Well-Posedness and Optimal Control of an Enhanced Caginalp System

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Summary of the presentation

Presentation of the problem

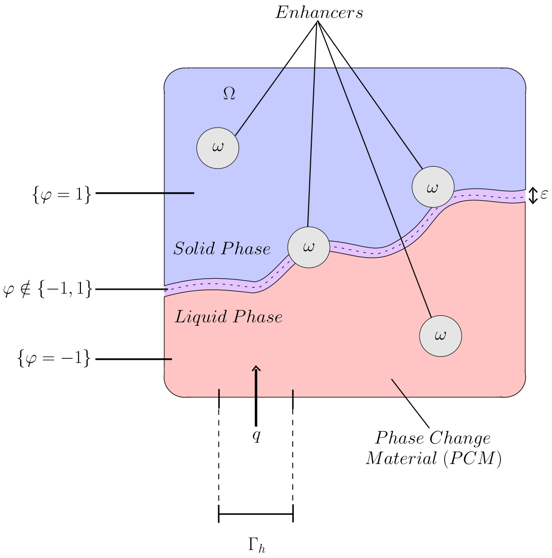

Figure: Representation of the melting

Three Unknowns

- Phase Field function $\varphi : [0, T] \times (\Omega \backslash \overline{\omega}) \to \mathbb{R}$
  $\rightsquigarrow$ Describes **Mushy Zone** evolution $\rightsquigarrow$ acts in $(\Omega \backslash \overline{\omega})$
- Relative Heat enhancers $u_\omega : [0, T] \times \omega \to \mathbb{R}$
  $\rightsquigarrow$ Acting in ω and through $\partial\omega$
- Relative heat Phase Change Material $u_\Omega : [0, T] \times (\Omega \backslash \overline{\omega}) \to \mathbb{R}$
  $\rightsquigarrow$ Acting in $(\Omega \backslash \overline{\omega})$ and through $\partial\omega$

Involved Quantities

- Thermal diffusivity $\beta > 0$

$$\beta := \frac{\kappa}{\rho c_p},$$

  κ thermal conductivity, ρ density, c_p isobaric specific heat capacity

- Mushy Zone thickness $\varepsilon > 0$
- Relaxation coefficient $\alpha > 0$
- Thermal Contact Resistance $\mathscr{R} > 0$

Assumption: closed system $\Leftrightarrow \partial_n u_{\Omega|\partial\Omega} = 0$, isobaric and isochoric process

Phase Field

Double-Well Potential $W : \mathbb{R} \to \mathbb{R}_+ \backslash \{0\}$

(W1) $W(\pm 1) = W'(\pm 1) = 0$;

(W2) $W(s) > 0$, $s \in \mathbb{R} \backslash \{-1, 1\}$;

(W3) $W \in L^\infty(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$.

Ginzburg-Landau type free energy functional

$$\mathcal{J}^\varepsilon(u_\Omega, \varphi) := \int_{\Omega \backslash \overline{\omega}} \left(\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} W(\varphi) - \lambda \varepsilon u_\Omega \left(\int_{-1}^{\varphi} W(s) \, \mathrm{d}s \right) \right) \mathrm{d}x$$

+ Allen-Cahn assumption

$$(\alpha \partial_t \varphi, \psi)_{L^2(\Omega \backslash \overline{\omega})} = -\nabla_\varphi \mathcal{J}^\varepsilon(u_\Omega, \varphi) \cdot \psi.$$

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$$\rightsquigarrow \begin{cases} \alpha \partial_t \varphi - \varepsilon \Delta \varphi = \varepsilon^{-1} W'(\varphi) + \lambda \varepsilon W(\varphi) u_\Omega \\ \partial_n \varphi = 0. \end{cases}$$

Relative Heat in PCM

(Fourier Law) $\rho(\varphi)c_p(\varphi)\partial_t u_\Omega = \mathrm{div}(\kappa(\varphi)\nabla u_\Omega) + f(\varphi, \partial_t\varphi)$

$\rightsquigarrow$ (Conservative form) $\partial_t u_\Omega = \mathrm{div}(\beta(\varphi)\nabla u_\Omega) + \frac{f(\varphi,\partial_t\varphi)}{\rho(\varphi)c_p(\varphi)}$

