega)\times L^2(\Omega)$, as $L^2(\Omega)$ is identified with its dual as it is done usually in the PDE context, the adjoint equation (2.4) is set on $L^2(\Omega)\times H^{-1}(\Omega)$, thus explaining the shift in the functional space and the duality product in (2.5). In this manuscript, we shall not give more details on it, which can be found in [EZ13] for instance.

Besides, as in the abstract case (recall (1.9)), $(Z_0, Z_1) \in L^2(\Omega)\times H^{-1}(\Omega)$ can be characterized through the Gramian operator $\Lambda_T^\chi : L^2(\Omega)\times H^{-1}(\Omega) \to H_0^1(\Omega)\times L^2(\Omega)$ defined by

$$\langle \Lambda_T^\chi(z_{0,a}, z_{1,a}), (z_{0,b}, z_{1,b})\rangle_{L^2\times H^{-1}} = \int_0^T\int_\omega \chi^2(t,x) z_a(t,x) z_b(t,x)\,dt\,dx, \tag{2.7}$$

where z_a and z_b correspond to the solutions of (2.4) with initial data $(z_{0,a}, z_{1,a})$, $(z_{0,b}, z_{1,b})$ respectively. Indeed, the minimizer $(Z_0, Z_1)\in L^2(\Omega)\times H^{-1}(\Omega)$ of the functional J^χ in (2.3) can be characterized by

$$\Lambda_T^\chi(Z_0, Z_1) = -(y_0, y_1). \tag{2.8}$$

The result in [DL09] then reads as follows: under the GCC for $(\{\chi_0 > 0\}, \Omega, T)$, for all $s \geq 0$, the Gramian operator Λ_T^χ is an isomorphism from $H^s_{(0)}(\Omega)\times H^{s-1}_{(0)}(\Omega)$ into $H^{s+1}_{(0)}(\Omega)\times H^s_{(0)}(\Omega)$. Obviously, this result also implies that the controlled trajectory y of (2.1) stays smooth.

Besides, if the equation is stated in a smooth compact manifold without boundary, the Gramian operator and its inverse actually are zero-order elliptic pseudo-differential operators. Some numerical experiments were done in [LN10] to support this idea in several geometric configurations. In particular, [LN10] conjectures that the inverse of the Gramian operator is a micro-local operator even in the case of a bounded domain, provided that there is no ray of Geometric Optics having infinite order of contact with the boundary.

Unfortunately, the work [DL09] does not apply to the case of boundary control, and this will be the main improvement of our analysis hereafter.

The result in [Ima02] can also be viewed as a particular instance of this regularity result with $s = 1$. Indeed, [Ima02] states a Carleman estimate with a source term in H^{-1}. Such estimate is performed by first deriving a Carleman estimate when the source term is in $L^2(L^2)$, and then using duality to obtain a control result for a source term in L^2. But direct duality only yields that the control function belongs to H^{-1}. The article [Ima02] thus provides a proof that the control function actually belongs to L^2, hence that the source term is in L^2, so that the controlled trajectory belongs to H^1. By duality, this yields a Carleman estimate in H^{-1}.

Let us also mention that some previous results were considering the case in which the abstract system (1.1) is exactly controllable in some time $T > 0$, and described the reachable set in the case of smoother controls, see in particular [Mil05, Lemma 4.2], [TW09, Theorem 11.3.6] and the more recent work [TW14]. But these works construct appropriate controls depending on the functional setting under consideration.

Our goal rather is to adapt the results in [DL09] to the case of unbounded control operators. In particular, we shall study if the control minimizing the L^2-norm (or some weighted version of it) will preserve the regularity of the initial data to be controlled.

2.1.2 Boundary control in the 1d setting

To better understand our study, let us start with the following simple setting of the 1d wave equation set on $(0,1)$ controlled from one side, say at $x = 1$:

$$\begin{cases} \partial_{tt}y - \partial_{xx}y = 0, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = 0, & t \in (0,T), \\ y(t,1) = v(t), & t \in (0,T), \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y_0, y_1), & x \in (0,1), \end{cases} \tag{2.9}$$

If we assume $v \in L^2(0,T)$, the corresponding functional setting corresponds to $(y_0, y_1) \in L^2(0,1) \times H^{-1}(0,1)$. Indeed, for boundary conditions in $L^2(0,T)$ ($L^2((0,T) \times \partial\Omega)$ in higher dimensions), the wave equation (2.9) is well-posed in $C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$ as a consequence of the hidden regularity for the adjoint equation observed from the boundary, see [LT83, Lio88a].

The classical Hilbert Uniqueness Method then proposes to minimize the functional

$$J(z_0, z_1) = \frac{1}{2} \int_0^T |\partial_x z(t,1)|^2 \, dt + \langle (z_0, z_1), (y_0, y_1) \rangle_{H_0^1 \times L^2, L^2 \times H^{-1}}, \tag{2.10}$$

over the set $(z_0, z_1) \in H_0^1(0,1) \times L^2(0,1)$, where z denotes the solution of the adjoint wave equation

$$\begin{cases} \partial_{tt}z - \partial_{xx}z = 0, & (t,x) \in (0,T) \times (0,1), \\ z(t,0) = z(t,1) = 0, & t \in (0,T), \\ (z(0,\cdot), \partial_t z(0,\cdot)) = (z_0, z_1) & x \in (0,1). \end{cases} \tag{2.11}$$

As the 1d wave equation can be solved explicitly using Fourier series and is 2-periodic, it is easily seen that the functional J in (2.10) can be completely decoupled mode by mode if the time T is even. In particular, for $T = 4$, explicit computations show that, if (y_0, y_1) is given by

$$(y_0, y_1) = \sum_{k=1}^{\infty} (\hat{y}_{0,k}, \hat{y}_{1,k}) \sin(k\pi x),$$

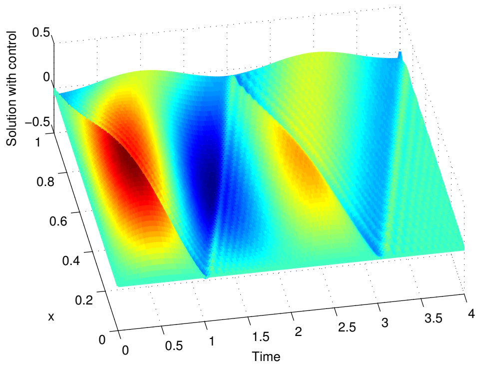

Figure 2.1: The controlled trajectory for the wave equation with initial data $(y_0(x), y_1(x)) = (0, \sin(\pi x))$ for the HUM control in time $T = 4$. A kick is introduced by the control function at $(t, x) = (0, 1)$ and travels in the domain, hence making the solution non-smooth.

then the minimizer (Z_0, Z_1) of J in (2.10) is given by

$$(Z_0, Z_1) = \sum_{k=1}^{\infty} \left(\frac{\hat{y}_{1,k}}{4k^2\pi^2}, -\frac{\hat{y}_{0,k}}{4} \right) \sin(k\pi x).$$

The control function v of minimal $L^2(0,T)$-norm then writes:

$$v(t) = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^k \left(\frac{\hat{y}_{1,k}}{k\pi} \cos(k\pi t) - \hat{y}_{0,k} \sin(k\pi t) \right).$$

We thus immediately conclude that the inverse of the Gramian operator in that case is an isomorphism from $H^s_{(0)}(0,1) \times H^{s-1}_{(0)}(0,1)$ to $H^{s+1}_{(0)}(0,1) \times H^s_{(0)}(0,1)$ for all $s \geq 0$. *But* the corresponding controlled trajectory may not be smooth. Indeed, if the initial data to be controlled is $(y_0, y_1) = (0, \sin(\pi x))$, the control function is given by $v(t) = -\cos(\pi t)/(4\pi)$, and the compatibility condition $y_0(1) = v(0)$ is violated. The control process therefore induces a kick in the controlled trajectory, localized on the characteristic emanating from $t = 0$, $x = 1$, as illustrated in Figure 2.1.

We shall therefore propose an alternate method to handle this case. Inspired by [DL09], even though we cannot "smooth up" the control operator as in [DL09], it is natural to introduce a smooth cut-off function in time in the control process.

What such a smooth cut-off function in time would do is intuitively clear on the above example, as it would simply avoid the creation of singularities when the control starts acting on the equations. In some sense, our results below show that this is the only reason for which the control operator may not maintain the degree of smoothness of the initial data to be controlled.

2.2 An alternate strategy: main results

We come back to the abstract setting and focus on the controllability problem (1.1)–(1.3). We assume from now on that solutions of the adjoint equation (1.4) satisfy the admissibility condition

(1.6) and the observability condition (1.5) in some time T^*: In particular, there exists a positive constant C_{obs} such that all z solution of (1.4) with initial datum $z_0 \in X$ satisfies

$$\|z_0\|_X^2 \leq C_{obs}^2 \int_0^{T^*} \|B^* z(t)\|_U^2 \, dt. \tag{2.12}$$

According to the counterexample exhibited above, we cannot expect regularity results on the controlled trajectory for smooth initial data to be controlled if we do not modify the way the control is computed.

We thus propose an alternate strategy based on a slightly modified controllability process in the spirit of [DL09]. Assuming (2.12), for $T > T^*$ and $\delta > 0$ with $T \geq T^* + 2\delta$, we introduce some smooth cut-off function $\eta = \eta(t) : \mathbb{R} \to [0,1]$ such that

$$\eta(t) = 1 \text{ for } t \in [\delta, T-\delta], \qquad \eta \text{ is flat close to } t = 0 \text{ and } t = T. \tag{2.13}$$

Thanks to (2.12), any solution z of (1.4) satisfies

$$\|z_0\|_X^2 \leq C_{obs}^2 \int_0^{T} \eta(t) \|B^* z(t)\|_U^2 \, dt. \tag{2.14}$$

We then introduce the functional J^η defined by

$$J^\eta(z_0) = \frac{1}{2} \int_0^T \eta(t) \|B^* z(t)\|_U^2 \, dt + \langle z_0, y_0 \rangle_X, \tag{2.15}$$

where z is the solution of the adjoint equation (1.4) with initial $z_0 \in X$. Again, the admissibility property (1.6) and the observability inequality (2.14) imply that the functional J^η is well-defined on X, coercive and strictly convex. Besides, similarly to the case in which no weight in time is involved, if we denote by Z_0 the minimizer of J^η, the function

$$v(t) = \eta(t) B^* Z(t), \quad t \in (0,T), \tag{2.16}$$

yields a null control for (1.1). In fact, this control is the one of minimal $L^2(0,T; dt/\eta; U)$-norm[1], where $L^2(0,T; dt/\eta; U)$ denotes the set of all functions w such that $w/\sqrt{\eta} \in L^2(0,T;U)$.

Let us also mention that, similarly as in (1.9), we can introduce the modified Gramian operator Λ_T^η defined by the bilinear form

$$\langle \Lambda_T^\eta z_{0,a}, z_{0,b} \rangle_X = \int_0^T \eta(t) \langle B^* z_a(t), B^* z_b(t) \rangle_U \, dt, \tag{2.17}$$

for $z_{0,a}$, $z_{0,b}$ in X, where z_a and z_b are the corresponding solutions of (1.4). The minimum Z_0 of J^η can then be characterized by the equation

$$\Lambda_T^\eta Z_0 + y_0 = 0. \tag{2.18}$$

The question then is to study the following control maps:

$$\mathbb{V} = (-\Lambda_T^\eta)^{-1} : \left\{ \begin{array}{c} X \longrightarrow X \\ y_0 \mapsto Z_0 \end{array} \right. \quad \text{and } \mathscr{V} : \left\{ \begin{array}{c} X \longrightarrow L^2\left(0, T, \dfrac{dt}{\eta(t)}; U\right) \\ y_0 \mapsto v = \eta B^* Z. \end{array} \right. \tag{2.19}$$

[1] If we were looking for the control of minimal $L^2(0,T;U)$-norm for $y' = Ay + \eta B v$, this could be obtained via formula (2.16) by minimizing $J^{\eta^2}(z_0) = \frac{1}{2}\int_0^T \eta(t)^2 \|B^* z(t)\|_U^2 \, dt + \langle z_0, y_0 \rangle_X$.

To state our results in their full generality, for $s \in \mathbb{N}$, we set $X_s = \mathscr{D}(A^s)$, and for $s \geq 0$, we define X_s as the Hilbert space obtained by suitably interpolating $\mathscr{D}(A^{\lfloor s \rfloor})$ and $\mathscr{D}(A^{\lceil s \rceil})$. In order to simplify the notation, we denote the corresponding norm on X_s simply by $\|\cdot\|_s$.

The main advantage of using the weight function η is that, with no further assumption on the control operator B, the control inherits the regularity of the data to be controlled. Indeed, with E. Zuazua we obtained the following result:

Theorem 2.1 ([EZ10]). *Assume that the adjoint equation* (1.4) *satisfies the admissibility condition* (1.6) *and the observability condition* (2.12)*. Let $s \geq 0$ and η as in* (2.13)*. If the initial datum y_0 to be controlled belongs to X_s, then the minimizer Z_0 of J^η and the control function v given by* (2.16)*, respectively, belong to X_s and $H_0^s(0,T;U)$.*

Besides, there exists a positive constant $C_s > 0$ independent of $y_0 \in X_s$ such that

$$\|Z_0\|_s^2 + \|v\|_{H_0^s(0,T;U)}^2 \leq C_s \|y_0\|_s^2 . \tag{2.20}$$

In other words, the maps $\mathbb{V}$ and $\mathscr{V}$ defined in (2.19) *satisfy:*

$$\mathbb{V} : X_s \longrightarrow X_s, \qquad \mathbb{V} : X_s \longrightarrow H_0^s(0,T;U). \tag{2.21}$$

Summing up, our strategy yields the control of minimal $L^2(0,T;dt/\eta;U)$ norm, and naturally reads the regularity of the initial data to be controlled, providing smoother controls for smoother initial data. Besides, if one is interested to the regularity in space of the controlled trajectory, it can be deduced as a consequence of Theorem 2.1:

Corollary 2.2 ([EZ10]). *Under the assumptions of Theorem 2.1, if the initial datum y_0 to be controlled belongs to X_s for $s \in \mathbb{N}$, then the controlled solution y of* (1.1) *with the control function v given by* (2.16) *belongs to*

$$C^s([0,T];X) \cap \left(\bigcap_{k=0}^{s-1} C^k([0,T];\mathcal{X}_{s-k}) \right), \tag{2.22}$$

where the spaces $(\mathcal{X}_j)_{j\in\mathbb{N}}$ are defined by induction by

$$\mathcal{X}_0 = X, \qquad \mathcal{X}_j = (A - \beta I)^{-1}(\mathcal{X}_{j-1} + BB^* X_j), \tag{2.23}$$

for β in the resolvent set of A.

Let us add some other remarks. When the operator B is bounded from X to U, the HUM functional J in (1.7), without the time cut-off function η, satisfies the same regularity results as the one in Theorem 2.1 for $s = 1$. For larger s, and if one furthermore assumes that $BB^* \in \mathscr{L}(X_j)$ for all $j \leq s-1$, then $\mathbb{V} = (-\Lambda_T^{\eta=1_{(0,T)}})^{-1}$ maps X_s into itself. One immediately deduces Corollary 2.2 as well. Of course, in this latter case, an easy induction argument shows that $\mathcal{X}_j = X_j$ for all $j \leq s$.

The spaces $\mathcal{X}_j$ are not explicit in general. However, there are several cases in which they can be shown to be included in Hilbert spaces of the form X_j, which in practical applications to PDE are constituted by functions that are smoother than X with respect to the space variable. This is in particular the case if BB^* maps X_j to itself for all $j \in \mathbb{N}$: the spaces $\mathcal{X}_j$ then simply coincide with X_j for all $j > 0$. Of course, this is sharp, since one cannot expect the controlled solution to be better than $C^0([0,T];X_s)$ for initial data $y_0 \in X_s$.

As we will see below, Theorem 2.1 mainly consists in suitable integration by parts in time. The main difference appearing in the proof when $\eta \equiv 1$ is that, when integrating by parts, boundary terms appear at $t = 0, T$, which can be suitably bounded when BB^* is bounded in the case $s = 1$. But when the cut-off function η is introduced, these boundary terms vanish and are transformed into time-integrated terms that are simply bounded by the weaker admissibility condition, therefore avoiding extra assumptions on the observation operator B^*.

Indeed, the main estimate allowing to prove Theorem 2.1 is the following commutator estimate:

$$\left\| A^2 \Lambda_T^\eta - A \Lambda_T^\eta A \right\|_{\mathscr{L}(X, \mathscr{D}(A)')} \leq C_\eta. \tag{2.24}$$

This can be proved as follows. For $z_{0,a} \in \mathscr{D}(A)$ and $z_{0,b} \in \mathscr{D}(A^2)$,

$$\begin{aligned} \langle (A^2 \Lambda_T^\eta - A \Lambda_T^\eta A) z_{0,a}, z_{0,b} \rangle_X &= \langle \Lambda_T^\eta z_{0,a}, A^2 z_{0,b} \rangle_X + \langle \Lambda_T^\eta A z_{0,a}, A z_{0,b} \rangle_X \\ &= \int_0^T \eta(t) \langle B^* z_a(t), B^* z_b''(t) \rangle_U \, dt + \int_0^T \eta(t) \langle B^* z_a'(t), B^* z_b'(t) \rangle_U \, dt \\ &= -\int_0^T \eta'(t) \langle B^* z_a(t), B^* z_b'(t) \rangle_U \, dt, \end{aligned}$$

where z_a and z_b are the corresponding solutions of (1.4) with initial data $z_{0,a}$, $z_{0,b}$. Thanks to (1.6), we therefore obtain

$$\left| \langle (A^2 \Lambda_T^\eta - A \Lambda_T^\eta A) z_{0,a}, z_{0,b} \rangle_X \right| \leq C \left\| z_{0,a} \right\|_X \left\| z_{0,b} \right\|_{\mathscr{D}(A)},$$

which proves (2.24).

Higher order estimates rely on

$$\left\| A^{2s} \Lambda_T^\eta - A^s \Lambda_T^\eta A^s \right\|_{\mathscr{L}(\mathscr{D}(A^{s-1}), \mathscr{D}(A^s)')} \leq C_{\eta,s}. \tag{2.25}$$

which can be proved similarly.

Let us finally mention that Theorem 2.1 is less precise than the results in [DL09]. However, the proof of the results in [DL09] requires the use of deep technical tools such as microlocal analysis and Littlewood-Paley decomposition and seems specific to the wave equation, while our approach is more robust and applies also for boundary control problems and any linear conservative equations, including for instance systems of coupled wave equations, or a wave equation coupled with an oscillator as in [EV14].

2.3 Application to the boundary controllability of the wave equation

Let us present the case of a boundary control for the wave equation. Let Ω be a smooth bounded domain of $\mathbb{R}^d$. Let $\chi : \partial\Omega \to [0,1]$ be a function defined on $\partial\Omega$, and set $\Gamma = \{\chi > 0\}$.

We now consider the following wave equation:

$$\begin{cases} \partial_{tt} y - \Delta y = 0, & (t,x) \in (0,\infty) \times \Omega \\ y(t,x) = \chi(x) v(t,x), & (t,x) \in (0,\infty) \times \partial\Omega, \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y_0, y_1) & \in L^2(\Omega) \times H^{-1}(\Omega), \end{cases} \tag{2.26}$$

where $v \in L^2((0,T) \times \Gamma)$ is the control function.

Setting $A_0 = -\Delta$ the Laplace operator defined on $H^{-1}(\Omega)$ with domain $\mathscr{D}(A_0) = H_0^1(\Omega)$, system (2.26) fits the abstract setting given above, once written as a first order system in the variable $Y = (y, \partial_t y) \in C([0,T]; L^2(\Omega) \times H^{-1}(\Omega))$, which satisfies

$$Y' = AY + Bv, \text{ with } A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \begin{cases} X = L^2(\Omega) \times H^{-1}(\Omega), \\ \mathscr{D}(A) = H_0^1(\Omega) \times L^2(\Omega), \end{cases}$$

and B defined by

$$Bv = \begin{pmatrix} 0 \\ A_0 \tilde{y} \end{pmatrix}, \quad \text{where} \quad \begin{cases} -\Delta \tilde{y} = 0, & \text{in } \Omega, \\ \tilde{y} = \chi v, & \text{on } \partial\Omega. \end{cases} \tag{2.27}$$

The map $v \in L^2(\Gamma) \mapsto \tilde{y} \in H^{1/2}(\Omega)$ is continuous (see [TW09, Chap. 10]) and then B is continuous from $U = L^2(\Gamma)$ to $\{0\} \times H^{-3/2}(\Omega) \subset \mathscr{D}(A^{1/2})'$. The control operator B is therefore unbounded. The fact that B is admissible follows from a hidden regularity result, proved for instance in [Lio88a].

Again, the exact controllability property for (2.26) in time T^* is equivalent to the *Geometric Control Condition* for (Γ, Ω, T^*), which asserts that all the rays of Geometric Optics in Ω touch Γ at a non-diffractive point in a time smaller than T^*, see [BLR92, BG97].

In the following, we assume the GCC in time T^* for Γ. Let $T > T^*$, choose $\delta > 0$ such that $T - 2\delta \geq T^*$ and fix a function η satisfying (2.13). The functional J^η introduced in (2.15) is now defined on $H_0^1(\Omega) \times L^2(\Omega)$ and reads as

$$J^\eta(z_0, z_1) = \frac{1}{2} \int_0^T \int_\Gamma \eta(t) \chi(x)^2 |\partial_n z(t,x)|^2 \, d\gamma dt + \langle (z_0, z_1), (y_0, y_1) \rangle_{H_0^1(\Omega) \times L^2(\Omega), L^2(\Omega) \times H^{-1}(\Omega)}, \tag{2.28}$$

where z is the solution of (2.4) with initial data (z_0, z_1).

Then Theorem 2.1 and Corollary 2.2 imply the following:

Theorem 2.3 ([EZ10]). *Assume that η is a smooth weight function satisfying* (2.13). *Given any $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a unique minimizer (Z_0, Z_1) of J^η over $H_0^1(\Omega) \times L^2(\Omega)$. Denoting Z the corresponding solution of* (2.4)*, the function*

$$v(t,x) = \eta(t) \chi(x) \partial_n Z(t,x), \tag{2.29}$$

is a control function for (2.26)*, which is characterized as the control function minimizing the $L^2(0,T; dt/\eta; L^2(\Gamma))$-norm.*

Furthermore, if (y_0, y_1) belongs to $H^s_{(0)}(\Omega) \times H^{s-1}_{(0)}(\Omega)$ for some $s > 0$, (Z_0, Z_1) belongs to $H^{s+1}_{(0)}(\Omega) \times H^s_{(0)}(\Omega)$ and, if the function χ is smooth, the control function v given by (2.29) *satisfies*

$$v \in H_0^s((0,T) \times \Gamma) \cap \left(\bigcap_{k=0}^{s-1} C^k([0,T]; H_0^{s-k-1/2}(\Gamma)) \right). \tag{2.30}$$

In particular, the controlled solution y of (2.26) *then satisfies*

$$y \in C^0([0,T]; H^s(\Omega)) \cap C^{s+1}([0,T]; H^{-1}(\Omega)). \tag{2.31}$$

Proof. Theorem 2.1 proves that $(Z_0, Z_1) \in H^{s+1}_{(0)}(\Omega) \times H^s_{(0)}(\Omega)$. The regularity property (2.30) on v is a consequence of the regularity property of the flux in [LLT86]. The regularity property (2.31) is a consequence of the fact that v has regularity (2.30), the initial data is in $H^s_{(0)}(\Omega) \times H^{s-1}_{(0)}(\Omega)$, and of the compatibility conditions of the data at $t = 0$, $x \in \partial\Omega$, see [LLT86]. $\square$

2.4 Sketch of the proof of Theorem 2.1

We now outline the proof of Theorem 2.1 in the case $s = 1$, the cases $s \in \mathbb{N}$ being completely similar, and the cases $s \geq 0$ being a consequence of interpolation theory. If Z_0 denotes the minimizer of J^η in (2.15), then it satisfies the following Euler-Lagrange equation: for all $z_0 \in X$,

$$0 = \langle \Lambda_T^\eta Z_0, z_0 \rangle_X + \langle z_0, y_0 \rangle_X. \tag{2.32}$$

Assuming temporarily that $A^2 Z_0 \in X$, we apply the Euler-Lagrange equation (2.32) to $z_0 = A^2 Z_0$:

$$0 = \langle \Lambda_T^\eta Z_0, A^2 Z_0 \rangle_X + \langle A^2 Z_0, y_0 \rangle_X.$$

But on one hand we have

$$\langle A^2 Z_0, y_0 \rangle_X = -\langle A Z_0, A y_0 \rangle_X.$$

On the other hand, according to (2.24),

$$\left| \langle \Lambda_T^\eta Z_0, A^2 Z_0 \rangle_X - \langle \Lambda_T^\eta A Z_0, A Z_0 \rangle_X \right| \leq C \left\| Z_0 \right\|_X \left\| Z_0 \right\|_{\mathscr{D}(A)},$$

while, according to (2.14),

$$\left\| A Z_0 \right\|_X^2 \leq C_{obs}^2 \langle \Lambda_T^\eta A Z_0, A Z_0 \rangle_X.$$

Therefore, assuming some regularity on Z_0, namely $Z_0 \in \mathscr{D}(A^2)$, we obtain

$$\left\| A Z_0 \right\|_X^2 \leq C \left\| Z_0 \right\|_X^2 + C \left\| A y_0 \right\|_X \left\| Z_0 \right\|_{\mathscr{D}(A)}, \tag{2.33}$$

so that, using that the map $\mathbb{V} = (-\Lambda_T^\eta)^{-1}$ in (2.19) is bounded in $\mathscr{L}(X)$, i.e. $\left\| Z_0 \right\|_X \leq C \left\| y_0 \right\|_X$, we derive

$$\left\| Z_0 \right\|_{\mathscr{D}(A)} \leq C \left\| y_0 \right\|_{\mathscr{D}(A)}.$$

Once such an estimate is proved, the estimate on $v \in H_0^1(0,T;U)$ is straightforward.

To make the formal proof given above rigorous, instead of choosing $z_0 = A^2 Z_0$ as a test function in (2.32), we take

$$z_{0,\tau} = \frac{1}{\tau^2} \left(Z(\tau) + Z(-\tau) - 2 Z_0 \right),$$

where Z is the solution of (1.4) with initial data Z_0, and we then pass to the limit in $\tau \to 0$. Indeed, $z_{0,\tau}$ approximates $Z''(0) = A^2 Z_0$, while for each $\tau > 0$, it belongs to X, and therefore is an admissible test function in (2.32). Details can be found in [EZ10].

Chapter 3

Application to the numerical computations of exact controls

3.1 Introduction

The goal of this chapter is to present an application of the results of Chapter 2 to the building of efficient numerical algorithms to compute null-controls for the abstract controllability problem (1.1)–(1.3).

Many articles have been devoted to this problem starting with the pioneering works by R. Glowinski, C.H. Li and J.-L. Lions [GL90, GLL90], see the recent book [GLH08] for a more complete account. The starting point is that when discretizing in space[1] the system (1.1) with a mesh size $h > 0$, the discretized systems take the form

$$y_h' = A_h y_h + B_h v_h, \quad t \in [0, T], \qquad y_h(0) = y_{0,h}. \tag{3.1}$$

One could then compute the null-controls v_h for the discrete systems (3.1). But the null-controls v_h computed for (3.1) do not necessarily converge to a control v for (1.1). Even worse, in general the norms of the controls v_h blow up as $h \to 0$. As identified in [GL90, GLL90], this is due to the presence of high-frequency spurious solutions created by the discretization process, and this suggested the use of some penalization of the high-frequency components of the solutions in order to re-establish a nice convergent behavior for the discrete controls v_h. These results have later received a thorough theoretical study starting from the work [IZ99], see also the recent survey articles [Zua05, EZ11c]. All these works underline that convergent numerical schemes for approximating (1.1) are not necessarily good for computing a convergent approximation of the null-control of (1.1). This is a serious warning on the use of naive numerical algorithms to compute controls.

We are then in a curious situation in which the discretization completely modifies the behavior of the control process, even though the continuous control problem (1.1)–(1.3) is well-understood. In the following, we explain how Theorem 2.1 can nevertheless yield a constructive approach to numerically build suitable convergent approximations of the control given by (2.16)–(2.18). As this algorithm will be based only on the understanding of the continuous control operator, we will call this approach the *continuous approach.*

[1]For simplifying the discussion, we focus on the case of a space semi-discretization, but the same phenomena appear for the fully-discrete case.

We shall then explain that if the adjoint systems $z_h' = A_h z_h$ are uniformly observable through B_h^* with respect to the discretization parameter $h > 0$, then the sequence of discrete controls v_h for (3.1) converges as $h \to 0$ to the null-control for (1.1) given by (2.16)–(2.18). As this algorithm is based on the understanding of the observability of all the discrete systems $z_h' = A_h z_h$, we will call this approach the *discrete approach*.

In the following, we first develop an iterative process to approximate the null-controls in the infinite-dimensional setting. The most natural strategy is to follow a minimization process for the functional J^η introduced in (2.15). We then approximate this iterative process using our numerical scheme at hand. In the continuous approach, we shall show that, provided that the number of iterations is not too large, these discrete iterates stay close to the continuous ones. In contrast, in the discrete approach, the approximation error do not deteriorate as the number of iterations goes to infinity.

The previous work [CMT11] based on Russell's technique [Rus78] to construct controls from stabilization results shares the same strategy as the continuous approach. This work actually inspired our study of the continuous approach. Very close ideas were also developed in the context of data assimilation, see e.g. [HR12] (see also [AB05, SU09]). Actually, as we will explain in Section 3.5, strong links exist between these two problems.

As a byproduct of our analysis, for both continuous and discrete approaches, we will derive rates of convergence for the controls computed that way. With the recent work [CMT11], these are to my knowledge the only results on the rate of convergence of the discrete controls.

3.2 The continuous setting: an algorithm

In the following, we assume that the adjoint equation (1.4) satisfies the admissibility property (1.6), the observability property (1.5), and we construct η as in (2.13) so that (2.14) is satisfied. In other words we assume that there exist constants C_{obs} and C_{ad} such that for all $z_0 \in X$,

$$\frac{1}{C_{obs}^2} \|z_0\|_X^2 \leq \langle \Lambda_T^\eta z_0, z_0 \rangle_X \leq C_{ad}^2 \|z_0\|_X^2 . \tag{3.2}$$

Based on the characterization of Z_0 as the minimizer of the functional J^η in (2.15), one can build an "algorithm" to approximate the minimizer in this infinite-dimensional setting. For instance, it suffices to apply a steepest descent or conjugate gradient iterative algorithm.

We focus on the steepest descent algorithm with fixed step:

- Initialization: Define
$$z_0^0 = 0. \tag{3.3}$$
- Iteration: For $z_0^k \in X$, define z_0^{k+1} by
$$z_0^{k+1} = z_0^k - \rho(\Lambda_T^\eta z_0^k + y_0), \tag{3.4}$$
where $\rho > 0$ is a fixed parameter, whose (small enough) value will be specified later on.

With E. Zuazua, we have obtained the following result:

Theorem 3.1 ([EZ13]). *Assume* (3.2). *Let $s \geq 0$, $y_0 \in X_s$ and $Z_0 \in X$ be the solution of* (2.18). *Setting $\rho_0 = 2/C_{ad}^4 C_{obs}^2$, for all $\rho \in (0, \rho_0)$, the sequence (z_0^k) defined by* (3.3)-(3.4) *satisfies for some constant $C > 0$ that for all $k \in \mathbb{N}$,*

$$\left\| z_0^k - Z_0 \right\|_X \leq C\delta^k \|y_0\|_X , \tag{3.5}$$

where $\delta \in (0,1)$ *is given by*

$$\delta(\rho) := \sqrt{1 - 2\frac{\rho}{C_{obs}^2} + \rho^2 C_{ad}^4}. \tag{3.6}$$

Besides, $Z_0 \in X_s$*, the sequence* (z_0^k) *belongs to* X_s*, and satisfies, for some constant* C_s *independent of* $Z_0 \in X_s$ *and* $k \in \mathbb{N}$*,*

$$\left\| z_0^k - Z_0 \right\|_s \leq C_s(1 + k^s)\delta^k \left\| y_0 \right\|_s, \quad k \in \mathbb{N}. \tag{3.7}$$

The first statement (3.5) in Theorem 3.1 is a direct consequence of the well-known results on the convergence rate for the steepest descent method when minimizing quadratic coercive continuous functionals in Hilbert spaces ([Cia82]). However, the result (3.7) is new and relies in an essential manner on the fact that the Gramian operator preserves the regularity properties of the data to be controlled, a fact that was proved in [EZ10], see also Chapter 2, and for which the weight function in time $\eta = \eta(t)$ plays a key role.

Actually, Theorem 3.1 is strongly based on Theorem 2.1, and relies on the commutator estimates

$$\|[\Lambda_T^\eta, A^s]\|_{\mathscr{L}(\mathscr{D}(A^{s-1}),X)} \leq C_s, \quad (s \in \mathbb{N}). \tag{3.8}$$

Note that this can be proved similarly as (2.24). In fact, both estimates (2.25) and (3.8) are equivalent.

Of course, these convergence results also imply that the sequence $(v^k) = (\eta B^* z^k)$, where z^k is the solution of (1.4) with initial data z_0^k, converge to the control v given by (2.16): for instance we have, for all $k \geq 0$,

$$\left\| v^k - v \right\|_{L^2(0,T;dt/\eta;U)} \leq C\delta^k \left\| y_0 \right\|_X. \tag{3.9}$$

Note that, in general, (3.7) also gives estimates on the convergence of v^k towards v in stronger norms when the data y_0 to be controlled lies in X_s for some $s \geq 0$.

Obviously, one could use other algorithms to minimize the functional J^η in (2.15). But in the following it will be important that the iterates z_0^k converge in X_s as $k \to \infty$ when the initial data to be controlled are in X_s. To our knowledge, this is an open problem when considering the iterates given by the conjugate gradient algorithm. This is due to the fact that it strongly uses orthogonality properties in the natural space X endowed with the scalar product adapted to the minimization problem $\langle \Lambda_T^\eta \cdot, \cdot \rangle_X$. This prevents us from using the iterates of the conjugate gradient algorithm in the following.

3.3 Numerical algorithms

3.3.1 The discretization process

Once the iterative algorithm (3.3)–(3.4) is built at the infinite-dimensional level one can mimic it for suitable numerical approximation schemes. In this way, combining the classical convergence properties of numerical schemes and the convergence properties (3.5)–(3.7) of the iterative algorithm (3.3)–(3.4), one can get quantitative convergence results towards the control.

To be more precise, let us consider a space semi-discrete approximation of equation (1.4) taking the form

$$z_h' = A_h z_h, \quad t \in (0,T), \qquad z_h(0) = z_{0h}, \tag{3.10}$$

where A_h is an approximation of the operator A in a finite dimensional vector space V_h embedded into X via a map $E_h : V_h \mapsto X$. We assume that it induces an Hilbert structure on V_h endowed by the norm $\|\cdot\|_h = \|E_h \cdot\|_X$, and that A_h is skew-adjoint with respect to that scalar product.

In practice one can think of finite-difference or finite-element approximations of the PDE under consideration, for instance, h being the characteristic length of the mesh size.

We also introduce B_h^*, an approximation of the operator B^*, defined on V_h with values in some Hilbert spaces U_h, and the discrete Gramian operator Λ_{Th}^η, corresponding to the discrete approximation of the Gramian operator (2.17), given by

$$\Lambda_{Th}^\eta = \int_0^T \eta(t) e^{-tA_h} B_h B_h^* e^{tA_h}\, dt.$$

By construction, Λ_{Th}^η is self-adjoint in V_h. We finally introduce a sequence of restriction operators $R_h : X \to V_h$, which will be chosen so that $(I - E_h R_h)z$ converges to 0 for smooth function $z \in \cap_{s \geq 0} X_s$, see Assumption 1 below.

We will not make precise our set of assumptions on the operators A_h and B_h, but rather directly assume some convergence estimates on the discrete Gramian operator Λ_{Th}^η:

Assumption 1. There exist $s > 0$, $\theta > 0$ and $C > 0$ so that for all $h > 0$

$$\begin{aligned} \|E_h R_h z\|_X &\leq C \|z\|_X, \quad z \in X, & (3.11)\\ \|E_h \Lambda_{Th}^\eta R_h z\|_X &\leq C \|z\|_X, \quad z \in X, & (3.12)\\ \|(E_h R_h - I_X)\, z\|_X &\leq C h^\theta \|z\|_s, \quad z \in X_s, & (3.13)\\ \|(E_h \Lambda_{Th}^\eta R_h - E_h R_h \Lambda_T^\eta)\, z\|_X &\leq C h^\theta \|z\|_s, \quad z \in X_s. & (3.14) \end{aligned}$$

Assumption 2. The norms of the operators Λ_{Th}^η in $\mathscr{L}(V_h)$ are uniformly bounded with respect to $h > 0$:

$$\mathcal{C}_{ad}^2 := \sup_{h \geq 0} \|\Lambda_{Th}^\eta\|_{\mathscr{L}(V_h)} < \infty, \tag{3.15}$$

where, when $h = 0$, we use the notation $V_0 = X$ and $\Lambda_{T0}^\eta = \Lambda_T^\eta$.

Assumption 3. There exists a constant $\mathcal{C}_{obs}$ such that for all $h \geq 0$ and $z_h \in V_h$,

$$\|z_h\|_h^2 \leq \mathcal{C}_{obs}^2 \langle \Lambda_{Th}^\eta z_h, z_h \rangle_h, \tag{3.16}$$

where, for $h = 0$, we use the notation $V_0 = X$ and $\Lambda_{T0}^\eta = \Lambda_T^\eta$.

