Least-squares pressure recovery in ROM for incompressible flows

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Contents

Outline

Motivation: ROMs for incompressible flow

Incompressible Navier Stokes-flow takes place during a time interval $[0, T]$:

Find a velocity field $\boldsymbol{u}(\mu) : \Omega \times (0, T) \to \mathbb{R}^d$ *and a pressure* $p(\mu) : \Omega \times (0, T) \to \mathbb{R}$ *such that*

$$\begin{cases} \partial_t \boldsymbol{u}(\mu) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \mu \Delta \boldsymbol{u}(\mu) + \nabla p(\mu) = \boldsymbol{f} & \text{in } \Omega \times (0, T), \\ \nabla \cdot \boldsymbol{u}(\mu) = 0 & \text{in } \Omega \times (0, T), \\ \boldsymbol{u}(\mu) = \boldsymbol{0} & \text{on } \partial\Omega \times (0, T), \\ \boldsymbol{u}(\mu, 0) = \boldsymbol{u}_0, & \text{in } \Omega, \end{cases} \tag{1}$$

Solution snapshots are obtained primarily by one of the following approaches :

- exact factorisation / coupled approach (ROM with TC, MO, IS)
- time splitting approach (ROM with SR, CX, YG)
- penalty methods with (ROM H. YAO)

Exact factorisation / coupled approach

Given the initialization $\boldsymbol{u}^0(\mu) = u_0$,

$$\begin{cases} \textit{Find } \boldsymbol{u}(\mu) \in \mathbf{H}^1_D(\Omega) \textit{ and } p(\mu) \in L^2_0(\Omega) \textit{ such that} \\ a(\boldsymbol{u}(\mu), \boldsymbol{u}(\mu), \boldsymbol{v}; \mu) - (\nabla \cdot \boldsymbol{v}, p(\mu)) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \quad \forall\, \boldsymbol{v} \in \mathbf{H}^1_D(\Omega), \\ (\nabla \cdot \boldsymbol{u}(\mu), q) = 0 \quad \forall\, q \in L^2_0(\Omega) \end{cases} \tag{2}$$

Remark

The reduced bases are built from snapshots and therefore satisfy: divergence free and boundary conditions

Motivation: the need of pressure recovery in ROMs

- In Reduced Order Modelling of incompressible flows it is a common practice to use weakly divergence-free velocities, thus dropping the pressure from the equations.
- Knowing the pressure is needed for a number of applications: calculation of forces on walls or immersed boundaries, code/model calibration with pressure data.
- There basically are two techniques to recover the reduced pressure once the reduced velocity is known:
  - **Poisson pressure equation (PPE)**. It is obtained as the divergence of the momentum conservation equation.
  - **Momentum equation recovery (MEQ)**. The momentum conservation equation is directly considered as an equation for the pressure.

Fast survey of pressure recovery procedures

Both methods come from the momentum equation written as

$$\nabla p = -\partial_t \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \mu \Delta \boldsymbol{u} + \boldsymbol{f}.$$

Poisson pressure equation (PPE) (Noack, Papas, Monkewitz, 2005):

$$-\Delta p = \nabla \cdot (\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \mu \Delta \boldsymbol{u} - \boldsymbol{f}), \text{ or } -\Delta p = \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u} - \boldsymbol{f}).$$

- Solved by Galerkin method on reduced pressure space.
- Boundary conditions are needed.
  - If no terms are dropped, the natural condition

$$\nabla p \cdot \boldsymbol{n} = -(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \mu \Delta \boldsymbol{u} + \boldsymbol{f}) \cdot \boldsymbol{n} \quad \text{holds.}$$

Fast survey of pressure recovery procedures

Notation: $\boldsymbol{X}_h$, M_h: FO spaces. $\boldsymbol{X}_r$, M_r: Reduced spaces.

Momentum equation recovery (MEQ) (Rovas &Patera, 2004, Kean & Schneier, 2020):
Look for $p_r \in M_r$ such that

$$(p_r, \nabla \cdot \boldsymbol{v}_r) = (\partial_t \boldsymbol{u}_r + \boldsymbol{u}_r \cdot \nabla \boldsymbol{u}_r, \nabla \boldsymbol{v}_r) + \mu(\nabla \boldsymbol{u}_r, \nabla \boldsymbol{v}_r) - \langle \boldsymbol{f}, \boldsymbol{v}_r \rangle, \quad \forall \boldsymbol{v}_r \in \boldsymbol{S}_r.$$

- The pressure gradient supremizers are added to the original velocity space to achieve the inf-sup condition for reduced (enriched velocity, pressure M_r) spaces:
  For any basis function q_r of M_r, find $\boldsymbol{s}_r \in \boldsymbol{X}_h$ such that

$$(\boldsymbol{s}_r, \boldsymbol{v}_h)_{H^1(\Omega)} = -(\nabla \cdot \boldsymbol{v}_r, q_r), \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.$$

  • $\boldsymbol{s}_r$ (the "supremizer") is the representation of ∇q_r (eventually + b. c.) on $\boldsymbol{X}_h$ by the Theorem of Riesz.