(Caginalp Assumption) $f(\varphi, \partial_t\varphi) = C(\varphi)l(\varphi)\partial_t\varphi$, where $l(\varphi)$ is the latent heat of fusion

(Conservation of L^2-norm) $C(\varphi) := -\lambda\varepsilon l(\varphi)^{-1}W(\varphi)$.

$$\rightsquigarrow \begin{cases} \alpha\partial_t\varphi - \varepsilon\Delta\varphi = \varepsilon^{-1}W'(\varphi) + \lambda\varepsilon W(\varphi)u_\Omega \\ \partial_n\varphi = 0 \\ \partial_t u_\Omega - \mathrm{div}(\beta(\varphi)\nabla u_\Omega) = -\lambda\varepsilon W(\varphi)\partial_t\varphi \\ \partial_n u_\Omega ? u_\omega ? \end{cases}$$

Thermal Contact Resistance

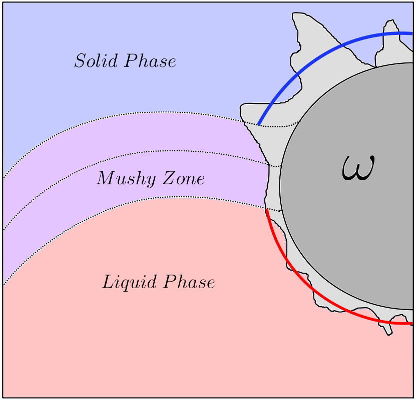

Figure: Thermal Contact Resistance and Phase Field

Thermal Contact Resistance

Assumptions:

$$\begin{cases} \beta(\varphi)\partial_n u_\Omega = \beta_\omega \partial_n u_\omega & \text{(Fourier Law)} \\ \beta_\omega \partial_n u_\omega = -\mathscr{R}^{-1}(u_\omega - u_\Omega) & \text{(Newton Cooling Law)} \end{cases}$$

Mathematical assumption: $\mathscr{R} > 0$ includes Thermal Contact Resistance

Physical relevance?

Thermal Contact Resistance

(Newton Cooling Law) $\beta(\varphi)\partial_n u_\Omega = \mathscr{R}(\varphi)^{-1}(u_\Omega - u_\omega)$

(Biot Number) $\mathscr{R}(\varphi)\kappa(\varphi) \approx B_{PCM}(\omega)^{-1}\frac{\mathcal{H}^d(\Omega\backslash\overline{\omega})}{\mathcal{H}^{d-1}(\partial\omega)}$

$$\rightsquigarrow \begin{cases} \beta(\varphi)\partial_n u_\Omega = \mathscr{R}(\varphi)^{-1}(u_\Omega - u_\omega) \\ \qquad = \kappa(\varphi)B_{PCM}(\omega)\left(\frac{\mathcal{H}^d(\Omega\backslash\overline{\omega})}{\mathcal{H}^{d-1}(\partial\omega)}\right)^{-1}(u_\Omega - u_\omega) \\ \beta_\omega\partial_n u_\omega = -\mathscr{R}_\omega^{-1}(u_\omega - u_\Omega) \\ \beta(\varphi)\partial_n u_\Omega = \beta_\omega\partial_n u_\omega \end{cases}$$

$\rightsquigarrow \kappa(\varphi) \approx \mathscr{R}_\omega^{-1} B_{PCM}(\omega)^{-1}\frac{\mathcal{H}^d(\Omega\backslash\overline{\omega})}{\mathcal{H}^{d-1}(\partial\omega)} \rightsquigarrow$ Dependence of Phase Field