Before going further, let us emphasize that Assumption 2, though straightforward when the observation operators are uniformly bounded on V_h, is not obvious when dealing with unbounded control operators, as it is the case for instance when considering boundary controllability of the wave equation. Indeed, in that case, one should be careful and prove a uniform admissibility result (here and in the following, "uniform" always refers to the dependence on the discretization parameter(s)). Also note that Assumption 2 together with (3.11) implies (3.12).

Assumption 3 states the uniform coercivity of the operators Λ_{Th}^η, or equivalently, the uniform observability of the discrete equations (3.10). This assumption often fails for classical convergent numerical approximations of the waves, see [GLL90, Zua05, EZ11c, GLH08], and this is why we develop the continuous approach without requiring Assumption 3. But several possibilities exist to reinforce the coercivity of Λ_{Th}^η when needed. Among the many available possibilities, let us quote for instance Tychonoff regularization (see Section 6.1.2) or filtering techniques. This modification has to be done "in the smallest possible way" to catch good convergence properties of the numerical approximations of the controls found that way. It therefore requires a careful analysis of the observability properties of the discrete dynamics, a fact which can be avoided when following the continuous approach.

3.3.2 The algorithm

Once the finite-dimensional approximations (3.10) of (1.4) have been set, we introduce the discrete functional

$$J_h^\eta(z_{0h}) = \frac{1}{2}\langle \Lambda_{Th}^\eta z_{0h}, z_{0h}\rangle_h + \langle z_{0h}, y_{0h}\rangle_h, \tag{3.17}$$

where y_{0h} is an approximation in V_h of $y_0 \in X$.

Of course, the functional J_h^η is a natural approximation of the continuous functional J^η defined by (2.15). One could then expect the minima of J_h^η to yield convergent approximations of the minima of the continuous functional J^η. As recalled in the introduction, this is in general not the case, see [GLH08, Zua05, EZ11c, EZ13]. This can be seen as an evidence of the lack of Γ-convergence of the functionals J_h^η towards J^η.

However, we can still focus on the following algorithm: For each $h > 0$, define the sequence z_{0h}^k by induction, inspired in the statement of Theorem 3.1, as follows:

$$z_{0h}^0 = 0, \quad \forall k \in \mathbb{N},\ z_{0h}^{k+1} = z_{0h}^k - \rho\left(\Lambda_{Th}^\eta z_{0h}^k + y_{0h}\right). \tag{3.18}$$

The idea is that the discrete iterates z_{0h}^k should stay close to the continuous ones z_0^k given by (3.3)–(3.4), at least if the number of iterations is not too large. Our results then simply are a trade-off between the errors done at each iteration induced by the discretization process, and the convergence estimates in Theorem 3.1.

3.3.3 The continuous approach

With E. Zuazua, we derived the following result:

Theorem 3.2 ([EZ13])**.** *Assume* (3.2) *and that Assumptions 1 and 2 hold. Let $y_0 \in X_s$ and $(y_{0h})_{h>0}$ be a sequence of functions such that for all $h > 0$, $y_{0h} \in V_h$. Setting $\rho_1 = \min\{\rho_0, 2/\mathcal{C}_{ad}^2\}$, where ρ_0 is given by Theorem 3.1, we choose $\rho \in (0, \rho_1)$ and let the sequence z_0^k, respectively z_{0h}^k, be defined by induction by* (3.3)-(3.4)*, respectively* (3.18)*.*

Then there exists a constant $C > 0$ independent of $h > 0$ such that for all $k \in \mathbb{N}$,

$$\left\|E_h z_{0h}^k - z_0^k\right\|_X \le k\rho \left\|E_h y_{0h} - y_0\right\|_X + Ckh^\theta \left\|y_0\right\|_s. \tag{3.19}$$

Using Theorems 3.1 and 3.2 together, we deduce the following convergence result:

Corollary 3.3 ([EZ13])**.** *Assume* (3.2) *and that Assumptions 1 and 2 hold. Let $y_0 \in X_s$ and $(y_{0h})_{h>0}$ be a sequence such that for all $h > 0$,*

$$\left\|E_h y_{0h} - y_0\right\|_X \le Ch^\theta \left\|y_0\right\|_s. \tag{3.20}$$

Then, choosing $\rho \in (0, \rho_1)$ with $\rho_1 = \min\{\rho_0, 2/\mathcal{C}_{ad}^2\}$, for all $h > 0$, setting

$$K_h^c = \lfloor \theta \frac{\log(h)}{\log(\delta)} \rfloor, \tag{3.21}$$

where δ is given by (3.6)*, we have, for some constant C independent of h,*

$$\left\|E_h z_{0h}^{K_h^c} - Z_0\right\|_X \le C|\log(h)|^{\max\{1,s\}} h^\theta \left\|y_0\right\|_s, \tag{3.22}$$

where $z_{0h}^{K_h^c}$ is the K_h^c-iterate of the sequence z_{0h}^k defined by (3.18)*.*

This is the *continuous approach* for building numerical approximations of the controls.

Let us emphasize that the above result yields a way to compute a good approximation of Z_0 provided that the iterative process (3.18) ends at some threshold K_h^c in (3.21). But this threshold can be difficult to estimate in practice as it involves the admissibility and observability constants in (3.2) via δ in (3.6), while nothing guarantees that the error estimate would not deteriorate when going beyond this threshold. Actually, the error estimate could indeed deteriorate as the number of iterations goes to infinity when the discrete systems are not uniformly observable, i.e. when Assumption 3 is not satisfied, as numerical evidences show, see Section 3.4. On the contrary, as we will explain below, if Assumption 3 holds, this error estimate cannot deteriorate.

The algorithm above and the error estimates we obtain are similar to those in [CMT11] where the iterative process proposed by D. Russell in [Rus78] to obtain controllability out of stabilization results is mimicked at the discrete level. The number of iterations in [CMT11] is of the order of $\lfloor \theta |\log(h)| m \rfloor$, where m is a constant that enters in the continuous stabilization property of the dissipative operator $A - BB^*$, and the error obtained that way is $h^\theta |\log(h)|^2$. But the results in [CMT11] apply only in the context of bounded control operators and they do not yield the control of minimal L^2-norm, whereas our approach applies under the weaker admissibility assumption on the control operator and yields effective approximations of the minimal norm controls (suitably weighted in time). To better underline the close links existing between [CMT11] and our approach, remark that the sequence z_0^k in (3.3)–(3.4) can be simply written as

$$z_0^k = -\sum_{j=0}^{k-1} (I - \rho \Lambda_T^\eta)^j (\rho y_0), \tag{3.23}$$

which indeed converges to $(I - (I - \rho\Lambda_T^\eta))^{-1}(-\rho y_0) = -(\Lambda_T^\eta)^{-1} y_0 = Z_0$ as $I - \rho\Lambda_T^\eta$ is of $\mathscr{L}(X)$-norm smaller than δ in (3.6). The approach in [CMT11], based on the construction in [Rus78], solves the control problem by computing the inverse of an operator of the form $I - L_T$, for some operator L_T of $\mathscr{L}(X)$-norm strictly smaller than 1, mainly consisting in solving subsequently one forward and one backward damped wave equation. The inversion of $I - L_T$ then can be done using approximations of the forms $\sum_{j=0}^{k-1} L_T^j$, which is indeed very similar to (3.23).

Note that estimates (3.22) also imply that the sequence

$$v_h^k(t) = \eta(t) B_h^* \exp(tA_h) z_{0h}^k,$$

defined for $k \geq 0$ satisfies that $v_h^{K_h^c}$ is close to v in (2.16), with some bounds (usually the same) on the error term. We do not state precisely the corresponding results since it would require to introduce further assumptions on the way the spaces U_h approximate U.

3.3.4 The discrete approach

The discrete approach differs from the continuous approach by the fact that we assume the uniform coercivity property Assumption 3 on the discrete Gramian operators, i.e. the uniform observability of the discrete adjoint equations (3.10). With E. Zuazua, we then obtained:

Theorem 3.4 ([EZ13]). *Assume* (3.2) *and that Assumptions 1, 2 and 3 hold. Let $y_0 \in X_s$ and $(y_{0h})_{h>0}$ be a sequence of functions such that for all $h > 0$, $y_{0h} \in V_h$. Setting $\rho_2 = 2/\mathcal{C}_{ad}^4 \mathcal{C}_{obs}^2$, we choose $\rho \in (0, \rho_2)$ and let the sequence z_0^k, respectively z_{0h}^k, be defined by induction by* (3.3)-(3.4), *respectively* (3.18).

Then there exists a constant $C > 0$ independent of $h > 0$ such that for all $k \in \mathbb{N}$,

$$\left\| E_h z_{0h}^k - z_0^k \right\|_X \leq C \left(\|E_h y_{0h} - y_0\|_X + h^\theta \|y_0\|_s \right). \tag{3.24}$$

Under Assumptions 1, 2 and 3, for all $y_{0h} \in V_h$, the discrete functional J_h^η in (3.17) is coercive, and the sequence z_{0h}^k is therefore convergent to the unique minimizer Z_{0h} of J_h^η, which can be characterized by the equation

$$\Lambda_{Th}^\eta Z_{0h} + y_{0h} = 0. \tag{3.25}$$

Since k can be made arbitrarily large in (3.24), we get the following result:

Corollary 3.5 ([EZ13]). *Assume* (3.2) *and that Assumptions 1, 2 and 3 hold. Let $y_0 \in X_s$ and $(y_{0h})_{h>0}$ be a sequence such that* (3.20) *holds. Let Z_0 be given by* (2.18). *Then there exists a constant $C > 0$ such that the sequence of minimizers Z_{0h} of the functionals J_h^η in* (3.17) *satisfies*

$$\|E_h Z_{0h} - Z_0\|_X \le C h^\theta \|y_0\|_s , \tag{3.26}$$

Corollary 3.5 is the convergence result obtained in [EZ11c] using another proof, directly based on the smoothness result obtained in Theorem 2.1. Indeed, using Theorem 2.1, Assumptions 1–3 and recalling the characterizations of Z_0 and Z_{0h} given in (2.18) and (3.25), we have

$$\begin{aligned}
&\|E_h Z_{0h} - Z_0\|_X \le \|Z_{0h} - R_h Z_0\|_h + \|(E_h R_h - I) Z_0\|_X \\
&\le C \|\Lambda_{Th}^\eta Z_{0h} - \Lambda_{Th}^\eta R_h Z_0\|_h + C h^\theta \|y_0\|_s \\
&\le C \|\Lambda_{Th}^\eta Z_{0h} - R_h \Lambda_T^\eta Z_0\|_h + C \|(E_h R_h \Lambda_T^\eta - E_h \Lambda_{Th}^\eta R_h) Z_0\|_X + C h^\theta \|y_0\|_s \\
&\le C \|E_h y_{0h} - E_h R_h y_0\|_X + C h^\theta \|y_0\|_s \le C h^\theta \|y_0\|_s .
\end{aligned}$$

We have preferred to present both continuous and discrete approaches similarly to underline where the modification occurs between the two approaches. It is easily seen that the main difference between (3.24) and (3.19) is that when considering the discrete approach, the accumulation of errors done at each iterate is bounded uniformly with respect to $k \in \mathbb{N}$. Looking at the proof more closely, this is due to the fact that, under Assumption 3, the errors come in a geometric series of ratio strictly smaller than 1.

We refer to [EZ11c] for numerical evidences of the fact that the convergence rates (3.26) are close to sharp. We will also briefly illustrate this fact in Section 3.4.

Let us emphasize here that Corollary 3.5 is new regarding the convergence rates (3.26). Otherwise, the strong (respectively weak) convergence in X of the discrete minimizers $E_h Z_{0h}$ to Z_0 when the discrete initial data $E_h y_{0h}$ converges strongly (resp. weakly) in X to y_0 as $h \to 0$, is a well-known consequence of Assumptions 1–3, going back for instance to [IZ99].

Thanks to Corollary 3.5, given $h > 0$, we can use any algorithm we wish to minimize J_h^η, which is a strictly convex and coercive functional. When Assumption 3 is satisfied, we will therefore use the conjugate gradient algorithm, for which we know that the minimum is attained in at most $\dim(V_h)$ iterations, and in general much faster than that. Besides, performing the conjugate gradient algorithm on J_h^η does not require anymore any estimate on the admissibility and observability constants in (3.2). Of course, the counterpart is that one first needs to prove Assumption 3, namely the uniform observability property for the space semi-discrete systems (3.10), and as mentioned earlier, this is not an easy problem.

3.3.5 Continuous versus Discrete approaches

When comparing the results in Corollaries 3.3 and 3.5 one may think that the continuous approach, which applies with a lot of generality, yields essentially the same convergence estimates as the discrete one, more intricate, making the latter irrelevant. This is not the case, and we list below some other relevant facts that may be used to compare the two approaches.

Implementation. To implement the continuous approach, as already mentioned, one immediately sees in Corollary 3.3 that estimating the threshold K_h^c in (3.21) is an intricate problem involving the estimates of the admissibility and observability constants in (3.2). This is a strong limitation of the continuous approach.

On the other hand, the discrete approach requires the proof of uniform observability property for the space semi-discrete systems (3.10), i.e. Assumption 3. This assumption fails in a number of examples, see e.g. [GLL90, IZ99, EZ11c], due to the presence of spurious high-frequency waves that do not travel - a phenomenon already mentioned in [Tre82]. Therefore, a natural strategy to re-establish uniform observability properties is to filter out these high-frequency waves, via filtering methods, Tychonoff regularization, or other suitable penalization techniques. We refer to the survey articles [Zua05, EZ11c] for some account on these strategies, and to the recent work [Mil12] for what is to my knowledge the most general setting in which Assumption 3 is proved.

Number of iterations. On one hand, the continuous approach uses a number of iterations given by (3.21), which is not very explicit and can be large. On the other hand, in the discrete approach, we can use a conjugate gradient algorithm - or any other algorithm - to minimize J_h^η in (3.17), which is fast and for which no estimate on the constant of observability or admissibility is needed.

3.4 Application to the wave equation

In this section, we present some numerical experiments to illustrate our results and discussions. We focus on the emblematic example of the 1d wave equation controlled from the boundary, in which case we can easily illustrate our results with some numerical simulations since the control function will simply be a function of time.

Let us consider the 1d wave equation (2.9) controlled from $x = 1$, and the adjoint problem (2.11). We take $T > 2$ and η as in (2.13) (with $T^* = 2$). The corresponding Gramian operator Λ_T^η is then given as follows: For $(z_0, z_1) \in H_0^1(0,1) \times L^2(0,1)$, solve (2.11) and then solve

$$\begin{cases} \partial_{tt}\psi - \partial_{xx}\psi = 0, & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = 0, \ \psi(t,1) = -\eta(t)\partial_x z(t,1), & t \in (0,T), \\ (\psi(T,\cdot), \partial_t\psi(T,\cdot)) = (0,0), & x \in (0,1). \end{cases} \tag{3.27}$$

Then

$$\Lambda_T^\eta(z_0, z_1) = ((-\partial_{xx,D})^{-1}\partial_t\psi(0,\cdot), -\psi(0,\cdot)), \tag{3.28}$$

where $-\partial_{xx,D}$ is the Laplace operator on $(0,1)$ with Dirichlet boundary conditions.

During the iterations (3.3)–(3.4), the control function is then approximated by the sequence

$$v^k(t) = \eta(t)\partial_x z^k(t,1), \quad t \in (0,T), \tag{3.29}$$

where z^k is the solution of (2.11) with initial data (z_0^k, z_1^k).

We consider the space semi-discrete wave equations of (2.11) using the finite difference approximation of the Laplace operator on a uniform mesh of size $h = 1/(N+1)$, with $N \in \mathbb{N}$. This yields an approximation Λ_{Th}^η of the Gramian operator Λ_T^η. Details on the discretization and notations used can be found in Part II Section 6.1.2, see also [EZ13].

One can check that this approximation scheme satisfies Assumption 1 with[2] $s \in (0,3]$, $\theta = 2s/3$. Assumption 2 is slightly more intricate as the observation operator is not bounded. But it

[2]Though this result would have been classical in the context of distributed controls, this did not seem to be known for boundary controls. We therefore derived a detailed proof of this rate of convergence in [EZ13, Chap.2–4].

is a consequence of the multiplier identity in [IZ99, Lemma 2.2], see also Section 6.1.2. Finally, Assumption 3 is not satisfied according to [IZ99], whatever $T > 0$ is. We are thus led to consider the continuous approach.

The continuous approach. To apply our numerical method, we need estimates on C_{obs} and C_{ad}. In this 1d context, it is rather easy to get good approximations using Fourier series and Parseval's identity. Therefore, we can take $T^* = 2$, and we choose $T = 4$. We then have the estimates $C_{obs}^2 = 1/2$ and $C_{ad}^2 = 4$. With $\rho = 1/8$, we have $\delta(\rho) = \sqrt{3}/2 \simeq 0.86$. From the multiplier identity in [IZ99, Lemma 2.2], we can also estimate $\mathcal{C}_{ad}^2 \leq 6$. Choosing $\rho = 1/8$ is then admissible for the continuous approach.

In order to test our numerical method, we fix the initial data to be controlled to $y_0 = 0$ and y_1 as in Figure 3.1, so that we obviously have $(y_0, y_1) \in H_0^1(0,1) \times L^2(0,1)$. We also compute a reference control v_{ref} obtained for a much smaller $h = 1/300$, plotted in Figure 3.1, right.

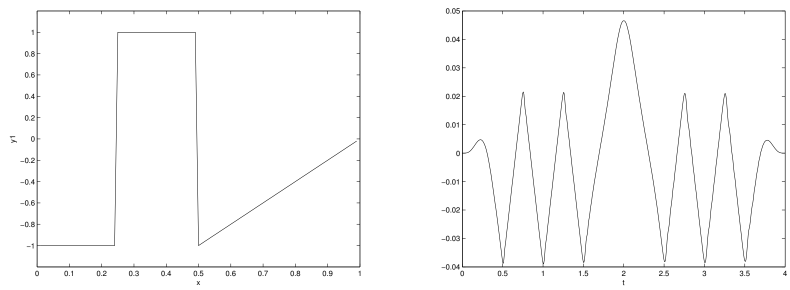

Figure 3.1: Left, the initial velocity y_1 to be controlled. Right, the reference control.

In the numerical simulations below, we represent the functions $v_h^{k,c}$ given, for $k \in \mathbb{N}$, by

$$v_h^{k,c}(t) = \eta(t) \left(\frac{z_{N+1,h}^{k,c}(t) - z_{N,h}^{k,c}(t)}{h} \right) = -\eta(t) \frac{z_{N,h}^{k,c}(t)}{h}, \quad t \in (0,4),$$

where $z_h^{k,c}$ is the solution of the numerical approximation of (2.11) with initial data $(z_{0h}^{k,c}, z_{1h}^{k,c})$, the k-th iterate of the algorithm (3.18). Indeed, this formula is the discrete counterpart of (3.29).

For $h = 1/100$, the number of iterations predicted by our method is 21. It corresponds to a relative error $\left\| v_h^{21,c} - v_{ref} \right\|_{L^2} / \left\| v_{ref} \right\|_{L^2}$ of 6.34%. It turns out that the smallest error (among the first thousands iterations) is of 6.24% and is achieved when $k = 13$, which is close to the predicted one. Figure 3.2 plots the corresponding controls.

To better illustrate how the iterative process evolves, we have run it during 50000 iterations and plot the relative error $\left\| v_h^{k,c} - v_{ref} \right\|_{L^2} / \left\| v_{ref} \right\|_{L^2}$. This is represented in Figure 3.3. As we see, the error does not reach zero but rather stays bounded from below. When looking more closely at the evolution of the error, we see that it first decays and then increases.

But the error increases rather slowly. This is due to the fact that the algorithm (3.18) follows the steepest descent algorithm for J_h^η in (3.17), which is convex quadratic. Therefore, the iterates always move forward in the direction of the gradient of the functional J_h^η. In particular, the evolution is very slow in the "bad directions" corresponding to the directions of weak coercivity of

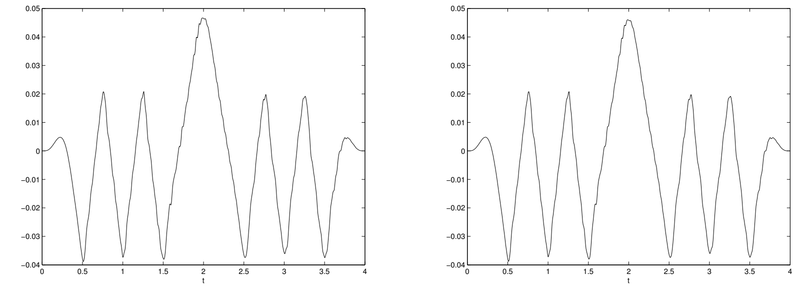

Figure 3.2: Left, the control obtained by the continuous approach at the predicted number of iterations, 21. Right, the control obtained by the continuous approach at the number of iterations 13 minimizing the relative error. In both cases, $h = 1/100$, $\rho = 1/8$.

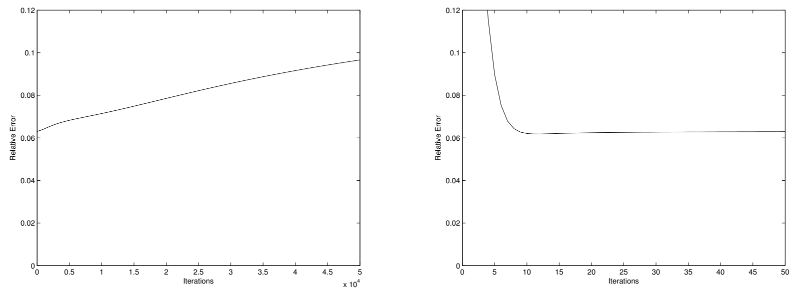

Figure 3.3: The relative error $\left\| v_h^{k,c} - v_{ref} \right\|_{L^2} / \|v_{ref}\|_{L^2}$ for the continuous approach at each iteration for $h = 1/100$, $\rho = 1/8$. Left: iterations from 0 to 50000. Right: zoom on the iterations between 0 and 50.

Λ_{Th}^{η}. In fact, from [Mic02], we know that the smallest eigenvalue in Λ_{Th} behaves like $\exp(-c/h)$, so that the algorithm indeed evolves very slowly in these directions.

However, when taking $N = 20$ - for larger N, the conjugate gradient algorithm does not even converge due to the strong degeneracy of the coercivity of Λ_{Th}^{η} -, we can compute the minimizer of the functional J_h^{η} using the conjugate gradient algorithm. The corresponding discrete exact control $v_{*,h}^c$ is plotted in Figure 3.4, right. As one sees, this exact control $v_{*,h}^c$ has a strong spurious oscillating behavior. The relative errors between the iterated controls $v_h^{k,c}$ and this limit oscillating control $v_{*,h}^c$ is plotted in Figure 3.4, left, exhibiting a slow convergence rate due to the bad conditioning of the Gramian matrix.

The continuous approach therefore requires some special care. In particular, if the algorithm is employed for a too large number of iterations k, something that can easily happen since the threshold in the number of iterations may be hard to compute in practice, the corresponding control may be very far away from the actual continuous one.

We also illustrate the rate of convergence of the continuous approach as $h \to 0$. In Figure 3.5, we plot $\log\left(\left\| v_h^{K_h^c,c} - v_{ref} \right\|\right)$ versus $|\log(h)|$. By linear regression we get the slope -1.01. This is due to the fact that y_1 is in $H^{-1+s}(0,1)$ for all $s < 3/2$, hence the convergence is expected to

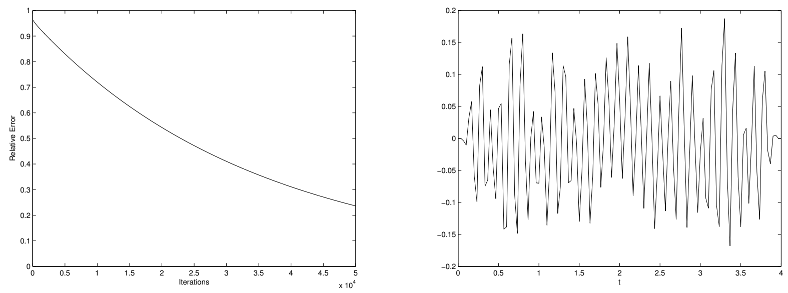

Figure 3.4: Left, the relative error $\left\| v_h^{k,c} - v_{*,h}^c \right\| / \left\| v_{*,h}^c \right\|$ for the continuous approach at each iteration k from $k = 0$ to $k = 50000$ for $h = 1/20$, $\rho = 1/8$. Right, the discrete exact control $v_{*,h}^c$ for $h = 1/20$.

be like $h^{2s/3}$ for all $s < 3/2$.

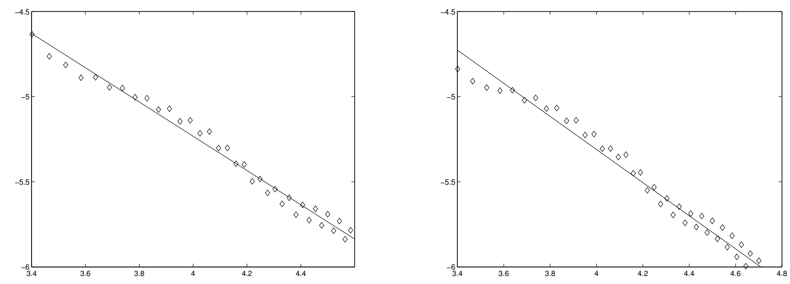

Figure 3.5: Left, convergence of the continuous approach: $\log\left(\left\| v_h^{K_h^c,c} - v_{ref} \right\|\right)$ versus $|\log(h)|$ (slope -1.01). Right, convergence of the discrete approach: $\left\| v_h^{\infty,d} - v_{ref} \right\|$ versus $|\log(h)|$ (slope -0.97).

The discrete approach. Recall that, according to [IZ99], the wave equation discretized using a finite-difference scheme on a uniform mesh does not satisfy Assumption 3. However, as noted in earlier works [GL90, Zua05], this is due to spurious high-frequency waves created by the discretization scheme, and one way to re-establish Assumption 3 simply consists in removing the frequencies higher than some threshold in the numerical scheme.

Following [IZ99], we can for instance consider the filtered spaces given for $\gamma \in (0,1)$ by

$$\mathcal{V}_h(\gamma/h) = \left\{ (z_{0h}, z_{1h}),\ s.t.\ z_{0h}, z_{1h} \in \operatorname*{Span}_{kh \leq \gamma} \left\{ (\sin(k\pi jh))_{j \in \{1,\cdots,N\}} \right\} \right\}.$$

Of course, $\mathcal{V}_h(\gamma/h)$ is a subspace of V_h. Since the functions $(w^k)_k$ defined by $w_j^k = \sqrt{2}\sin(k\pi jh)$ are eigenfunctions of the discrete Laplace operator, we can introduce the orthogonal projection

P_h^γ of V_h onto $\mathcal{V}_h(\gamma/h)$ (with respect to the scalar product of V_h) and the Gramian operator should then be replaced by

$$\Lambda_{Th}^\gamma = P_h^\gamma \Lambda_{Th}^\eta P_h^\gamma. \tag{3.30}$$

Assumptions 1 and 2 still hold for any $\gamma \in (0,1)$, with proofs similar to those in the continuous approach. Furthermore, according to [IZ99], Assumption 3 also holds when the time T is greater than $T_\gamma := 2/\cos(\pi\gamma/2)$. Note that this is not a consequence of the convergence of the numerical schemes, and this requires a thorough study of the discrete dynamics. The proof of [IZ99] uses a spectral decomposition of the solutions of the discrete wave equation and the Ingham inequality for nonharmonic Fourier series.

For $\gamma \in (0,1)$ and $T > T_\gamma$, we can then perform a conjugate gradient algorithm on the discrete functionals J_h^γ defined by $\mathcal{V}_h(\gamma/h)$:

$$J_h^\gamma(z_{0h}, z_{1h}) = \frac{1}{2}\langle \Lambda_{Th}^\gamma(z_{0h}, z_{1h}), (z_{0h}, z_{1h})\rangle_h + \langle (z_{0h}, z_{1h}), (y_{0h}, y_{1h})\rangle_h. \tag{3.31}$$

Doing this, we do not need any estimate on the admissibility and observability constants to run the algorithms.

We therefore run the algorithms for $h = 1/100$ and $h = 1/300$, $\gamma = 1/3$, and the initial data (y_0, y_1) with $y_0 = 0$ and y_1 as in Figure 3.1, see Figure 3.6. The algorithm converges very fast and it requires only 10 and 9 iterations for $h = 1/100$ and $h = 1/300$, respectively.

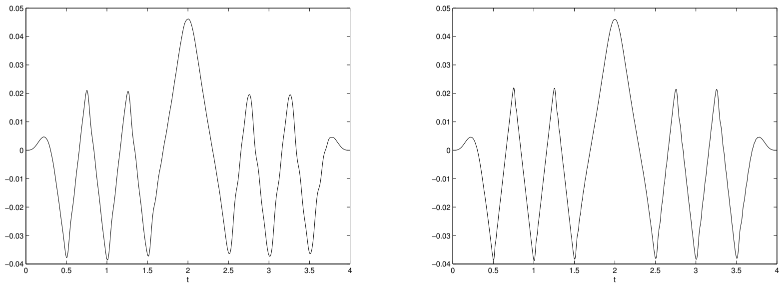

Figure 3.6: The controls $v_h^{\infty,d}$ for $h = 1/100$ (left) and $h = 1/300$ (right). We have set $v_{ref} = v_h^{\infty,d}$ for $h = 1/300$.

In the previous simulations, the quantity $v_h^{k=\infty,d}$ has been computed using the conjugate gradient method as indicated above. The reference control is the one computed for $h = 1/300$.

In Figure 3.5 right, we finally represent the rate of convergence of the discrete controls (to be compared with Figure 3.5 left). Here again, the slope is -0.97, due to the fact that y_1 is almost lying in $H^{1/2}(0,1)$.

3.5 A data assimilation problem

In this section, we discuss a data assimilation problem that can be treated by the techniques developed in this chapter.

Under the same notations as before, we consider a system driven by the equation

$$Z' = AZ, \quad t \geq 0, \qquad Z(0) = Z_0, \qquad m(t) = B^*Z(t). \tag{3.32}$$

We assume that Z_0 is not known but, instead, we know the partial measurement on the solution given by $m(t) = B^* Z(t)$ for $t \in (0,T)$. The question then is the following: Given $m \in L^2(0,T;U)$, can we reconstruct Z_0?

This problem is of course related to the study of the observation map:

$$\begin{aligned} \mathcal{O}: \quad X &\longrightarrow L^2(0,T;U) \\ z_0 &\mapsto B^* z, \end{aligned} \tag{3.33}$$

where z is the solution of (1.4) with initial data z_0.

This map $\mathcal{O}$ is well-defined in these spaces under the condition (1.6). Besides, the observability inequality (1.5) for (1.4) is completely equivalent to the fact that the map $\mathcal{O}$ has continuous inverse from $L^2(0,T;U) \cap \mathrm{Ran}(\mathcal{O})$ to X.

It is therefore natural to assume the admissibility and observability estimates (1.6) and (1.5) in order to guarantee that $\mathcal{O}$ is well-defined and invertible on its range. In order to obtain an efficient reconstruction algorithm, the most natural idea is to introduce the functional

$$\tilde{J}^\eta(z_0) = \frac{1}{2} \int_0^T \eta \left\| B^* z - m \right\|_U^2 \, dt - \frac{1}{2} \int_0^T \eta \left\| m \right\|_U^2 \, dt, \tag{3.34}$$

where z is the solution of (1.4) with initial data z_0. But $\tilde{J}^\eta$ can be rewritten as

$$\tilde{J}^\eta(z_0) = \frac{1}{2} \int_0^T \eta \left\| B^* z \right\|_U^2 \, dt - \int_0^T \eta \langle B^* z, m \rangle_U \, dt = \frac{1}{2} \int_0^T \eta \left\| B^* z \right\|_U^2 \, dt + \langle z_0, y(0) \rangle_X, \tag{3.35}$$

where $y(0)$ is given by the backward resolution of

$$y' = Ay + \eta B m, \quad t \in (0,T), \qquad y(T) = 0. \tag{3.36}$$

Under the form (3.35), the functional $\tilde{J}^\eta$ appears as a particular case of the functional J^η in (2.15), and therefore the analysis developed in this chapter applies. The continuous approach then provides similar results as the ones developed in [HR12].

3.6 Comments

The question of getting uniform observability properties for numerical approximations of wave equations is still challenging. As we have seen, this property is helpful to derive efficient algorithms to compute good approximations of the continuous controls.

In dimension higher than 2, there are a few results which prove uniform observability estimates for the wave equation: we refer to [Zua99] for the 2d case on a uniform mesh, which yields a sharp result on the scale of filtering needed to recover uniform observability properties. We also refer to [Mil12] for the d dimensional case under general approximation conditions. To our knowledge, the result in [Mil12] is the best one when considering general meshes in any dimension. Still, a precise time estimate for the uniform observability result is missing and whether the filtering scales obtained in [Mil12] are sharp is an open problem.

We also refer to the recent work [CFCM13] for another approach based on the Carleman estimate for the continuous wave equation. This approach does not produce the control of minimal L^2 norm, but rather the one minimizing the sum of the L^2-norm of the controlled trajectory and of the L^2-norm of the control. The key idea then is to use a smooth enough approximation space for which the Carleman estimate for the continuous wave equation applies, guaranteeing that way the coercivity of the method. We refer to [CFCM13] for more precise details. Though this might be costly, this computation method is interesting as it also allows to develop adaptive strategies [Mün].

So far, we have only focused on the computation of controls when considering space semi-discrete approximations of (1.1), (1.4). Of course, similar ideas apply when considering fully discrete approximations of the equations (1.1), (1.4). Let us also point out that the observability properties for fully discrete approximations of (1.1), (1.4) can be studied in two steps, first analyzing the observability properties of the time-continuous and space semi-discrete approximations of (1.1), (1.4), and then use the results presented in Chapter 9, see Section 9.3.1.

More recently, with A. Marica and E. Zuazua, we started to think again of the problem of uniform observability of the approximations of the wave equation in a different manner. The idea is that one should perhaps construct the approximation mesh depending on the controllability/observability problem under consideration instead of first discretizing and then try to compute the discrete controls on it. We already have some preliminary results in that case, showing that in 1d, if we consider a mesh which is the image of a uniform mesh by a strictly concave diffeomorphism, observability from $x = 1$ - which corresponds to the side on which the cells get smaller and smaller - holds uniformly with respect to the mesh size. This is a very natural strategy as one has more information close to the observation set, and the mesh can then be refined there. However, when going away from the observation set, there is no more reason to keep an accurate description of the solution and the mesh can be larger and larger. Note that this fact was already remarked and used in the context of inverse problems, see for instance [BDK05] where a similar idea has been discussed for recovering the conductivity of an elliptic problem from the knowledge of the Neumann-to-Dirichlet map.

Part II

On an inverse problem for the wave equation

Chapter 4

Introduction

The goal of this part is to present several results related to an inverse problem for the waves.

Roughly speaking, an inverse problem arises when one wishes to estimate some parameter(s) of a system through a measurement on the physical system. The study of the problem then usually follows the following steps:

- Uniqueness: the measurement determines uniquely the parameters.
- Stability: if two measurements are close, then the parameters are close.
- Reconstruction: given a measurement, how to reconstruct the parameters?

We refer for instance to [KT04, Isa06, Cho09, Yam09, Uhl99] for several examples of inverse problems.

Hereafter, I will focus on the following inverse problem for the waves. Given Ω a smooth bounded domain of $\mathbb{R}^d$ and $T > 0$, we consider the wave equation:

$$\begin{cases} \partial_{tt} y - \Delta y + p y = f, & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = f_\partial(t,x), & (t,x) \in (0,T) \times \partial\Omega, \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y_0, y_1), & x \in \Omega. \end{cases} \tag{4.1}$$

We assume that the source terms f, f_∂ and the initial data (y_0, y_1) are known, but the potential p is unknown. To underline the dependence of the solution y of (4.1) with respect to p, we denote it by $y[p]$.

Our goal is to understand if one can determine and reconstruct the potential p from the knowledge of the flux on some open subset Γ of the boundary defined by

$$\mathscr{M}(p) = \partial_\nu y[p](t,x), \quad (t,x) \in (0,T) \times \Gamma, \tag{4.2}$$

where ∂_ν denotes the normal derivative on the boundary.

In the following, we shall always assume that $f \in L^2(0,T; L^2(\Omega))$, $f_\partial \in H^1((0,T) \times \Gamma)$ and $(y_0, y_1) \in H^1(\Omega) \times L^2(\Omega)$ with the compatibility condition $y_0(x) = f_\partial(0,x)$ for all $x \in \partial\Omega$. Under these assumptions, $y[p]$ belongs to $C^0([0,T]; H^1(\Omega))$ and its normal derivative on the boundary belongs to $L^2((0,T) \times \partial\Omega)$. The definition (4.2) therefore makes sense in $L^2((0,T) \times \Gamma)$.

Uniqueness for the inverse problem (4.1)–(4.2) has already been studied in the literature in [BK81, Kli92], and stability results were obtained in [PY97, Yam99, IY01, IY03b, Bau] for several variants of the inverse problems (4.1)–(4.2).