Outline

Least squares procedure : Tools

- Gradient operator "au sens faible/distribution" $G : M_h \mapsto \mathbf{H}^{-1}(\Omega)$ given by

$$\langle G\, q_h, \boldsymbol{v}_h \rangle = -(\nabla \cdot \boldsymbol{v}_h, q_h)_0, \quad \forall q_h \in M_h, \forall \boldsymbol{v}_h \in \boldsymbol{X}_h. \tag{3}$$

Note that $\langle G\, q_h, \boldsymbol{v}_h \rangle = \langle \nabla q_h,\ \boldsymbol{v}_h \rangle - \int_{\Gamma_N} q_h\, \boldsymbol{v}_h \cdot \boldsymbol{n}$ and thus, if $\Gamma_N = \emptyset$, the operator G would be the gradient operator "au sens faible".

- Riesz representation : $\Pi_h^{(k)} : \mathbf{H}^{-1}(\Omega) \mapsto \boldsymbol{X}_h$ defined by

$$(\Pi_h^{(k)} \varphi, \boldsymbol{v}_h)_k = \langle \varphi, \boldsymbol{v}_h \rangle, \quad \forall \varphi \in \mathbf{H}^{-1}(\Omega), \forall \boldsymbol{v}_h \in \boldsymbol{X}_h, \text{ for either } k = 0 \text{ or } k = 1. \tag{4}$$

Observe that from (4) and (3),

$$(\Pi_h^{(k)}(G\, q_h), \boldsymbol{v}_h)_k = -(\nabla \cdot \boldsymbol{v}_h, q_h)_0, \quad \forall q_h \in M_h, \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.$$

Least squares procedure : Formulation

We propose to obtain the pressure by minimisation of the dual discrete norm of the residual,

$$p_r = \arg\min_{q_r \in M_r} J(q_r) := \| G q_r - T(\boldsymbol{u}_r) \|^2_{\boldsymbol{X}'_h},$$

where $T(\boldsymbol{u}) = -(\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \mu \Delta \boldsymbol{u} - \boldsymbol{f})$.

- The norm in $\boldsymbol{X}'_h$ is computed via a Riesz representation operator:

$$p_r = \arg\min_{q_r \in M_r} J(q_r) := \| \Pi_h^{(k)} \left(G\, q_r - T(\boldsymbol{u}_r) \right) \|^2_k, \tag{5}$$

- $\Pi_h^{(k)}$ is a supremizer (rather maximizer) operator:

$$\| \Pi_h^{(k)} \varphi \|_{H^k(\Omega)} = \max_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{(\Pi_h^{(k)} \varphi, \boldsymbol{v}_h)_{H^k(\Omega)}}{\| \boldsymbol{v}_h \|_{H^k(\Omega)}} = \max_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \frac{\langle \varphi, \boldsymbol{v}_h \rangle}{\| \boldsymbol{v}_h \|_{H^k(\Omega)}}$$

$$\Longrightarrow \quad \| \Pi_h^{(k)} \varphi \|_{H^k(\Omega)} = \| \varphi \|_{\boldsymbol{X}'_h}$$

Least squares procedure

If the FOM (velocity, pressure) spaces satisfy an inf-sup condition, then the pressure recovery problem admits a unique solution.

- It satisfies the normal equations,

$$(\Pi_h^{(k)}(\nabla p_r) - \Pi_h^{(k)} T(\boldsymbol{u}_r), \Pi_h^{(k)}(\nabla q_r))_{H^k(\Omega)} = 0, \ \forall\, q_r \in M_r.$$

- It is equivalent to a linear system, that is assembled in the off-line stage.

Algebraic expression of pressure recovery problem

- Let $\{\phi_i\}_{i=1}^{n_h}$ be a basis of $\boldsymbol{X}_h$,
- Let $\{\psi_i\}_{i=1}^{n_r}$ be a basis of M_r,

then problem (12) is equivalent to : *Find* $p_r = \sum_{i=1}^{n_r} p_i \psi_i \in M_r$ *s. t.*

$\vec{p} \in \mathbb{R}^{n_r}$ with $(\vec{p})_i = p_i$ is the solution of the linear system:

$$\mathcal{M}\vec{p} = \vec{r} \quad \text{with} \quad \mathcal{M} = \mathcal{B}\,\mathcal{G}^{-1}\,\mathcal{B}^t \text{ and } \vec{r} = \mathcal{B}\,\mathcal{G}^{-1}\,\vec{R}, \tag{6}$$

where

$$\mathcal{B} \in \mathbb{R}^{n_r \times n_h} : \mathcal{B}_{ij} = -(\nabla \cdot \phi_j, \psi_i),$$

$$\mathcal{G} \in \mathbb{R}^{n_h \times n_h} : \mathcal{G}_{ij} = (\phi_j, \phi_i)_{H^k(\Omega)}$$

$$\vec{R} \in \mathbb{R}^{n_h} : \vec{R}_i = \langle T(\boldsymbol{u}_r), \phi_i \rangle.$$