The General Model

$$
\begin{cases}
\alpha\partial_t\varphi - \varepsilon\Delta\varphi = -\varepsilon^{-1}W'(\varphi) + \lambda\varepsilon u_\Omega W(\varphi) & \text{in } (0,T]\times(\Omega\backslash\overline{\omega}) \\
\partial_n\varphi = 0 & \text{on } [0,T]\times\partial(\Omega\backslash\overline{\omega}) \\
\partial_t u_\Omega - \mathrm{div}(\beta(\varphi)\nabla u_\Omega) = -\lambda\varepsilon W(\varphi)\partial_t\varphi & \text{in } (0,T]\times\Omega\backslash\overline{\omega} \\
\partial_t u_\omega - \beta_\omega\Delta u_\omega = 0 & \text{in } (0,T]\times\omega \\
-\beta_\omega\partial_n u_\omega = \frac{1}{\mathscr{R}}(u_\Omega - u_\omega) & \text{on } [0,T]\times\partial\omega \\
\beta(\varphi)\partial_n u_\Omega = \beta_\omega\partial_n u_\omega & \text{on } [0,T]\times\partial\omega \\
\beta(\varphi)\partial_n u_\Omega = \mathbb{1}_{\Gamma_h} q \text{ and } \partial_n\varphi = 0 & \text{on } [0,T]\times\partial\Omega \\
u_{\Omega|t=0} = u_0 \text{ and } \varphi_{|t=0} = \varphi_0 & \text{in } \Omega \\
u_{\omega|t=0} = v_0 & \text{in } \omega,
\end{cases}
$$

$\rightsquigarrow$ Quasilinear System

Norm Conservation

Assume $q = 0$:

$$\alpha\|\partial_t\varphi(t)\|^2_{L^2(\Omega\setminus\overline{\omega})} + \varepsilon\|\nabla\varphi(t)\|^2_{L^2(\Omega\setminus\overline{\omega})} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\Omega(t)\|^2_{L^2(\Omega\setminus\overline{\omega})}$$
$$+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\omega(t)\|^2_{L^2(\omega)} + \int_{\Omega\setminus\overline{\omega}} \beta(\varphi)|\nabla u_\Omega(t)|^2 \, \mathrm{d}x + \beta_\omega\|\nabla u_\omega(t)\|^2_{L^2(\omega)}$$
$$+ \frac{1}{\mathscr{R}}\|u_\Omega(t) - u_\omega(t)\|^2_{L^2(\partial\omega)} = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega\setminus\overline{\omega}} W(\varphi)\, \mathrm{d}t\right)$$

$$\rightsquigarrow \begin{cases} u_\omega \in L^\infty([0,T], L^2(\omega)) \cap L^2([0,T], H^1(\omega)) \\ u_\Omega \in L^\infty([0,T], L^2(\Omega\setminus\overline{\omega})) \cap L^2([0,T], H^1(\Omega\setminus\overline{\omega})) \\ \varphi \in L^\infty([0,T], \dot{H}^1(\Omega\setminus\overline{\omega})) \cap \dot{H}^1([0,T], L^2(\Omega\setminus\overline{\omega})) \end{cases}$$

+ estimates for $\varphi \in L^\infty([0,T], L^2(\Omega\setminus\overline{\omega}))$.

A Simplified Case: Enhanced Caginalp

Same physical properties = Caginalp model. Adding enhancers:

$$\begin{cases} \alpha\partial_t\varphi - \varepsilon\Delta\varphi = -\varepsilon^{-1}W'(\varphi) + \frac{l}{2}u_\Omega & \text{in } (0,T]\times(\Omega\backslash\overline{\omega}) \\ \partial_n\varphi = 0 & \text{on } [0,T]\times\partial(\Omega\backslash\overline{\omega}) \\ \partial_t u_\Omega - \beta_\Omega\Delta u_\Omega = -\frac{l}{2}\partial_t\varphi & \text{in } (0,T]\times\Omega\backslash\overline{\omega} \\ \partial_t u_\omega - \beta_\omega\Delta u_\omega = 0 & \text{in } (0,T]\times\omega \\ -\mathscr{R}\beta_\omega\partial_n u_\omega = (u_\Omega - u_\omega) & \text{on } [0,T]\times\partial\omega \\ \beta_\Omega\partial_n u_\Omega = \beta_\omega\partial_n u_\omega & \text{on } [0,T]\times\partial\omega \\ \beta_\Omega\partial_n u_\Omega = \mathbb{1}_{\Gamma_h}q \text{ and } \partial_n\varphi = 0 & \text{on } [0,T]\times\partial\Omega \\ u_{\Omega|t=0} = u_0 \text{ and } \varphi_{|t=0} = \varphi_0 & \text{in } \Omega \\ u_{\omega|t=0} = v_0 & \text{in } \omega, \end{cases}$$

$\rightsquigarrow$ semilinear system

A Simplified Case: Enhanced Caginalp

Results

Semigroup method + fixed point

Theorem (Well-Posedness)

Assume $(u_0, v_0, \varphi_0) \in L^2(\Omega \backslash \overline{\omega}) \times L^2(\omega) \times H^1(\Omega \backslash \overline{\omega})$, $q \in H^{\frac{1}{4}}([0,T], L^2(\Gamma_h)) \cap L^2([0,T], H^{\frac{1}{2}}(\Gamma_h))$. *Then, there exists a unique strong solution* $(\varphi, u_\omega, u_\Omega)$.