I will therefore report on works focusing on the reconstruction step, i.e. the reconstruction of p from $\mathscr{M}(p)$. As in Part I, I will start by studying the inverse problem (4.1)–(4.2) and explain how we can design an algorithm to estimate the potential p from the measurement of the flux $\mathscr{M}(p)$, see Chapter 5. In Chapter 6, I will then explain the difficulties arising when trying to follow this algorithm numerically, and in particular present a theoretical convergence result. We thus achieve, at least theoretically, our goal of designing a convergent numerical method to compute p from $\mathscr{M}(p)$. However, the numerical implementation of our strategy is still challenging, as we will explain in Sections 5.3.2 and 6.5.1.

Chapter 5

Stability and reconstruction

5.1 Stability of the inverse problem (4.1)–(4.2)

The goal of this section is to explain the strategy used for the stability of the inverse problem set in (4.1)–(4.2), and how this can be used to derive a reconstruction algorithm for inverting $p \mapsto \mathscr{M}(p)$.

One of the difficulty of the inverse problem (4.1)–(4.2) is that it is non-linear in the potential p. In particular, if $y[p^a]$ and $y[p^b]$ denote the corresponding solutions of (4.1), the difference $z = y[p^a] - y[p^b]$ solves

$$\begin{cases} \partial_{tt} z - \Delta z + p^b z = y[p^a](p^b - p^a), & (t,x) \in (0,T) \times \Omega, \\ z(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ (z(0,\cdot), \partial_t z(0,\cdot)) = (0,0), & x \in \Omega, \end{cases} \tag{5.1}$$

and its flux on the part Γ of the boundary is

$$\partial_\nu z(t,x) = \mathscr{M}(p^a) - \mathscr{M}(p^b), \quad (t,x) \in (0,T) \times \Gamma. \tag{5.2}$$

5.1.1 A simplified model

In order to give some insights on our strategy, we start by focusing on the following simplified inverse problem. For z the solution of

$$\begin{cases} \partial_{tt} z - \Delta z = g, & (t,x) \in (0,T) \times \Omega, \\ z(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ (z(0,\cdot), \partial_t z(0,\cdot)) = (0,0), & x \in \Omega, \end{cases} \tag{5.3}$$

find the source term g from the measurement of

$$\partial_\nu z(t,x) \quad \text{for } (t,x) \in (0,T) \times \Gamma. \tag{5.4}$$

The first remark is that this inverse problem cannot be solved in general if g depends on both variables t and x. Indeed, if $\tilde{z}$ is a smooth function compactly supported in $(0,T) \times \Omega$, then the measurements of z and $z + \tilde{z}$ are the same while they correspond to different source terms, namely g and $g + (\partial_{tt} - \Delta)\tilde{z}$.

It is then natural to restrict ourselves to source terms g having a particular form, for instance g given with separated variables:

$$g = \alpha(t)\beta(x). \tag{5.5}$$

In the particular case $\alpha = 1$, differentiating (5.3) with respect to time, one easily sees that we are back to an observability problem as $\partial_t z$ solves:

$$\begin{cases} (\partial_{tt} - \Delta)(\partial_t z) = 0, & (t,x) \in (0,T) \times \Omega, \\ \partial_t z(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ (\partial_t z(0,\cdot), \partial_t(\partial_t z)(0,\cdot)) = (0, \beta(x)), & x \in \Omega. \end{cases} \tag{5.6}$$

Therefore, in the case $\alpha = 1$, the possibility of reconstructing g - equivalently β - from the Neumann boundary measurement, is equivalent to an observability inequality for the wave equation when observed from $(0,T) \times \Gamma$, recall Section 3.5.

In the following, we shall therefore assume that the wave equation in Ω is observable through Γ in time $T > 0$. Thanks to [BLR92, BG97], this is equivalent to the geometric control condition for (Γ, Ω, T).

For general $\alpha \in W^{1,1}(0,T)$, time differentiation of (5.3) yields

$$\begin{cases} (\partial_{tt} - \Delta)(\partial_t z) = (\partial_t \alpha)\beta, & (t,x) \in (0,T) \times \Omega, \\ \partial_t z(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega, \\ (\partial_t z(0,\cdot), \partial_t(\partial_t z)(0,\cdot)) = (0, \alpha(0)\beta(x)), & x \in \Omega. \end{cases} \tag{5.7}$$

Therefore, using the observability inequality for the wave equation in Ω, we get

$$\begin{aligned} |\alpha(0)| \, \|\beta\|_{L^2(\Omega)} = \|\alpha(0)\beta\|_{L^2(\Omega)} &\leq C \|\partial_\nu \partial_t z\|_{L^2((0,T)\times\Gamma)} + C \|(\partial_t \alpha)\beta\|_{L^1(0,T;L^2(\Omega))} \\ &\leq C \|\partial_\nu \partial_t z\|_{L^2((0,T)\times\Gamma)} + C \|\partial_t \alpha\|_{L^1(0,T)} \|\beta\|_{L^2(\Omega)} . \end{aligned}$$

Therefore, if $\|\partial_t \alpha\|_{L^1(0,T)}$ is small enough compared to $|\alpha(0)|$, i.e.

$$|\alpha(0)| > C \|\partial_t \alpha\|_{L^1(0,T)} , \tag{5.8}$$

we obtain an estimate on the $L^2(\Omega)$-norm of β immediately.

Remark 5.1. *When α is a given function in $W^{1,1}(0,T)$, then one can identify β in* (5.5) *from the measurement* (5.3) *under the geometric control condition. We refer for instance to the works [Yam95, ASTT09, TW14] for more detailed discussions of related cases.*

5.1.2 The full model: preliminaries

Following Section 5.1.1, we would like to differentiate the equation (5.1) with respect to the time variable. We therefore assume the potentials $p = p(x)$ to be independent of the time variable t.

In that case, differentiating the equation (5.1) with respect to time, we get

$$\begin{cases} (\partial_{tt} - \Delta + p^b)(\partial_t z) = \partial_t y[p^a](p^b - p^a), & (t,x) \in (0,T) \times \Omega, \\ \partial_t z(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ (\partial_t z(0,\cdot), \partial_t(\partial_t z)(0,\cdot)) = (0, y_0(p^b - p^a)), & x \in \Omega. \end{cases} \tag{5.9}$$

As before, we assume that the wave equation with potential p^a is observable through Γ in time T. To be more precise, we shall consider a class $\mathscr{C}$ of potentials $p = p(x)$ such that for all $p \in \mathscr{C}$, there exist two constants $C_{obs}(p)$, $C_0(p)$ such that all solutions of

$$\begin{cases} (\partial_{tt} - \Delta + p)w = g, & (t,x) \in (0,T) \times \Omega, \\ w(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ (w(0,\cdot), \partial_t w(0,\cdot)) = (w_0, w_1), & x \in \Omega, \end{cases} \tag{5.10}$$

satisfy

$$\|(w_0, w_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq C_{obs}(p) \|\partial_\nu w\|_{L^2((0,T)\times\Gamma)} + C_0(p) \|g\|_{L^1(0,T;L^2(\Omega))} . \tag{5.11}$$

Working in such class $\mathscr{C}$ and applying (5.11) to the function $\partial_t z$ solving (5.9), we obtain

$$\left\|y^0(p^b - p^a)\right\|_{L^2(\Omega)} \leq C_{obs}(p^b) \|\partial_\nu \partial_t z\|_{L^2((0,T)\times\Gamma)} + C_0(p^b) \left\|\partial_t y[p^a](p^b - p^a)\right\|_{L^1(0,T;L^2(\Omega))} .$$

In particular, this yields:

$$\left(\inf_\Omega |y_0| - C_0(p^b) \|\partial_t y[p^a]\|_{L^1(0,T;L^\infty(\Omega))}\right) \left\|p^b - p^a\right\|_{L^2(\Omega)} \leq C_{obs}(p^b) \|\partial_\nu \partial_t z\|_{L^2((0,T)\times\Gamma)} . \tag{5.12}$$

In other words, we have a stability estimate in the class $\mathscr{C}$ of potentials $p = p(x)$ if

$$\inf_\Omega |y_0| > C_0(p) \|\partial_t y[p^a]\|_{L^1(0,T;L^\infty(\Omega))} , \tag{5.13}$$

to be compared with (5.8).

Of course, for the stability condition (5.13) and the stability estimate (5.12) to be valid for all potentials in the class $\mathscr{C}$, one should have observability estimates (5.11) uniformly in the class $\mathscr{C}$. In that case, introducing

$$C_{obs}(\mathscr{C}) = \sup_{\mathscr{C}} C_{obs}(p), \quad C_0(\mathscr{C}) = \sup_{\mathscr{C}} C_0(p), \tag{5.14}$$

we get the following result: if $p^a, p^b \in \mathscr{C}$ and p^a is such that $y[p^a]$ belongs to $W^{1,1}(0,T;L^\infty(\Omega))$ with

$$\inf_\Omega |y_0| > C_0(\mathscr{C}) \|\partial_t y[p^a]\|_{L^1(0,T;L^\infty(\Omega))} , \tag{5.15}$$

then we have the stability estimate

$$\begin{aligned}\left(\inf_\Omega |y_0| - C_0(\mathscr{C}) \|\partial_t y[p^a]\|_{L^1(0,T;L^\infty(\Omega))}\right) \left\|p^b - p^a\right\|_{L^2(\Omega)}& \\ \leq C_{obs}(\mathscr{C}) \left\|\partial_t \mathscr{M}(p^b) - \partial_t \mathscr{M}(p^a)\right\|_{L^2(0,T;L^2(\Gamma))} .&\end{aligned} \tag{5.16}$$

The class of potentials $\mathscr{C}$ could be for instance the class

$$L^\infty_{\leq m}(\Omega) = \{p \in L^\infty(\Omega) \mid \|p\|_{L^\infty(\Omega)} \leq m\},$$

see [BLR92]. Let us also emphasize that for $\mathscr{C} = L^\infty_{\leq m}(\Omega)$ for some $m > 0$, $y_0 \in L^\infty(\Omega)$ and $y[p^a] \in W^{1,1}(0,T;L^\infty(\Omega))$, estimate (5.16) is necessarily sharp, as hidden-regularity estimates for $\partial_t z$ in (5.9) imply

$$\begin{aligned}&\left\|\partial_t \mathscr{M}(p^b) - \partial_t \mathscr{M}(p^a)\right\|_{L^2(0,T;L^2(\Gamma))} = \|\partial_\nu \partial_t z\|_{L^2(0,T;L^2(\Gamma))} \\ &\leq C_m \left\|y_0(p^b - p^a)\right\|_{L^2(\Omega)} + C_m \left\|\partial_t y[p^a](p^b - p^a)\right\|_{L^1(0,T;L^2(\Omega))} \\ &\leq C_m \left(\|y_0\|_{L^\infty(\Omega)} + \|\partial_t y[p^a]\|_{L^1(0,T;L^\infty(\Omega))}\right) \left\|p^b - p^a\right\|_{L^2(\Omega)} .\end{aligned}$$

5.1.3 The full model: main stability result

In this section, we present the results obtained in [Bau], closely related to the works [IY01, IY03b]. For other related results, we refer to [BK81, Kli92, PY96, PY97, Yam99].

These results require the following assumptions, originally due to [Ho86]. We say that the triplet (Γ, Ω, T) satisfy the Gamma-conditions (see [Lio88a]) if

- (Γ, Ω) satisfies the geometric condition:

$$\exists x_0 \in \mathbb{R}^N \setminus \overline{\Omega}, \quad \{x \in \partial\Omega, \text{ s.t. } (x - x_0) \cdot \nu(x) \geq 0\} \subset \Gamma, \tag{5.17}$$

  where $\nu = \nu(x)$ is the outgoing normal at $x \in \Omega$.

- T satisfies the lower bound:

$$T > \sup_{x \in \Omega} |x - x_0|. \tag{5.18}$$

Under these geometric conditions, the following stability result was proved:

Theorem 5.2 ([Bau]). *Let $m > 0$, and consider a potential $p^a \in L^\infty_{\leq m}(\Omega)$ such that for some $K > 0$*

$$y[p^a] \in H^1(0, T; L^\infty(\Omega)) \quad \textit{with} \quad \|y[p^a]\|_{H^1(0,T;L^\infty(\Omega))} \leq K, \tag{5.19}$$

where $y[p^a]$ denotes the solution of (4.1) *with potential p^a. Also assume the following positivity condition:*

$$\exists \alpha > 0, \quad \inf_{x \in \Omega} |y_0(x)| \geq \alpha. \tag{5.20}$$

Let us further assume that (Γ, Ω, T) satisfies the Gamma-conditions (5.17)–(5.18)*. Then there exists a constant $C > 0$ depending on m, K and α such that for all $p^b \in L^\infty_{\leq m}(\Omega)$, $\mathscr{M}(p^a) - \mathscr{M}(p^b) \in H^1(0, T; L^2(\Gamma))$ and*

$$\frac{1}{C} \left\| p^a - p^b \right\|_{L^2(\Omega)} \leq \left\| \mathscr{M}(p^a) - \mathscr{M}(p^b) \right\|_{H^1(0,T;L^2(\Gamma))} \leq C \left\| p^a - p^b \right\|_{L^2(\Omega)}. \tag{5.21}$$

Let us give some comments on Theorem 5.2.

First, Theorem 5.2 does not give any precise assumption on the smoothness of the data y^0, y^1, f, f_∂ directly. They rather appear through the bound K in (5.19) in an intricate way.

Second, the geometric condition (5.17)–(5.18) is stronger than the Geometric Control Condition of C. Bardos, G. Lebeau and J. Rauch [BLR92]. To our knowledge, only the recent result [SU13] gets stability under this sharp geometric condition and under the assumption that one can cover the domain with a foliation of strictly convex surfaces (this is needed to get local Carleman estimates, see e.g. [Tat96]). We refer to [SU13] for a more accurate description of the assumptions required, see also Section 5.3.1 for more comments. However, as we did not manage to get a reconstruction algorithm from it, we focus our presentation on Theorem 5.2.

Below, we rapidly present the proof of (5.17)–(5.18). As we will see, these conditions come from the derivation of a Carleman estimate for the wave equation. There is a huge literature on that topic, starting from the pioneering works of Klibanov [BK81, Kli92]. Regarding Carleman estimates, we refer to the works [FI96, Zha00, Ima02, FYZ07] among many others.

5.1.4 Sketch of the proof of Theorem 5.2

Let us briefly give a sketch of the proof of Theorem 5.2. The main idea is to modify the norms appearing in (5.16) so that the coefficient in the right hand side can be made positive by suitably tuning them.

In order to do that, Carleman estimates are used. Assume that Γ satisfies (5.17) for some $x_0 \notin \overline{\Omega}$. According to (5.18), we can choose $\beta \in (0,1)$ such that

$$\beta T > \sup_{x \in \Omega} |x - x_0|. \tag{5.22}$$

We then introduce the Carleman weights defined for $(t,x) \in (-T,T) \times \Omega$ by

$$\psi(t,x) = |x - x_0|^2 - \beta t^2 + C_0, \quad \text{and for } \lambda > 0, \quad \varphi(t,x) = e^{\lambda \psi(t,x)}, \tag{5.23}$$

where $C_0 > 0$ is chosen such that $\psi \geq 1$ in $(-T,T) \times \Omega$.

We then have the following estimate:

Theorem 5.3. *Assume the multiplier condition* (5.17) *and the time condition* (5.22)*. Then there exists $\lambda > 0$ such that for φ as in* (5.23)*, there exist $s_0 = s_0(m) > 0$ and a positive constant $M = M(m)$ such that for all $p \in L^\infty_{\leq m}(\Omega)$, for all $s \geq s_0$ and for all $w \in L^2(-T,T;H^1_0(\Omega))$ satisfying $\partial_t^2 w - \Delta w + pw \in L^2((-T,T) \times \Omega)$ and $\partial_\nu w \in L^2((-T,T) \times \Gamma)$,*

$$s \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} \left(|\partial_t w|^2 + |\nabla w|^2 \right) dxdt + s^3 \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |w|^2 \, dxdt$$
$$\leq M \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |\partial_{tt} w - \Delta w + pw|^2 \, dxdt + Ms \int_{-T}^{T} \int_{\Gamma} e^{2s\varphi} \left| \partial_\nu w \right|^2 \, d\sigma dt, \tag{5.24}$$

If w furthermore satisfies $w(0,\cdot) = 0$ in Ω, one also has

$$s^{1/2} \int_{\Omega} e^{2s\varphi(0)} |\partial_t w(0)|^2 \, dx \leq M \int_{-T}^{T} \int_{\Omega} e^{2s\varphi} |\partial_t^2 w - \Delta w + pw|^2 \, dxdt$$
$$+ Ms \int_{-T}^{T} \int_{\Gamma} e^{2s\varphi} \left| \partial_\nu w \right|^2 \, d\sigma dt. \tag{5.25}$$

As stated, Theorem 5.3 can be found in our recent work [BDBE13], but it is used in several places in the literature. Estimate (5.24) can be found in the articles [FI96, Ima02] and is also deeply related to other Carleman estimates for hyperbolic equations, see e.g. [Zha00, FYZ07]. The additional estimate (5.25) is hidden in [IY01, IY03b, Bau].

Theorem 5.3 gives Carleman estimates for the wave operator with a potential in $L^\infty(\Omega)$. This actually follows easily from the Carleman estimates (5.24)–(5.25) with potential $p = 0$ by choosing $s_0(m) = \max\{s_0(0), s_1(m)\}$ with $s_1(m)^3 = 2M(0)m^2$.

One can apply (5.25) to $w = \partial_t z = \partial_t (y[p^a] - y[p^b])$ in (5.9) extended oddly on $(-T,T)$. We then obtain,

$$s^{1/2} \left\| e^{s\varphi(0)} y_0 (p^a - p^b) \right\|^2_{L^2(\Omega)} \leq 2M \left\| e^{s\varphi} \partial_t y[p^a](p^b - p^a) \right\|^2_{L^2(0,T;L^2(\Omega))}$$
$$+ 2Ms \left\| e^{s\varphi} \partial_t (\mathscr{M}(p^a) - \mathscr{M}(p^b)) \right\|^2_{L^2(0,T;L^2(\Gamma))} .$$

Using that for all x, $t \mapsto \varphi(t,x)$ decays on $[0,T]$, we get, instead of (5.16),

$$\left(s^{1/2}\inf_{\Omega}|y_0|^2 - 2M\left\|\partial_t y[p^a]\right\|^2_{L^2(0,T;L^\infty(\Omega))}\right)\left\|e^{s\varphi(0)}(p^a-p^b)\right\|^2_{L^2(\Omega)} \leq 2Ms\left\|e^{s\varphi}\partial_t(\mathscr{M}(p^a)-\mathscr{M}(p^b))\right\|^2_{L^2(0,T;L^2(\Gamma))}. \quad (5.26)$$

Using (5.19) and (5.20), we can take s large enough so that the left hand-side becomes positive, thus yielding (5.21).

Remark that this proof mainly consists in considering (5.15) in the weighted norms corresponding to the Carleman estimates. Namely, looking at the Carleman norms with parameter s, $C_0(\mathscr{C}_m)$ in the corresponding version of (5.14) would be of the form $2M(m)/\sqrt{s}$, which can be made arbitrarily small by taking s large enough, so that the corresponding weighted version of (5.15) is satisfied.

5.2 A reconstruction algorithm

5.2.1 Statement

The idea now is to take advantage of the Carleman estimate of Theorem 5.3 to design a reconstruction algorithm for the potential from the knowledge of the flux of the solution.

To be more precise, we assume that we want to recover the potential $P \in L^\infty(\Omega)$, on which we have an a priori estimate of the form

$$\|P\|_{L^\infty(\Omega)} \leq m. \quad (5.27)$$

We denote by Y the corresponding solution to (4.1) with potential P, and we call $\mathscr{M}(P)$ the corresponding measurement of the flux on $(0,T)\times\Gamma$, still under the geometric conditions (5.17)–(5.18).

We also assume that Y satisfies the regularity assumption

$$Y \in H^1(0,T;L^\infty(\Omega)), \quad (5.28)$$

and that the initial condition y_0 enjoys the positivity condition (5.20).

An iterative process. To reconstruct P from $\mathscr{M}(P)$ $(= \partial_\nu Y[P]|_{(0,T)\times\Gamma})$, we propose the following algorithm:

- Initialization: $p^0 = 0$.
- Iteration. Given p^k, we set $\mu_k = \partial_t\left(\partial_\nu y[p^k] - \partial_\nu Y[P]\right)$ on $(0,T)\times\Gamma$, where $y[p^k]$ denotes the solution of (4.1) with potential p^k.
  We then introduce the functional $J_{s,p^k}[\mu_k,0]$ defined, for some $s \geq 1$ that will be chosen independently of k, by

$$J_{s,p^k}[\mu_k,0](w) = \frac{1}{2s}\int_0^T\int_\Omega e^{2s\varphi}|\partial_{tt}w - \Delta w + p^k w|^2\,dxdt + \frac{1}{2}\int_0^T\int_\Gamma e^{2s\varphi}|\partial_\nu w - \mu_k|^2\,d\sigma dt, \quad (5.29)$$

on the trajectories $w \in L^2(0,T;H_0^1(\Omega))$ such that $\partial_{tt}w - \Delta w + p^k w \in L^2((0,T)\times\Omega)$, $\partial_\nu w \in L^2((0,T)\times\Gamma)$ and $w(0,\cdot) = 0$ in Ω.
Let W^k be the unique minimizer of the functional $J_{s,p^k}[\mu_k, 0]$, and then set

$$\tilde{p}^{k+1} = p^k + \frac{\partial_t W^k(0,\cdot)}{y_0}, \tag{5.30}$$

where y_0 is the initial condition in (4.1).
Finally, set

$$\forall x \in \Omega,\ p^{k+1}(x) = T_m(\tilde{p}^{k+1}(x)), \qquad \text{where } T_m(\rho) = \begin{cases} \rho, & \text{if } |\rho| \leq m, \\ \operatorname{sign}(\rho)m, & \text{if } |\rho| \geq m. \end{cases} \tag{5.31}$$

According to Theorem 5.3, the above algorithm is well-posed provided that we choose β in the construction of the weight function (5.23) such that (5.22) is satisfied. Indeed, in that case, Theorem 5.3 implies that for all $k \geq 0$, the functional $J_{s,p^k}[\mu_k, 0]$ in (5.29) is coercive and strictly convex and therefore possess a unique minimizer.

With L. Baudouin and M. de Buhan we obtained the following convergence result:

Theorem 5.4 ([BDBE13]). *Assuming the multiplier condition* (5.17)*, the time condition* (5.22)*, the positivity condition* (5.20)*, the regularity condition* (5.28) *and the a priori assumption* (5.27)*, there exists a constant $M > 0$ such that for all $s \geq s_0(m)$ and $k \in \mathbb{N}$,*

$$\int_\Omega e^{2s\varphi(0)}(p^{k+1} - P)^2\,dx \leq \frac{M}{\sqrt{s}} \int_\Omega e^{2s\varphi(0)}(p^k - P)^2\,dx. \tag{5.32}$$

In particular, when s is large enough, the above algorithm converges.

Let us emphasize that this algorithm is constructive. Indeed, each iteration consists in a minimization of a quadratic strictly convex functional. Besides, the algorithm converges with no initial a priori guess. This is a great advantage compared to the classical methods for solving this inverse problems usually based on a non-linear least square method, see for instance [GGS13], i.e. on the minimization of the functional

$$J(p) = \int_0^T \int_\Gamma |\partial_\nu y[p] - \partial_\nu Y[P]|^2 d\sigma dt, \tag{5.33}$$

$y[p]$ being the solution of (4.1) corresponding to potential p. But this minimization problem is not convex and may have several local minima. Therefore, classical minimization algorithms are not guaranteed to converge to the global minimum of J if the initial guess is not close to the target potential P. We will later comment on the possible applicability of the above algorithm into numerics. But one can already see that, due to the presence of double exponentials in the functionals $J_{s,p_k}[\mu_k, 0]$, the numerical implementation of the above algorithm is not straightforward.

5.2.2 Sketch of the proof of Theorem 5.4

The idea beyond the algorithm presented in Section 5.2.1 is to try to estimate

$$w_k = \partial_t \left(y[p^k] - Y[P] \right),$$

which satisfies:

$$\begin{cases} \partial_{tt} w_k - \Delta w_k + p^k w_k = \partial_t Y[P](P - p^k), & (t,x) \in (0,T) \times \Omega, \\ w_k(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ (w_k(0,\cdot), \partial_t w_k(0,\cdot)) = (0, y_0(P - p^k)), & x \in \Omega, \end{cases} \tag{5.34}$$

while its flux on the part Γ of the boundary is

$$\partial_\nu w_k(t,x) = \mu_k(t,x), \quad (t,x) \in (0,T) \times \Gamma. \tag{5.35}$$

The idea then is to try to estimate w_k solution of (5.34) from the knowledge of its flux μ_k on the boundary Γ. The problem is that we do not know a priori the source term in (5.34). As the proof of the stability result given in Section 5.1.4 indicates that the source term in (5.34) brings less informations than the normal derivative on $(0,T) \times \Gamma$, we can therefore simply try to approximate w_k by minimizing the functional $J_{s,p^k}[\mu_k, 0]$ in (5.29), i.e. by trying to fit w with the flux condition $\partial_\nu w \simeq \mu_k$ and the source term $\partial_{tt} w - \Delta w + p^k w \simeq 0$.

Indeed, by construction, setting $g_k = \partial_t Y[P](P - p^k)$, w_k minimizes the functional

$$J_{s,p^k}[\mu_k, g_k](w) = \frac{1}{2s} \int_0^T \int_\Omega e^{2s\varphi} |\partial_{tt} w - \Delta w + p^k w - g_k|^2 \, dxdt + \frac{1}{2} \int_0^T \int_\Gamma e^{2s\varphi} |\partial_\nu w - \mu_k|^2 \, d\sigma dt, \tag{5.36}$$

over the set of $w \in L^2(0,T; H_0^1(\Omega))$, $\partial_{tt} w - \Delta w + p^k w \in L^2((0,T) \times \Omega)$, $\partial_\nu w \in L^2((0,T) \times \Gamma)$ and $w(\cdot, 0) = 0$ in Ω.

Therefore, comparing w_k to W_k amounts to compare the minimizers of $J_{s,p^k}[\mu_k, g_k]$ and $J_{s,p^k}[\mu_k, 0]$. But using the Carleman estimates (5.24)–(5.25), one easily shows that for all $p \in L^\infty_{\leq m}(\Omega)$, there exist positive constants $s_0(m)$ and $M = M(m)$ such that for $s \geq s_0(m)$, $\mu \in L^2((0,T) \times \Gamma)$ and $g^a, g^b \in L^2((0,T) \times \Omega)$, the minimizers w^a of $J_{s,p}[\mu, g^a]$ and w^b of $J_{s,p}[\mu, g^b]$ satisfy:

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t w^a(0) - \partial_t w^b(0)|^2 \, dx \leq M \int_0^T \int_\Omega e^{2s\varphi} |g^a - g^b|^2 \, dxdt. \tag{5.37}$$

Using (5.37) for $p = p^k$ and $g^a = g_k$, $g^b = 0$, we readily obtain

$$s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t w_k(0) - \partial_t W_k(0)|^2 \, dx \leq M \int_0^T \int_\Omega e^{2s\varphi} |g_k|^2 \, dxdt. \tag{5.38}$$

Therefore, using $\partial_t w_k(0) = y_0(P - p^k)$, setting $\tilde{p}^{k+1}$ as in (5.30) and bounding the right hand-side, we obtain

$$s^{1/2} \left\| e^{s\varphi(0)} (P - \tilde{p}^{k+1}) \right\|^2_{L^2(\Omega)} \leq \frac{M \left\| \partial_t Y[P] \right\|^2_{L^2(L^\infty)}}{\inf_\Omega |y_0|^2} \left\| e^{s\varphi(0)} (P - p^k) \right\|^2_{L^2(\Omega)}.$$

The last step of the iteration argument only guarantees that p^{k+1} stays in the set $L^\infty_{\leq m}(\Omega)$. Of course, due to the a priori estimate (5.27), the last estimate also holds for p^{k+1}, thus concluding the proof of Theorem 5.4.

5.3 Comments

5.3.1 On the geometric control condition

Designing a convergent algorithm similar to the one in Section 5.2.1 under the sharp geometric control condition of [BLR88, BLR92] is an open problem, though, as we have said, [SU13] obtained a stability result under this sharp geometric control condition and the assumption that the domain can be foliated with strictly convex surfaces. Basically, this condition is needed to get local Carleman estimates guaranteeing uniqueness.

This condition therefore reminds the one in [Tat96] for the unique continuation of the wave equation when lower order terms are allowed. Though analyticity of the operator implies unique continuation thanks to the classical Holmgren's uniqueness theorem, see e.g. [Hör63], the situation is way more intricate when the coefficients in the lower order terms are only assumed to be $C^\infty_{t,x}$. In that case, unique continuation across non-characteristic surfaces holds if the surface is strictly pseudo-convex [Tat96], while it may fail otherwise [AB95].

A typical problem where this plays a critical role is for the observability of the wave equation with potentials p depending on both time and space variables. When the potential $p \in L^\infty(\Omega)$ depends on the space variable only, observability holds under the GCC, thanks to the propagation of singularities [Leb94] and the propagation of regularity [MS78, MS82]. Actually, in [DE], B. Dehman and myself have slightly improve that result to potentials $L^q(\Omega)$ for $q > \max\{d, 2\}$ in the case of a distributed observation, showing in particular that the observability constant is uniform in balls of $L^q(\Omega)$ for $q > \max\{d, 2\}$.

But when the potentials depend on both time and space variables, the situation differs drastically: when the geometric control condition of [BLR88] is satisfied, observability is known for potentials p which are analytic in the time variable with values in $L^\infty(\Omega)$, see [RZ98, Tat95]; when the Gamma conditions (5.17)–(5.18) are satisfied, observability holds for potentials in $L^\infty(0, T; L^d(\Omega))$ as an immediate consequence of the Carleman estimate in Theorem 5.3.

Back to the observability problem (4.1)–(4.2), one could argue that the potential p only depends on the space variable. However, the source term depends on both time and space variables, thus requiring the use of local Carleman estimates to get uniqueness.

Keeping that in mind, in [DE], we nevertheless considered the problem of recovering an initial data from a measurement when doing an error on the potentials, assumed to depend only on the space variable, with an algorithm in the spirit of Section 3.5. There, we managed to show that the errors on the potentials only create minor errors on the high-frequency of the recovered initial data, with quantified estimates on the error, see [DE, Theorem 1.3] for precise statements. However, these ideas do not seem to be adaptable to solve the inverse problem (4.1)–(4.2).

5.3.2 On the numerical approximation of the algorithm

As we said, our proposed algorithm is based on the minimization of quadratic convex functionals at each iterations. We therefore expect it to be well-suited for numerical implementations. However, the functionals (5.36) contains weight functions involving double exponentials. This is numerically intractable.

With L. Baudouin and M. de Buhan, we are currently investigating the possibility to derive efficient numerical algorithms to compute the potential P based on the proposed algorithm in Section 5.2.1. In order to do that, we have first removed one of the exponentials and replaced the function φ in the functionals by ψ. This can be done following basically the same ideas as the one developed here, to the price of adding some new terms in the functional supported in the set $\{\psi \leq 0\}$. Actually, this was the weight function used to prove stability results for a similar inverse problem in [IY01].

But it is still not enough to guarantee the good behavior of the algorithm as we prove that it converges for sufficiently large parameters. We therefore propose to work only on the conjugated variable $e^{s\psi}w$. This has the advantage of removing all the exponentials from the equations of $\tilde{w} = e^{s\psi}w$, acting as a preconditioner. Also note that we could try to derive a more "step by step" algorithm by trying to first get good approximations of the potential P close to the observation set. We are currently exploring these ideas numerically.

Let us also mention that here we do not mention the difficulties arising from the discretization effects, which provide several new phenomena that should be handled as well, see next chapter.

Chapter 6

Convergence issues for the inverse problem (4.1)–(4.2)

6.1 Introduction

6.1.1 Presentation of the problem

In this chapter, following the works [BE13, BEO], our goal is to discuss convergence issues for the inverse problem (4.1)–(4.2) and its numerical approximations.

To be more precise, we assume that we have a potential P satisfying the bound (5.27) with some parameter $m > 0$, for which we know that the corresponding trajectory $Y = Y[P]$ of (4.1) satisfies the regularity assumption (5.28) and we know the measurement $\mathscr{M}(P)$. We also assume that the initial condition y_0 satisfies the positivity condition (5.20), and we furthermore assume that we are in the geometric setting of Theorem 5.2, i.e. that (Γ, Ω, T) satisfies the Gamma-conditions (5.17)–(5.18).

When trying to numerically compute the potential P, it is natural to discretize the wave equation (4.1) and then to try to fit the approximation of the flux given numerically with $\mathscr{M}(P)$ by suitably choosing the discrete potential. To be more specific, we consider a space semi-discrete approximation of the wave equation (4.1), that we write for simplicity

$$\left\{ \begin{array}{ll} \partial_{tt} y_h - \Delta_h y_h + p_h y_h = f_h, & (t, x_h) \in (0,T) \times \Omega_h, \\ y_h(t, x_h) = f_{\partial,h}, & (t, x_h) \in (0,T) \times \partial\Omega_h, \\ (y_h(0,\cdot), \partial_t y_h(0,\cdot)) = (y_{0h}, y_{1h}), & x_h \in \Omega_h, \end{array} \right. \tag{6.1}$$

where Ω_h is a mesh approximating Ω of mesh size $h > 0$, Δ_h is an approximation of the Laplace operator, and y_h, p_h, f_h, $f_{\partial,h}$, y_{0h}, y_{1h} denote suitable approximations of y, p, f, f_∂, y_0, y_1. Precise statements will be given later in the chapter.

We also introduce an approximation of the flux on the boundary $\partial_{\nu,h} y_h$, and define

$$\mathscr{M}_h(p_h) = \partial_{\nu,h} y_h(t, x_h) \text{ for } (t, x_h) \in (0,T) \times \Gamma_h, \tag{6.2}$$

where Γ_h is an approximation of Γ.

In the sequel, we address the following issue: if p_h is a sequence of discrete potentials in $L^\infty(\Omega_h)$ with $\|p_h\|_{L^\infty(\Omega_h)} \leq m$ such that $\mathscr{M}_h(p_h)$ converges to $\mathscr{M}(P)$ as $h \to 0$ in a suitable topology, do we have the convergence $p_h \to P$ as $h \to 0$?

Though such a statement would be very natural, it turns out that it is difficult to verify. This is due to the possible presence of spurious high-frequency waves in the numerical approximations (6.1), see e.g. [Tre82], [Zua05] for the consequences on the observability problems and Section 6.1.2 afterwards for more comments. Also note that there is a priori no reason for this convergence issue to be independent of the numerical scheme employed for discretizing (4.1).

In order to study the convergence of the discrete inverse problems, we will follow the classical strategy of Lax and rely on the two following properties:

- Stability of the discrete inverse problems. The discrete counterpart of the stability estimate (5.21) has to hold uniformly with respect to the discretization parameter h.
- Consistency of the discrete inverse problems. For any potential P satisfying our assumption, there exists a sequence of potentials p_h such that $\mathscr{M}_h(p_h)$ converges to $\mathscr{M}(P)$ (in a suitable topology).

Of course, the most difficult part of the argument is the uniform stability of the discrete inverse problems. This will be done following the strategy developed in the continuous case, recall Sections 5.1.3–5.1.4. In particular, this will led us to develop Carleman estimates for space semi-discrete wave equations, which have to hold uniformly with respect to the discretization parameter. In order to do this, we will follow the computations made in [Bau] in the discrete setting, and develop discrete computations in the spirit of the discrete Carleman estimates derived in [BHLR10a] for elliptic equations, and further developed in [BHLR10b, BHLR11, BR13, EdG11] (see also [KS91] for an earlier elliptic discrete Carleman estimate, but with a non-optimal restriction on the Carleman parameter).

Similarly as in the continuous case, see Section 5.1.1, we start by studying the observability properties of the discrete wave equation, that we briefly recall hereafter. We then present discrete Carleman estimates for the 1d wave equation discretized in space using the finite-difference method.