The pressure gradient supremisers revisited

The supremisers procedure consists in solving the mixed problem (we consider the Stokes problem with homogeneous Dirichlet b. c.)

$$\begin{cases} \textit{Find } \boldsymbol{u}_r \in \boldsymbol{X}_r \text{ and } p_r \in M_r \textit{ such that} \\ (\nabla \boldsymbol{u}_r, \nabla \boldsymbol{v}_r) - (\nabla \cdot \boldsymbol{v}_r, p_r) = \langle \boldsymbol{f}, \boldsymbol{v}_r \rangle, \quad \forall\, \boldsymbol{v}_r \in \boldsymbol{X}_r, \\ (\nabla \cdot \boldsymbol{u}_r, q_r) = 0, \quad \forall\, q_r \in M_r, \end{cases}$$

where the velocity space $\boldsymbol{X}_r$ has been enriched with the pressure gradient supremisers:

$$\boldsymbol{X}_r = \boldsymbol{S}_r \oplus \boldsymbol{X}_{0r},$$

with

$$\boldsymbol{S}_r = \{\Pi_h^{(k)}(\nabla q_r) \text{ such that } q_r \in M_r\} \subset \boldsymbol{X}_h,$$

$$\boldsymbol{X}_{0r} = \{\boldsymbol{v}_h \in \boldsymbol{X}_r \text{ such that } (\nabla \cdot \boldsymbol{v}_r, q_r) = 0, \forall q_r \in M_r\}.$$

The pressure gradient supremizers revisited

The mixed problem is equivalent to the sequence of two problems

$$(\nabla \boldsymbol{u}_r, \nabla \boldsymbol{v}_r) = \langle \boldsymbol{f}_r,\ \boldsymbol{v}_{0r} \rangle, \quad \forall\, \boldsymbol{v}_{0r} \in \boldsymbol{X}_{0r};$$

$$(\Pi_h^{(k)}(\nabla p_r),\ \Pi_h^{(k)}(\nabla q_r))_{H^k(\Omega)} = (\Pi_h^{(k)}(T(\boldsymbol{u}_r)), \Pi_h^{(k)}(\nabla q_r))_{H^k(\Omega)},\ \forall\, q_r \in M_r,$$

with $\boldsymbol{u}_r \in \boldsymbol{X}_{0r}$, $p_r \in M_r$.

That is, **the pressure obtained by the supremisers procedure is the one obtained by the least-squares procedure**.

- This also occurs for the full-order solution.

Outline

1 Motivation

2 Least squares pressure recovery

3 Error estimates

4 Numerical results

Error estimates

Theorem

Assuming that the discrete inf-sup condition holds, then the pressure obtained by the least-squares recovery procedure satisfies

$$\|p_h - p_r\|_0 \leq C_h \left(d_{L^2(\Omega)}(p_h, M_r) + \| T(\boldsymbol{u}_h) - T(\boldsymbol{u}_r)\|_{\mathbf{H}^{-1}(\Omega)} \right),$$

where

- *If* $k = 0$, C_h *is the smallest constant appearing in the inverse estimate* $\|\nabla \boldsymbol{v}_h\|_0 \leq C_h \|\boldsymbol{v}_h\|_0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h.$
- *If* $k = 1$, C_h *is independent of* h

Error estimates for unsteady Navier-Stokes

Assuming that the incompressible flow takes place during a time interval $[0, T]$, we consider the problem

Find a velocity field $\boldsymbol{u}(\mu) : \Omega \times (0, T) \to \mathbb{R}^d$ *and a pressure* $p(\mu) : \Omega \times (0, T) \to \mathbb{R}$ *such that*

$$\begin{cases} \partial_t \boldsymbol{u}(\mu) + \boldsymbol{u}(\mu) \cdot \nabla \boldsymbol{u}(\mu) - \mu \Delta \boldsymbol{u}(\mu) + \nabla p(\mu) = \boldsymbol{f} & \text{in } \Omega \times (0, T), \\ \nabla \cdot \boldsymbol{u}(\mu) = 0 & \text{in } \Omega \times (0, T), \\ \boldsymbol{u}(\mu) = \boldsymbol{0} & \text{on } \Gamma_D \times (0, T), \\ -\mu \nabla \boldsymbol{u}(\mu) \cdot \boldsymbol{n} + p(\mu)\boldsymbol{n} = 0 & \text{on } \Gamma_N \times (0, T), \\ \boldsymbol{u}(\mu, 0) = \boldsymbol{u}_0, & \text{in } \Omega, \end{cases} \tag{7}$$

where $\boldsymbol{u}_0$ is a initial field velocity given.