A Simplified Case: Enhanced Caginalp

Results

Maximal Regularity + Bootstrap

Theorem (Hölder Regularity)

Assume (u_0, v_0, φ_0) *regular enough. Then, the solution* $(\varphi, u_\omega, u_\Omega)$ *satisfies*

$$(\varphi, u_\omega, u_\Omega) \in C^{0,\alpha}\Big([0,T], C^{0,\frac{1}{2}+\alpha}(\Omega\backslash\overline{\omega}) \times C^{0,\frac{1}{2}+\alpha}(\omega) \times C^{0,\frac{1}{2}+\alpha}(\Omega\backslash\overline{\omega})\Big),$$

where $\alpha := \frac{1}{2} - \frac{d}{r}$ *for some* $r \in (2d, +\infty)$.

Summary of the presentation

Setting of the problem

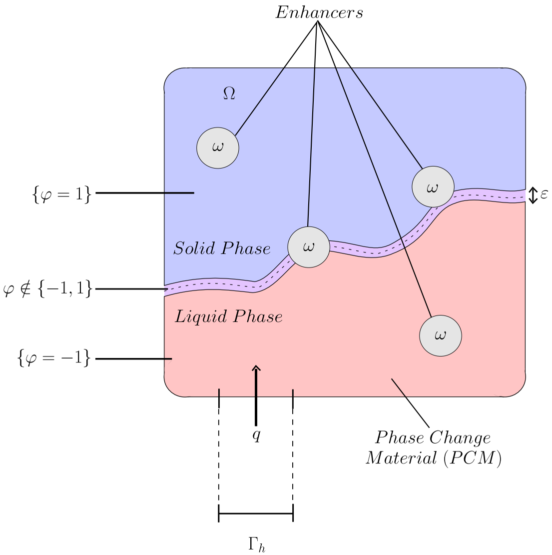

Figure: Find the optimal heat flux q?

Setting of the Problem

Minimize functional for $q \in L^2(\Gamma_h)$

$$\mathcal{J}^\gamma(q) := \frac{1}{2}\|\varphi(T) - 1\|^2_{L^2(\Omega\backslash\varpi)} + \frac{1}{2}\|\varphi - 1\|^2_{L^2([0,T]\times(\Omega\backslash\varpi))} + \gamma\|q\|^2_{L^2([0,T]\times\Gamma_h)}.$$

→ Approach in optimal way melting at final time

→ Optimize the cost of L^2 norm melting process

→ penalization term for coercivity

Issue: solution for $q \in L^2$?

Variational Solution and Well-Posedness

Result

Proposition (Weak Well-Posedness)

Let $(u_0, v_0, \varphi_0) \in L^2(\Omega \backslash \overline{\omega}) \times L^2(\omega) \times H^1(\Omega \backslash \overline{\omega})$ and $q \in L^2([0, T] \times \Gamma_h)$. Then, the system has a unique (variational) weak solution.

Sketch: Approximation by $(\varphi^m, u_\omega^m, u_\Omega^m)$ for q^m in case of strong solution $\rightsquigarrow$ satisfy variational formulation $\rightsquigarrow$ uniform bounds $\rightsquigarrow$ weak limit + uniqueness by a priori estimates and Grönwall.

Existence of an Optimal Control

Result

Proposition (Existence of a Control)

The functional $\mathcal{J}^\gamma$ has a minimum over $L^2(\Gamma_h)$.

Sketch: Continuity of functional by estimates + Grönwall + coercivity, then use minimizing sequence and weak lower semicontinuity.

Summary of the presentation

Objectives

- Uniqueness Control?
- Differentiability functional and characterization minimizer?
- Simulations solution with optimal control
- Quasilinear setting.

Thank you for your attention!