6.1.2 Observability properties of space semi-discrete wave equations

Let us focus on the case of the 1d wave equation

$$\begin{cases} \partial_{tt} w - \Delta w = 0, & (t,x) \in (0,T) \times (0,1), \\ w(t,0) = w(t,1) = 0, & t \in (0,T), \\ (w(0,\cdot), \partial_t w(0,\cdot)) = (w_0, w_1), & x \in (0,1), \end{cases} \tag{6.3}$$

observed from the boundary $x = 1$, i.e. observed from $\partial_x w(t,1)$, $t \in (0,T)$. It is well-known that this observation operator is admissible and that observability holds for $T \geq 2$, see [Lio88a]. This can be proved (for $T > 2$) using for instance the multiplier technique, i.e. multiplying equation (6.3) by $x\partial_x w$ and integrating. We then obtain

$$2 \int_0^1 x \partial_x w(t,x) \partial_t w(t,x)\, dx \bigg|_0^T + \int_0^T \|(w(t), \partial_t w(t))\|^2_{H^1_0(0,1) \times L^2(0,1)}\, dt = \int_0^T |\partial_x w(t,1)|^2\, dt. \tag{6.4}$$

Using that the energy of solutions of (6.3) is preserved, we immediately obtain the observability inequality

$$(T-2)\, \|(w_0, w_1)\|^2_{H^1_0(0,1) \times L^2(0,1)} \leq \int_0^T |\partial_x w(t,1)|^2\, dt. \tag{6.5}$$

Let us then present what happens for the space semi-discrete wave equation, discretized using the finite difference method. For $N \in \mathbb{N}^*$, we set $h = 1/(N+1)$ and $(0,1)_h = \{jh,\, j \in \{1, \cdots, N\}\}$, $[0,1)_h = \{jh,\, j \in \{0, \cdots, N\}\}$. For $x_h \in (0,1)_h$, we define

$$\Delta_h w_h(x_h) = \frac{1}{h^2}\left(w_h(x_h + h) + w_h(x_h - h) - 2w_h(x_h)\right).$$

The semi-discrete approximation of the wave equation (6.3) then reads as follows:

$$\begin{cases} \partial_{tt} w_h - \Delta_h w_h = 0, & (t, x_h) \in (0,T) \times (0,1)_h, \\ w_h(t,0) = w_h(t,1) = 0, & t \in (0,T), \\ (w_h(0,\cdot), \partial_t w_h(0,\cdot)) = (w_{0h}, w_{1h}), & x_h \in (0,1)_h. \end{cases} \tag{6.6}$$

To go further, it is convenient to introduce some notations. For $x_h \in [0,1)_h$, we define

$$\partial_h^+ w_h(x_h) = \frac{w_h(x_h + h) - w_h(x_h)}{h} = \partial_h^- w_h(x_h + h).$$

We also introduce $L^2((0,1)_h)$ (respectively $L^2([0,1)_h)$) the set of discrete functions w_h defined for $x_h \in (0,1)_h$ (resp. $x_h \in [0,1)_h$), which we endow with the norms

$$\|w_h\|^2_{L^2((0,1)_h)} = h \sum_{x_h \in (0,1)_h} |w_h(x_h)|^2, \Big(\text{resp. } \|w_h\|^2_{L^2([0,1)_h)} = h \sum_{x_h \in [0,1)_h} |w_h(x_h)|^2\Big).$$

We similarly introduce $L^\infty((0,1)_h)$ and $L^\infty([0,1)_h)$ endowed with the norms $\|w_h\|_{L^\infty((0,1)_h)} = \sup_{(0,1)_h} |w_h(x_h)|$ and $\|w_h\|_{L^\infty([0,1)_h)} = \sup_{[0,1)_h} |w_h(x_h)|$, respectively.

As in the continuous case, solutions w_h of (6.6) have constant energy:

$$E_h(t) = \|\partial_t w_h(t)\|^2_{L^2((0,1)_h)} + \left\|\partial_h^+ w_h(t)\right\|^2_{L^2([0,1)_h)} = E_h(0). \tag{6.7}$$

Following [IZ99], we can then mimic the multiplier method in (6.4)–(6.5) and derive:

$$(T-2)E_h(0) \le \int_0^T |\partial_h^- w(t,1)|^2\, dt + \frac{1}{2}\int_0^T \left\|h \partial_h^+ \partial_t w_h\right\|^2_{L^2([0,1)_h)}\, dt. \tag{6.8}$$

The newly created term

$$\frac{1}{2}\int_0^T \left\|h \partial_h^+ \partial_t w_h\right\|^2_{L^2([0,1)_h)}\, dt \tag{6.9}$$

cannot be absorbed by the left hand-side of (6.8) uniformly with respect to $h > 0$. To be more specific, [IZ99] proves that for all $T > 0$,

$$\liminf_{h \to 0} \inf_{w_h \text{ solving (6.6)}} \frac{\displaystyle\int_0^T |\partial_h^- w(t,1)|^2\, dt}{E_h(0)} = 0. \tag{6.10}$$

To show that it goes to 0 faster than any polynomial (Actually, it goes to 0 as $\exp(-c/h)$, see [Mic02].), one can construct Gaussian beam solutions of the equation (6.1) traveling at a velocity close to 0 and concentrated away from the observation set $x = 1$, see e.g. [MZ10], as illustrated on Figure 6.1.

This can also be seen from (6.8). As high-frequencies in a mesh of size h are of the order of $1/h$, the operator $h\partial_h^+$ is of the order of 1 at frequencies $\xi \simeq 1/h$, so that for high-frequencies, the term (6.9) is of the order of the energy.

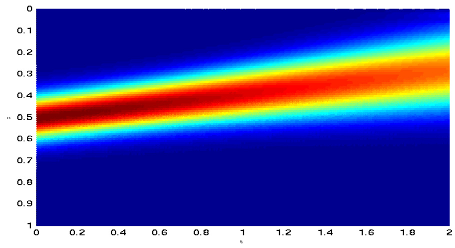

Figure 6.1: A discrete wave packet and its propagation. In the horizontal axis we represent the time variable, varying between 0 and 2, and in the vertical one the space variable x ranging from 0 to 1.

However, at frequencies $\xi = o(1/h)$, the operator $h\partial_h^+$ is of the order of $o(1)$ and the term (6.9) can be absorbed from the left hand-side of (6.8) for T large enough.

For this reason, the operators $h\partial_h^+$ weakly converge to 0 as $h \to 0$. In particular, considering the term (6.9) as a reinforcement of the observability operator, the observability inequalities (6.8) for (6.6) converge to the observability inequality (6.5) for (6.3).

Therefore, in the following, we will consider that the discrete version of the observability inequality (6.5) for (6.3) is the observability inequalities (6.8) for (6.6) containing the additional term (6.9).

6.2 Discrete Carleman estimates

This section aims at presenting the counterpart of Theorem 5.3 for the discrete wave equations (6.3), obtained with L. Baudouin in [BE13].

Theorem 6.1 ([BE13]). *Let $\Omega = (0,1)$, $\Gamma = \{1\}$, and assume the multiplier condition* (5.17) *and the time condition* (5.22). *There exists $\lambda > 0$, such that for φ as in* (5.23), *there exist $s_0 > 0$, $h_0 > 0$, a positive constant M and $\varepsilon > 0$ such that for all $h \in (0, h_0)$, for all $s \in (s_0, \varepsilon/h)$, for all $w_h \in H^2(-T,T; L^2((0,1)_h))$ with $w_h(t,0) = w_h(t,1) = 0$, we have*

$$s\int_{-T}^{T}\left(\|e^{s\varphi}\partial_t w_h\|^2_{L^2((0,1)_h)} + \left\|e^{s\varphi}\partial_h^+ w_h\right\|^2_{L^2([0,1)_h)}\right) dt$$
$$+ s^3\int_{-T}^{T}\|e^{s\varphi}w_h\|^2_{L^2((0,1)_h)}\, dt \le M\int_{-T}^{T}\|e^{s\varphi}(\partial_{tt}w_h - \Delta_h w_h)\|^2_{L^2((0,1)_h)}\, dt$$
$$+ Ms\int_{-T}^{T}|e^{s\varphi(t,1)}\partial_h^- w_h(t,1)|^2 dt + Ms\int_{-T}^{T}\left\|e^{s\varphi}h\partial_h^+\partial_t w_h\right\|^2_{L^2([0,1)_h)}\, dt. \quad (6.11)$$

If w_h furthermore satisfies $w_h(0,\cdot) = 0$ in $(0,1)_h$, one also has

$$s^{1/2}\left\|e^{s\varphi(0,\cdot)}\partial_t w_h(0,\cdot)\right\|^2_{L^2((0,1)_h)} \le M\int_{-T}^{T}\|e^{s\varphi}(\partial_{tt}w_h - \Delta_h w_h)\|^2_{L^2((0,1)_h)}\, dt$$
$$+ Ms\int_{-T}^{T}|e^{s\varphi(t,1)}\partial_h^- w_h(t,1)|^2 dt + Ms\int_{-T}^{T}\left\|e^{s\varphi}h\partial_h^+\partial_t w_h\right\|^2_{L^2([0,1)_h)}\, dt. \quad (6.12)$$

The proof of Theorem 6.1 follows the one of Theorem 5.3 and mainly consists in discrete integration by parts, carefully following the computations similarly as in [BHLR10a]. However, the discrete integrations by parts create a new term compared to Theorem 5.3, namely

$$s \int_{-T}^{T} \left\| e^{s\varphi} h \partial_h^+ \partial_t w_h \right\|_{L^2([0,1)_h)}^2 \, dt, \tag{6.13}$$

which has to be compared with (6.9).

Due to the results presented in Section 6.1.2, this term is needed. Otherwise, one would have a contradiction with (6.10). Besides, it is at the right scale. Indeed, at frequency ξ, $h\partial_h^+ \simeq \xi h$, so that the additional term (6.13) can be absorbed by the left hand side of (6.11) if $\xi = o(1/h)$, while it cannot be absorbed for frequencies of the order of $1/h$.

Another difference with Theorem 5.3 is that the parameter s in Theorem 6.1 cannot be taken arbitrarily large and is limited to some threshold ε/h. This is due to the fact that

$$e^{-s\varphi} \partial_h^+ (e^{s\varphi}) = s\partial_x \varphi + \mathcal{O}_\lambda(sh). \tag{6.14}$$

Therefore, to mimic the computations of the continuous case, one should assume $sh \leq \varepsilon_\lambda$, where ε_λ depends on λ exponentially. Also note that these limitations on the size of the Carleman parameters also appear in the discrete Carleman estimates derived in [BHLR10a, BHLR10b, BHLR11, BR13, EdG11].

As in the continuous case, the discrete Carleman estimates (6.12)–(6.11) can be used as well for the wave operator with bounded potentials $p_h \in L^\infty((0,1)_h)$ with $\|p_h\|_{L^\infty((0,1)_h)} \leq m$ by taking s larger than $(2Mm^2)^{1/3}$ if necessary.

Let us also point out again that, as $h \to 0$, the discrete operator $h\partial_h^+$ weakly converges to 0 (in some appropriate sense). Therefore, one can derive Theorem 5.3 by passing to the limit $h \to 0$ in the discrete Carleman estimates in Theorem 6.1.

Finally, we would like to point out the remark used in [CFCM13] that if we were considering other discretization methods for which the discrete solutions automatically embed into $H^2((-T,T) \times \Omega)$, then the Carleman estimate of Theorem 5.3 can be used directly. But this strategy requires the use of very regular spaces of discretization, and therefore makes the numerical implementations possibly costly.

6.3 Convergence and stability results

Based on the discrete Carleman estimates derived in Theorem 6.1, one can follow the arguments developed in the continuous case, recall Section 5.1.4, to deduce stability results for the discrete inverse problem (6.1)–(6.2).

However, we have to take into account the additional term (6.13). This naturally leads us to introduce an extended observation operator $\tilde{\mathscr{M}}_h$ defined for $p_h \in L^\infty((0,1)_h)$ by

$$\tilde{\mathscr{M}}_h(p_h) = (\partial_h^- y_h(\cdot, 1), h\partial_h^+ \partial_t y_h) \in L^2(0,T) \times L^2(0,T; L^2([0,1)_h)), \tag{6.15}$$

where $y_h = y_h[p_h]$ denotes the solution of

$$\begin{cases} \partial_{tt} y_h - \Delta_h y_h + p_h y_h = f_h, & (t, x_h) \in (0,T) \times (0,1)_h, \\ y_h(t,0) = f_{0h}(t), \quad y_h(t,1) = f_{1h}(t), & t \in (0,T), \\ (y_h(0,\cdot), \partial_t y_h(0,\cdot)) = (y_{0h}, y_{1h}), & x_h \in (0,1)_h. \end{cases} \tag{6.16}$$

We then derive the following stability result:

Theorem 6.2 ([BE13]). *Let $\gamma > 0$, $m > 0$, $K > 0$, $T > 1$, and assume that the initial data y_{0h} satisfies*

$$\inf_{(0,1)_h} |y_{0h}| \geq \gamma, \tag{6.17}$$

the potential $p_h^a \in L^\infty((0,1)_h)$ satisfies

$$\|p_h^a\|_{L^\infty((0,1)_h)} \leq m, \tag{6.18}$$

and the corresponding solution $y_h[p_h^a]$ of (6.16) *belongs to $H^1(0,T;L^\infty((0,1)_h))$ and satisfies*

$$\|y_h[p_h^a]\|_{H^1(0,T;L^\infty((0,1)_h))} \leq K. \tag{6.19}$$

Then there exists a constant C depending only on γ, m, K, and T such that for any potential $p_h^b \in L^\infty((0,1)_h)$ with $\left\|p_h^b\right\|_{L^\infty((0,1)_h)} \leq m$,

$$\frac{1}{C} \left\|p_h^a - p_h^b\right\|_{L^2((0,1)_h)} \leq \left\|\tilde{\mathscr{M}}_h(p_h^a) - \tilde{\mathscr{M}}_h(p_h^b)\right\|_{H^1(0,T)\times L^2(0,T;L^2([0,1)_h))}. \tag{6.20}$$

The new discrete measurement operators $\tilde{\mathscr{M}}_h$ contains an additional observation of the discrete solution (6.16) supported everywhere in the domain. But as we have said, the second component of the observation operator $\tilde{\mathscr{M}}_h$ weakly converges to 0 as $h \to 0$ and this will therefore not significantly modify the expected convergence result. Consequently, for $p \in L^\infty(0,1)$, we introduce the operator

$$\tilde{\mathscr{M}}_0(p) = (\partial_x y(\cdot,1), 0) \in L^2(0,T) \times L^2(0,T;L^2(0,1)),$$

where $y = y[p]$ is the solution of

$$\begin{cases} \partial_{tt} y - \Delta y + py = f, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = f_0(t), \quad y(t,1) = f_1(t), & (t,x) \in (0,T), \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y_0, y_1), & x \in (0,1). \end{cases} \tag{6.21}$$

To state our convergence result, we need to be able to compare $L^2([0,1)_h)$ with $L^2(0,1)$. We therefore introduce the extension operator e_h extending discrete functions w_h of $L^2([0,1)_h)$ as piecewise constant functions taking value $w_h(jh)$ on the intervals $(jh,(j+1)h)$ for $j \in \{0,\cdots,N\}$. For simplicity, we still denote by $\tilde{\mathscr{M}}_h$ the operator defined by

$$\tilde{\mathscr{M}}_h(p_h) = (\partial_h^- y_h(\cdot,1), h\mathrm{e}_h(\partial_h^+ \partial_t y_h)) \in L^2(0,T) \times L^2(0,T;L^2(0,1)),$$

where $y_h = y_h[p_h]$ denotes the solution of (6.16).

Our convergence result then reads as follows:

Theorem 6.3 ([BE13]). *Let $T > 1$, y_0 such that $\inf |y_0| > 0$, and assume that $(y_0,y_1) \in H^1(0,1) \times L^2(0,1)$, $f \in L^1(0,T;L^2(0,1))$, f_0, $f_1 \in H^1(0,T)$ with $y_0(0) = f_0(0)$, $y_0(1) = f_1(0)$. Let P in $L^\infty(0,1)$, and assume that $y[P]$ belongs to $H^1(0,T;L^\infty(0,1))$. Then there exists a sequence of discrete data (y_{0h},y_{1h}), f_h, (f_{0h},f_{1h}) only depending on (y_0,y_1), f and (f_0,f_1) with*

$$\inf_{h>0} \inf_{x_h \in [0,1]_h} |y_{0h}(x_h)| > 0,$$

and such that:

- **Consistency:** *there exists a sequence* $p_h \in L^\infty((0,1)_h)$ *for which*

$$\begin{cases} \limsup_{h\to 0} \|p_h\|_{L^\infty((0,1)_h)} < \infty, \\ \lim_{h\to 0} \left\| \tilde{\mathscr{M}}_h(p_h) - \tilde{\mathscr{M}}_0(P) \right\|_{H^1(0,T)\times L^2(0,T;L^2(0,1))} = 0, \\ \limsup_{h\to 0} \|y_h[p_h]\|_{H^1(0,T;L^\infty((0,1)_h))} < \infty \end{cases}$$

- **Convergence:** *any sequence* $p_h \in L^\infty((0,1)_h)$ *for which*

$$\begin{cases} \limsup_{h\to 0} \|p_h\|_{L^\infty((0,1)_h)} < \infty, \\ \lim_{h\to 0} \left\| \tilde{\mathscr{M}}_h(p_h) - \tilde{\mathscr{M}}_0(P) \right\|_{H^1(0,T)\times L^2(0,T;L^2(0,1))} = 0, \end{cases} \tag{6.22}$$

converges to P *in* $L^2(0,1)$ *in the following sense:*

$$\lim_{h\to 0} \|\mathrm{e}_h(p_h) - P\|_{L^2(0,1)} = 0. \tag{6.23}$$

Theorem 6.3 states that if we manage to find discrete potentials p_h such that $\tilde{\mathscr{M}}_h(p_h)$ converges to $\tilde{\mathscr{M}}(P)$ with $\mathrm{e}_h(p_h)$ uniformly bounded in $L^\infty(0,1)$, then $\mathrm{e}_h(p_h)$ converges to P in $L^2(0,1)$. The consistency part of Theorem 6.3 shows that it is indeed possible to find discrete potentials p_h such that $\tilde{\mathscr{M}}_h(p_h)$ converges to $\tilde{\mathscr{M}}(P)$, so that the convergence result is not empty.

Let us also point out that the discrete systems (6.16) involve choices of discrete data (y_{0h}, y_{1h}), f_h, (f_{0h}, f_{1h}). In Theorem 6.3, we only say that such suitable data exist. But these discrete data can be constructed explicitly from the knowledge of the continuous data (y_0, y_1), f and (f_0, f_1), following for instance the proof of [BEO, Lemma 4.3], basically projecting on the discrete mesh the continuous solution of (6.21) with $p = 0$. Another construction was proposed in [BE13] requiring more assumptions on the set of data (y_0, y_1), f and (f_0, f_1), guaranteeing in particular that $y[P]$ belongs to $C^1([0,T]; H^1(0,1))$ so that Sobolev's embedding automatically implies $y[P] \in H^1(0,T; L^\infty(0,1))$, see Section 4 in [BE13].

6.4 More general geometric setting

In [BEO], in collaboration with L. Baudouin and A. Osses, we extended the analysis carried out above to the multidimensional case for rectangular domains when the wave equation is space semi-discretized using the finite difference method on a uniform mesh. The new difficulty is that now the geometry plays an important role, as there are observation sets Γ which do not satisfy the geometric condition (5.17). We therefore discuss two cases:

- When (Γ, Ω) satisfies the geometric condition (5.17). In that case, we obtain Lipschitz stability results similar to the ones in Theorem 6.2.
- When (Γ, Ω) does not satisfy the geometric condition (5.17). In this case, we develop arguments similar to the ones in [BY06, Bel04].

6.4.1 Under the geometric condition (5.17)

This case is very similar to the one presented above in the 1d setting. The main difficulty is to derive a Carleman estimate for the space semi-discrete wave equation discretized using the finite-difference method on a uniform mesh. This is done similarly as in Theorem 6.1 and presents the same features:

- The presence of an additional term similar to (6.13) taking care of the high-frequency components of the solutions.
- The limitation of the parameter s of the form $sh \leq \varepsilon$ for some $\varepsilon > 0$.

We do not write the explicit details as it would require the introduction of several notations, and we refer to [BEO, Section 2] for precise statements and proofs.

Once this is done, we can derive convergence results similar to the ones in Theorem 6.3. But the condition $y[p] \in H^1(0,T;L^\infty(\Omega))$ in (5.19) is not easy to deal with, as the regularity of a solution $y[P]$ of the wave equation (4.1) is rather quantified in spaces of $C([0,T];H^{s+1}_{(0)}(\Omega)) \cap C^1([0,T];H^s_{(0)}(\Omega))$. In dimension d, it is therefore natural to assume $y[P] \in C^1([0,T];H^{d/2+\varepsilon}_{(0)}(\Omega))$ for some $\varepsilon > 0$. In dimension 2, this leads us to assume that $y[P] \in C^1([0,T];H^2(\Omega))$ (here we made the choice of restricting ourselves to integer regularity levels). This corresponds to a regularity of the form $y[P] \in C^0([0,T];H^3(\Omega)) \cap C^3([0,T];L^2(\Omega))$ corresponding to $(y_0, y_1) \in H^3(\Omega) \times H^2(\Omega)$, $f \in \cap_{k=0}^2 W^{k,1}(0,T;H^{2-k}(\Omega))$ and $f_\partial \in H^3((0,T)\times\partial\Omega)$ with compatibility conditions at $t = 0$, $x \in \partial\Omega$ and $P \in H^1(\Omega)$. Under these regularity assumptions, the compatibility conditions at time $t = 0$ implies that

$$\partial_{tt} f_0(0,x) - \Delta y_0(x) + P y_0(x) = f(0,x), \quad \forall x \in \partial\Omega.$$

so that $P|_{\partial\Omega}$ is also known (recall that $\inf_\Omega y_0 > 0$).

Therefore, to state a convergence result similar to Theorem 6.3 in 2d, it is natural to assume $P \in H^1(\Omega) \cap L^\infty(\Omega)$ and that its trace on the boundary $P|_{\partial\Omega}$ is also known. Remark that this latter condition is not too demanding in practice when the boundary is accessible and it still makes sense for the inverse problem (4.1)–(4.2). When Ω is a 2d rectangle, we indeed obtain such convergence result in [BEO, Theorem 1.6] under the following assumptions on the potential P and the observed trajectory $y[P]$ of (4.1):

- y_0 is strictly positive on Ω, $\inf_\Omega |y_0| > 0$,
- $(y_0, y_1) \in H^1(\Omega) \times L^2(\Omega)$, $f \in L^1(0,T;L^2(\Omega))$, $f_\partial \in H^1((0,T)\times\Omega)$ with $y_0(0) = f_0(0)$, $y_0(1) = f_1(0)$,
- $y[P] \in H^1(0,T;H^2(\Omega)) \cap H^2(0,T;H^1(\Omega))$,
- $P \in H^1(\Omega) \cap L^\infty(\Omega)$ and P is known on the boundary $\partial\Omega$: $P|_{\partial\Omega} = p_\partial$.

Under these assumptions, we obtain the same convergence result as in Theorem 6.3. Details can be found in [BEO].

6.4.2 When the geometric condition (5.17) is not satisfied

We still focus on the case of Ω being a 2d rectangle, and now Γ is a part of the boundary which does not satisfy the geometric condition (5.17). In that case, we shall first recall what can be done in the continuous setting. To our knowledge, the best result available is due to M. Bellassoued in [Bel04]. Below we state a slightly improved version of it:

Theorem 6.4 ([Bel04], revisited in [BEO])**.** *Assume that there exists an open subset $\Gamma_1 \subset \partial\Omega$ of the boundary $\partial\Omega$ and an open subset ω of Ω such that:*

- *$\Gamma \subset \Gamma_1$ and (Γ_1, Ω) satisfies the condition (5.17);*

- *ω contains a neighborhood of Γ_1 in Ω, i.e. for some $\delta > 0$,*

$$\Gamma_{1,\delta} := \{x \in \Omega, \text{ with } d(x, \Gamma_1) < \delta \ \} \ \subset \omega. \tag{6.24}$$

Let p^a be a potential lying in the class $\Lambda(P_\omega, m)$ defined for $P_\omega \in L^\infty(\omega)$ and $m > 0$ by

$$\Lambda(P_\omega, m) = \{p \in L^\infty(\Omega), \text{ s.t. } p|_\omega = P_\omega \ \text{ and } \ \|p\|_{L^\infty(\Omega)} \le m\}. \tag{6.25}$$

Let $y_0 \in H^1(\Omega)$ satisfying the positivity condition (5.20) and assume that $y[p^a]$ solving (4.1) satisfies the regularity condition

$$y[p^a] \in H^1(0,T; L^\infty(\Omega)) \cap W^{2,1}(0,T; L^2(\Omega)). \tag{6.26}$$

Let $\alpha > 0$ and $M > 0$. Then there exist $C > 0$ and $T > 0$ large enough such that for all $p^b \in \Lambda(P_\omega, m)$ satisfying

$$p^a - p^b \in H_0^1(\Omega) \text{ and } \left\|p^a - p^b\right\|_{H_0^1(\Omega)} \le M, \tag{6.27}$$

we have $\mathscr{M}[p^a] - \mathscr{M}[p^b] \in H^1(0,T; L^2(\Gamma))$ and

$$\left\|p^a - p^b\right\|_{L^2(\Omega)} \le C \left[\log\left(2 + \frac{C}{\|\mathscr{M}[p^a] - \mathscr{M}[p^b]\|_{H^1(0,T;L^2(\Gamma))}}\right)\right]^{-\frac{1}{1+\alpha}}. \tag{6.28}$$

The geometric setting of the stability result in Theorem 6.4 is important. The set Γ is an arbitrary (non-empty) open subset of the boundary $\partial\Omega$. This observation set is included in a part Γ_1 of the boundary satisfying the geometric condition (5.17), and in a neighborhood of which the potential is known.

The idea underlying Theorem 6.4 is that in ω, as the potential is known as $p^a = p^b = P_\omega$ in ω, we can derive an estimate on $y[p^a]-y[p^b]$ in ω from the measurement $\mathscr{M}[p^a]-\mathscr{M}[p^b]$. This is done using the Fourier-Bros-Iagoniltzer (FBI) transform which links solutions of the wave equation with solutions of an elliptic PDE. The critical ingredient of the FBI transform is the following: taking the convolution of a compactly supported function $w(t)$ with an analytic approximation of the identity given by $\rho_\lambda(t) = \lambda\rho(\lambda t)$, the distribution

$$W_\lambda(t) = \rho_\lambda \star w(t) = \int_{\mathbb{R}} \rho_\lambda(\tau) w(t-\tau)\, d\ \tau,$$

converges to w in $\mathscr{D}'(\mathbb{R})$, while it can be extended to a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ by the formula

$$W_\lambda(t + \mathbf{i}s) = \int_{\mathbb{R}} \rho_\lambda(\tau + \mathbf{i}s) w(t-\tau)\, d\ \tau. \tag{6.29}$$

The interest of this formula is given by the following computation:

$$\begin{aligned}
\partial_s W_\lambda(t+\mathbf{i}s) &= \int_{\mathbb{R}} \partial_s \left(\rho_\lambda(\tau + \mathbf{i}s)\right) W(t-\tau)\, d\tau \\
&= \mathbf{i} \int_{\mathbb{R}} \partial_\tau \left(\rho_\lambda(\tau + \mathbf{i}s)\right) w(t-\tau)\, d\tau \\
&= \int_{\mathbb{R}} \rho_\lambda(\tau + \mathbf{i}s)(\mathbf{i}\partial_t) w(t-\tau)\, d\tau.
\end{aligned}$$

In particular, two derivatives of w in t correspond to two derivatives of $W_\lambda(t+\mathbf{i}s)$ in s but with a change of sign. Using this property for $w = \partial_t(y[p^a]-y[p^b])$ (actually, it has to be suitably cut off in time, but we omit this for simplifying our presentation), and considering the time variable t as a parameter, we are then back to an observability problem for W_λ solving an elliptic type PDE

$$-\partial_{ss}W_\lambda - \Delta W_\lambda + p^a W_\lambda = 0 \text{ in } \omega, \tag{6.30}$$

with W_λ vanishing on Γ_1 and with known estimates on $\partial_\nu W_\lambda$ for $s\in\mathbb{R}$ and $x\in\Gamma$. We can therefore use classical Carleman estimates (see e.g. [Hör85]) to deduce estimates on W_λ close to $s=0$ from the estimates of its normal derivative in Γ and an a priori estimate on W_λ. We then use that the trace of W_λ at $s=0$ contains some informations on w thanks to the convergence $W_\lambda(\cdot+\mathbf{i}0)\to w$ as $\lambda\to\infty$. Optimizing the parameters in the elliptic Carleman estimate, and the parameter λ quantifying the proximity of W_λ to w, one obtains a logarithmic estimate on w on $(-T,T)\times\omega$ from estimates on $(-2T,2T)\times\Gamma$. Now, to conclude the proof of Theorem 6.4, it is sufficient to combine this last estimate with the ones obtained in Theorem 5.2 (slightly modified to the case of a distributed observation).

Therefore, everything amounts to suitably choose the kernel ρ in (6.29). In [Bel04], M. Bellassoued used the classical gaussian kernel

$$\rho(t) = \exp(-t^2/2),$$

also used in the previous works by [Rob91, Rob95] to quantify the unique continuation property for the wave equation. This choice proves (6.28) with $\alpha = 1$.

To obtain the proof of (6.28) with arbitrary $\alpha>0$, we use a more singular kernel ρ - first used in [LR97] to estimate the decay rate of the damped wave equation when the damped region does not satisfy the geometric control condition - and defined as the inverse Fourier transform of $\exp(-|\xi|^{2n})$ for some suitable choice of $n\in\mathbb{N}$. We also refer to [Phu10] where such kernel was used to improve the quantification of the unique continuation property for the wave equation proposed in [Rob95].

When trying to adapt the above proof to the case of a semi-discrete wave equation in a square observed from a part of the boundary Γ which does not satisfy the geometric condition (5.17), we will follow the same strategy. For the clarity of the exposition, we focus on the 2d case, when the wave equation (4.1) is set on the square $\Omega=(0,1)^2$ and observed from one non-empty open subset Γ of the boundary and is discretized using the finite-difference method on a uniform mesh.

We do not introduce all the notations needed for the precise statement of the next result, but we hope that they will be transparent enough for the reader to understand the main idea[1].

Theorem 6.5 ([BEO]). *Assume the existence of a neighborhood $\omega\subset\Omega$ of Γ such that*

- *ω contains a neighborhood of Γ_1 in Ω, in the sense that it contains some $\Gamma_{1,\delta}$ for some $\delta>0$ (recall (6.24) for its definition).*
- *the potential p_h is known on $\partial\Omega_h$ and in ω_h, where it takes the value $P_{\omega,h}\in L^\infty(\omega_h)$.*

Let p_h^a be a potential lying in the class $\Lambda_h(P_{\omega,h},m)$ defined for $P_{\omega,h}\in L^\infty(\omega_h)$ and $m>0$ by

$$\Lambda_h(P_{\omega,h},m) = \{p_h\in L^\infty(\Omega_h),\ s.t.\ p_h|_{\omega_h} = P_{\omega,h} \text{ and } \|p_h\|_{L^\infty(\Omega_h)}\le m\}. \tag{6.31}$$

[1] The discrete sets are $\Omega_h=\{(ih,jh),\ i\in\{1,\cdots,N\},\ j\in\{1,\cdots,N\}\}$, $\Omega_{h,1}^-=\{(ih,jh),\ i\in\{0,\cdots,N\},\ j\in\{1,\cdots,N\}\}$, $\Omega_{h,2}^-=\{(ih,jh),\ i\in\{1,\cdots,N\},\ j\in\{0,\cdots,N\}\}$, $\omega_h=\omega\cap\Omega_h$, $\partial\Omega_h=\{(ih,jh),\ (i,j)\in\{0,N+1\}\times\{1,\cdots,N\}$ or $(i,j)\in\{1,\cdots,N\}\times\{0,N+1\}\ \}$, $\Gamma_h=\Gamma\cap\partial\Omega_h$. The discrete derivation operators are $\partial_{h,k}^+y_h(x_h)=(y_h(x_h+h\vec{e}_k)-y_h(x_h))/h$, $k=1,2$. The discrete spaces $L^2(\Omega_h)$, $L^\infty(\Omega_h)$ and $H^1(\Omega_h)$ are discrete versions of $L^2(\Omega)$, $L^\infty(\Omega)$ and $H^1(\Omega)$. We refer to [BEO] for precise definitions.

Let $M > 0$ and $\alpha > 0$. Assume that $y_{0h} \in H^1(\Omega_h)$ with $\inf_{\Omega_h} |y_{0h}| > 0$ and that the solution $y_h[p_h^a]$ of (6.1) *with potential p_h^a satisfies the conditions*

$$y_h[p_h^a] \in H^1(0,T;L^\infty(\Omega_h)) \cap W^{2,1}(0,T;L^2(\Omega_h)). \tag{6.32}$$

Then there exist $C > 0$ and $h_0 > 0$ such that for $T > 0$ large enough, for all $h \in (0, h_0)$, for all $p_h^b \in \Lambda_h(P_{\omega,h}, m)$ satisfying

$$p_h^a = p_h^b \text{ on } \partial\Omega_h, \quad \text{and} \quad \left\|p_h^a - p_h^b\right\|_{H^1(\Omega_h)} \leq M, \tag{6.33}$$

we have

$$\begin{aligned} \left\|p_h^a - p_h^b\right\|_{L^2(\Omega_h)} \leq C &\left[\log\left(2 + \frac{C}{\left\|\mathscr{M}_h[p_h^a] - \mathscr{M}_h[p_h^b]\right\|_{H^1(0,T;L^2(\Gamma_h))}}\right)\right]^{-\frac{1}{1+\alpha}} \\ &+ Ch^{1/(1+\alpha)} + Ch \sum_{k=1,2} \left\|\partial_{h,k}^+ \partial_{tt} y_h[p_h^a] - \partial_{h,k}^+ \partial_{tt} y_h[p_h^b]\right\|_{L^2(0,T;L^2(\Omega_{h,k}^-))}. \end{aligned} \tag{6.34}$$

The proof of Theorem 6.5 follows the one in the continuous case, and in particular uses the above FBI transform for space semi-discrete approximations $y_h[p_h]$ of the solutions $y[p]$ of (4.1). At this step, one strongly uses the fact that the FBI transform only acts on the time variable. It therefore does not induce any modification on the space semi-discrete approximation, and we can use elliptic Carleman estimates for the semi-discrete operator $-\partial_{ss} - \Delta_h + p_h^a$ in ω_h instead of (6.30). We can therefore use the Carleman estimates in [BHLR10a, BHLR10b] and follow the same arguments as for Theorem 6.4, combining a logarithmic stability result to estimate $w_h = \partial_t(y_h[p_h^a] - y_h[p_h^b])$ in ω_h from the difference of the measurements $\mathscr{M}_h[p_h^a] - \mathscr{M}_h[p_h^b]$, and the discrete Carleman estimate obtained in Theorem 6.1.

Nevertheless, let us emphasize that new terms appear in the stability estimate (6.34) compared to the stability estimate (6.28): a penalization term corresponding to (6.13), already appearing under the geometric condition (5.17), and the new term $Ch^{1/(1+\alpha)}$.

Of course, the penalization term corresponding to (6.13) is remanent from the Carleman estimate obtained in Theorem 6.1 and needed to handle the spurious high-frequency solutions of the space semi-discrete approximation of the wave equation. The term $Ch^{1/(1+\alpha)}$, however, is less clear at first glance. It actually comes from the restriction on the parameter in the discrete Carleman estimates obtained in [BHLR10a, BHLR10b]. If τ denotes the Carleman parameter used for the semi-discrete elliptic equation $-\partial_{ss} - \Delta_h + p_h^a$, it is limited to the threshold ε/h for some $\varepsilon > 0$. We already have seen this condition in Theorem 6.1, and already explained its origin. But in the proof of Theorem 6.5, this restricts the possible choices when optimizing the parameters. We therefore remark that when the optimization of the parameters would require a choice of the Carleman parameter τ larger than ε/h, we can simply choose it as ε/h, and this gives (6.34).

Let us finally point out that the stability estimate (6.34) does not produce strictly speaking a stability estimate, but rather an approximate stability estimate due to the term $Ch^{1/(1+\alpha)}$. But this is not too important as this term goes to zero as $h \to 0$. We can therefore still use the stability estimate (6.34) to deduce convergence results for the corresponding discrete inverse problems (6.1)–(6.2) in the geometric setting mentioned above, when the potential P is known not only on the boundary $\partial\Omega$ but also on a set ω satisfying the conditions of Theorem 6.4. We refer to [BEO, Theorem 1.6] for the precise assumptions and requirement in that case.

6.5 Comments

6.5.1 Towards a discrete algorithm to reconstruct the potential

In this chapter, we have derived convergence results for the inverse problem (4.1)–(4.2). This led us in particular to introduce Carleman estimates for space semi-discrete approximations of the wave equation (4.1), for which an additional term (6.13) corresponding to a reinforcement of the observation is needed.

Following the reconstruction result obtained in Theorem 5.4, we can then derived an explicit algorithm to construct discrete potentials p_h satisfying the convergence conditions (6.22) guaranteeing that p_h indeed approximates the potential P corresponding to a measurement $\mathscr{M}(P)$. In order to do this, for instance in 1d with $\Omega = (0,1)$ and $\Gamma = \{1\}$, the algorithm presented in Section 5.2.1 should be slightly modified as follows.

• Initialization: $p_h^0 = 0$.

• Iteration. Given p_h^k, we set $\mu_{k,h}(t) = \partial_t \left(\mathscr{M}_h(p_h^k)(t) - \mathscr{M}(P)(t)\right)$ on $(0,T)$. We then introduce the functional $J_{s,p_h^k}[\mu_{k,h}, 0]$ defined, for some $s \geq 1$ that will be chosen independently of k, by

$$\begin{aligned} J_{s,p_h^k}[\mu_{k,h}, 0](w_h) = \frac{1}{2s} \int_0^T \left\| e^{s\varphi}(\partial_{tt} w_h - \Delta_h w_h + p_h^k w_h)\right\|^2_{L^2((0,1)_h)} \, dt \\ + \frac{1}{2} \int_0^T \left(|e^{s\varphi(t,1)}(\partial_h^- w_h(t,1) - \mu_{k,h}(t))|^2 + \left\| e^{s\varphi} h \partial_h^+ \partial_t w_h \right\|^2_{L^2([0,1)_h)} \right) dt, \end{aligned} \tag{6.35}$$

on the trajectories $w_h \in H^2(0,T; L^2((0,1)_h))$ with $w_h(t,0) = w_h(t,1) = 0$ for all $t \in (0,T)$ and $w_h(0,\cdot) = 0$ in $(0,1)_h$.