Error estimates for the evolutionary model

We consider the implicit Euler time discretization of problem (7) with constant time-step size $\Delta t = T/N$:

Given the initialization $\boldsymbol{u}^0(\mu) = u_0$,

$$\begin{cases} \textit{Find } \boldsymbol{u}^n(\mu) \in \mathbf{H}^1_D(\Omega) \textit{ and } p^n(\mu) \in L^2_0(\Omega) \textit{ such that} \\ a(\boldsymbol{u}^n(\mu), \boldsymbol{u}^n(\mu), \boldsymbol{v}; \mu) - (\nabla \cdot \boldsymbol{v}, p^n(\mu)) = \langle \boldsymbol{f}^n, \boldsymbol{v} \rangle, \quad \forall\, \boldsymbol{v} \in \mathbf{H}^1_D(\Omega), \\ (\nabla \cdot \boldsymbol{u}^n(\mu), q) = 0 \quad \forall\, q \in L^2_0(\Omega), \\ \forall\, n = 1, 2, \ldots N. \end{cases} \tag{8}$$

where $\boldsymbol{f}^n = \boldsymbol{f} + \frac{1}{\Delta t}\boldsymbol{u}^{n-1}$.

Error estimates for the evolutionary model

We consider FE FOM and ROM approximations,

– FOM approximation : $(\boldsymbol{u}_h^n(\mu), p_h^n(\mu)) \in \boldsymbol{X}_h \times M_h$ such that

$$\left\{ \begin{array}{l} a(\boldsymbol{u}_h^n(\mu), \boldsymbol{u}_h^n(\mu), \boldsymbol{v}_h; \mu) - (\nabla \cdot \boldsymbol{v}_h, p_h^n(\mu)) = \langle \boldsymbol{f}_h^n, \boldsymbol{v}_h \rangle, \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{X}_h, \\ (\nabla \cdot \boldsymbol{u}_h^n(\mu), q_h) = 0 \quad \forall\, q_h \in M_h, \end{array} \right. \tag{9}$$

– ROM aproximations : $(\boldsymbol{u}_r^n(\mu), p_r^n(\mu)) \in \boldsymbol{X}_r \times M_r$, where the reduced velocity $\boldsymbol{u}_r^n(\mu)$ is computed by problem

$$a(\boldsymbol{u}_r^n(\mu), \boldsymbol{u}_r^n(\mu), \boldsymbol{v}_r; \mu) = \langle \boldsymbol{f}_r^n, \boldsymbol{v}_r \rangle, \quad \forall\, \boldsymbol{v}_r \in \boldsymbol{X}_r, \tag{10}$$

where the reduced pressure $p_r^n(\mu)$ is recovered from the reduced velocity by the L-S method (12).

Error estimates for the evolutionary model

To obtain error estimates for the reduced pressure we introduce the following discrete functions (we omit the dependency on the parameters for brevity):

- $\boldsymbol{u}_h : [0, T] \to \boldsymbol{X}_h$ is the piecewise linear in time function such that $\boldsymbol{u}_h(t_n) = \boldsymbol{u}_h^n$.
- $p_h : [0, T] \to M_h$ is the piecewise constant in time function that takes the value p_h^n on (t_{n-1}, t_n).
- $\boldsymbol{u}_r : [0, T] \to \boldsymbol{X}_r$ is the piecewise linear in time function such that $\boldsymbol{u}_r(t_n) = \boldsymbol{u}_r^n$.
- $p_r : [0, T] \to M_r$ is the piecewise constant in time function that takes the value p_r^n on (t_{n-1}, t_n).

Error estimates for the evolutionary model

Theorem

Assume that the pair of spaces $(\boldsymbol{X}_h, M_h)$ satisfy the discrete inf-sup condition, then

- *If $k = 0$,*

$$\|p_h - p_r\|_{L^1(L^2(\Omega))} \leq C_h \left(d_{L^1(L^2(\Omega))}(p_h, M_r) + \|\boldsymbol{u}_h - \boldsymbol{u}_r\|_{L^2(\mathbf{H}^1_D(\Omega))} \right),$$

- *If $k = 1$,*

$$\|p_h - p_r\|_{L^1(L^2(\Omega))} \leq C \left(d_{L^1(L^2(\Omega))}(p_h, M_r) + \|\boldsymbol{u}_h - \boldsymbol{u}_r\|_{L^2(\mathbf{H}^1_D(\Omega))} + \right.$$
$$\left. + \|D_t(\boldsymbol{u}_h - \boldsymbol{u}_r)\|_{L^1(\mathbf{H}^{-1}(\Omega))} \right),$$

where $D_t \boldsymbol{v} = \frac{1}{\Delta t}(\boldsymbol{v}(t_n) - \boldsymbol{v}(t_{n-1}))$.