Let W_h^k be the unique minimizer of the functional $J_{s,p_h^k}[\mu_{k,h}, 0]$, and then set

$$\tilde{p}_h^{k+1} = p_h^k + \frac{\partial_t W_h^k(0,\cdot)}{y_{0h}}, \tag{6.36}$$

where y_{0h} is an approximation of the initial condition in (6.1) approximating y_0, and

$$\forall x_h \in \Omega_h,\ p_h^{k+1}(x_h) = T_m(\tilde{p}_h^{k+1}(x_h)), \quad \text{where } T_m(\rho) = \begin{cases} \rho, & \text{if } |\rho| \leq m, \\ \text{sign}(\rho) m, & \text{if } |\rho| \geq m. \end{cases} \tag{6.37}$$

Using this algorithm, and combining the results in Chapter 5 and Chapter 6, we can prove that for all $h > 0$, the discrete potentials p_h^k given by the above algorithm converge to some potential p_h in $L^2((0,1)_h)$, and the sequence of potentials $\mathrm{e}_h(p_h)$ converges to P in $L^2(0,1)$.

We are therefore close to a convergent reconstruction algorithm for solving the inverse problem (4.1)–(4.2). However, as explained in Section 5.3.2, this is not sufficient numerically due to the presence of many exponential terms in the functional J_{s,p_h} in (6.35) and the requirement s large to guarantee the convergence of the algorithm. With L. Baudouin and M. de Buhan, we are currently working on trying to solve this difficulty, to design first a continuous algorithm based on some Carleman type estimate following the leads presented in Section 5.3.2, and to adapt it to the new phenomena created by the discretization process to finally design an efficient numerical algorithm to solve the reconstruction problem for the inverse problem (4.1)–(4.2).

6.5.2 More general approximation schemes

In this chapter, we have presented several results related to the convergence of the inverse problem (4.1)–(4.2) when solutions of (4.1) are approximated by the finite difference method. As we have seen, with this method, we need to introduce a penalization term coming from (6.13) to obtain uniform stability results for the discrete inverse problems corresponding to (6.1)–(6.2).

But this result depends on the approximation scheme we have used. The same discussion should therefore be done for other approximation schemes. In view of the literature on the observability inequality (6.5) for solutions of the wave equation (6.3) and the corresponding discrete observability inequalities, it may for instance happen that this penalization term is not needed if one uses the mixed finite element method [CM06, CMM08, Erv10]. But most of the approximation schemes used in practice seems to require a suitable penalization term similar to the one in (6.9), see [Zua05, EZ11c, Erv09, Mil12].

We did not mention either the effects of time-discretization. For what concerns the effects of time discretization on observability issues for wave-type operators, or more generally for abstract conservative systems, we refer to Chapter 9. We think that this should give us some insights on what should be done to handle fully discrete cases. Very likely, the same results as presented above will hold if an additional viscous penalization term in time is added in the discrete measurement operator. But this issue still needs to be carefully analyzed.

6.5.3 More general inverse problems

This part only focused on the inverse problem (4.1)–(4.2), but the question we addressed concerns all inverse problems. It seems that a systematic theoretical study of reconstruction algorithms and the possible difficulties induced by the discretization effects is missing. For instance, the same discussion could be done with the Calderón problem, consisting in recovering a conductivity from the Dirichlet-to-Neumann map, for which uniqueness and stability is known thanks to the works [SU87, Ale88]. We did a first attempt on that question in [EdG11], but we only managed to get a stability result - or rather an approximate stability result - for the discrete Calderón problems, leaving open the convergence issue. The problem in that latter case is that the usual strategy to prove stability results is based on the construction of complex geometric optics solutions, which are high-frequency solutions. Therefore, one needs to get suitable convergence results for the Dirichlet-to-Neumann map - which involves a large set of data - and especially at high-frequencies.

6.5.4 Controllability of semilinear wave equations

Carleman estimates for the wave equation are in the heart of many results on the controllability of the semilinear wave equation, because it provides a way to explicitly quantify how the constant of controllability depends on the lower order terms. We refer for instance to the works [Zha00, LZ05, DZZ08, FYZ07] for several works related to the analysis of the dependence of the observability constant with respect to potentials and its consequences on the controllability of semilinear wave equations. Our investigations suggest that a similar strategy could be applied for the numerical approximation of controls for semi-linear wave equations, but this requires further work. This idea has already been developed in the context of semilinear heat equations in the recent work [BR13].

Part III

Integral transforms

Chapter 7

Introduction

The goal of this part is to explain some consequences of integral transforms in the context of control and observability issues.

In Chapter 8, we focus on the study of the reachable set of the heat equation assuming the wave equation is observable. Our study, mainly based on [EZ11b, EZ11a], will rely on the use of an integral transform in the spirit of previous works by L. Miller [Mil04, Mil06a]. In these works, the integral transform is based on the so-called Kannai transform [Kan77], which constructs solutions of the heat equation in terms of solutions of the wave equation. We shall rather do the opposite, writing some trajectories of the wave equation in terms of solutions of the heat equation. This approach is somehow dual to the one in [Mil04, Mil06a] and yields some new insights on the reachable set (not on the cost of controllability in small time, though).

In Chapter 9, following [EZ], we shall then explain how similar ideas can be adapted for studying observability properties of conservative systems when discretized in time. As we explained in Part I, this question arises naturally when designing numerical algorithms to compute the controls. Our idea basically consists in transforming solutions of the time-discrete approximations of the equation in solutions of the continuous equation. This transform is less singular as above. In particular, as we will see, this transform can also be inverted, and its inverse has a form very similar to the direct transform. This allows to derive uniform observability results for time-discrete systems provided that the solutions are filtered at a suitable scale, in the spirit of the earlier work [EZZ08], but with an explicit and optimal time estimate.

Chapter 8

On the reachable set of the heat equation

8.1 Introduction

The problem we would like to address here is the description of the reachable set for the heat equation.

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain, and assume that the control acts on the open subset $\omega \subset \Omega$. We consider the heat equation

$$\begin{cases} \partial_t y - \Delta y = v\chi_\omega, & (t,x) \in (0,\infty) \times \Omega, \\ y(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ y(0,x) = 0, & x \in \Omega, \end{cases} \tag{8.1}$$

and we define the reachable set at time $T > 0$ by

$$\mathscr{R}(T) = \{y[v](T), \text{ for } v \in L^2((0,T) \times \omega)\}. \tag{8.2}$$

Note that though the reachable set depends *a priori* on the time horizon $T > 0$, the null-controllability of the heat equation in arbitrarily small time implies that it is independent of the time $T > 0$, (see [Sei79, Mil06b]) and we then denote the reachable set simply by $\mathscr{R}$.

To my knowledge, the description of the reachable set for the heat equation has been completely clarified only for the heat equation in one space dimension in the work [FR71] by H. Fattorini and D. Russell. There, the authors used a description in terms of the coefficients on the basis of the eigenfunctions of the Laplacian and very precise estimates on bi-orthogonal functions. Our goal is to generalize these results in the spirit of the works [Mil04, Mil06b] by L. Miller.

8.1.1 Observability results for the heat equation

To study control issues for (8.1), in general the idea rather consists in working on the observability of the adjoint heat equation[1]

$$\begin{cases} \partial_t z - \Delta z = 0, & (t,x) \in (0,\infty) \times \Omega, \\ z(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega, \end{cases} \tag{8.3}$$

[1] We do the change of variable $t \mapsto T - t$ in order to consider the forward in time heat equation.

observed through an open subset $\omega \subset \Omega$.

The first observability results for (8.3) obtained from arbitrary non-empty open subsets ω are the ones by A. Fursikov and O. Imanuvilov in [FI96] and by G. Lebeau and L. Robbiano in [LR95]. They proved that for any $T > 0$, there exists a constant $C(T)$ such that for any smooth solution z of (8.3),

$$\|z(T)\|_{L^2(\Omega)} \leq C(T) \|z\|_{L^2((0,T)\times\omega)} . \tag{8.4}$$

Note that, in both cases, the observability estimate (8.4) is obtained via suitable Carleman estimates, and both strategies provide an estimate on the cost of the observability in time $T > 0$, i.e. the best constant $C(T)$ in (8.4), of the form:

$$C(T) \leq C \exp\left(\frac{\gamma}{T}\right). \tag{8.5}$$

Here, γ is related to the exponential rate of blow up of the cost of observability in small times. It has later been proved in [FCZ00a] that this cost of observability indeed blows up with such rates, namely

$$\sup_{B_r \subset \Omega \setminus \overline{\omega}} \frac{r^2}{4} \leq \liminf_{T\to 0} T \log(C(T)). \tag{8.6}$$

L. Miller in [Mil04] then improved this bound to

$$\sup_{x\in\Omega} \frac{d(x,\overline{\omega})^2}{4} \leq \liminf_{T\to 0} T \log(C(T)). \tag{8.7}$$

There, the control transmutation method was proposed to study the cost of observability in small time for the heat equation (8.3), when the wave equation on Ω is observable in time $2S_*$ through the cylinder $(0, 2S_*) \times \omega$, i.e. if $(\omega, \Omega, 2S_*)$ satisfies the Geometric Control Condition of C. Bardos, G. Lebeau and J. Rauch [BLR92]. That way, [Mil04] gave a bound from above on $C(T)$, later improved by G. Tenenbaum and M. Tucsnak in [TT07] as follows:

$$\limsup_{T\to 0} T \log(C(T)) \leq \alpha \frac{S_*^2}{4}, \text{ for all } \alpha > 3. \tag{8.8}$$

This is the last result I know estimating the cost of observability in small times, thereby still leaving a gap between the bounds from below (8.7) and the bounds from above (8.8). The bound from above (8.8) in [TT07] relies on the construction of bi-orthogonal families for solving a control problem for the 1d heat equation, and the so-called control transmutation method to write controlled trajectories of the heat equation in terms of controlled trajectories of the wave equation, that we briefly recall hereafter.

Given $y_0 \in H_0^1(\Omega)$, denote by w the solution of the wave equation

$$\begin{cases} \partial_{ss} w - \Delta w = v_w \chi_\omega, & (s,x) \in (-\infty,\infty) \times \Omega, \\ w(s,x) = 0, & (s,x) \in (-\infty,\infty) \times \partial\Omega, \\ (w(0,x), \partial_s w(0,x)) = (y_0(x), 0), & x \in \Omega, \end{cases} \tag{8.9}$$

controlled through ω on the time interval $(-2S_*, 2S_*)$ so that for all s with $|s| \geq 2S_*$,

$$(w(s,\cdot), \partial_s w(s,\cdot)) = (0,0), \text{ in } \Omega.$$

Now, let ρ be a solution of the controllability problem

$$\begin{cases} (\partial_t - \partial_{ss})\rho(t,s) = 0, & (t,s) \in (0,T) \times (-2S_*, 2S_*), \\ \rho(0,s) = \delta_0(s), & s \in (-2S_*, 2S_*), \\ \rho(T,s) = 0, & s \in (-2S_*, 2S_*), \\ \partial_s \rho(t,0) = 0, & t \in (0,T), \\ \rho(t, \pm 2S_*) = v_\pm(t), & t \in (0,T), \end{cases} \tag{8.10}$$

where δ_0 is the Dirac function, and the controls act in $s = 2S_*$ and $s = -2S_*$. Then the function y defined for $(t, x) \in [0, T] \times \Omega$ by

$$y(t,x) = \int_{\mathbb{R}} \rho(t,s) w(s,x)\, ds, \tag{8.11}$$

solves the control problem

$$\begin{cases} \partial_t y - \Delta y = v_y \chi_\omega, & (t,x) \in (0,\infty) \times \Omega, \\ y(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \\ y(0,x) = y_0(x), & x \in \Omega, \\ y(T,x) = 0, & x \in \Omega, \end{cases} \tag{8.12}$$

where $v_y = \int_{\mathbb{R}} \rho(t,s) v_w(s)\, ds$.

Therefore, estimate (8.8) in [TT07] mainly requires to get precise estimates on a kernel ρ solving the control problem (8.10). Note that once this is done, the above control transmutation method yields an estimate on the cost of controllability in small times. By duality, this also estimates the cost of observability in small times.

8.1.2 Back to the reachability set

Using (8.4)–(8.5), one easily checks that all solutions z of (8.3) satisfy

$$\left\| e^{-\gamma/t} z(t) \right\|_{L^2((0,T)\times\Omega)} \leq CT \left\| z \right\|_{L^2((0,T)\times\omega)},$$

so that for all $\delta > \gamma$, all solutions z of (8.3) satisfy

$$\int_0^\infty \int_\Omega e^{-2\delta/t} |z(t,x)|^2 \, dt dx \leq C \int_0^\infty \int_\omega |z(t,x)|^2 \, dt dx. \tag{8.13}$$

Estimate (8.13) gives a description of the reachable set for the heat equation (8.1). Indeed, if (8.13) holds for some $\delta > 0$, the reachable set $\mathscr{R}$ of the heat equation (8.1) contains $e^{-2\sqrt{\delta}\sqrt{-\Delta_D}} L^2(\Omega)$, where $-\Delta_D$ is the Laplace operator with Dirichlet homogeneous boundary conditions with domain $H^2 \cap H_0^1(\Omega)$ on $L^2(\Omega)$, see Corollary 8.2 and [EZ11b] for further details.

We are thus interested in the best δ in (8.13), that is

$$\delta_* = \inf\{\delta, \text{ such that (8.13) holds}\}. \tag{8.14}$$

According to the above remark, we immediately have:

$$\delta_* \leq \limsup_{T \to 0} T \log(C(T)). \tag{8.15}$$

Therefore, the estimate (8.8) in [TT07] already provides a bound from above on δ_*.

Besides, the lower bound stated in (8.6) (due to [FCZ00a]), based on the function

$$z_r(t,x) = \frac{1}{(4\pi t)^{d/2}} \sin\left(\frac{r|x|}{2t}\right) \exp\left(\frac{r^2 - |x|^2}{4t}\right), \tag{8.16}$$

also applies to estimate δ_* from below:

$$\sup_{B_r \subset \Omega \setminus \overline{\omega}} \frac{r^2}{4} \leq \delta_*. \tag{8.17}$$

In the following, we will show that we can improve the estimates from above on δ_* obtained as a consequence of (8.8) and (8.15) by using an integral transform mapping solutions of the heat equation (8.3) to trajectories of the wave equation.

8.2 Main results

8.2.1 Statement

As our method will rely on an integral transform on the evolution variable, it is convenient to introduce an abstract operator A_0, defined on some Hilbert X with dense domain $\mathscr{D}(A_0)$ and compact resolvent. We further assume that A_0 is self-adjoint and positive definite.

We shall thus consider the following abstract heat equation instead of (8.3):

$$\partial_t z + A_0 z = 0, \quad t \geq 0, \qquad z(0) = z_0 \in X. \tag{8.18}$$

The observation is done through an operator[2] $B^* \in \mathscr{L}(\mathscr{D}(A_0), U)$, where U is a Hilbert space.

We also introduce the corresponding wave equation

$$\partial_{ss} w + A_0 w = 0, \quad s \in \mathbb{R}, \quad (w(0), \partial_s w(0)) = (w_0, w_1) \in \mathscr{D}(A_0^{1/2}) \times X. \tag{8.19}$$

Note that the time in (8.19) is denoted by the variable s. This helps distinguishing it from the time evolution parameter t for (8.18).

Our main assumption is the following one, corresponding to the observability of the wave equation in time $2S_*$: there exists a constant C_w such that all solutions w of (8.19) with initial data $(w_0, w_1) \in \mathscr{D}(A_0) \times \mathscr{D}(A_0^{1/2})$ satisfy

$$\|(w_0, w_1)\|_{\mathscr{D}(A_0^{1/2}) \times X} \leq C_w \|B^* w(s)\|_{L^2(-S_*, S_*; U)}. \tag{8.20}$$

Remark that under Assumption (8.20) it is well known [Rus73] that the abstract heat equation (8.18) is observable through B^* in any time $T > 0$: for all solutions z of (8.18),

$$\|z(T)\|_X \leq C(T) \|B^* z\|_{L^2(0,T;U)}. \tag{8.21}$$

We can therefore focus on the abstract version of (8.13). With E. Zuazua we obtained:

Theorem 8.1 ([EZ11b]). *Let us assume that the abstract wave equation* (8.19) *is observable in time* $2S_*$ *in the sense of* (8.20). *Then there exists a constant* $C > 0$ *such that any solution* z *of* (8.18) *with initial data* $z_0 \in \mathscr{D}(A_0)$ *satisfies*

$$\int_0^\infty \exp\left(-\frac{S_*^2}{2t}\right) \|z(t)\|_X^2 \, dt \leq C \int_0^\infty \|B^* z(t)\|_U^2 \, dt. \tag{8.22}$$

First remark that since the operator A_0 is strictly positive, the solution z of (8.18) is exponentially decaying and the integrals in (8.22) are finite.

Theorem 8.1 implies that δ_* in (8.14) satisfies

$$\delta_* \leq \frac{S_*^2}{4}. \tag{8.23}$$

This estimate on δ_* is better than the one obtained by combining the remark (8.15) and the estimate from below for $C(T)$ in (8.8) obtained in [TT07].

Let us emphasize that there are some cases in which the above estimate (8.23) is sharp, in particular when

$$S_* = \sup_{B_r \subset \Omega \setminus \overline{\omega}} r.$$

[2] This operator comes from the control problem $\partial_t y + A_0 y = Bv$, where $B \in \mathscr{L}(U, \mathscr{D}(A_0)')$ is the control operator, thus justifying the notation B^* for the observation operator.

This occurs in particular when the set $\Omega \setminus \overline{\omega}$ is a ball. In 1d, this yields another proof of the results obtained by H. Fattorini and D. Russell in [FR71] derived using careful estimates on bi-orthogonals. Besides, using a symmetrization argument, one easily gets that solutions z of the 1d heat equation

$$\begin{cases} \partial_t z - \partial_{xx} z = 0, & (t,x) \in (0,\infty) \times (0,L), \\ z(t,0) = z(t,L) = 0, & t \in (0,\infty), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{8.24}$$

with initial data $z_0 \in H^2 \cap H_0^1(0,L)$ satisfy

$$\int_0^\infty \exp\left(-\frac{L^2}{2t}\right) \|z(t)\|_{L^2(0,L)}^2 \, dt \leq C_{\mathscr{R}} \int_0^\infty |\partial_x z(t,L)|^2 \, dt. \tag{8.25}$$

We finally mention the following corollary, whose proof is classical:

Corollary 8.2 ([EZ11b]). *Under the assumptions of Theorem 8.1, the reachable set $\mathscr{R}$ for the heat type equation* (8.18) *satisfies:*

$$A_0^{-1/4} \exp(-S_* \sqrt{A_0}) X \subset \mathscr{R}. \tag{8.26}$$

Remark that Corollary 8.2 is much more precise than the usual exact controllability to the trajectories for (8.18) in any arbitrary time $T > 0$, which only provides $\cup_{T>0} \exp(-TA_0)X \subset \mathscr{R}$.

8.2.2 An integral transform

The main idea of the proof of Theorem 8.1 is to derive an integral transform linking solutions of (8.18) to solutions of (8.19), namely:

Theorem 8.3 ([EZ11b]). *Let $S > 0$. If z denotes a solution of* (8.18)*, then the function w defined by*

$$w(s) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} \sin\left(\frac{sS}{2t}\right) \exp\left(\frac{s^2 - S^2}{4t}\right) z(t) \, dt \tag{8.27}$$

solves the abstract wave equation (8.19) *in $(-S,S)$.*

The idea underlying Theorem 8.3 is the following one. If z denotes a solution of (8.18), we look for w which can be written as

$$w(s) = \int_0^\infty k(t,s) z(t) \, dt, \tag{8.28}$$

such that w solves the abstract wave equation (8.19).

It is not difficult to check that w in (8.28) will be a solution of the wave equation (8.19) for $s \in (-S,S)$ provided that the kernel k satisfies:

$$\begin{cases} \partial_t k + \partial_{ss} k = 0, & (t,s) \in (0,\infty) \times (-S,S), \\ k(0,s) = 0, & s \in (-S,S), \\ \lim_{t\to\infty} k(t,s) = 0, & \text{for all } s \in (-S,S). \end{cases} \tag{8.29}$$

But it turns out that we explicitly know such a solution: namely we can take

$$k_S(t,s) = \frac{1}{\sqrt{4\pi t}} \sin\left(\frac{sS}{2t}\right) \exp\left(\frac{s^2 - S^2}{4t}\right). \tag{8.30}$$

To show that it is indeed a solution of (8.29), remark that $\exp(s^2/4t)/\sqrt{4\pi t}$ is a solution of the backward heat equation in $(8.29)_1$. The kernel k_S simply is the imaginary part of the translation of that function in the complex plane $s \mapsto s + iS$.

Note also that this is precisely the same function as the one in (8.16) used in [FCZ00a, Zua01] to derive the lower bound on δ_*.

Using the kernel k_S, w in (8.28) solves the wave equation (8.19) on $(-S, S)$, with initial data

$$w(0) = 0, \quad \partial_s w(0) = \int_0^\infty \frac{S}{4\sqrt{\pi} t^{3/2}} \exp\left(-\frac{S^2}{4t}\right) z(t)\, dt. \tag{8.31}$$

The proof of Theorem 8.1 then comes naturally. For z solution of (8.18), we choose w as in (8.27) with $S = S_*$ and we apply the observability estimate (8.20) to it. We then simply perform estimates from below on $\|\partial_s w(0)\|_X$ and from above for $\|B^* w\|_{L^2(-S_*, S_*; U)}$, from which we derive (8.22). In this step, we strongly use the explicit form of the kernel k_{S_*}.

8.3 Comments

8.3.1 Links with the vanishing viscosity transport equation

Starting from the observability inequality (8.25) for the 1d heat equation (8.24), one can deduce by compactness that for any $T > 0$, there exists $C_{\mathscr{R}}(T) > 0$ such that any solution z of (8.24) with initial data $z_0 \in H^2 \cap H_0^1(0, L)$ satisfies

$$\int_0^\infty \exp\left(-\frac{L^2}{2t}\right) \|z(t)\|_{L^2(0,L)}^2 \, dt \leq C_{\mathscr{R}}(T) \int_0^T |\partial_x z(t, L)|^2 \, dt. \tag{8.32}$$

However this does not provide any hint on the dependence of $C_{\mathscr{R}}(T)$ as a function of $T > 0$.

Actually, it was observed by P. Lissy in [Lis] that if $C_{\mathscr{R}}(T)$ is smaller than $\exp(\alpha/T)$ for any $\alpha > 0$ as $T \to 0$, this would imply that, for any $T > L$, the systems

$$\begin{cases} \partial_t y + \partial_x y - \varepsilon \partial_{xx} y = 0, & (t, x) \in (0, T) \times (0, L), \\ y(t, 0) = v(t), \quad y(t, L) = 0, & t \in (0, T), \\ y(0, x) = y_0(x), & x \in (0, L), \end{cases} \tag{8.33}$$

are null controllable with a cost of controllability bounded uniformly with respect to $\varepsilon \in (0, 1)$.

The controllability properties of the vanishing viscosity transport equation (8.33) were first studied in [CG05]. It turns out that, though a reasonable conjecture is that the systems (8.33) are uniformly null-controllable in time $T > L$, this is still an open problem. In [CG05], systems (8.33) were proved to be uniformly controllable provided that the time T satisfies $T > 4.3L$. It was improved to $T > 4.2L$ in [Gla10]. Recently, doing the link with the cost of controllability in small time (8.4) for the 1d heat equation and using the results in [TT07], P. Lissy in [Lis12] managed to show the uniform controllability of (8.33) in any time $T > 2\sqrt{3}L$. But still, getting uniform controllability results for (8.33) with respect to ε in any time $T > L$ is an open problem.

8.3.2 Finite time observability

Our approach can also be applied to study the cost of finite time observability $C(T)$ in (8.21), by introducing a kernel $k_{T,S}$ satisfying

$$\begin{cases} \partial_t k_{T,S} + \partial_{ss} k_{T,S} = 0, & (t, s) \in (0, T) \times (-S, S), \\ k_{T,S}(0, s) = 0, & s \in (-S, S), \\ k_{T,S}(T, s) = 0, & s \in (-S, S). \end{cases} \tag{8.34}$$

However, though we can construct solutions $k_{T,S}$ of this system using a power series expansion, following for instance [Joh82, p.211], we were not able to construct a solution of (8.34) with suitable bounds to improve the results in [TT07].

8.3.3 Robustness of the method

Among the advantages of our approach with respect to the existing control transmutation method is that it also applies in contexts in which the wave equation (8.19) observed through B^* during a time $2S_* > 0$ only enjoys the following unique continuation property: if w is a solution of the wave equation (8.19),

$$B^* w(s) = 0 \quad \forall s \in (-S_*, S_*) \Rightarrow w \equiv 0. \tag{8.35}$$

Indeed, in that case, $\|B^* w(s)\|_{L^2(-S_*,S_*;U)}$ is a norm on the trajectories of the wave equation. Therefore, there exists a norm $\|\cdot\|_*$ and a constant $C > 0$ such that any solution w of the wave equation (8.19) with initial data $(0, w_1)$ satisfies

$$\|w_1\|_* \le C \|B^* w\|_{L^2(-S_*,S_*)} .$$

Arguing as in the proof of Theorem 8.1, we can derive

$$\left\| \int_0^\infty \frac{1}{t^{3/2}} \exp\left(-\frac{S_*^2}{4t}\right) z(t)\, dt \right\|_*^2 \le C \int_0^\infty \log(t+2)^2 \|B^* z(t)\|_U^2 \, dt. \tag{8.36}$$

Of course, for this result to be relevant, one should be able to give a precise description of the norm $\|\cdot\|_*$. Note that this can be done in a certain number of situations, among which the 1d wave equation observed from one point which is not a node of any eigenfunctions, or a square with an observation from one side.

Actually, in the most general geometric settings, the unique continuation property (8.35) for the wave equation

$$\begin{cases} \partial_{ss} w - \Delta w = 0, & (s,x) \in (-\infty, \infty) \times \Omega, \\ w(s,x) = 0, & (s,x) \in (-\infty,\infty) \times \partial\Omega, \\ (w(0,x), \partial_s w(0,x)) = (w_0(x), w_1(x)), & x \in \Omega, \end{cases} \tag{8.37}$$

is difficult to quantify. However, some quantifications of the unique continuation property of the wave equation (8.37) exist provided that the time $2S_* > 0$ is (possibly much) larger than the time given by Holmgren's uniqueness theorem. In that case, one gets the following result: given any $\beta > 1$, there exist $S_* > 0$ large enough and $C > 0$ such that any solution w of (8.37) with initial data $(w_0, w_1) \in (H^2 \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ satisfies

$$\|(w_0, w_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \le C \exp(C\Lambda^\beta) \|\partial_s w\|_{L^2(-S_*,S_*;L^2(\omega))} ,$$
$$\text{with } \Lambda = \frac{\|(w_0,w_1)\|_{(H^2\cap H_0^1(\Omega))\times H_0^1(\Omega)}}{\|(w_0,w_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}}. \tag{8.38}$$

With $\beta = 2$, this result was proved in [Rob95] using the Fourier-Bros-Iagoniltzer (FBI) transform. It was later improved to any $\beta > 1$ in [Phu10] using a more singular kernel in the FBI transform first used in [LR97], see Section 6.4.2 for more details on the FBI transform.

For $\beta \in (1, 2)$, estimate (8.38) actually is sufficient to prove the observability estimate (8.4) using our transmutation technique, as we explained in the article [EZ11a]. In order to show that (8.38) implies the observability estimate (8.4) for the heat equation (8.3), we use the observability estimate (8.38), a transmutation kernel solving (8.34), and an iterative argument to involve more and more frequencies with the help of the dissipation of the heat semigroup based on the argument in [Mil10]. This latter argument is a counterpart, from the observability point of view, of the iterative argument produced in the original article [LR95] by G. Lebeau and L. Robbiano. Though, our approach does not give any improvement on the best constant δ_* in (8.14) in these general configurations.

More than the result, it is interesting to notice that the article [LR95] proving the observability inequality (8.4) was obtained through an analysis of the quantification of a unique continuation problem for an elliptic equation:

$$\begin{cases} -\partial_{ss} w - \Delta w = f, & (s,x) \in (-\infty, \infty) \times \Omega, \\ w(s,x) = 0, & (s,x) \in (-\infty, \infty) \times \partial\Omega. \end{cases} \tag{8.39}$$

Similarly, as explained in Section 6.4.2, the FBI transform used in [Rob95, Phu10] links the wave equation to an elliptic equation of the form (8.39). On the other hand, the control transmutation method developed in [Mil04] expresses solutions of the heat equation (8.1) in terms of solutions of the wave equation (8.9), while our approach writes solutions of the wave equation (8.19) in terms of solutions of the heat equations (8.18). All these results and remarks point out the interest of relating semigroups of different types, and in particular with the help of integral transforms.

8.3.4 A Carleman type estimate

Together with J. Dardé, we recently realized that estimate (8.22) can be proved for the 1d heat equation observed from one side directly using a Carleman type approach in the spirit of the work [FI96] by A. Fursikov and O. Imanuvilov. Indeed, we can prove the following:

Proposition 8.4 (Joint work with J. Dardé)**.** *There exists a constant $C > 0$ such that for any smooth solution z of* (8.24)*, we have the observability inequality:*

$$\int_0^\infty \int_0^L |z(t,x)|^2 \exp\left(\frac{x^2 - L^2}{2t}\right) dt\, dx \leq C \int_0^\infty t |\partial_x z(t,L)|^2\, dt. \tag{8.40}$$

Let us emphasize here that Proposition 8.4 states a more precise result than the one obtained in (8.25) due to the presence of the weight function in x in (8.40). Besides, see below, the proof of Proposition 8.4 is more direct than the proof of (8.25) and closely follows the one of the classical Carleman estimates for the heat equation derived for instance in [FI96]. In that context, the corresponding weight function $(x^2 - L^2)/4t$ corresponds to the exponential envelop of the kernel k_L in (8.30). It also has the particularity to be scale invariant for the parabolic scaling $(t,x) \mapsto (\lambda^2 t, \lambda x)$ (with L scaled as $L \mapsto \lambda L$).

Proof. We introduce the new unknown (the conjugated variable)

$$\tilde{z}(t,x) = z(t,x) t \exp\left(\frac{x^2 - L^2}{4t}\right), \quad (t,x) \in (0,\infty) \times (0,L). \tag{8.41}$$

It satisfies the equations

$$\begin{cases} \partial_t \tilde{z} + \dfrac{x}{t}\partial_x \tilde{z} - \dfrac{1}{2t}\tilde{z} - \partial_{xx}\tilde{z} - \dfrac{L^2}{4t^2}\tilde{z} = 0, & (t,x) \in (0,\infty) \times (0,L), \\ \tilde{z}(t,0) = \tilde{z}(t,L) = 0, & t \in (0,\infty), \\ \tilde{z}(0,x) = 0, & x \in (0,L). \end{cases} \tag{8.42}$$

We then introduce the energy $E(t)$ and the dissipation $D(t)$:

$$E(t) = \int_0^L |\tilde{z}(t,x)|^2\,dx, \tag{8.43}$$

$$D(t) = \int_0^L |\partial_x \tilde{z}(t,x)|^2\,dx - \frac{L^2}{4t^2}\int_0^L |\tilde{z}(t,x)|^2\,dx. \tag{8.44}$$

Easy computations show that they satisfy the following ODE:

$$\frac{dE}{dt}(t) - \frac{2}{t}E(t) + 2D(t) = 0, \tag{8.45}$$

$$\frac{dD}{dt}(t) + 2\int_0^L \left|-\partial_{xx}\tilde{z}(t,x) - \frac{L^2}{4t^2}\tilde{z}(t,x)\right|^2 dx = \frac{L}{t}|\partial_x \tilde{z}(t,L)|^2. \tag{8.46}$$

But, by Poincaré's inequality, $D(t)$ is non-negative for $t \geq L^2/(2\pi)$. Therefore, for $T > L^2/(2\pi)$, integrating (8.46) between 0 and T, we get

$$\int_0^T \int_0^L \left|-\partial_{xx}\tilde{z}(t,x) - \frac{L^2}{4t^2}\tilde{z}(t,x)\right|^2 dtdx \leq \int_0^T \frac{L}{2t}|\partial_x \tilde{z}(t,L)|^2\,dt. \tag{8.47}$$

Using the boundary conditions on $\tilde{z}$ and the explicit parametrix for the operator $-\partial_{xx} - \frac{L^2}{4t^2}$, we derive

$$\int_0^T \int_0^L \frac{1}{t^2}|\tilde{z}(t,x)|^2\,dtdx \leq C\int_0^T \frac{1}{t}|\partial_x \tilde{z}(t,L)|^2\,dt,$$

for a constant independent of $T > L^2/(2\pi)$. Besides, from (8.45), again using Poincaré estimate, for all $t \geq T > L^2/2\pi$,

$$\frac{d}{dt}\left(\frac{E(t)}{t^2}\right) + \frac{L^2}{2T^2}\left(\frac{4\pi^2 T^2}{L^4} - 1\right)\frac{E(t)}{t^2} \leq 0,$$

while $t \mapsto E(t)/t^2$ is decreasing on $(L^2/2\pi, T)$:

$$\frac{E(T)}{T^2} \leq \frac{1}{T - L^2/(2\pi)}\int_{L^2/2\pi}^T \frac{E(t)}{t^2}\,dt.$$

Therefore,

$$\int_0^\infty \int_0^L \frac{1}{t^2}|\tilde{z}(t,x)|^2\,dtdx \leq C\int_0^T \frac{1}{t}|\partial_x \tilde{z}(t,L)|^2\,dt,$$

Using (8.41), we immediately obtain (8.40). □

Proposition 8.4 can be easily generalized to higher dimension when $\Omega \setminus \overline{\omega}$ is a ball and ω is a neighborhood of the whole boundary $\partial\Omega$. It then yields a sharp result in that case due to the bound from below (8.17). If such direct approach could be developed in more general geometric setting, one could possibly address the problem of characterizing δ_* in situations in which the geometric control condition is not satisfied. But so far, this is still an open problem.

Proposition 8.4 also indicates that it is possible to derive Carleman type estimates with weights in time that do not blow up as $t \to T$. This is precisely what has been done in Theorem 11.1 to the price of convexifying close to T (Theorem 11.1 is stated for the backward heat equation so that it corresponds to t close to $t = 0$ in Part IV).

Chapter 9

Observability of time-discrete conservative equations

9.1 Introduction

The goal of this chapter is to discuss the observability properties of time discrete approximations of abstract conservative equations.

To be more precise, let A be a skew-adjoint operator defined on a Hilbert space X, with dense domain $\mathscr{D}(A)$ and compact resolvent. We consider the abstract conservative system given by

$$\partial_t z = Az, \quad t \in \mathbb{R}, \qquad z(0) = z_0 \in X. \tag{9.1}$$

Such abstract equation encompasses several models of interest, and among them the wave equation, Schrödinger equation or the plates equations. Note in particular that the system (9.1) is well-posed forward and backward in time, and we can therefore consider the solution on the whole time interval $t \in \mathbb{R}$. Besides, the energy $\|z(t)\|_X$ is preserved among time, thus justifying our terminology “abstract conservative equations”.

Our goal is to discuss some properties of the time-discrete approximations of (9.1), with a particular focus on observability properties. We will only consider time discretization schemes preserving the energy of the solutions, as it is an important feature of the continuous model (9.1).

The time-discrete approximations we will consider are as follows. Let $\tau > 0$ representing the time-discretization parameter, and define the time-discrete approximation schemes by

$$z_\tau^{k+1} = \mathbb{T}_\tau z_\tau^k, \quad k \in \mathbb{Z}, \qquad z_\tau^0 = z_0, \tag{9.2}$$

where $\mathbb{T}_\tau$ is an approximation of $\exp(\tau A)$ and z_τ^k approximates the solution z of (9.1) at time $k\tau$.

To describe more accurately the time-discrete models (9.2), we shall specify the assumptions on $\mathbb{T}_\tau$. We will assume that $\mathbb{T}_\tau$ is given by

$$\mathbb{T}_\tau = \exp(if(-i\tau A)) \tag{9.3}$$

for some function f satisfying the following properties:

(H1) $f : (-R, R) \to (-\pi, \pi)$ is smooth on some interval $(-R, R)$ for some $R \in \mathbb{R}_+^* \cup \{+\infty\}$, (H2) $f(0) = 0$, $f'(0) = 1$,

(H3) for all $\delta < R$, $\inf\{f'(\alpha), |\alpha| \leq \delta\} > 0$.

In Assumption (H1), R stands for the range for the stability of the numerical scheme. We assume f to be real valued to guarantee that the time discrete approximation schemes (9.2) are energy preserving. The fact that its image is in $(-\pi, \pi)$ comes from the fact that one cannot measure frequencies larger than π/τ in a mesh of mesh-size τ. Assumption (H2) enforces the consistency of the scheme.

Assumption (H3) is the main assumption allowing to do our construction hereafter. In particular, it implies that we can define the inverse of f as a function $g : (-f(R), f(R)) \to (-R, R)$.

For sake of simplicity, and because all the examples we encountered in the literature satisfy it, we also impose:

(H4) f is odd.

Note that this abstract setting is satisfied for several time-discrete approximation schemes, among which the midpoint scheme, the Newmark method for discretizing second order systems $z'' + A_0 z = 0$ with positive self-adjoint operator A_0, etc.

Let us also emphasize that due to assumption (H1), the time-discrete equations (9.2) are well-posed when z_0 is in some filtered class, that we now introduce. As A is skew-adjoint with compact resolvent, its spectrum is given by a sequence of eigenvalues $(i\mu_j)_{j\in J}$, with $J = \mathbb{N}$ or $\mathbb{Z}$, and of eigenvectors $(\Phi_j)_{j\in J}$ which form an orthonormal basis of X. For all $\delta > 0$, we introduce the filtered class

$$\mathscr{C}(\delta) = \mathrm{Span}\{\Phi_j, \text{ with } |\mu_j| \leq \delta\}. \tag{9.4}$$

It is then easy to check that (9.2) is well-posed for initial data $z_0 \in \mathscr{C}(\delta/\tau)$ for any $\delta < R$.

In the following, our goal is to derive some uniform observability properties for the time discrete models (9.2) from the observability properties of the time continuous model (9.1), which usually are better understood.

We will proceed as in the previous chapter, i.e. to any given discrete solution z_τ of (9.2), we associate a continuous trajectory z of (9.1).

9.2 Main results

9.2.1 A transmutation technique

With E. Zuazua, we proved the following transmutation formula:

Theorem 9.1 ([EZ]). *Assume (H1)–(H4), and fix $\delta \in (0, R)$. Let χ be smooth function compactly supported in $(-f(R), f(R))$ and taking value 1 in $[-f(\delta), f(\delta)]$. For $\tau > 0$, define the kernel*

$$\rho_\tau(t, s) = \frac{1}{2\pi\tau} \int_{-\pi}^{\pi} \exp\left(\frac{i}{\tau}(\alpha s - g(\alpha)t)\right) \chi(\alpha)\, d\alpha. \tag{9.5}$$

Then, if $z_0 \in \mathscr{C}(\delta/\tau)$ and z_τ denotes the corresponding solution of (9.2), the function

$$z(t) = \tau \sum_{k\in\mathbb{Z}} \rho_\tau(t, k\tau) z_\tau^k \tag{9.6}$$

solves (9.1) with initial data z_0.

Note that for all $t \in \mathbb{R}$ and $\tau > 0$, the sum in (9.6) converges as

$$\rho_\tau(t,s) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\mu s} \exp\left(-\frac{i}{\tau} g(\mu\tau)t\right)\chi(\mu\tau)\,d\mu. \tag{9.7}$$

is the inverse Fourier transform of the smooth function $e^{-ig(\mu\tau)t/\tau}\chi(\mu\tau)$ and thus decays faster than any polynomial.

Let us give some insights on the construction of the kernel ρ_τ. If we expand the initial data as

$$z_0 = \sum_J a_j \Phi_j, \tag{9.8}$$

then the discrete solutions z_τ of (9.2) satisfies

$$z_\tau^k = \sum_J a_j \exp\left(i\frac{f(\mu_j\tau)}{\tau}k\tau\right)\Phi_j. \tag{9.9}$$

It is therefore natural to introduce the time continuous functions

$$z_\tau(s) = \sum_J a_j \exp\left(i\frac{f(\mu_j\tau)}{\tau}s\right)\Phi_j, \tag{9.10}$$

for which we have $z_\tau(k\tau) = z_\tau^k$ for all $k \in \mathbb{Z}$. If one wants to find a kernel ρ_τ such that

$$z(t) = \int_{\mathbb{R}} \rho_\tau(t,s) z_\tau(s)\,ds, \tag{9.11}$$

solves (9.1), we should have for all $\mu \in \mathbb{R}$ with $|\mu|\tau \le f(\delta)$,

$$\exp\left(i\frac{g(\mu\tau)}{\tau}t\right) = \int_{\mathbb{R}} \rho_\tau(t,s)\exp(i\mu s)\,ds. \tag{9.12}$$

Multiplying by $\chi(\mu\tau)$, which equals 1 for $|\mu|\tau \le f(\delta)$, and taking the inverse Fourier transform, formula (9.7) comes out naturally. The proof of Theorem 9.1 follows a similarly strategy, using discrete Fourier transforms instead of the continuous Fourier transform.

9.2.2 Estimates on the kernel

As we have already mentioned in the previous chapter, one of the main difficulties of the transmutation based techniques is to get suitable estimates on the kernel function ρ_τ in (9.5).

In order to give some insights on these estimates, note that the function z_τ in (9.10) satisfies the equation

$$\partial_s^\tau z_\tau = A z_\tau, \quad s \in \mathbb{R}, \tag{9.13}$$

where the operator ∂_s^τ is defined in Fourier as

$$\forall \mu \in \mathbb{R}, \quad \partial_s^\tau(e^{i\mu s}) = i\frac{g(\mu\tau)}{\tau}e^{i\mu s}. \tag{9.14}$$

Therefore, the kernel ρ_τ in (9.11) should satisfy the transport type equation

$$\begin{cases} \partial_t \rho_\tau + \partial_s^\tau \rho_\tau = 0, & (t,s) \in \mathbb{R}\times\mathbb{R}, \\ \rho_\tau(0,s) = \delta_0(s), & s \in \mathbb{R}, \end{cases} \tag{9.15}$$

δ_0 being the Dirac function in the point $s = 0$. If we think to (9.15) as a transport type equation, we should expect that ρ_τ should stay concentrated on the cone of light of the equations. This cone of light can actually be seen more efficiently on the formula (9.5) by an oscillatory phase argument: $\rho_\tau(t, s)$ is smaller than any polynomial rate in τ outside the set of critical points of the phase $\alpha \mapsto \alpha s - g(\alpha)t$. For instance, for any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exists $C_{\varepsilon,n} > 0$ independent of τ such that

$$|\rho_\tau(t,s)| \leq C_{\varepsilon,n}\tau^n, \text{ for all } (t,s) \in [0,T] \times \mathbb{R}, \text{ with } \begin{cases} s < \inf\limits_{\alpha \in \mathrm{Supp}\chi} \{g'(\alpha)\}t - \varepsilon, \\ \text{or} \\ s > \sup\limits_{\alpha \in \mathrm{Supp}\chi} \{g'(\alpha)\}t + \varepsilon. \end{cases} \tag{9.16}$$

We refer to [EZ, Proposition 3.4] for precise statements.

9.2.3 Observability of the time-discrete models (9.2)

Theorem 9.1 can be used to derive observability properties for the time-discrete equations (9.2) from the observability of the time-continuous equation (9.1).

We consider the observation problem for (9.1) from an observation operator[1] B^* being in $\mathscr{L}(\mathscr{D}(A^p), U)$ for some $p \geq 0$ and Hilbert space U. In particular, we assume that there exist $T_* > 0$ and a constant $C_* > 0$ such that any solution z of (9.1) with initial data $z_0 \in \mathscr{D}(A^p)$ satisfies

$$\|z_0\|_X \leq C_* \|B^* z\|_{L^2(0,T_*;U)}. \tag{9.17}$$

As a byproduct of Theorem 9.1, we get the following result:

Theorem 9.2 ([EZ]). *Assume that the equation* (9.1) *is observable through* $B^* \in \mathscr{L}(\mathscr{D}(A^p), U)$ *in time* T_* *in the sense of* (9.17)*, and consider time-discrete approximations schemes given by* (9.2)–(9.3) *satisfying assumptions (H1)–(H4).*

For any $\delta < R$*, for any* T_δ *satisfying*

$$T_\delta > \frac{T_*}{\inf\limits_{|\alpha| \leq \delta} f'(\alpha)}, \tag{9.18}$$

there exist a constant $C > 0$ *and* $\tau_0 > 0$ *such that for all* $\tau \in (0, \tau_0)$*, any solution* z_τ *of* (9.2) *with initial data* $z_0 \in \mathscr{C}(\delta/\tau)$ *satisfies the following time discrete observability inequalities:*

$$\|z_0\|_X^2 \leq C\tau \sum_{k\tau \in [0, T_\delta]} \left\|B^* z_\tau^k\right\|_U^2. \tag{9.19}$$

Theorem 9.2 states a uniform observability property for the time-discrete models (9.2) provided that the initial data is filtered at a scale δ/τ. This is the property which is needed to derive convergence results for the discrete controls (i.e. of what we called the discrete approach in Part I). For more comments on this fact, we refer to the discussion in Part I, see also [Zua05].

The proof of Theorem 9.2 is actually rather easy once we have Theorem 9.1. Indeed, given a discrete solution z_τ of (9.2), we can associate by (9.6) a solution of (9.1), for which we can use (9.17). It then simply remains to estimate the right hand side of (9.17), where we used the fact that the kernel is small outside the cone of light (9.16). We refer the reader to [EZ] for extensive details.

[1] Here again, the observation operator is denoted by B^* as our study is motivated by the control problem $y' = Ay + Bv$.

Let us also note that Theorem 9.2 is new only for the time estimate (9.18). Otherwise, the uniform observability for the time-discrete models (9.2) in filtered class was proved in [EZZ08], but with no control on the time of uniform observability. The idea in [EZZ08] was to use the characterization of the observability in terms of the resolvent estimate in the spirit of [BZ04, Mil05], but this strategy does not provide any explicit estimate on the time of observability.

9.2.4 Optimality of the time estimate in (9.18)

The time estimate (9.18) is sharp in general. To see that, we can simply invert the transmutation method as follows:

Theorem 9.3 ([EZ]). *Assume (H1)–(H4), and fix $(\delta_1, \delta_2) \subset [0, R)$.*

Let χ be smooth even function compactly supported in $(-R, R)$ and taking value 1 in $[\delta_1, \delta_2]$. For $\tau > 0$, define the kernel

$$q_\tau(t, s) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(\frac{i}{\tau}(f(\mu\tau)s - \mu t)\right) \chi(\mu\tau)\, d\mu. \tag{9.20}$$

Then, if $z_0 \in \mathscr{C}(\delta_2/\tau) \cap \mathscr{C}(\delta_1/\tau)^\perp$ and z denotes the corresponding solution of (9.1)*, the discrete function z_τ defined by*

$$z_\tau^k = \int_{\mathbb{R}} q_\tau(t, k\tau) z(t)\, dt \tag{9.21}$$

solves (9.2) *with initial data z_0.*

The proof is of course very similar to the one of Theorem 9.1.

This result can be applied to derive uniform admissibility results for the time-discrete models (9.2) when the data belong to the filtered class $\mathscr{C}(\delta/\tau)$ for $\delta < R$, but this was already proved in [EZZ08]. The proof of admissibility in [EZZ08] was based on a spectral characterization of admissibility in terms of packets of eigenfunctions, in the spirit of the spectral characterization of observability obtained in [RTTT05].

Theorem 9.3 can also be used to show that the time condition (9.18) is sharp. Indeed, if for some time T_δ, the time-discrete equations (9.2) are uniformly observable for data in the filtered class $\mathscr{C}(\delta/\tau)$, then for all $(\delta_1, \delta_2) \subset (0, R)$, we can proceed as before and show that the abstract continuous equation (9.1) are uniformly observable in the class $\mathscr{C}(\delta_2/\tau) \cap \mathscr{C}(\delta_1/\tau)^\perp$ in any time larger than $T_\delta \sup_{\alpha \in [\delta_1, \delta_2]} f'(\alpha)$. It is then straightforward to construct some particular examples for which the time T_* of observability of the time continuous model (9.1) should necessarily satisfy $T_* \leq T_\delta \inf_{|\alpha| \leq \delta} f'(\alpha)$, thus proving the optimality of the time estimate (9.18).

9.3 Further comments

9.3.1 Fully discrete approximation schemes

When discretizing the equation (9.1), one should consider both space and time discrete approximations of the model under consideration.

While our study only focuses on the time-discretization process, it also gives some insights in the case of fully discrete equations. Indeed, as our method applies in an abstract setting, it depends on the space operator A very weakly. For instance, consider a sequence of operators A_h corresponding to space discrete approximations of the operator A on a mesh of size $h > 0$, such that:

- for all $h > 0$, A_h is skew-adjoint on an Hilbert space X_h, with dense domain $\mathscr{D}(A_h)$ and compact resolvent;
- there exist $p > 0$ and $C_p > 0$ such that for all $h > 0$, $B_h^* \in \mathscr{L}(\mathscr{D}(A_h^p), U_h)$ for some Hilbert spaces U_h, with $\|B_h^*\|_{\mathscr{L}(\mathscr{D}(A_h^p),U_h)} \leq C_p$;
- the (time-continuous) equations $\partial_t z_h = A_h z_h$ observed through B_h^* are observable in the time T_*, uniformly with respect to $h > 0$.

Under these assumptions, we choose a time discretization process given by a function f as in (9.3) satisfying (H1)–(H4). Our result proves that the fully discrete models given by

$$z_{\tau,h}^{k+1} = \mathbb{T}_{\tau,h} z_{\tau,h}^k \, k \in \mathbb{Z}, \quad z_{\tau,h}(0) = z_0, \quad \text{with } \mathbb{T}_{\tau,h} = \exp(if(-i\tau A_h)), \tag{9.22}$$

are uniformly observable in the class $\mathscr{C}_h(\delta/\tau)$ for any $\delta < R$ in any time T_δ satisfying (9.18), where $\mathscr{C}_h(\delta/\tau)$ is the filtered class associated to the operator A_h.

Therefore, Theorem 9.2 implies that the problem of studying the observability properties of fully discrete approximations of an abstract conservative equation (9.1) can mainly be reduced to the study of the uniform observability properties of the space semi-discrete approximation schemes $\partial_t z_h = A_h z_h$ observed through B_h^*. But, as recalled in Section 3.6, this latter issue is a difficult question, and it has not yet received a full answer. We refer to [Mil12] for the most general result in that context, and to [EZ11c] for several references related to this topic.

Also note that the filtering class $\mathscr{C}_h(\delta/\tau)$ can be simply reduced to the whole space X_h if $\|A_h\|_{\mathscr{L}(X_h)} \tau \leq \delta$. This condition can be interpreted as a kind of Courant-Friedrichs-Lax condition since the norm of the operator A_h is usually of the order of $1/h^\alpha$ for some $\alpha > 0$, where h is the size of the spatial mesh.

Very likely, this strategy could also be used to derive Carleman estimates for fully discrete approximations of the wave equation based on the Carleman estimates in Theorem 6.1 developed for the space semi-discrete wave equation, but due to the presence of weight functions depending on both time and space variables, this requires more work.

9.3.2 Dissipative schemes

Note that our approach applies only in the context of time-discrete approximation schemes preserving the energy. Though this is a natural assumption as the time-continuous equation (9.1) preserves the energy, it is sometimes useful to add some small dissipation to ensure the stability of the numerical schemes. But we were not able to derive a transmutation technique in that case. This is due to the fact that the discrete systems are no longer time-reversible, and the discrete models thus displays some properties of the classical heat equation. Therefore, in the spirit of the previous chapter, it might be possible to construct some more singular kernels to link the time-discrete models to their continuous counterpart. But so far, this is still an open problem.

Part IV

Controllability of non-homogeneous viscous fluids

Chapter 10

Introduction

In this part, we present some controllability results obtained for non-homogeneous viscous fluids.

In Chapter 11, we will focus on the local exact controllability to constant states for the compressible Navier-Stokes equations, mainly following the work [EGGP12]. As we will see, the main difficulty comes from the coupling of the parabolic equation satisfied by the fluid velocity and of the transport equation satisfied by the density. This requires to introduce new Carleman type estimates, with a weight function which is in particular constant along the flow of the target velocity.

In Chapter 12, we explain how these ideas can be adapted to obtain local exact controllability to trajectories for the non-homogeneous incompressible Navier-Stokes equations, following [BEG]. In that case, the coupling between the equations of the density and of the velocity is weaker. However, the pressure term, appearing as the Lagrange multiplier of the divergence free condition, introduces new difficulties.

Chapter 11

Controllability of compressible Navier-Stokes equations

11.1 Introduction

The goal of this chapter is to present some controllability results for compressible Navier-Stokes equations.

Let $T > 0$ and Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \in \{1, 2, 3\}$. The motion of a barotropic compressible viscous fluid is described by the following set of equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div}(\mathbf{u}) + \nabla p(\rho) = 0, & \text{in } (0,T) \times \Omega. \end{cases} \tag{11.1}$$

Here ρ is the density of the fluid, $\mathbf{u}$ its velocity and $p = p(\rho)$ is the pressure law. It is usually given by the standard polytropic law:

$$p(\rho) = \kappa \rho^\gamma, \quad \text{with } \gamma \geq 1 \text{ and } \kappa > 0. \tag{11.2}$$

The parameters μ and λ correspond to constant viscosity parameters and are assumed to satisfy $\mu > 0$ and $d\lambda + 2\mu > 0$.

To be well-posed, system (11.1) has to be completed with initial data

$$(\rho(0, \cdot), \mathbf{u}(0, \cdot)) = (\rho_0, \mathbf{u}_0), \quad \text{in } \Omega, \tag{11.3}$$

and suitable boundary conditions in which the controls act. For (11.1), these boundary conditions are of the form

$$\begin{cases} \mathbf{u}(t,x) = \mathbf{v}_{\mathbf{u}}(t,x), & \text{for } (t,x) \in (0,T) \times \partial\Omega, \\ \rho(t,x) = v_\rho(t,x), & \text{for } (t,x) \in (0,T) \times \partial\Omega, \text{ with } \mathbf{u}(t,x) \cdot \mathbf{n}(x) < 0, \end{cases} \tag{11.4}$$

where $\mathbf{v}_{\mathbf{u}}, v_\rho$ are the controls. Though, they will never explicitly in our construction.

In the following, we study local exact controllability of (11.1) to constant trajectories in time $T > 0$. To be more precise, given $(\overline{\rho}, \overline{\mathbf{u}}) \in \mathbb{R}_+^* \times \mathbb{R}^d$ and $(\rho_0, \mathbf{u}_0)$ close to $(\overline{\rho}, \overline{\mathbf{u}})$, we ask whether it is possible or not to find controls such that the corresponding solution of (11.1)–(11.4) satisfies

$$(\rho(T, \cdot), \mathbf{u}(T, \cdot)) = (\overline{\rho}, \overline{\mathbf{u}}), \quad \text{in } \Omega. \tag{11.5}$$

There were no result on the local exact controllability of the compressible Navier-Stokes equations before our works, except for [Amo11] which proved local exact controllability for compressible Navier-Stokes equations in 1d written in Lagrangian coordinates and when the initial density coincides with the target density. But local exact controllability for homogeneous incompressible Navier-Stokes equations has been widely studied starting from the pioneering work [FI96], in particular in the works [Ima01, FCGIP04]. It has even been shown that one can use controls with some vanishing components, see [FCGIP06, CG05, CL]. Let us also mention that there are some global controllability results when the control acts on the whole boundary, see the article [CF96] based on the return method developed in [Cor96] for global controllability of incompressible perfect fluids in 2d.

Before studying the compressible Navier-Stokes equations, we should also quote the results obtained for the 1d compressible Euler equations, namely the ones obtained in [LR03, CBdN08] concerning control and local stabilization results of classical solutions. Local exact controllability results were also obtained in the context of weak entropy solutions in [Gla07], later extended to the 1d non-isentropic Euler equation [Gla14].

Going back to the control problem (11.1)–(11.5), it is natural to follow the approach developed by A. Fursikov and O. Imanuvilov in [FI96] and try to adapt it to our case. The main idea in [FI96] is to develop Carleman estimates for the heat equation. These Carleman estimates are weighted observability estimates which degenerate close to the initial time (this is necessary due to the ill-posedness of the heat equation when solving it backward in time), and which are parametrized by some parameter allowing to handle lower order terms easily (recall for instance the Carleman estimate developed in Theorem 5.3 for the wave equation). We refer for instance to [FCZ00b] for the study of the global controllability of semi-linear heat equations, where it is shown that the critical blow up rate of the semi-linearity allowing global controllability is intimately linked to the power of the parameter in the Carleman estimates.

We shall therefore begin by revisiting and generalizing slightly the Carleman estimates in [FI96] for the heat equation. As we will see, this will be one of the elements needed in the proofs of controllability results for non-homogeneous viscous fluids.

11.2 Carleman estimates for the heat equation

Let Ω be a smooth bounded domain of $\mathbb{R}^d$, and consider the time-backward heat equation

$$\begin{cases} -\partial_t z - \Delta z = g, & (t,x) \in (0,T) \times \Omega, \\ z(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega. \end{cases} \tag{11.6}$$

We consider the case of a distributed observation through an non-empty open subset $Q_{obs} \subset [0,T] \times \Omega$.

We assume that the observation set Q_{obs} is such that there exists a function $\tilde{\psi} = \tilde{\psi}(t,x) \in C^2([0,T] \times \overline{\Omega})$ satisfying

$$\begin{cases} \forall (t,x) \in [0,T] \times \overline{\Omega}, \quad \tilde{\psi}(t,x) \in [0,1], \\ \forall (t,x) \in [0,T] \times \partial\Omega, \quad \partial_{\mathbf{n}} \tilde{\psi}(t,x) \leq 0, \\ \inf_{[0,T] \times (\overline{\Omega} \setminus Q_{obs})} \{|\nabla \tilde{\psi}|\} > 0. \end{cases} \tag{11.7}$$

For $m \geq 1$, we define

$$\psi(t,x) = \tilde{\psi}(t,x) + 6m. \tag{11.8}$$

For any $T_0 > 0$ and $T_1 > 0$ with $T_0 + 2T_1 < T$, we choose a weight function in time $\theta(t)$

depending on the parameters $m \geq 1$ and $\mu \geq 2$ as follows:

$$\theta = \theta(t) \text{ such that } \begin{cases} \forall t \in [0, T_0],\ \theta(t) = 1 + \left(1 - \dfrac{t}{T_0}\right)^{\mu}, \\ \forall t \in [T_0, T - 2T_1],\ \theta(t) = 1, \\ \forall t \in [T - T_1, T),\ \theta(t) = \dfrac{1}{(T-t)^m}, \\ \theta \text{ is increasing on } [T - 2T_1, T - T_1], \\ \theta \in C^2([0, T)). \end{cases} \tag{11.9}$$

We finally introduce the following weight functions $\varphi = \varphi(t, x)$ and $\xi = \xi(t, x)$:

$$\varphi(t, x) = \theta(t)\left(\lambda e^{6\lambda(m+1)} - e^{\lambda\psi(t,x)}\right), \quad \xi(t, x) = \theta(t) e^{\lambda\psi(t,x)}, \tag{11.10}$$

where μ is chosen as

$$\mu = s\lambda^2 e^{\lambda(6m-4)}, \tag{11.11}$$

and s, λ are positive parameters with $s \geq 1$, $\lambda \geq 1$ (Hence $\mu \ \geq 2$, thus being compatible with the condition $\theta \in C^2([0, T))$.) The weight functions φ, ξ, θ depend on s, λ, m, and should rather be denoted by $\varphi_{s,\lambda,m}$, resp. $\xi_{s,\lambda,m}$, $\theta_{s,\lambda,m}$, but we drop these indexes for simplicity of notations.

We derive the following L^2-Carleman estimate for the heat equation (11.6):

Theorem 11.1 (Theorem 2.5 in [BEG]). *Let $m \geq 1$, $T_0 > 0$ and $T_1 > 0$ with $T_0 + 2T_1 < T$, and consider the weights functions φ, ξ, θ introduced in* (11.7)–(11.11). *Then there exist constants $C_0 > 0$, $s_0 \geq 1$ and $\lambda_0 \geq 1$ such that for all smooth functions z on $[0, T] \times \Omega$ satisfying $z = 0$ on $(0, T) \times \partial\Omega$, for all $s \geq s_0$, $\lambda \geq \lambda_0$, we have*[1]

$$\begin{aligned} \int_\Omega |\nabla z(0)|^2 e^{-2s\varphi(0)} &+ s^2\lambda^3 e^{2\lambda(6m+1)} \int_\Omega |z(0)|^2 e^{-2s\varphi(0)} \\ &+ s\lambda^2 \iint_{(0,T)\times\Omega} \xi |\nabla z|^2 e^{-2s\varphi} + s^3\lambda^4 \iint_{(0,T)\times\Omega} \xi^3 |z|^2 e^{-2s\varphi} \\ &\leq C_0 \iint_{(0,T)\times\Omega} |(-\partial_t - \Delta) z|^2 e^{-2s\varphi} + C_0 s^3\lambda^4 \iint_{Q_{obs}} \xi^3 |z|^2 e^{-2s\varphi}. \end{aligned} \tag{11.12}$$

This L^2-Carleman estimate for (11.6) is very close to the one in [FI96]. The powers of the parameters s and of the functions ξ are the same as in [FI96]. However, this result differs from [FI96] on some points.

First, the weight function in time θ in (11.9) does not blow up as $t \to 0$, while the work [FI96] considered a weight function of the form $1/(t(T-t))^m$ for $m \geq 1$. This is achieved to the price of strongly convexifying the weight function θ close to $t = 0$ at some rate depending on the parameters s and λ, see (11.9), (11.11). This will be particularly helpful to handle more easily the dual controllability problem, see Corollary 11.2 below.

Second, the weight function in time θ in (11.9) is constant on a (possibly large) interval of time. This is not critical for Theorem 11.1, but it will be helpful when considering the coupling with the transport phenomena appearing in non-homogeneous fluids.

Third, the weight function $\tilde{\psi}$ in (11.7) depends on both time and space variables. As we will see in the following, in order to handle the equation of the density, we will need to consider

[1] In this part, we deliberately omit the terms $dxdt$, dx in the integrals to simplify the notations.

weight functions $\tilde{\psi}$ which are constant along the flow of the target trajectory. We shall therefore pay a particular attention to the possibility of considering a function $\tilde{\psi}$ depending on both time and space variables. Recent results were developed in the same spirit in the recent work [CSRZ] for the control of a visco-elasticity model with moving controls.

The proof of Theorem 11.1 can be done similarly as in [FI96], except for the terms involving time-derivatives of the weight function θ which have to be considered very cautiously, as each derivative in θ close to $t = 0$ increases the power of s. But it turns out that the most singular terms have good signs. For instance, we could add in the left hand-side of (11.12) the term

$$s^2\lambda^2 \iint_{(0,T_0)\times\Omega} |\partial_t\theta| \xi\varphi |z|^2 e^{-2s\varphi}.$$

Also note that, similarly as for the Carleman estimate in [FI96], the Carleman estimate in Theorem 11.1 applies to more general operators of the form:

$$\begin{cases} -\sigma\partial_t z - \operatorname{div}(A\nabla z) = g, & (t,x) \in (0,T)\times\Omega, \\ z(t,x) = 0, & (t,x) \in (0,T)\times\partial\Omega, \end{cases} \tag{11.13}$$

for $\sigma \in W^{1,\infty}((0,T)\times\Omega)$ and $A \in W^{1,\infty}((0,T)\times\Omega; M_{d\times d}(\mathbb{R}))$ such that there exists $\alpha > 0$ with

$$\forall (t,x) \in (0,T)\times\Omega,\ \forall \xi \in \mathbb{R}^d, \quad \begin{cases} \sigma(t,x) \geq \alpha, \\ \langle \xi, A(t,x)\xi\rangle_{\mathbb{R}^d} \geq \alpha \left\|\xi\right\|_{\mathbb{R}^d}^2, \end{cases}$$

which are the natural conditions for the well-posedness of (11.13).

Under the above assumption, using duality and the Carleman estimate in Theorem 11.1, we can study the following null-controllability problem: For $f \in L^2((0,T)\times\Omega)$ and $y_0 \in H_0^1(\Omega)$, find a function $v \in L^2(Q_{obs})$ such that the solution y of

$$\begin{cases} \sigma\partial_t y - \operatorname{div}(A(t,x)\nabla y) = f + v\chi_{obs}, & (t,x) \in (0,T)\times\Omega, \\ y(t,x) = 0, & (t,x) \in (0,T)\times\partial\Omega, \\ y(0,x) = y_0(x), & x \in \Omega, \end{cases} \tag{11.14}$$

where χ_{obs} is the indicator function of Q_{obs}, satisfies

$$y(T) = 0 \quad \text{in } \Omega. \tag{11.15}$$

Indeed, we get:

Corollary 11.2 (Theorem 2.6 in [BEG]). *Let $y_0 \in H_0^1(\Omega)$ and assume that $f \in L^2((0,T)\times\Omega)$ satisfies, for some parameters $s \geq s_0$ and $\lambda \geq \lambda_0$, $\xi^{-3/2} f e^{s\varphi} \in L^2((0,T)\times\Omega)$. Then there exists a control function $v \in L^2(Q_{obs})$ such that the controlled trajectory y solution of (11.14) satisfies (11.15). Besides, we can construct v and y in such a way that $(y_0, f) \mapsto (y, v)$ is linear and satisfies*

$$\begin{aligned} s^{3/2}\lambda^2 \left\| y e^{s\varphi}\right\|_{L^2(L^2)} + s^{1/2}\lambda \left\|\xi^{-1}\nabla y e^{s\varphi}\right\|_{L^2(L^2)} + s^{-1/2}\left\|\xi^{-2}\partial_t y e^{s\varphi}\right\|_{L^2(L^2)} \\ + s^{-1/2}\left\|\xi^{-2} D^2 y e^{s\varphi}\right\|_{L^2(L^2)} + \left\|\xi^{-3/2} v e^{s\varphi}\right\|_{L^2(Q_{obs})} \\ \leq C \left\|\xi^{-3/2} f e^{s\varphi}\right\|_{L^2(L^2)} + C\left\| y_0 e^{2s\varphi(0)}\right\|_{H_0^1(\Omega)}, \end{aligned} \tag{11.16}$$

where $\|\cdot\|_{L^2(L^2)} = \|\cdot\|_{L^2((0,T)\times\Omega)}$.

We can be more precise on the dependence of the controlled trajectory with respect to the initial data and get in the right-hand side of (11.16) the term

$$Cs^{1/2}\lambda \left\| \xi(0,\cdot)^{-1} y_0 e^{s\varphi(0)} \right\|_{L^2(\Omega)} + Cs^{-1/2} \left\| \xi(0,\cdot)^{-2} \nabla y_0 e^{s\varphi(0)} \right\|_{L^2(\Omega)}$$

instead of $C \left\| y_0 e^{2s\varphi(0)} \right\|_{H_0^1(\Omega)}$, but this better estimate is of little interest in practice.

Below, we present the applications of these results to fluids.

11.3 The 1-dimensional case

11.3.1 Main result

In [EGGP12], we focused on the compressible Navier-Stokes equations (11.1) in the 1-dimensional case corresponding to $\Omega = (0, L)$. Setting $\nu = \lambda + 2\mu$ ($\nu > 0$), (11.1) simply reduces to

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & \text{in } (0,T) \times (0,L), \\ \rho(\partial_t u + u\partial_x u) - \nu \partial_{xx} u + \partial_x(p(\rho)) = 0, & \text{in } (0,T) \times (0,L), \end{cases} \tag{11.17}$$

still considered with controls on the boundary as in (11.4). The constant trajectories $(\overline{\rho}, \overline{u}) \in \mathbb{R}_+^* \times \mathbb{R}$ are stationary solutions of (11.17). It is therefore natural to ask if one can drive solutions of (11.17) to $(\overline{\rho}, \overline{u})$ exactly in some time T.

To better understand this problem, we start by linearizing equations (11.17) around $(\overline{\rho}, \overline{u})$:

$$\begin{cases} \partial_t \rho + \overline{u}\partial_x \rho + \overline{\rho}\partial_x u = 0, & \text{in } (0,T) \times (0,L), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u + p'(\overline{\rho})\partial_x \rho = 0, & \text{in } (0,T) \times (0,L). \end{cases} \tag{11.18}$$

For $\overline{u} = 0$, this system reduces to the equation

$$\overline{\rho}\partial_{tt} u - \nu \partial_{xx}\partial_t u - p'(\overline{\rho})\overline{\rho}\partial_{xx} u = 0, \text{ in } (0,T) \times (0,L), \tag{11.19}$$

with controls on the boundary. Up to a normalization, this corresponds to the wave equation with structural damping studied in [RR07]. According to [RR07], (11.19) is not spectrally controllable in any time $T > 0$ due to the presence of an accumulation point in the spectrum. We also refer to [CRR12] for another aspect of this phenomenon in the context of the stabilization of linearized compressible Navier-Sokes equations (11.18) around the velocity $\overline{u} = 0$.

Another way to understand this lack of controllability is to construct solutions of the adjoint equations which violates the observability estimate. This can be done using a Gaussian beam approach in the spirit of the work of J. Ralston [Ral69] for the construction of solutions of the wave equations localized around the rays of Geometric Optics, see [Deb]. Indeed, this approach constructs solutions to the adjoint of (11.18) in the case $\overline{u} = 0$ - equivalently of (11.19) - which do not travel and therefore cannot be observed from the boundary. This can be done by considering the adjoint of (11.18) in the extended domain $x \in \mathbb{R}$ and taking the Fourier transform in space. For each frequency $\xi \in \mathbb{R}$, the adjoint equation of (11.18) (still with $\overline{u} = 0$, and stated in (σ, z)), after the change of variable $t \mapsto T - t$, writes

$$\frac{d}{dt} \begin{pmatrix} \hat{\sigma}(t,\xi) \\ \hat{z}(t,\xi) \end{pmatrix} + A(\xi)^* \begin{pmatrix} \hat{\sigma}(t,\xi) \\ \hat{z}(t,\xi) \end{pmatrix} = 0, \text{ where } A(\xi) = \begin{pmatrix} 0 & \overline{\rho} i\xi \\ \dfrac{p'(\overline{\rho})}{\overline{\rho}} i\xi & \dfrac{\nu}{\overline{\rho}}|\xi|^2 \end{pmatrix}.$$

For large ξ, the matrix $A(\xi)$ has two real eigenvalues $X_+(\xi)$ and $X_-(\xi)$. The largest branch $X_+(\xi)$ goes to infinity as $|\xi| \to \infty$ in a quadratic way, and corresponds to parabolic effects, while the smallest branch $X_-(\xi)$ converges to a fixed real number as $|\xi| \to \infty$. Therefore, constructing data localized in space and whose Fourier transform has a vanishing projection on the eigenfunction corresponding to $X_+(\xi)$, one can construct solutions of (11.18) which have small traces on the boundary, with a size of order 1, so that these solutions contradict the observability property.

At this point, it is interesting to remark that the eigenvectors corresponding to $X_\pm(\xi)$, to the main order in ξ, are related to the data (in the physical space)

$$p'(\overline{\rho})\rho - \nu \partial_x u, \quad \text{and} \quad u + \frac{\nu}{\overline{\rho}^2}\partial_x \rho, \tag{11.20}$$

which are the 1d linear counterparts of the *effective viscous flux* $p(\rho) - \nu \mathrm{div}\, \mathbf{u}$ introduced in [Lio98] (see also [FNP01]) and of the *effective velocity* $\mathbf{u} + \nu \nabla \rho/\rho^2$, see [BDL03, BD03]. Both quantities were proved to be important to get some compactness results needed to prove global existence results for several models of compressible fluids.

When $\overline{u} \neq 0$, the change of variable $x \mapsto x + \overline{u}t$ allows to apply immediately the above analysis as well. Therefore, the time of controllability for (11.17) necessarily is larger than $L/|\overline{u}|$. This is also consistent with the works [MRR13] studying the linear equation (11.19) with a control set moving during the evolution: roughly speaking, it is then proved that if the control set covers the whole domain, the linear equation (11.19) is null-controllable.

In this context, with O. Glass, S. Guerrero and J.-P. Puel, we obtained the following result:

Theorem 11.3 ([EGGP12]). *Let $(\overline{\rho}, \overline{u}) \in \mathbb{R}_+^* \times \mathbb{R}_*$, $\Omega = (0, L)$ and let $T > L/|\overline{u}|$. There exists $\varepsilon > 0$ such that for all $(\rho_0, u_0) \in H^1(0, L) \times H^1(0, L)$ with*

$$\|(\rho_0 - \overline{\rho}, u_0 - \overline{u})\|_{H_0^1(0,L) \times H_0^1(0,L)} \leq \varepsilon, \tag{11.21}$$

there exists a controlled trajectory (ρ, u) of (11.17) with initial condition (ρ_0, u_0) satisfying

$$(\rho(T), u(T)) = (\overline{\rho}, \overline{u}) \text{ in } (0, L). \tag{11.22}$$

Besides, (ρ, u) enjoys the following regularity:

$$\rho \in C^0([0, T]; H^1(0, L)),\ u \in L^2(0, T; H^2(0, L)) \cap H^1(0, T; L^2(0, L)). \tag{11.23}$$

Theorem 11.3 holds in any time $T > L/|\overline{u}|$. According to [Deb], this is the sharp time when considering strategies based on the linearization of the constant trajectory $(\overline{\rho}, \overline{u})$. But the use of the non-linear effects of (11.17) could possibly remove this assumption on the time, as it is the case for instance for the Euler equation [Cor96].

Theorem 11.3 was first proved in [EGGP12] under the assumption that (ρ_0, u_0) is close to $(\overline{\rho}, \overline{u})$ in $H^3(0, L) \times H^3(0, L)$. This was due to the fact that we were working with the original weight function ($\theta(t) \simeq 1/t$ close to $t = 0$) of [FI96]. In particular, we needed to consider initial data (ρ_0, u_0) corresponding to rather smooth solutions of (11.17), so that it could be subtract to (11.17) and the newly created source term could be handled. As we did not find any reference for this precise problem, we used the work [MN80] considering a far more general problem (compressible heat conductive Navier-Stokes equations in 3d), but to the price of adding stronger regularity assumptions on the initial data.

Let us also point out that the 1d linearized compressible Navier-Stokes equations have later been analyzed on the torus when the control acts only in the equation of the velocity. Using a spectral approach as in [MRR13], it was proved in [CDRR] that this system is exactly controllable to the trajectories in any time $T > L/|\overline{u}|$ provided that the initial and target densities have the same mass - this is a natural assumption as in the torus, when the control acts only on the velocity equation, this quantity is preserved.

Let us also remark that, following the proof carefully, one can impose that the controlled velocity u given by Theorem 11.3 satisfies $u(t,x) = \overline{u}$ at $x = L$ if $\overline{u} > 0$ and at $x = 0$ if $\overline{u} < 0$. In other words, the controls can all be taken in the incoming (upwind) side, i.e. the side in which the fluid enters the domain.

11.3.2 Ideas of the proof

To prove Theorem 11.3, we use a fixed point argument. By symmetry, we can choose $\overline{u} > 0$ without loss of generality.