Outline

Treatment of small amplitude modes

- Unconstrained LSpROM : find $\vec{p} \in \mathbb{R}^{n_r}$ solution of problem

$$\vec{p} = \underset{\vec{q} \in \mathbb{R}^{n_r}}{\text{arg min}} \|\mathcal{D}^t\, \vec{q} - \mathcal{L}^t\, \vec{R}\|^2_{\mathbb{R}^{n_r}}\,. \tag{11}$$

- Constrained LSpROM approach 1 : find $\vec{p} \in \mathbb{R}^{n_r}$ solution of the constrained problem :

$$\vec{p} = \underset{\vec{q} \in \mathbb{R}^{n_r} \ \text{s.t}\ q_j^2 \leq \varepsilon \lambda_j}{\text{arg min}} \|\mathcal{D}^t\, \vec{q} - \mathcal{L}^t\, \vec{R}\|^2_{\mathbb{R}^{n_r}}\,. \tag{12}$$

- Constrained LSpROM approach 2 : find $\vec{p} \in \mathbb{R}^{n_r}$ solution of the orthogonality constrained LSpROM :

$$[\vec{p}] = \underset{[\vec{q_*}][\vec{q_*}]^t = I_{n_r}}{\text{arg min}} \|\mathcal{D}^t\, [\lambda]\, [\vec{q_*}] - \mathcal{L}^t\, [\vec{R}]\|^2_F, \tag{13}$$

where $||\cdot||_F$ denotes the Frobenius norm.

Flow past a cylinder Re $= 100$.

We have tested the LS pressure recovery using the velocity computed with a POD solution of two academic flows.

- **Test 1:** Flow past a cylinder at $Re = 100$.
  - The flow is considered in a channel of rectangular shape with height $H = 30D$ and length $45D$, with a cylindric hole of diameter D placed at $L_1 = 10D$ from the left boundary and $H/2$ from the bottom wall.
  - At the inflow boundary, a horizontal velocity is imposed. On the remaining boundaries, we set a free-slip condition on the horizontal walls, a no-slip condition on the cylinder, and a normal stress free condition on the outflow boundary to allow the fluid to exit through the outlet of the channel.

Flow past a cylinder Re = 100.

- Regarding the computational aspects, we use a space-time discretizations consisting of a non-uniform triangular mesh made of 21174 cells, and a first order semi-implicit Euler integration scheme of step $\Delta t = 10^{-2}$.
- The resulting flow shows a creation of alternating low-high pressure vortices downstream the cylinder, triggering the generation of periodic Von Karman vortex pattern in the wake region.

Flow past a cylinder Re = 100 : Example of snapshot

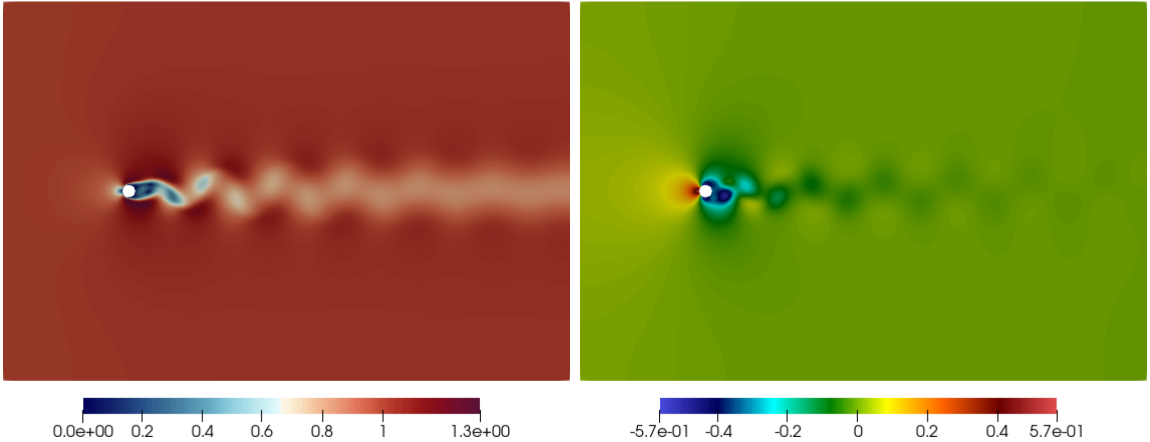
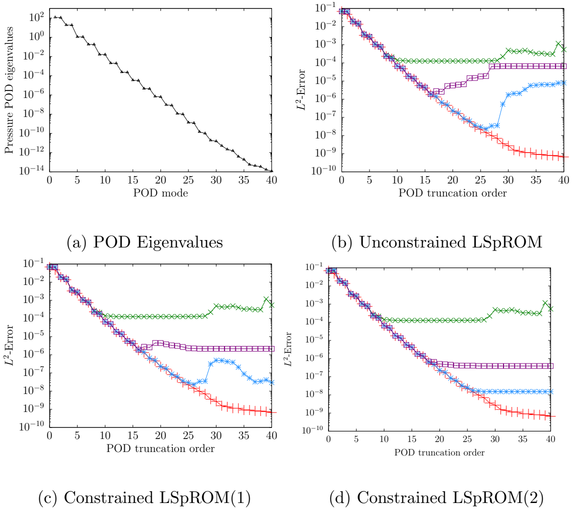