Doing the change of unknown $(\rho, u) \mapsto (\rho - \overline{\rho}, u - \overline{u})$, solving the problem (11.17) with initial data (ρ_0, u_0) satisfying (11.21), (11.22) is equivalent to solve the null-controllability problem

$$(\rho(T,\cdot), u(T,\cdot)) = (0,0) \text{ in } (0,L), \tag{11.24}$$

for the system

$$\begin{cases} \partial_t \rho + (\overline{u} + u)\partial_x \rho + \overline{\rho}\partial_x u = f_\rho, & \text{in } (0,T)\times(0,L), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = -p'(\overline{\rho})\partial_x \rho + f_u, & \text{in } (0,T)\times(0,L), \\ (\rho(0,\cdot), u(0,\cdot)) = (\rho_0, u_0) \text{ small in } H^1_0(0,L)^2, & \end{cases} \tag{11.25}$$

where $f_\rho = f_\rho(\rho, u)$ and $f_u = f_u(\rho, u)$ are quadratic terms in (ρ, u).

Given $(\hat{\rho}, \hat{u})$ in a suitable class, we are thus led to solve the null-controllability problem (11.24) for the equation

$$\begin{cases} \partial_t \rho + (\overline{u} + u)\partial_x \rho + \overline{\rho}\partial_x u = f_\rho(\hat{\rho}, \hat{u}), & \text{in } (0,T)\times(0,L), \\ \overline{\rho}(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = -p'(\overline{\rho})\partial_x \hat{\rho} + f_u(\hat{\rho}, \hat{u}), & \text{in } (0,T)\times(0,L), \\ (\rho(0,\cdot), u(0,\cdot)) = (\rho_0, u_0) \text{ small in } H^1_0(0,L)^2. & \end{cases} \tag{11.26}$$

Given $(\hat{\rho}, \hat{u})$, the null-controllability problem for the velocity u is a classical problem for the heat equation and fits into the setting of Corollary 11.2. To be more precise, we first extend the domain to $(0,T)\times(-5\overline{u}T, L)$, on which we extend the source term and the initial datum by 0, and we consider a weight function $\tilde{\psi}$ such that all its critical points are in $(0,T)\times(-4\overline{u}T, -2\overline{u}T)$. We then apply Corollary 11.2[2] with this weight function and simply truncate the controlled solution on $(0, L)$ to get the control $u(t, 0)$ and the controlled trajectory u.

Once u is constructed, the equation of the density ρ is a simple transport equation with a given source term, with a velocity $\overline{u} + u$. As u is supposed to be small (in $L^2(0,T;H^2(\Omega))$ and thus in $L^1(0,T;L^\infty(\Omega))$), the flow corresponding to $\overline{u} + u$ is close to the one corresponding to $\overline{u}$. It is then easy to check that one can construct a suitable controlled density ρ simply by gluing the solutions ρ_f and ρ_b of the forward transport equation

$$\begin{cases} \partial_t \rho_f + (\overline{u} + u)\partial_x \rho_f = f_\rho(\hat{\rho}, \hat{u}) - \overline{\rho}\partial_x u, & \text{in } (0,T)\times(0,L), \\ \rho_f(0,\cdot) = \rho_0, & \text{in } (0,L), \\ \rho_f(t,0) = 0, & \text{in } (0,T), \end{cases} \tag{11.27}$$

and the backward transport equation

$$\begin{cases} \partial_t \rho_b + (\overline{u} + u)\partial_x \rho_b = f_\rho(\hat{\rho}, \hat{u}) - \overline{\rho}\partial_x u, & \text{in } (0,T)\times(0,L), \\ \rho_b(T,\cdot) = 0, & \text{in } (0,L), \\ \rho_b(t,L) = 0, & \text{in } (0,T). \end{cases} \tag{11.28}$$

[2] Here, the parameter m in the weight functions is of no use and we simply take $m = 1$.

Indeed, one can then simply set

$$\rho = \eta\rho_f + (1-\eta)\rho_b, \tag{11.29}$$

where η solves

$$\begin{cases} \partial_t\eta + (\overline{u}+u)\partial_x\eta = 0, & \text{in } (0,T)\times(0,L), \\ \eta(0,\cdot) = 1, & \text{in } (0,L), \\ \eta(t,0) = \eta_0(t), & \text{in } (0,T), \end{cases} \tag{11.30}$$

$\eta_0(t)$ being a smooth cut-off function taking value 1 in a neighborhood of $t=0$ and vanishing after some time small enough.

Now, the problem is the following. To perform the fixed point argument, we need to work within suitable weighted Sobolev spaces, with the weights given by the Carleman estimate (11.16), both for u and ρ. We shall then be able to derive weighted estimates on the density ρ. It is then natural to impose that the weight function $\tilde{\psi}$ in (11.7) solves the transport equation

$$\partial_t\tilde{\psi} + \overline{u}\partial_x\tilde{\psi} = 0, \text{ in } (0,T)\times(0,L), \tag{11.31}$$

i.e. $\tilde{\psi}(t,x) = \tilde{\psi}_0(x-\overline{u}t)$, $\tilde{\psi}_0$ being the initial data of the transport equation in (11.31). To be compatible with the assumptions required for the control of the equation of the velocity, we extend $\tilde{\psi}_0$ to $(-6\overline{u}T, L)$ and choose it such that its critical points are in $(-4\overline{u}T, -3\overline{u}T)$. Setting then $\tilde{\psi}(t,x) = \tilde{\psi}_0(x-\overline{u}t)$ in the whole set $(0,T)\times(-5\overline{u}T, L)$, we easily check that all the critical points of $\tilde{\psi}$ belong to the set $(0,T)\times(-4\overline{u}T, -2\overline{u}T)$.

But this is not enough. Indeed, deriving weighted energy estimates on ρ_f (namely estimate on $\rho_f e^{s\varphi}$ for instance) requires

$$\partial_t\varphi + \overline{u}\partial_x\varphi \leq 0,$$

while, for ρ_b, it requires

$$\partial_t\varphi + \overline{u}\partial_x\varphi \geq 0.$$

With (11.31), one can then estimate ρ_f on sets $(0,T_f)\times(0,L)$ for which $\partial_t\theta \leq 0$ on $(0,T_f)$, and ρ_b on sets $(T_b,T)\times(0,L)$ for which $\partial_t\theta \geq 0$.

We are therefore led to choose T_0 and T_1 in (11.9) in such a way that $T - 2T_0 - 2T_1 > L/|\overline{u}|$, $\theta(t) = 1$ on $(T_0, T-2T_1)$ as in (11.9), and we choose η_0 in (11.30) so that $\eta_0(t) = 1$ for $t \in (0,T_0)$ and $\eta_0(t) = 0$ for $t \geq 2T_0$. From (11.29), this guarantees, for u small enough, that $\rho = \rho_f$ for $t \in (0,T_0)$ and $\rho = \rho_b$ for $t \in (T-2T_1, T)$.

Another difficulty then occurs when trying to perform the fixed point argument, due to the strength of the coupling. Indeed, equation (11.27) implies that ρ_f has the same regularity as $\partial_x u$, so that at first glance, $\partial_x\rho$ will have the same strength as $\partial_{xx}u$. But $\partial_x\rho$ appears as a source term in the controllability problem for u. It will therefore be too singular to be handled directly.

This is why we introduce the so-called effective velocities (recall (11.20)):

$$\tilde{u}_f = u + \frac{\nu}{\overline{\rho}^2}\partial_x\rho_f, \quad \tilde{u}_b = u + \frac{\nu}{\overline{\rho}^2}\partial_x\rho_b. \tag{11.32}$$

The equation on $\tilde{u}_f$ then becomes

$$\overline{\rho}(\partial_t + (\overline{u}+u)\partial_x + \partial_x u)\tilde{u}_f + p'(\overline{\rho})\partial_x\hat{\rho} = 2\overline{\rho}u\partial_x u + \frac{\nu}{\overline{\rho}}\partial_x(f_\rho(\hat{\rho},\hat{u})) + f_u(\hat{\rho},\hat{u}). \tag{11.33}$$

To gain some better decoupling, we therefore replace the controllability problem for ρ in $(11.26)_{(1)}$ by

$$\partial_t\rho + (\overline{u}+u)\partial_x\rho + \frac{p'(\overline{\rho})\overline{\rho}}{\nu}\rho + \overline{\rho}\partial_x u = f_\rho(\hat{\rho},\hat{u}) + \frac{p'(\overline{\rho})\overline{\rho}}{\nu}\hat{\rho} \text{ in } (0,T)\times(0,L). \tag{11.34}$$

Solving this controllability problem in the same way as before, we obtain for $\tilde{u}_f$ the equation

$$\overline{\rho}(\partial_t + (\overline{u} + u)\partial_x + \partial_x u)\tilde{u}_f + \frac{p'(\overline{\rho})\overline{\rho}^2}{\nu}\tilde{u}_f$$
$$= \frac{p'(\overline{\rho})\overline{\rho}^2}{\nu}u + 2\overline{\rho}u\partial_x u + \frac{\nu}{\overline{\rho}}\partial_x(f_\rho(\hat{\rho}, \hat{u})) + f_u(\hat{\rho}, \hat{u}). \quad (11.35)$$

The coupling between $\tilde{u}_f$ and u is now much better: $\tilde{u}_f$ has the same regularity as u, so does $\partial_x \rho_f$. Using this effective velocity $\tilde{u}_f$, we are then able to derive suitable estimates on $\partial_x \rho_f$ (and similarly for $\partial_x \rho_b$ by using $\tilde{u}_b$) allowing to perform Schauder's fixed point theorem.

11.4 Extensions

11.4.1 Extension to the multi-dimensional setting

With O. Glass and S. Guerrero, we are presently trying to extend this result to the multi-dimensional setting. We already have some promising result, and in particular the following one:

Theorem 11.4. *Let $\overline{\rho} > 0$, $\overline{\mathbf{u}} \in \mathbb{R}^d \setminus \{0\}$ $(d \leq 3)$ and let $L_0 > 0$ be larger than the thickness of Ω in the direction $\overline{u}/|\overline{u}|$, and assume*

$$T > L_0/|\overline{u}|. \quad (11.36)$$

Then there exists $\varepsilon > 0$ such that for all $(\rho_0, \mathbf{u}_0) \in H^2(\Omega) \times H^2(\Omega)$ satisfying

$$\|(\rho_0, \mathbf{u}_0) - (\overline{\rho}, \overline{\mathbf{u}})\|_{H^2(\Omega)\times H^2(\Omega)} \leq \varepsilon, \quad (11.37)$$

there exists a solution (ρ, u) of (11.1) with initial data $(\rho_0, \mathbf{u}_0)$ and satisfying the control requirement

$$\rho(T, x) = \overline{\rho}, \quad \mathbf{u}(T, x) = \overline{\mathbf{u}} \quad \text{in } \Omega. \quad (11.38)$$

Besides, the controlled trajectory $(\rho, \mathbf{u})$ has the following regularity:

$$(\rho, \mathbf{u}) \in C([0, T]; H^2(\Omega)) \times (L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega))).$$

Theorem 11.4 requires a strong smoothness assumption on $(\rho, \mathbf{u})$. Actually, the regularity of the velocity field $\mathbf{u}$ is important, as $\mathbf{u} \in L^2(0, T; H^3(\Omega))$ implies (in dimension ≤ 3) that $\mathbf{u} \in L^1(0, T; W^{1,\infty}(\Omega))$, and therefore the corresponding flow is well-defined by Cauchy-Lipschitz.

To prove Theorem 11.4, we first extend the domain Ω into a big torus $\mathbb{T}_L$ and, for $\hat{\mathbf{u}}$ small enough in $L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega))$, we consider the null controllability problem for the linear system

$$\begin{cases} \partial_t \rho + (\overline{\mathbf{u}} + \hat{\mathbf{u}}) \cdot \nabla \rho + \overline{\rho} \operatorname{div} \mathbf{u} = f_\rho + v_\rho \chi, & \text{in } (0, T) \times \mathbb{T}_L, \\ \overline{\rho}(\partial_t \mathbf{u} + \overline{\mathbf{u}} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + p'(\overline{\rho}) \nabla \rho = \mathbf{f_u} + \mathbf{v_u} \chi, & \text{in } (0, T) \times \mathbb{T}_L, \end{cases} \quad (11.39)$$

where χ is the indicator function of $\mathbb{T}_L \setminus \overline{\Omega}$. Indeed, after the change of unknown $(\rho, \mathbf{u}) \mapsto (\rho - \overline{\rho}, \mathbf{u} - \overline{\mathbf{u}})$, this system corresponds to the linearization of the non-linear system (11.1) (except for the term $\hat{\mathbf{u}} \cdot \nabla \rho$ in $(11.39)_{(1)}$ which is quadratic, but cannot be handled as a source term due to regularity issues, similarly as in the 1d setting).

To be more precise, for $(\rho_0, \mathbf{u}_0)$ small in $H^2(\mathbb{T}_L) \times H^2(\mathbb{T}_L)$, $f_\rho \in L^2(0, T; H^2(\mathbb{T}_L))$ and $\mathbf{f_u} \in L^2(0, T; H^1(\mathbb{T}_L))$, we want to find controls v_ρ, $\mathbf{v_u}$ such that the solution of (11.39) with initial data $(\rho_0, \mathbf{u}_0)$ satisfies $(\rho(T), \mathbf{u}(T)) = (0,0)$ in $\mathbb{T}_L$ and $(\rho, \mathbf{u})$ enjoys the following regularity: $\rho \in C^0([0,T]; H^2(\mathbb{T}_L))$ and $\mathbf{u} \in L^2(0,T; H^3(\Omega)) \cap H^1(0,T; H^1(\Omega))$.

By duality, we focus on the observability of the adjoint problem

$$\begin{cases} -\partial_t \sigma - \operatorname{div}((\overline{\mathbf{u}} + \hat{\mathbf{u}})\sigma) - p'(\overline{\rho}) \operatorname{div} \mathbf{z} = g_\sigma, & \text{in } (0,T) \times \mathbb{T}_L, \\ -\overline{\rho}(\partial_t \mathbf{z} + \overline{\mathbf{u}} \cdot \nabla \mathbf{z}) - \mu \Delta \mathbf{z} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{z} - \overline{\rho} \nabla \sigma = \mathbf{g}_{\mathbf{z}}, & \text{in } (0,T) \times \mathbb{T}_L, \end{cases} \tag{11.40}$$

for $g_\sigma \in L^2(0,T; H^{-2}(\mathbb{T}_L))$ and $\mathbf{g}_{\mathbf{z}} \in L^2(0,T; H^{-3}(\mathbb{T}_L))$, observed through $(0,T) \times (\mathbb{T}_L \setminus \overline{\Omega})$.

Similarly as in the 1d case, we set $\nu = \lambda + 2\mu$. It is then interesting to introduce the quantity

$$q = \nu \operatorname{div} \mathbf{z} + \overline{\rho} \sigma, \tag{11.41}$$

which is the counterpart of the effective viscous flux $p(\rho) - \nu \operatorname{div} \mathbf{u}$ for the adjoint equation (11.40) (recall (11.20)). Indeed, if $(\sigma, \mathbf{z})$ solves (11.40), the variables (σ, q) solve a closed subsystem

$$\begin{cases} -\partial_t \sigma - \operatorname{div}((\overline{\mathbf{u}} + \hat{\mathbf{u}})\sigma) + \dfrac{p'(\overline{\rho})\overline{\rho}}{\nu} \sigma = g_\sigma + \dfrac{p'(\overline{\rho})}{\nu} q, & \text{in } (0,T) \times \mathbb{T}_L, \\ -\dfrac{\overline{\rho}}{\nu}(\partial_t q + \overline{\mathbf{u}} \cdot \nabla q) - \Delta q - \dfrac{p'(\overline{\rho})\overline{\rho}^2}{\nu^2} q & \\ \qquad = \operatorname{div} \mathbf{g}_{\mathbf{z}} + \dfrac{\overline{\rho}^2}{\nu} g_\sigma + \dfrac{\overline{\rho}^2}{\nu} \operatorname{div}(\hat{\mathbf{u}}\sigma) - \dfrac{p'(\overline{\rho})\overline{\rho}^3}{\nu^2} \sigma, & \text{in } (0,T) \times \mathbb{T}_L. \end{cases} \tag{11.42}$$

As in the 1d case, we see that this subsystem decreases the strength of the coupling of both equations.

We then deduce the observability of (11.42) from the study of the null controllability problem for the adjoint equation

$$\begin{cases} \partial_t r + (\overline{\mathbf{u}} + \hat{\mathbf{u}}) \cdot \nabla r + \dfrac{p'(\overline{\rho})\overline{\rho}}{\nu} r = f_r - \dfrac{\overline{\rho}^2}{\nu} \hat{\mathbf{u}} \cdot \nabla y - \dfrac{p'(\overline{\rho})\overline{\rho}^3}{\nu^2} y + v_r \chi, & \text{in } (0,T) \times \mathbb{T}_L, \\ \dfrac{\overline{\rho}}{\nu}(\partial_t y + \overline{\mathbf{u}} \cdot \nabla y) - \Delta y - \dfrac{p'(\overline{\rho})\overline{\rho}^2}{\nu^2} y = f_y + \dfrac{p'(\overline{\rho})}{\nu} r + v_y \chi, & \text{in } (0,T) \times \mathbb{T}_L, \\ (r(0,\cdot), y(0,\cdot)) = (r_0, y_0),\ (r(T,\cdot), y(T,\cdot)) = (0,0), & \text{in } \mathbb{T}_L, \end{cases} \tag{11.43}$$

where $(r_0, y_0) \in H^2(\mathbb{T}_L) \times H^3(\mathbb{T}_L)$ and $f_r, f_y \in L^2(0,T; H^2(\mathbb{T}_L))$. As the coupling between the equations in (11.43) is rather weak, the null-controllability problem (11.43) can be solved by a fixed point argument and the good understanding of the null-controllability problem

$$\begin{cases} \partial_t r + (\overline{\mathbf{u}} + \hat{\mathbf{u}}) \cdot \nabla r + \dfrac{p'(\overline{\rho})\overline{\rho}}{\nu} r = \hat{f}_r + v_r \chi, & \text{in } (0,T) \times \mathbb{T}_L, \\ \dfrac{\overline{\rho}}{\nu} \partial_t y - \Delta y = \hat{f}_y + v_y \chi, & \text{in } (0,T) \times \mathbb{T}_L, \\ (r(0,\cdot), y(0,\cdot)) = (r_0, y_0), & \text{in } \mathbb{T}_L, \\ (r(T,\cdot), y(T,\cdot)) = (0,0), & \text{in } \mathbb{T}_L, \end{cases} \tag{11.44}$$

which completely decouples the controllability problem for both equations.

Based on Corollary 11.2 and the construction in 1d, one can easily get a solution (r, y) of the controllability problem (11.43) with the regularity $(r, y) \in L^2(0,T; L^2(\mathbb{T}_L)) \times L^2(0,T; H^2(\mathbb{T}_L)) \cap H^1(0,T; L^2(\mathbb{T}_L))$. However, one needs to get suitable regularity results for the controllability problem (11.39). We shall then prove that for $(r_0, y_0) \in H^2(\mathbb{T}_L) \times H^3(\mathbb{T}_L)$ and $f_r, f_y \in L^2(0,T; H^2(\mathbb{T}_L))$, one can find a trajectory (r, y) solving (11.43) with $r \in L^2(0,T; H^2(\mathbb{T}_L))$ and $y \in L^2(0,T; H^4(\mathbb{T}_L)) \cap H^1(0,T; H^2(\mathbb{T}_L))$. This is done by a bootstrap argument and the analysis of the regularity of the control maps $(r_0, \hat{f}_r) \mapsto (r, v_r)$ and $(y_0, \hat{f}_y) \mapsto (y, v_y)$ in (11.44).

Once this is done, we get an observability estimate for the system (σ, q) in (11.42), and then on $\mathbf{z}$ from $(11.40)_{(2)}$. By duality, we obtain a controllability result for system (11.39). We can then perform a Schauder's fixed point argument to show Theorem 11.4.

Note that for this method to apply, it is very helpful to assume that the controls act on the whole boundary $(0, T) \times \partial\Omega$. Otherwise, q does not satisfy homogeneous boundary conditions and extra work is required to handle the newly created terms.

11.4.2 Extension to more general target trajectories

Of course, another natural question concerns the local exact controllability to target trajectories of (11.17). In other words, given a trajectory $(\overline{\rho}, \overline{u})$ of (11.17), and an initial data (ρ_0, u_0) close to $(\overline{\rho}(0,\cdot), \overline{u}(0,\cdot))$ in a suitable functional setting, find a controlled trajectory (ρ, u) solution of (11.17) such that at time $T > 0$,

$$(\rho(T,\cdot), u(T,\cdot)) = (\overline{\rho}(T,\cdot), \overline{u}(T,\cdot)) \quad \text{in } (0, L). \tag{11.45}$$

This issue is currently analyzed by M. Savel, J.-P. Raymond and myself. The difficulty is twofold. First, several new terms are created involving derivatives of the target trajectory, and this requires several additional technicalities. Second, the condition for controlling the density should be modified. We should at least assume that the flow $\overline{X}$ corresponding to the velocity field $\overline{u}$ should push away all the points of the domain between the times $t = 0$ and $t = T$. To be more precise, introducing the flow[3]

$$\frac{d\overline{X}}{dt}(t,x) = \overline{u}(t, \overline{X}(t,x)), \quad t \geq 0, \qquad \overline{X}(0,x) = x \in \mathbb{R},$$

we should assume that

$$\forall x \in [0, L], \, \exists t_x \in (0, T), \text{ s.t. } \overline{X}(t_x, x) \notin [0, L]. \tag{11.46}$$

In the construction of the controlled density, one should then be particularly cautious about possible oscillations of the trajectories around the boundary. Note that this difficulty is solved in the case of incompressible fluids, see next chapter, but the controlled density in that case is not smooth. It is therefore not fully adequate for the case of compressible fluids, and the construction should be adapted to smooth up the controlled density adequately.

[3]For sake of simplicity, we extend $\overline{u}$ to $[0, T] \times \mathbb{R}$.

Chapter 12

Controllability of the incompressible non-homogeneous Navier-Stokes equations

12.1 Main result

In this section, we consider the incompressible non-homogeneous Navier-Stokes equation. Let $T > 0$ and Ω be a smooth bounded domain of $\mathbb{R}^2$. The motion of an incompressible viscous fluid is then described by the following set of equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, & \text{in } (0,T) \times \Omega, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p = 0, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } (0,T) \times \Omega. \end{cases} \tag{12.1}$$

Here ρ is the density of the fluid, $\mathbf{u}$ its velocity and p its pressure. Note that in the case of incompressible flows, the pressure is not given in terms of the density as in (11.2) in the compressible case. Actually, in the context of incompressible fluids, the pressure is the Lagrange multiplier of the divergence free condition $\operatorname{div} \mathbf{u} = 0$.

As it is classically done in the context of incompressible fluid (see e.g. [BF13]), we introduce the space $V^1(\Omega)$ of vector fields in $H^1(\Omega)$ satisfying the divergence free condition, and $V_0^1(\Omega)$ for elements of $V^1(\Omega)$ vanishing on the boundary $\partial\Omega$.

Our goal is to study the local exact controllability problem to trajectories. Again, we consider controls acting on the boundary, as in (11.4). We then introduce a trajectory $(\overline{\rho}, \overline{\mathbf{u}})$ solving (12.1) in $(0,T) \times \Omega$. Extending $\overline{\mathbf{u}}$ to $[0,T] \times \mathbb{R}^2$, we introduce the corresponding flow

$$\frac{d\overline{X}}{dt}(t, t_0, x) = \overline{\mathbf{u}}(t, \overline{X}(t, t_0, x)), \quad t \geq 0, \qquad \overline{X}(t_0, t_0, x) = x \in \mathbb{R}^2, \tag{12.2}$$

and, for $T > 0$, the set

$$\Omega_{out}^T = \{x \in \overline{\Omega} \,|\, \exists t_x \in (0,T), \text{ s.t. } \overline{X}(t_x, 0, x) \notin \overline{\Omega}\}. \tag{12.3}$$

This set corresponds to all the points that are transported outside $\overline{\Omega}$ before the time $T > 0$. Note that it does not depend on the extension chosen for $\overline{\mathbf{u}}$. Our main geometric assumption, coming from the transport equation satisfied by the density, is the following one:

$$\Omega_{out}^T = \overline{\Omega}. \tag{12.4}$$

As we would like to control on the smallest possible set of the boundary and to describe what our approach gives at best, we assume that there exists an open subset Γ_0 of the boundary $\partial\Omega$ satisfying the following assumptions:

$$\begin{array}{ll} \text{(i).} & \Gamma_0 \text{ has a finite number of connected components,} \\ \text{(ii).} & \inf_{[0,T]\times\overline{\Gamma_0}} \overline{\mathbf{u}}\cdot\mathbf{n} > 0. \end{array} \tag{12.5}$$

Under these conditions, we get the following result:

Theorem 12.1 ([BEG]). *Let Ω be a smooth bounded domain of $\mathbb{R}^2$. Assume that $(\overline{\rho}, \overline{\mathbf{u}})$ solves* (12.1) *and satisfies*

$$(\overline{\rho}, \overline{\mathbf{u}}) \in C^2([0,T]\times\overline{\Omega}) \times C^2([0,T]\times\overline{\Omega}) \quad \text{and} \quad \inf_{[0,T]\times\overline{\Omega}} \overline{\rho} > 0. \tag{12.6}$$

Assume that the condition (12.4) *is satisfied for the time T and that there exists Γ_0 (possibly empty) satisfying* (12.5).

Then there exists $\varepsilon > 0$ such that for all $(\rho_0, \mathbf{u}_0) \in L^\infty(\Omega)\times V^1(\Omega)$ satisfying

$$\|\rho_0 - \overline{\rho}(0,\cdot)\|_{L^\infty(\Omega)} + \|\mathbf{u}_0 - \overline{\mathbf{u}}(0,\cdot)\|_{H^1_0(\Omega)} \leq \varepsilon, \tag{12.7}$$

there exists a controlled trajectory $(\rho, \mathbf{u})$ solving (12.1) *with initial data*

$$(\rho(0,\cdot), \mathbf{u}(0,\cdot)) = (\rho_0, \mathbf{u}_0) \text{ in } \Omega, \tag{12.8}$$

with the boundary condition $\mathbf{u}(t,x) = \overline{\mathbf{u}}(t,x)$ for all $(t,x) \in (0,T)\times\Gamma_0$, and satisfying the controllability property

$$(\rho(T,\cdot), \mathbf{u}(T,\cdot)) = (\overline{\rho}(T,\cdot), \overline{\mathbf{u}}(T,\cdot)) \text{ in } \Omega. \tag{12.9}$$

Besides, one can construct $(\rho, \mathbf{u})$ with the following regularity property:

$$(\rho, \mathbf{u}) \in L^\infty((0,T)\times\Omega) \times (H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))). \tag{12.10}$$

Note that Condition (ii) in (12.5) implies that for all $t > 0$, the velocity field $\overline{\mathbf{u}}(t,\cdot)$ is directed to the exterior of Ω on Γ_0. Actually, it even implies that there exists $\gamma > 0$ such that $\inf_{[0,T]\times\overline{\Gamma_0}} \overline{\mathbf{u}}\cdot\mathbf{n} \geq \gamma$. In this case, choosing $\varepsilon > 0$ in (12.7) smaller if necessary, we can impose that the controlled velocity $\mathbf{u}$ of Theorem 12.1 stays in a neighborhood of $\overline{\mathbf{u}}$ in $C^0([0,T];V^1(\Omega))$ small enough to guarantee $\mathbf{u}(t,x)\cdot\mathbf{n} > 0$ for all $(t,x) \in (0,T)\times\Gamma_0$. Therefore, there is no control on the density ρ on the set Γ_0 either, as Γ_0 corresponds to an outflow part of the boundary for $\mathbf{u}$ as well.

To our knowledge, the study of the control properties of (12.1) has only been done in [FC12], mainly from the point of view of optimal control. In the context of controllability, however, there was to our knowledge no previous result.

12.2 Sketch of the proof

As before, our proof is based on a suitable fixed point argument.

Working on the unknown $(\rho - \overline{\rho}, \mathbf{u} - \overline{\mathbf{u}})$, still denoted $(\rho, \mathbf{u})$ for simplicity, we are back to study the boundary null-controllability property of the system

$$\begin{cases} \partial_t\rho + (\overline{\mathbf{u}} + \mathbf{u})\cdot\nabla\rho = -\mathbf{u}\cdot\nabla\overline{\rho}, & \text{in } (0,T)\times\Omega, \\ \overline{\rho}\partial_t\mathbf{u} + \overline{\rho}(\overline{\mathbf{u}}\cdot\nabla)\mathbf{u} + \overline{\rho}(\mathbf{u}\cdot\nabla)\overline{\mathbf{u}} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(\rho,\mathbf{u}), & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } (0,T)\times\Omega, \\ (\rho(0), \mathbf{u}(0)) = (\rho_0, \mathbf{u}_0), \ \text{small in } L^\infty(\Omega)\times V^1_0(\Omega), & \end{cases} \tag{12.11}$$

where $\mathbf{f}(\rho, \mathbf{u})$ contains quadratic terms in $(\rho, \mathbf{u})$ and the linear term $-\rho(\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}})$.

The coupling between the transport equation $(12.11)_{(1)}$ and the Stokes problem $(12.11)_{(2)}$ is thus rather weak. Indeed, $(12.11)_{(1)}$ implies that the density has the same strength (in terms of powers of the Carleman parameter s) as the velocity field $\mathbf{u}$, while the density appears linearly in the source term of $(12.11)_{(2)}$.

One could therefore think that the local null-controllability problem for (12.11) is easy. This would be the case if the Stokes operator in $(12.11)_{(2)}$ was replaced by the heat operator. Unfortunately, the situation for the Stokes operator $(12.11)_{(2)}$ is much more intricate, and this is the main difficulty of our work.

Note that the situation drastically simplifies when either $\nabla\overline{\rho}$ or $(\partial_t \overline{\mathbf{u}} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}})$ vanishes. In this case indeed, one of the two linear couplings between the equation is cancelled. In particular, in this case the problem reduces to the study of the null-controllability of the Stokes operator, which can be found in the literature in several works, among which [Ima01, FCGIP04, IPY09]. But still, the non-constant coefficient $\overline{\rho}$ in front of $\partial_t \mathbf{u}$ in $(12.11)_{(2)}$ creates some difficulty as it introduces a new term in the Laplacian of the pressure Δp of the form $\nabla\overline{\rho} \cdot \partial_t \mathbf{u}$.

We shall therefore focus our study on the null-controllability of the Stokes operator. We begin by considering an extension $\mathcal{O}$ of Ω, $\mathcal{O}$ being a smooth bounded domain of $\mathbb{R}^2$ such that $\mathcal{O} \setminus \overline{\Omega}$ is non-empty and $\Gamma_0 \subset \partial\mathcal{O} \cap \partial\Omega$. We also extend $(\overline{\rho}, \overline{\mathbf{u}})$ to $[0, T] \times \mathcal{O}$ such that it satisfies (12.6) on $[0, T] \times \overline{\mathcal{O}}$ and $\mathbf{u}_0$ and the source term to be controlled by 0 outside Ω.

We then consider the null-controllability problem for the system

$$\begin{cases} \overline{\rho}(\partial_t \mathbf{u} + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\overline{\mathbf{u}}) - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} + \mathbf{v}1_{\mathcal{O}\setminus\overline{\Omega}}, & \text{in } (0, T) \times \mathcal{O}, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } (0, T) \times \mathcal{O}, \\ \mathbf{u} = \mathbf{0}, & \text{on } (0, T) \times \partial\mathcal{O}, \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \mathcal{O}. \end{cases} \tag{12.12}$$

where $\mathbf{v} \in L^2((0, T) \times (\mathcal{O} \setminus \overline{\Omega}))$ is the control function.

As for the heat equation (recall Section 11.2), this null-controllability property is linked to the observability of the adjoint system:

$$\begin{cases} -\partial_t(\overline{\rho}\mathbf{z}) - 2D(\overline{\rho}\mathbf{z})\overline{\mathbf{u}} - \overline{\rho}\mathbf{z}\operatorname{div} \overline{\mathbf{u}} - \nu\Delta\mathbf{z} + \nabla p = \mathbf{g}, & \text{in } (0, T) \times \mathcal{O}, \\ \operatorname{div} \mathbf{z} = 0, & \text{in } (0, T) \times \mathcal{O}, \\ \mathbf{z} = \mathbf{0}, & \text{on } (0, T) \times \partial\mathcal{O}, \end{cases} \tag{12.13}$$

where $D\mathbf{z} = (\nabla\mathbf{z} + {}^t\nabla\mathbf{z})/2$ is the symmetrized gradient. In a first approximation, we can consider, instead of (12.13), the equations

$$\begin{cases} -\overline{\rho}\partial_t \mathbf{z} - \nu\Delta\mathbf{z} + \nabla p = \mathbf{g}, & \text{in } (0, T) \times \mathcal{O}, \\ \operatorname{div} \mathbf{z} = 0, & \text{in } (0, T) \times \mathcal{O}, \\ \mathbf{z} = 0, & \text{on } (0, T) \times \partial\mathcal{O}. \end{cases} \tag{12.14}$$

Taking the curl of the equation (12.14), $w = \operatorname{curl} \mathbf{z}$ satisfies:

$$-\overline{\sigma}\partial_t w - \nu\Delta w = \operatorname{curl} \mathbf{g} + \partial_t \mathbf{z} \cdot \nabla^{\perp}\overline{\rho}, \quad \text{in } (0, T) \times \mathcal{O}, \tag{12.15}$$

but w does not satisfy homogeneous boundary conditions.

Still, as w solves a backward heat equation, by duality with the null controllability problem dealt with in Corollary 11.2, in the spirit of [IY03a], one can derive

$$\iint_{(0,T)\times\mathcal{O}} \xi^3|w|^2e^{-2s\varphi} \leq C\left(\iint_{(0,T)\times(\mathcal{O}\setminus\overline{\Omega})} \xi^3|w|^2e^{-2s\varphi} + s\iint_{(0,T)\times\mathcal{O}} \xi^4|\mathbf{z}|^2e^{-2s\varphi}\right.$$
$$\left. +\lambda^{-1}\iint_{(0,T)\times\partial\mathcal{O}} \xi^3|w|^2e^{-2s\varphi} + s^{-1}\lambda^{-2}\iint_{(0,T)\times\mathcal{O}} \xi^2|\mathbf{g}|^2e^{-2s\varphi}\right), \quad (12.16)$$

where the weights functions φ, ξ, θ are defined as in (11.7)–(11.11) with Ω replaced by $\mathcal{O}$ and $Q_{obs} = (0,T)\times(\mathcal{O}\setminus\overline{\Omega})$.

Using then the fact that $-\Delta\mathbf{z}(t,\cdot) = \mathbf{curl}\, w(t,\cdot)$ in $\mathcal{O}$ and $\mathbf{z}(t,\cdot) = 0$ on $\partial\mathcal{O}$, one can use elliptic Carleman estimates as in [IP03] to deduce estimates on $\mathbf{z}$ from w: for all $t\in(0,T)$,

$$s\lambda^2\int_{\mathcal{O}} \xi^4|\mathbf{z}|^2e^{-2s\varphi} \leq C\int_{\mathcal{O}} \xi^3|w|^2e^{-2s\varphi} + Cs\lambda^2\int_{\mathcal{O}\setminus\overline{\Omega}} \xi^4|\mathbf{z}|^2e^{-2s\varphi}. \quad (12.17)$$

We can then absorb the term in $\mathbf{z}$ in (12.16) by taking λ large enough, up to the addition of a new observation term in $\mathbf{z}$.

But one still needs to absorb the boundary term in (12.16). In order to do that, we first remark that w on the boundary $\partial\mathcal{O}$ is bounded by $\partial_{\mathbf{n}}\mathbf{z}$ on $\partial\mathcal{O}$. The goal therefore is to estimate $\xi^{3/2}\partial_{\mathbf{n}}\mathbf{z}e^{-s\varphi}$ from the source term $\mathbf{g}$ in (12.14) and observation terms.

In order to do that, we introduce the stream function ζ associated to $\mathbf{z}$, which satisfies $\Delta\zeta(t,\cdot) = w(t,\cdot)$ in $\mathcal{O}$ and $\zeta(t,\cdot)$ is constant on each connected component of the boundary of $\partial\mathcal{O}$ (at this step, we strongly use the fact that we are in 2d). Elliptic Carleman estimates then also apply to ζ, provided that $\partial\mathcal{O}$ is a level set of $\psi(t,\cdot)$ for all $t\in(0,T)$. We then get, for all $t\in(0,T)$,

$$s^3\lambda^4\int_{\mathcal{O}} \xi^6|\zeta|^2e^{-2s\varphi} \leq C\int_{\mathcal{O}} \xi^3|w|^2e^{-2s\varphi} + s^3\lambda^4\int_{\mathcal{O}\setminus\overline{\Omega}} \xi^4|\zeta|^2e^{-2s\varphi}. \quad (12.18)$$

Remark that here, we use that the elliptic Carleman estimate applies to ζ without introducing new boundary terms as ζ and $\psi(t,\cdot)$ both are constant on each connected component of the boundary of $\partial\mathcal{O}$.