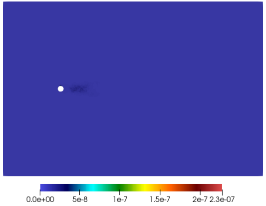
Figure: High fidelity velocity (left) and pressure (right) solutions of the flow past a cylinder, $Re = 100$



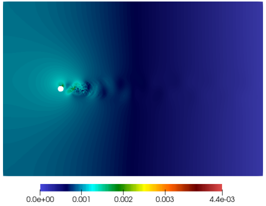
(a) POD Eigenvalues
(b) Unconstrained LSpROM
(c) Constrained LSpROM(1)
(d) Constrained LSpROM(2)

Figure: Pressure errors for the flow past a cylinder at $Re = 100$.

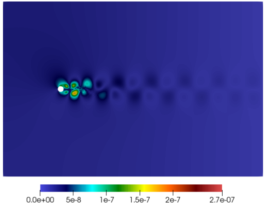
Flow past a cylinder Re = 100 : Errors behaviors



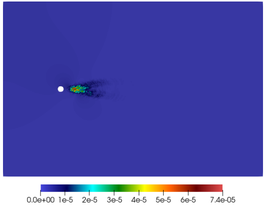
(a) POD reconstruction



(b) Chacon et al



(c) (k=0)



(d) (k=1)

Flow in a lid driven cavity $\mathbf{Re} = \mathbf{10000}$

- **Test 2**: Lid-driven cavity flow at $Re = 10.000$.
  - The flow is considered in a cavity of square shape $]0, D[\times]0, D[$ where the fluid is driven by a tangential velocity of magnitude acting on its top wall. No-slip conditions are imposed on the remaining walls.
  - To perform the numerical computations, we used a triangular mesh composed of 32928 cells and a first order semi-implicit Euler scheme of step $\Delta t = 10^{-3}$ for time integration.
  - The resulting flow is cyclic, where in the lower and upper left corners, the secondary vortex separates into two small vortices that periodically reincorporate.

Flow in a lid driven cavity **Re = 10000** : Example of snapshot

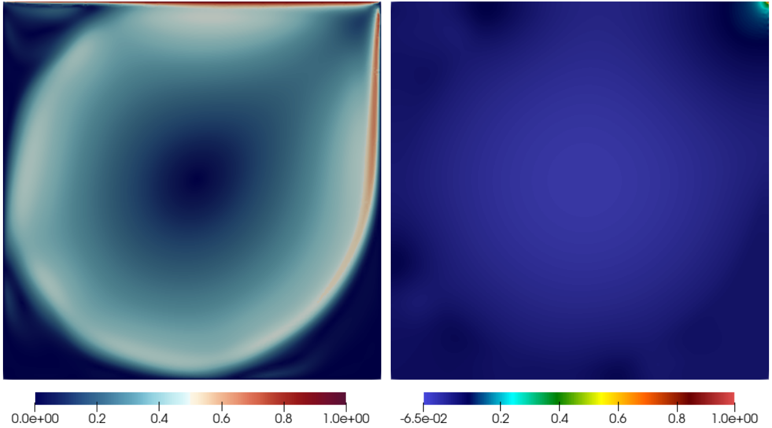
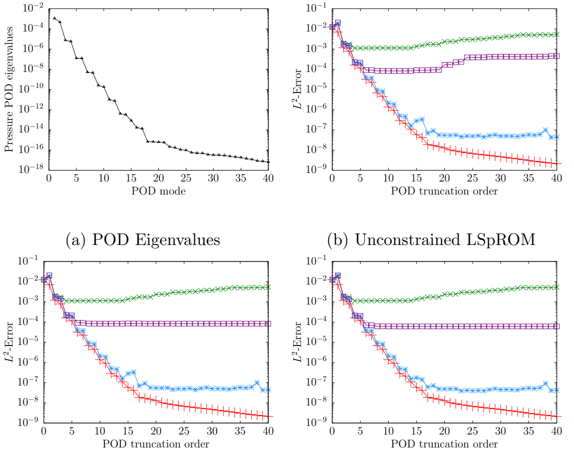


Figure: High fidelity velocity (left) and pressure (right) solutions of the cavity flow, $Re = 10000$



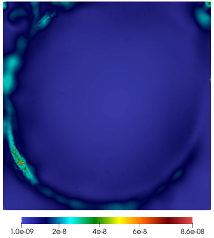
(a) POD Eigenvalues

(b) Unconstrained LSpROM

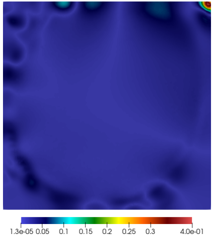
(c) Constrained LSpROM approach 1

(d) Constrained LSpROM approach 2

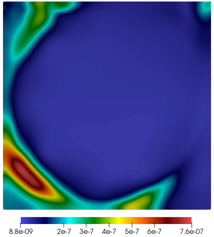
Figure: Pressure errors for the flow in a Lid Driven Cavity $Re = 10000$.



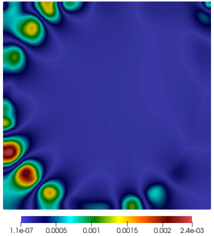
(a) POD



(b) Chacon et al



(c) (k=0)



(d) (k=1)

Figure: Pressure isovalues errors in the cavity flow $Re = 10000$, by taking 40 modes for velocity and pressure

Conclusions & work in progress

We have

- Introduced a least-squares method to recover the reduced pressure for incompressible flows.
- Given some fundamental theoretical results concerning the existence and uniqueness of the solution whenever the full-order pair of velocity-pressure spaces is inf-sup stable.
- Proved an optimal error estimate for the reduced pressure.
- Proved that our method is equivalent to the pressure gradient supremizers and to the Momentum Equation Recovery techniques.

We intend to

- Apply the method to snapshots including time differentiation.
- Extend to solving general saddle point problems.
- Recover the pressure for full model by LS procedures.

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Proper orthogonal decomposition reduced order model (see Kunish [8])

Considering the time as parameter,

- the ensembles of velocity snapshots $\chi^v = \text{span}\left\{\boldsymbol{u}_h^1, \ldots, \boldsymbol{u}_h^N\right\}$
- and pressure snapshots $\chi^p = \text{span}\left\{p_h^1, \ldots, p_h^N\right\}$.

POD method seeks low-dimensional bases $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_{r_v}\}$ and $\{\psi_1, \ldots, \psi_{r_p}\}$ in real Hilbert spaces $\mathcal{H}_v$, $\mathcal{H}_p$ that optimally approximate the velocity and pressure snapshots with respect to the discrete $L^2(\mathcal{H}_v)$, $L^2(\mathcal{H}_p)$ norms, respectively (cf. [8]).

- T is a given function in the Lebesgue space $L^2(X \times Y)$. $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^s$
- The integral operator B with kernel T and its adjoint operator B^* expressed as

$$\varphi \mapsto B\,\varphi, \qquad (B\,\varphi)(y) = \int_X T(x,y)\varphi(x)\;dx.$$

$$v \mapsto B^*\,v, \qquad (B^*\,v)(x) = \int_Y T(x,y)v(y)dy.$$

- $A = B^* B$ is an integral operator whose kernel K is

$$K(x,\xi) = \int_Y T(x,y)\;T(\xi,y)\;dy.$$

$\exists$ a Hilbert basis $(\varphi_m)_{m\geq 0}$ in $L^2(X)$ where φ_m is an eigenvector of A related to a non-negative eigenvalue λ_m, such as

$$A\,\varphi_m = \lambda_m\,\varphi_m, \qquad \forall m \geq 0. \tag{5}$$

Using Mercer's theorem yields the following decomposition

$$K(x,\xi) = \sum_{m\geq 0} \lambda_m \varphi_m(x)\varphi_m(\xi), \qquad \forall (x,\xi) \in X \times X.$$

- There exists a system $(\varphi_m, v_m, \sigma_m)_{m\geq 0}$ such that $(\varphi_m)_{m\geq 0}$ is an orthonormal basis in $L^2(X)$, $(v_m)_{m\geq 0}$ an orthonormal system in $L^2(Y)$ and $(\sigma_m)_{m\geq 0}$ a sequence of nonnegative real numbers such that

$$B\,\varphi_m = \sigma_m\, v_m, \qquad B^*\, v_m = \sigma_m\, \varphi_m, \qquad \sigma_m = \sqrt{\lambda_m}.$$

The sequence $(\sigma_m)_{m\geq 0}$ is ordered decreasingly and decays toward zero.