Now, to absorb the boundary term in (12.16), the idea is to do energy estimates on (12.14) with a weight function independent of the space variable. As it has to coincide with φ on the boundary to estimate the boundary term in (12.16), we assume

$$\forall t\in[0,T],\ \tilde{\psi}(t)_{|\partial\mathcal{O}} \text{ is constant, and } \forall t\in[0,T],\ \inf_{\mathcal{O}}\tilde{\psi}(t,\cdot) = \tilde{\psi}(t)_{|\partial\mathcal{O}}, \quad (12.19)$$

so that

$$\varphi_*(t) = \max_{x\in\mathcal{O}}\varphi(t,x) = \varphi(t)_{|\partial\mathcal{O}}, \quad \xi_*(t) = \min_{x\in\mathcal{O}}\xi(t,x) = \xi(t)_{|\partial\mathcal{O}}.$$

One can then derive $L^2(0,T;H^2(\mathcal{O}))$ estimates on $\xi_*^{3/2}\mathbf{z}e^{-s\varphi_*}$ by energy estimates for (12.12), using the estimate on $s^{3/2}\lambda^2\xi_*^3\zeta e^{-s\varphi_*}$ in (12.18). In this step, we need the parameter m in (11.8)–(11.9) to be greater than 5. This allows to estimate $\xi^{3/2}\partial_{\mathbf{n}}\mathbf{z}e^{-s\varphi_*}$ in $L^2((0,T)\times\partial\mathcal{O})$ from the source term $\mathbf{g}$ of (12.14) and the right-hand side of (12.18). This yields an observability estimate for $\mathbf{z}$ in (12.14), and by duality a controllability result for $\mathbf{u}$ in (12.12).

Summing up, in order to get a null-controllability result for (12.12), we need to choose

$\tilde{\psi} = \tilde{\psi}(t, x) \in C^2([0,T] \times \overline{\mathcal{O}})$ satisfying

$$\begin{cases} \forall (t,x) \in [0,T] \times \overline{\mathcal{O}}, \quad \tilde{\psi}(t,x) \in [0,1], \\ \forall (t,x) \in [0,T] \times \partial\mathcal{O}, \quad \partial_{\mathbf{n}} \tilde{\psi}(t,x) \leq 0, \\ \forall t \in [0,T], \ \tilde{\psi}(t)_{|\partial\mathcal{O}} \text{ is constant}, \\ \forall t \in [0,T], \ \inf_{\mathcal{O}} \tilde{\psi}(t, \cdot) = \tilde{\psi}(t)_{|\partial\mathcal{O}}, \\ \inf_{[0,T]\times\overline{\Omega}} \{|\nabla \tilde{\psi}|\} > 0. \end{cases} \tag{12.20}$$

We choose $m \geq 5$ and let ψ as in (11.8). For $T_0 > 0$ and $T_1 > 0$ with $T_0 + 2T_1 < T$, we choose a weight function in time $\theta(t)$ as in (11.9) with μ as in (11.11). We finally introduce the weight functions $\varphi = \varphi(t,x)$ and $\xi = \xi(t,x)$ as in (11.10).

Using these weight functions, we then obtain the following result for the controllability problem (12.12):

Theorem 12.2 (Theorem 2.2 in [BEG])**.** *Under the above assumptions, there exists a constant $C > 0$ such that for all $s \geq s_0$ and $\lambda \geq \lambda_0$, if $\mathbf{u}_0 \in V_0^1(\mathcal{O})$ and $\mathbf{f} \in L^2((0,T) \times \mathcal{O})$ satisfies*

$$\iint_{(0,T)\times\mathcal{O}} \xi^{-4} |\mathbf{f}|^2 e^{2s\varphi} < \infty, \tag{12.21}$$

then there exists a control function $\mathbf{v} \in L^2((0,T) \times (\mathcal{O} \setminus \overline{\Omega}))$ and a controlled trajectory $\mathbf{u} \in L^2((0,T) \times \mathcal{O})$ such that $\mathbf{u}$ solves the control problem (12.12) *and $\mathbf{u}$ satisfies the estimate*

$$\begin{aligned} \|\mathbf{u} e^{\frac{3}{4} s \varphi^*}\|^2_{L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))} + s^{1/2} \lambda^{5/2} \iint_{(0,T)\times\mathcal{O}} \xi^{2/m-4} |\mathbf{u}|^2 e^{2s\varphi} \\ \leq C \left(\iint_{(0,T)\times\mathcal{O}} \xi^{-4} |\mathbf{f}|^2 e^{2s\varphi} + \|\mathbf{u}_0 e^{\frac{5}{4} s \varphi(0,\cdot)}\|^2_{V_0^1(\mathcal{O})} \right). \end{aligned} \tag{12.22}$$

The important point in Theorem 12.2 is the fact that we derive an estimate on $\xi^{-2+1/m} \mathbf{u} e^{s\varphi}$ in $L^2((0,T) \times \mathcal{O})$ (hence on $\xi^{-2} \mathbf{u} e^{s\varphi}$) in terms of an estimate on $\xi^{-2} \mathbf{f} e^{s\varphi}$, while most of the literature only produces estimates on $\mathbf{u} e^{(1-\varepsilon)s\varphi}$ in $L^2((0,T) \times \mathcal{O})$ for some $\varepsilon > 0$ in terms of $\mathbf{f} e^{s\varphi}$ in $L^2((0,T) \times \mathcal{O})$. Though such a difference in the exponential is harmless when $\mathbf{f}$ only involves quadratic terms, it cannot be sufficient in our case as the source terms in (12.11) contain linear coupling terms. Nevertheless, let us mention that Theorem 12.2 can be derived from [IPY09] when $\overline{\rho}$ is constant and the weight function ψ does not depend on the time variable.

Let us also remark that performing weighted energy estimates starting from (12.17) instead of (12.18) does not allow to absorb the boundary term in (12.16), thus limiting Theorem 12.2 to the 2d case when there is a part of the boundary on which no control is active. However, when the control is active on the whole boundary, the situation is much better, see Section 12.3.1 below.

Once Theorem 12.2 is proved, we have to control the density. This can be done as in the compressible case by gluing forward and backward solutions with the correct initial, respectively final, conditions (in dimension 2, this can be done in the context of classical solutions by [Zua02] as the velocity field belongs to $L^2(0,T; H^2(\mathcal{O}))$). Indeed, from assumption (12.4), we can choose T_0 and $T_1 > 0$ small enough so that for all $x \in \overline{\Omega}$, there exists $t_x \in (T_0, T - 2T_1)$ such that $\overline{X}(t_x, T_0, x) \notin \overline{\Omega}$. Using the cut-off function η solving

$$\begin{cases} \partial_t \eta + (\overline{\mathbf{u}} + \mathbf{u}) \cdot \nabla \eta = 0, & \text{in } (0,T) \times \Omega, \\ \eta(0, \cdot) = 1, & \text{in } \Omega, \\ \eta(t,x) = 1_{t \leq T_0}, & \text{in } (0,T) \times \partial\Omega, \text{ with } (\overline{\mathbf{u}} + \mathbf{u}) \cdot \mathbf{n} < 0, \end{cases} \tag{12.23}$$

the construction in (11.27)–(11.30) easily adapts.

Of course, to get estimates on the density constructed that way, one should also assume that the weight function $\tilde{\psi}$ satisfies

$$\partial_t \tilde{\psi} + \overline{\mathbf{u}} \cdot \nabla \tilde{\psi} = 0, \text{ in } (0,T) \times \Omega, \tag{12.24}$$

as in (11.31).

The construction of $\tilde{\psi}$ satisfying the conditions (12.20) and (12.24) can be done when the controls act in the complementary of Γ_0 satisfying (12.5), see [BEG, Lemma 4.1].

The last difficulty left is in the application of Schauder's fixed point theorem, as the density we constructed is not smooth. Therefore, getting compactness of the fixed point map we consider is not straightforward. Fortunately, using the concept of renormalized solutions introduced in [DL89] and adapting it to the case of transport equations with non-tangential velocity fields, the results in [Boy05] (see also [BF07, Theorem 4]) proved that our construction of the density enjoys some compactness properties. We can therefore use Schauder's fixed point theorem to conclude the proof of Theorem 12.1.

12.3 Comments

12.3.1 The 3d case

Our approach is limited to the 2d case as we used the stream function to absorb the boundary term in (12.16). Therefore, it would be better to be able to estimate the boundary term in (12.16) by avoiding the use of the stream function. It turns out that this is precisely what is done in [IPY09] for a function ψ independent of t (and a constant coefficient $\overline{\rho}$, but this is not an important issue). But [IPY09] requires the use of a precise Carleman estimate for the heat equation with non-homogeneous boundary conditions with optimal powers of the Carleman parameters s. Whether this can be done or not with a weight function ψ depending on time satisfying (12.20)–(12.24) is an open problem.

However, when the control acts on the whole boundary, similarly as in Section 11.4.1, we can put the domain Ω into a large torus $\mathcal{O} = \mathbb{T}_L$, extending the vector field $\overline{\mathbf{u}}$ to $\mathbb{T}_L$ smoothly such that it vanishes away from $\overline{\Omega}$ (this extended vector field will not necessarily be divergence free in $\mathbb{T}_L$, but it will be divergence free in Ω, which is the only thing that matters for our purpose). The same proof as the one presented above then becomes simpler as there is no more boundary conditions.

In this case, we can therefore omit the difficulties coming from the boundary and generalize Theorem 12.1 to the 3d case:

Theorem 12.3 ([BEG]). *Let Ω be a smooth bounded domain of $\mathbb{R}^3$. Assume that $(\overline{\rho}, \overline{\mathbf{u}})$ solves* (12.1) *and satisfies* (12.6)*. Assume that the condition* (12.4) *is satisfied for the time T.*

There exists $\varepsilon > 0$ such that for all $(\rho_0, \mathbf{u}_0) \in L^\infty(\Omega) \times V^1(\Omega)$ satisfying (12.7)*, there exists a controlled trajectory $(\rho, \mathbf{u})$ solving* (12.1) *with initial data $(\rho_0, \mathbf{u}_0)$, satisfying the controllability property* (12.9)*, and enjoying the regularity* (12.10)*.*

The only difference in the proof comes from the fact that the transport equations in the construction of the controlled density have to be solved using renormalized solutions, see [Boy05], as now the velocity fields are not regular enough to produce classical flows.

12.3.2 On the return method

Our results also allow the use of non-trivial trajectories. For instance, if we control from the whole boundary locally around the trajectory $(\overline{\rho}, \overline{\mathbf{u}}) = (1, \mathbf{0})$, it would be inefficient to consider the constant trajectory $(\overline{\rho}(t), \overline{\mathbf{u}}(t)) = (1, \mathbf{0})$, as condition (12.4) could not be satisfied in any time $T > 0$. But instead, given $T > 0$, one can consider the trajectory $(\overline{\rho}(t), \overline{\mathbf{u}}(t)) = (1, \eta(t)\mathbf{U})$ with $\eta = \eta(t)$ being a non-negative smooth cut-off function in time taking value 1 on an interval of size $T/2$ and $\mathbf{U} \in \mathbb{R}^d$. For $\mathbf{U}$ large enough, this produces a trajectory satisfying Assumption (12.4). Theorem 12.1 or 12.3 applies and yields local exact controllability to the constant state $(\overline{\rho}, \overline{\mathbf{u}}) = (1, \mathbf{0})$ in any time $T > 0$.

This example follows the spirit of the return method introduced by J.-M. Coron in [Cor96] in the context of Euler equations, extended in [Cor96, CF96] in the context of Navier-Stokes equations on a bounded domain with Navier boundary conditions and on a manifold without boundary. The main idea is to construct suitable trajectories of the system around which the system is locally exactly controllable. Our above results, Theorem 12.1 and Theorem 12.3, only provide local exact controllability of (12.1) to trajectories satisfying the regularity and geometric conditions (12.6), (12.4).

Following this idea, this yields the following application of [Cor96, CF96] when the control acts on the whole boundary in dimension 2. First, extend the domain Ω into a large torus. Arguing as above, we can rewrite the problem as a controllability problem in the torus with distributed controls supported outside Ω. Now, if $\overline{\mathbf{u}}$ is a smooth trajectory of (12.1) corresponding to a constant positive density $\overline{\rho}$, using [CF96], starting from any smooth initial data $\mathbf{u}_0$, we can construct controls such that the controlled trajectory reaches exactly $\overline{\mathbf{u}}(T)$ at time T, while ρ stays constant $\rho(t, x) = \overline{\rho}$. Besides, we can impose that this controlled trajectory equals $(\overline{\rho}, \mathbf{U})$ for large constant vector fields $\mathbf{U} \in \mathbb{R}^2$ during the time interval $(T/3, 2T/3)$ so that (12.4) is satisfied. That way, we construct trajectories of (12.1) linking $(\overline{\rho}, \mathbf{u}_0)$ to $(\overline{\rho}, \overline{\mathbf{u}}(T))$ satisfying (12.4) for any smooth function $\mathbf{u}_0$ with $\operatorname{div} \mathbf{u}_0 = 0$ and smooth solutions $(\overline{\rho}, \overline{\mathbf{u}})$ of (12.1) for which the density stays constant (in that case, $\overline{\mathbf{u}}$ simply is a smooth solution of the incompressible Navier-Stokes equation for a fluid of constant density $\overline{\rho}$). Note however that this construction is limited to trajectories for which $\overline{\rho}$ is constant. In that case, as mentioned earlier, the linear coupling in $(12.11)_1$ disappears and the controllability result in Theorem 12.1 becomes much simpler.

Whether this type of strategy can be extended to other trajectories and to other geometric settings is an open problem.

Bibliography

[AB95] S. Alinhac and M. S. Baouendi. A nonuniqueness result for operators of principal type. *Math. Z.*, 220(4):561–568, 1995.

[AB05] D. Auroux and J. Blum. Back and forth nudging algorithm for data assimilation problems. *C. R. Math. Acad. Sci. Paris*, 340(12):873–878, 2005.

[Ale88] G. Alessandrini. Stable determination of conductivity by boundary measurements. *Appl. Anal.*, 27:153–172, 1988.

[Amo11] E.V. Amosova. Exact local controllability for the equations of viscous gas dynamics. *Differentsial'nye Uravneniya*, 47(12):1754–1772, 2011.

[ASTT09] C. Alves, A. L. Silvestre, T. Takahashi, and M. Tucsnak. Solving inverse source problems using observability. Applications to the Euler-Bernoulli plate equation. *SIAM J. Control Optim.*, 48(3):1632–1659, 2009.

[Bau] L. Baudouin. Lipschitz stability in an inverse problem for the wave equation. http://hal.archives-ouvertes.fr/hal-00598876/fr/.

[BD03] D. Bresch and B. Desjardins. Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostrophic model. *Comm. Math. Phys.*, 238(1-2):211–223, 2003.

[BDBE13] L. Baudouin, M. De Buhan, and S. Ervedoza. Global Carleman estimates for waves and applications. *Comm. Partial Differential Equations*, 38(5):823–859, 2013.

[BDK05] L. Borcea, V. Druskin, and L. Knizhnerman. On the continuum limit of a discrete inverse spectral problem on optimal finite difference grids. *Comm. Pure Appl. Math.*, 58(9):1231–1279, 2005.

[BDL03] D. Bresch, B. Desjardins, and C.-K. Lin. On some compressible fluid models: Korteweg, lubrication, and shallow water systems. *Comm. Partial Differential Equations*, 28(3-4):843–868, 2003.

[BE13] L. Baudouin and S. Ervedoza. Convergence of an inverse problem for a 1-D discrete wave equation. *SIAM J. Control Optim.*, 51(1):556–598, 2013.

[BEG] M. Badra, S. Ervedoza, and S. Guerrero. Local controllability to trajectories for non-homogeneous 2-d incompressible Navier-Stokes equations. *Preprint, in revision*, 2014.

[Bel04] M. Bellassoued. Global logarithmic stability in inverse hyperbolic problem by arbitrary boundary observation. *Inverse Problems*, 20(4):1033–1052, 2004.

[BEO] L. Baudouin, S. Ervedoza, and A. Osses. Stability of an inverse problem for the discrete wave equation and convergence results. *J. Math. Pures Appl., to appear.*

[BF07] F. Boyer and P. Fabrie. Outflow boundary conditions for the incompressible non-homogeneous Navier-Stokes equations. *Discrete Contin. Dyn. Syst. Ser. B*, 7(2):219–250 (electronic), 2007.

[BF13] F. Boyer and P. Fabrie. *Mathematical tools for the study of the incompressible Navier-Stokes equations and related models*, volume 183 of *Applied Mathematical Sciences*. Springer, New York, 2013.

[BG97] N. Burq and P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(7):749–752, 1997.

[BHLR10a] F. Boyer, F. Hubert, and J. Le Rousseau. Discrete Carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations. *J. Math. Pures Appl. (9)*, 93(3):240–276, 2010.

[BHLR10b] F. Boyer, F. Hubert, and J. Le Rousseau. Discrete carleman estimates for elliptic operators in arbitrary dimension and applications,. *SIAM J. Control Optim.*, 48:5357–5397, 2010.

[BHLR11] F. Boyer, F. Hubert, and J. Le Rousseau. Uniform controllability properties for space/time-discretized parabolic equations. *Numer. Math.*, 118(4):601–661, 2011.

[BK81] A. L. Bukhgeım and M. V. Klibanov. Uniqueness in the large of a class of multidimensional inverse problems. *Dokl. Akad. Nauk SSSR*, 260(2):269–272, 1981.

[BLR88] C. Bardos, G. Lebeau, and J. Rauch. Un exemple d'utilisation des notions de propagation pour le contrôle et la stabilisation de problèmes hyperboliques. *Rend. Sem. Mat. Univ. Politec. Torino*, (Special Issue):11–31 (1989), 1988. Nonlinear hyperbolic equations in applied sciences.

[BLR92] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *SIAM J. Control and Optim.*, 30(5):1024–1065, 1992.

[Boy05] F. Boyer. Trace theorems and spatial continuity properties for the solutions of the transport equation. *Differential Integral Equations*, 18(8):891–934, 2005.

[BR13] F. Boyer and J. Le Rousseau. Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, (0):–, 2013.

[BY06] M. Bellassoued and M. Yamamoto. Logarithmic stability in determination of a coefficient in an acoustic equation by arbitrary boundary observation. *J. Math. Pures Appl. (9)*, 85(2):193–224, 2006.

[BZ04] N. Burq and M. Zworski. Geometric control in the presence of a black box. *J. Amer. Math. Soc.*, 17(2):443–471 (electronic), 2004.

[CBdN08] J.-M. Coron, G. Bastin, and B. d'Andréa Novel. Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems. *SIAM J. Control Optim.*, 47(3):1460–1498, 2008.

[CDRR] S. Chowdhury, M. Debanjana, M. Ramaswamy, and M. Renardy. Null controllability of the linearized compressible Navier Stokes system in one dimension. *Preprint*, 2013.

[CEG09] J.-M. Coron, S. Ervedoza, and O. Glass. Uniform observability estimates for the 1-d discretized wave equation and the random choice method. *Comptes Rendus Mathematique*, 347(9-10):505 – 510, 2009.

[CF96] J.-M. Coron and A. V. Fursikov. Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary. *Russian J. Math. Phys.*, 4(4):429–448, 1996.

[CFCM13] N. Cîndea, E. Fernández-Cara, and A. Münch. Numerical controllability of the wave equation through primal methods and Carleman estimates. *ESAIM Control Optim. Calc. Var.*, 19(4):1076–1108, 2013.

[CG05] J.-M. Coron and S. Guerrero. Singular optimal control: a linear 1-D parabolic-hyperbolic example. *Asymptot. Anal.*, 44(3-4):237–257, 2005.

[Cho09] M. Choulli. *Une introduction aux problèmes inverses elliptiques et paraboliques*, volume 65 of *Mathématiques & Applications (Berlin)*. Springer-Verlag, Berlin, 2009.

[Cia82] P. G. Ciarlet. *Introduction à l'analyse numérique matricielle et à l'optimisation.* Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1982.

[CL] J.-M. Coron and P. Lissy. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components. *Invent. Math., to appear.*

[CM06] C. Castro and S. Micu. Boundary controllability of a linear semi-discrete 1-d wave equation derived from a mixed finite element method. *Numer. Math.*, 102(3):413–462, 2006.

[CMM08] C. Castro, S. Micu, and A. Münch. Numerical approximation of the boundary control for the wave equation with mixed finite elements in a square. *IMA J. Numer. Anal.*, 28(1):186–214, 2008.

[CMT11] N. Cîndea, S. Micu, and M. Tucsnak. An approximation method for exact controls of vibrating systems. *SIAM J. Control Optim.*, 49(3):1283–1305, 2011.

[Cor96] J.-M. Coron. On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl. (9)*, 75(2):155–188, 1996.

[Cor96] J.-M. Coron. On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions. *ESAIM Control Optim. Calc. Var.*, 1:35–75 (electronic), 1995/96.

[CRR12] S. Chowdhury, M. Ramaswamy, and J.-P. Raymond. Controllability and stabilizability of the linearized compressible Navier-Stokes system in one dimension. *SIAM J. Control Optim.*, 50(5):2959–2987, 2012.

[CSRZ] F. W. Chaves-Silva, L. Rosier, and E. Zuazua. Null controllability of a system of viscoelasticity with a moving control. http://arxiv.org/abs/1303.3452.

[DE] B. Dehman and S. Ervedoza. Dependence of high-frequency waves with respect to potentials. *SIAM J. Control Optim., to appear.*

[Deb] M. Debayan. Some controllability results for linearized compressible Navier-Stokes system. *Preprint*, 2014.

[DL89] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[DL09] B. Dehman and G. Lebeau. Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time. *SIAM J. Control and Optim.*, 48(2):521–550, 2009.

[DZZ08] T. Duyckaerts, X. Zhang, and E. Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(1):1–41, 2008.

[EdG11] S. Ervedoza and F. de Gournay. Uniform stability estimates for the discrete Calderón problems. *Inverse Problems*, 27(12):125012, 37, 2011.

[EGGP12] S. Ervedoza, O. Glass, S. Guerrero, and J.-P. Puel. Local exact controllability for the one-dimensional compressible Navier-Stokes equation. *Arch. Ration. Mech. Anal.*, 206(1):189–238, 2012.

[EHL14] S. Ervedoza, M. Hillairet, and C. Lacave. Long-time behavior for the two-dimensional motion of a disk in a viscous fluid. *Communications in Mathematical Physics*, pages 1–58, 2014.

[Erv09] S. Ervedoza. Spectral conditions for admissibility and observability of wave systems: applications to finite element schemes. *Numer. Math.*, 113(3):377–415, 2009.

[Erv10] S. Ervedoza. Observability properties of a semi-discrete 1D wave equation derived from a mixed finite element method on nonuniform meshes. *ESAIM Control Optim. Calc. Var.*, 16(2):298–326, 2010.

[EV10] S. Ervedoza and J. Valein. On the observability of abstract time-discrete linear parabolic equations. *Rev. Mat. Complut.*, 23(1):163–190, 2010.

[EV14] S. Ervedoza and M. Vanninathan. Controllability of a simplified model of fluid-structure interaction. *ESAIM: Control, Optimisation and Calculus of Variations*, 20:547–575, 4 2014.

[EZ] S. Ervedoza and E. Zuazua. Transmutation techniques and observability for time-discrete approximation schemes of conservative systems. *Numer. Math., to appear.*

[EZ10] S. Ervedoza and E. Zuazua. A systematic method for building smooth controls for smooth data. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1375–1401, 2010.

[EZ11a] S. Ervedoza and E. Zuazua. Observability of heat processes by transmutation without geometric restrictions. *Math. Control Relat. Fields*, 1(2):177–187, 2011.

[EZ11b] S. Ervedoza and E. Zuazua. Sharp observability estimates for heat equations. *Arch. Ration. Mech. Anal.*, 202(3):975–1017, 2011.

[EZ11c] S. Ervedoza and E. Zuazua. The wave equation: Control and numerics. In P. M. Cannarsa and J. M. Coron, editors, *Control of Partial Differential Equations*, Lecture Notes in Mathematics, CIME Subseries. Springer Verlag, 2011.

[EZ13] S. Ervedoza and E. Zuazua. *Numerical approximation of exact controls for waves.* Springer Briefs in Mathematics. Springer, New York, 2013.

[EZZ08] S. Ervedoza, C. Zheng, and E. Zuazua. On the observability of time-discrete conservative linear systems. *J. Funct. Anal.*, 254(12):3037–3078, June 2008.

[FC12] E. Fernández-Cara. Motivation, analysis and control of the variable density Navier-Stokes equations. *Discrete Contin. Dyn. Syst. Ser. S*, 5(6):1021–1090, 2012.

[FCGIP04] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.

[FCGIP06] E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel. Some controllability results for the N-dimensional Navier-Stokes and Boussinesq systems with $N-1$ scalar controls. *SIAM J. Control Optim.*, 45(1):146–173 (electronic), 2006.

[FCZ00a] E. Fernández-Cara and E. Zuazua. The cost of approximate controllability for heat equations: the linear case. *Adv. Differential Equations*, 5(4-6):465–514, 2000.

[FCZ00b] E. Fernández-Cara and E. Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(5):583–616, 2000.

[FI96] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[FNP01] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 3(4):358–392, 2001.

[FR71] H. O. Fattorini and D. L. Russell. Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.*, 43:272–292, 1971.

[FYZ07] X. Fu, J. Yong, and X. Zhang. Exact controllability for multidimensional semilinear hyperbolic equations. *SIAM J. Control Optim.*, 46(5):1578–1614 (electronic), 2007.

[GGS13] L. Gaudio, M.J. Grote, and O. Schenck. Interior point method for time-dependent inverse problems. In Tunisia Gammarth, editor, *Proc. of 11th Internat. Conf. on Math. and Numerical Aspects of Wave Propagation (WAVES 2013)*, pages 121–122, 2013.

[GL90] R. Glowinski and C. H. Li. On the numerical implementation of the Hilbert uniqueness method for the exact boundary controllability of the wave equation. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(2):135–142, 1990.

[Gla07] O. Glass. On the controllability of the 1-D isentropic Euler equation. *J. Eur. Math. Soc. (JEMS)*, 9(3):427–486, 2007.

[Gla10] O. Glass. A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit. *J. Funct. Anal.*, 258(3):852–868, 2010.

[Gla14] O. Glass. On the controllability of the non-isentropic 1-D Euler equation. *J. Differential Equations*, 257(3):638–719, 2014.

[GLH08] R. Glowinski, J.-L. Lions, and J. He. *Exact and approximate controllability for distributed parameter systems*, volume 117 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2008. A numerical approach.

[GLL90] R. Glowinski, C. H. Li, and J.-L. Lions. A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls: description of the numerical methods. *Japan J. Appl. Math.*, 7(1):1–76, 1990.

[Ho86] L. F. Ho. Observabilité frontière de l'équation des ondes. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(12):443–446, 1986.

[Hör63] L. Hörmander. *Linear partial differential operators*. Die Grundlehren der mathematischen Wissenschaften, Bd. 116. Academic Press Inc., Publishers, New York, 1963.

[Hör85] L. Hörmander. *The analysis of linear partial differential operators. III*, volume 274 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1985. Pseudodifferential operators.

[HR12] G. Haine and K. Ramdani. Reconstructing initial data using observers: error analysis of the semi-discrete and fully discrete approximations. *Numer. Math.*, 120(2):307–343, 2012.

[Ima01] O. Y. Imanuvilov. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6:39–72 (electronic), 2001.

[Ima02] O. Y. Imanuvilov. On Carleman estimates for hyperbolic equations. *Asymptot. Anal.*, 32(3-4):185–220, 2002.

[IP03] O. Y. Imanuvilov and J.-P. Puel. Global Carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems. *Int. Math. Res. Not.*, 16:883–913, 2003.

[IPY09] O. Y. Imanuvilov, J.-P. Puel, and M. Yamamoto. Carleman estimates for parabolic equations with nonhomogeneous boundary conditions. *Chin. Ann. Math. Ser. B*, 30(4):333–378, 2009.

[Isa06] V. Isakov. *Inverse problems for partial differential equations*, volume 127 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2006.

[IY01] O. Y. Imanuvilov and M. Yamamoto. Global uniqueness and stability in determining coefficients of wave equations. *Comm. Partial Differential Equations*, 26(7-8):1409–1425, 2001.

[IY03a] O. Y. Imanuvilov and M. Yamamoto. Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations. *Publ. Res. Inst. Math. Sci.*, 39(2):227–274, 2003.

[IY03b] O. Y. Imanuvilov and M. Yamamoto. Determination of a coefficient in an acoustic equation with a single measurement. *Inverse Problems*, 19(1):157–171, 2003.

[IZ99] J.A. Infante and E. Zuazua. Boundary observability for the space semi discretizations of the 1-d wave equation. *Math. Model. Num. Ann.*, 33:407–438, 1999.

[Joh82] F. John. *Partial differential equations*, volume 1 of *Applied Mathematical Sciences*. Springer-Verlag, New York, fourth edition, 1982.

[Kan77] Y. Kannai. Off diagonal short time asymptotics for fundamental solutions of diffusion equations. *Commun. Partial Differ. Equations*, 2(8):781–830, 1977.

[Kli92] M. V. Klibanov. Inverse problems and Carleman estimates. *Inverse Problems*, 8(4):575–596, 1992.

[KS91] M. V. Klibanov and F. Santosa. A computational quasi-reversibility method for Cauchy problems for Laplace's equation. *SIAM J. Appl. Math.*, 51(6):1653–1675, 1991.

[KT04] M. V. Klibanov and A. Timonov. *Carleman estimates for coefficient inverse problems and numerical applications.* Inverse and Ill-posed Problems Series. VSP, Utrecht, 2004.

[Leb94] G. Lebeau. Équations des ondes amorties. *Séminaire sur les Équations aux Dérivées Partielles, 1993–1994,École Polytech.*, 1994.

[Lio88a] J.-L. Lions. *Contrôlabilité exacte, Stabilisation et Perturbations de Systèmes Distribués. Tome 1. Contrôlabilité exacte*, volume RMA 8. Masson, 1988.

[Lio88b] J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, 30(1):1–68, 1988.

[Lio98] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2*, volume 10 of *Oxford Lecture Series in Mathematics and its Applications.* The Clarendon Press, Oxford University Press, New York, 1998. Compressible models, Oxford Science Publications.

[Lis] P. Lissy. An application of a conjecture due to Ervedoza and Zuazua concerning the observability of the heat equation in small time to a conjecture due to Coron and Guerrero concerning the uniform controllability of a convection-diffusion equation in the vanishing viscosity limit. *Systems Control Lett., to appear.*

[Lis12] P. Lissy. A link between the cost of fast controls for the 1-D heat equation and the uniform controllability of a 1-D transport-diffusion equation. *C. R. Math. Acad. Sci. Paris*, 350(11-12):591–595, 2012.

[LLT86] I. Lasiecka, J.-L. Lions, and R. Triggiani. Nonhomogeneous boundary value problems for second order hyperbolic operators. *J. Math. Pures Appl. (9)*, 65(2):149–192, 1986.

[LN10] G. Lebeau and M. Nodet. Experimental study of the HUM control operator for linear waves. *Experiment. Math.*, 19(1):93–120, 2010.

[LR95] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20(1-2):335–356, 1995.

[LR97] G. Lebeau and L. Robbiano. Stabilisation de l'équation des ondes par le bord. *Duke Math. J.*, 86(3):465–491, 1997.

[LR03] T.-T. Li and B.-P. Rao. Exact boundary controllability for quasi-linear hyperbolic systems. *SIAM J. Control Optim.*, 41(6):1748–1755 (electronic), 2003.

[LT83] I. Lasiecka and R. Triggiani. Regularity of hyperbolic equations under $L_2(0, T; L_2(\Gamma))$-Dirichlet boundary terms. *Appl. Math. Optim.*, 10(3):275–286, 1983.

[LZ05] W. Li and X. Zhang. Controllability of parabolic and hyperbolic equations: toward a unified theory. In *Control theory of partial differential equations*, volume 242 of *Lect. Notes Pure Appl. Math.*, pages 157–174. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[Mic02] S. Micu. Uniform boundary controllability of a semi-discrete 1-D wave equation. *Numer. Math.*, 91(4):723–768, 2002.

[Mil04] L. Miller. Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time. *J. Differential Equations*, 204(1):202–226, 2004.

[Mil05] L. Miller. Controllability cost of conservative systems: resolvent condition and transmutation. *J. Funct. Anal.*, 218(2):425–444, 2005.

[Mil06a] L. Miller. The control transmutation method and the cost of fast controls. *SIAM J. Control Optim.*, 45(2):762–772 (electronic), 2006.

[Mil06b] L. Miller. On exponential observability estimates for the heat semigroup with explicit rates. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 17(4):351–366, 2006.

[Mil10] L. Miller. A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1465–1485, 2010.

[Mil12] L. Miller. Resolvent conditions for the control of unitary groups and their approximations. *J. Spectr. Theory*, 2(1):1–55, 2012.

[MN80] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.*, 20(1):67–104, 1980.

[MRR13] P. Martin, L. Rosier, and P. Rouchon. Null controllability of the structurally damped wave equation with moving control. *SIAM J. Control Optim.*, 51(1):660–684, 2013.

[MS78] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31(5):593–617, 1978.

[MS82] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math.*, 35(2):129–168, 1982.

[Mün] A. Münch. Personal communication.

[MZ10] A. Marica and E. Zuazua. Localized solutions for the finite difference semi-discretization of the wave equation. *C. R. Math. Acad. Sci. Paris*, 348(11-12):647–652, 2010.

[Phu10] K. D. Phung. Waves, damped wave and observation. In Ta-Tsien Li, Yue-Jun Peng, and Bo-Peng Rao, editors, *Some Problems on Nonlinear Hyperbolic Equations and Applications*, Series in Contemporary Applied Mathematics CAM 15, 2010.

[PY96] J.-P. Puel and M. Yamamoto. On a global estimate in a linear inverse hyperbolic problem. *Inverse Problems*, 12(6):995–1002, 1996.

[PY97] J.-P. Puel and M. Yamamoto. Generic well-posedness in a multidimensional hyperbolic inverse problem. *J. Inverse Ill-Posed Probl.*, 5(1):55–83, 1997.

[Ral69] J. V. Ralston. Solutions of the wave equation with localized energy. *Comm. Pure Appl. Math.*, 22:807–823, 1969.

[Rob91] L. Robbiano. Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques. *Comm. Partial Differential Equations*, 16(4-5):789–800, 1991.

[Rob95] L. Robbiano. Fonction de coût et contrôle des solutions des équations hyperboliques. *Asymptotic Anal.*, 10(2):95–115, 1995.

[RR07] L. Rosier and P. Rouchon. On the controllability of a wave equation with structural damping. *Int. J. Tomogr. Stat.*, 5(W07):79–84, 2007.

[RTTT05] K. Ramdani, T. Takahashi, G. Tenenbaum, and M. Tucsnak. A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator. *J. Funct. Anal.*, 226(1):193–229, 2005.

[Rus73] D. L. Russell. A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. *Studies in Appl. Math.*, 52:189–211, 1973.

[Rus78] D. L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Rev.*, 20(4):639–739, 1978.

[RZ98] L. Robbiano and C. Zuily. Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients. *Invent. Math.*, 131(3):493–539, 1998.

[Sei79] T. I. Seidman. Time-invariance of the reachable set for linear control problems. *J. Math. Anal. Appl.*, 72(1):17–20, 1979.

[SU87] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[SU09] P. Stefanov and G. Uhlmann. Thermoacoustic tomography with variable sound speed. *Inverse Problems*, 25(7):075011, 16, 2009.

[SU13] P. Stefanov and G. Uhlmann. Recovery of a source term or a speed with one measurement and applications. *Trans. Amer. Math. Soc.*, 365(11):5737–5758, 2013.

[Tat95] D. Tataru. Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. Partial Differential Equations*, 20(5-6):855–884, 1995.

[Tat96] D. Tataru. Carleman estimates and unique continuation for solutions to boundary value problems. *J. Math. Pures Appl. (9)*, 75(4):367–408, 1996.

[Tre82] L. N. Trefethen. Group velocity in finite difference schemes. *SIAM Rev.*, 24(2):113–136, 1982.

[TT07] G. Tenenbaum and M. Tucsnak. New blow-up rates for fast controls of Schrödinger and heat equations. *J. Differential Equations*, 243(1):70–100, 2007.

[TW09] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[TW14] M. Tucsnak and G. Weiss. From exact observability to identification of singular sources. *Mathematics of Control, Signals, and Systems*, pages 1–21, 2014.

[Uhl99] G. Uhlmann. Developments in inverse problems since Calderón's foundational paper. In *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Math., pages 295–345. Univ. Chicago Press, Chicago, IL, 1999.

[Yam95] M. Yamamoto. Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method. *Inverse Problems*, 11(2):481–496, 1995.

[Yam99] M. Yamamoto. Uniqueness and stability in multidimensional hyperbolic inverse problems. *J. Math. Pures Appl. (9)*, 78(1):65–98, 1999.

[Yam09] M. Yamamoto. Carleman estimates for parabolic equations and applications. *Inverse Problems*, 25(12):123013, 2009.

[Zha00] X. Zhang. Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities. *SIAM J. Control Optim.*, 39(3):812–834 (electronic), 2000.

[Zua99] E. Zuazua. Boundary observability for the finite-difference space semi-discretizations of the 2-D wave equation in the square. *J. Math. Pures Appl. (9)*, 78(5):523–563, 1999.

[Zua01] E. Zuazua. Some results and open problems on the controllability of linear and semilinear heat equations. In *Carleman estimates and applications to uniqueness and control theory (Cortona, 1999)*, volume 46 of *Progr. Nonlinear Differential Equations Appl.*, pages 191–211. Birkhäuser Boston, Boston, MA, 2001.

[Zua02] E. Zuazua. Log-Lipschitz regularity and uniqueness of the flow for a field in $(W^{n/p+1,p}_{\mathrm{loc}}(\mathbb{R}^n))^n$. *C. R. Math. Acad. Sci. Paris*, 335(1):17–22, 2002.

[Zua05] E. Zuazua. Propagation, observation, and control of waves approximated by finite difference methods. *SIAM Rev.*, 47(2):197–243 (electronic), 2005.