- A direct result is the Karhunen-Loève/POD expansion (POD)

$$T(x,y) = \sum_{m\geq 0} \sigma_m\, \varphi_m(x) v_m(y), \qquad \forall (x,y) \in X \times Y.$$

- The *(KL-approximation)* of function T of order M is denoted T_M and is

$$T_M(x,y) = \sum_{m=0}^{M} \sigma_m\, \varphi_m(x) v_m(y), \qquad \forall (x,y) \in X \times Y.$$

Approximation errors

$$\frac{\|T - T_M\|_{L^2(X\times Y)}}{\|T\|_{L^2(X\times Y)}} = \sqrt{\frac{\sum_{m\geq M+1}\lambda_m}{\sum_{m\geq 0}\lambda_m}}. \quad (6)$$

Let $(\psi_m)_{m\geq 0}$ be a Hilbertian basis in $L^2(X)$. We set $u_m(y) = \int_X T(x,y)\psi_m(x)\,dx$ and $S_M = \sum_{0\leq m\leq M}\psi_m \otimes u_m$.

$$\|T - T_M\|_{L^2(X\times Y)} \leq \|T - S_M\|_{L^2(X\times Y)}. \quad (7)$$

Approximation/Rate of convergence

$X = I =]-1,1[$. Assume that $T \in H^\tau(I, L^2(Y))$ for $\tau \geq 0$. Then, the following bound holds

$$\|T - T_M\|_{L^2(I\times Y)} \leq C_T\, M^{-\tau}.$$

Proper orthogonal decomposition reduced order model (see Kunish [8])

It can be shown that the following POD projection error formulas hold [7, 8]:

$$\Delta t \sum_{n=1}^{N} \left\| \boldsymbol{u}_h^n - \sum_{i=1}^{r_v} (\boldsymbol{u}_h^n, \boldsymbol{\varphi}_i)_{\mathcal{H}_v} \boldsymbol{\varphi}_i \right\|_{\mathcal{H}_v}^2 = \sum_{i=r_v+1}^{M_v} \lambda_i, \tag{8}$$

and

$$\Delta t \sum_{n=1}^{N} \left\| p_h^n - \sum_{i=1}^{r_p} (p_h^n, \psi_i)_{\mathcal{H}_p} \psi_i \right\|_{\mathcal{H}_p}^2 = \sum_{i=r_p+1}^{M_p} \gamma_i, \tag{9}$$

- M_v, M_p are the rank of χ^v and χ^p, respectively, and λ_i, γ_i are the associated eigenvalues.
- $\mathcal{H}_v, \mathcal{H}_p$ can be any real Hilbert spaces, (here $\mathcal{H}_v = \mathbf{L}^2$ and $\mathcal{H}_p = L^2$).
- In what follows, we are going to take $r_v = r_p = r$.

Proper orthogonal decomposition reduced order model

We respectively consider the following velocity and pressure spaces for the POD setting:

$$\boldsymbol{X}_r = \text{span}\{\varphi_1, \ldots, \varphi_r\} \subset \boldsymbol{X}_h,$$

and

$$M_r = \text{span}\{\psi_1, \ldots, \psi_r\} \subset M_h$$

Remark

Since the POD velocity modes are linear combinations of the snapshots obtained from solving (4), they satisfy the boundary conditions in and are solenoidal. Thus, the POD velocity modes belong to $\boldsymbol{X}_{0h}$*, which yields* $\boldsymbol{X}_{0r} \subset \boldsymbol{X}_{0h}$.

Galerkin projection-based POD-ROM

$$\boldsymbol{u}(\boldsymbol{x},t) \approx \boldsymbol{u}_r(\boldsymbol{x},t) = \sum_{i=1}^{r} a_i(t)\boldsymbol{\varphi}_i(\boldsymbol{x}), \quad p(\boldsymbol{x},t) \approx p_r(\boldsymbol{x},t) = \sum_{i=1}^{r} b_i(t)\psi_i(\boldsymbol{x}), \quad r << \mathcal{N} \tag{10}$$

where $\{a_i(t)\}_{i=1}^r$ and $\{b_i(t)\}_{i=1}^r$ are the sought time-varying coefficients representing the POD-Galerkin velocity and pressure trajectories.
Galerkin Navier-Stokes-POD-ROM problem reads :

- **Initialization.** Set: $\boldsymbol{u}_r^0 = \sum_{i=1}^r(\boldsymbol{u}^0, \boldsymbol{\varphi}_i)\boldsymbol{\varphi}_i$.
- **Iteration.** For $n = 0, 1, \ldots, N-1$: *Given* $\boldsymbol{u}_r^n \in \boldsymbol{X}_r$, *find* $\boldsymbol{u}_r^{n+1} \in \boldsymbol{X}_r$ *such that:*

$$a(\boldsymbol{u}_r, \boldsymbol{u}_r, \boldsymbol{\varphi}; \mu) = (\boldsymbol{f}^{n+1}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in \boldsymbol{X}_r. \tag{11}$$

In (11), **the pressure term vanishes** due to the fact that $\boldsymbol{u}_r^n$ belongs to $\boldsymbol{X}_{0h}$.
$\Longrightarrow$ We than need a procedure to recover it p

Thanks for your attention!