Bernstein inequalities via the heat semigroup

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- Approximation theory (see e.g. the book of G.G. Lorentz),
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More on the Bernstein inequality can be found in the survey paper:
- H. Queffelec and R. Zarrouf, Arxiv March 2019.

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$$\sup_{B(x,r)} |\nabla u| \leq \frac{C\lambda^{1+m/2}}{r} \sup_{B(x,r)} |u| \qquad (r < r_0,\ m = dim\, M).$$

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$$\|\nabla u\|_\infty \leq C\lambda \|u\|_\infty.$$

- F. Filbir and H.N. Mhaskar, 2010 for a more general setting. However, they had to assume rather restrictive regularity assumptions on the semigroup $e^{t\Delta}$ (or the corresponding heat kernel).

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- M Riemannian manifold, d and dx: Riemannian distance and measure,
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We make the following assumptions:

- doubling condition:

$$v(x,2r) \leq Cv(x,r)$$

- Gaussian upper bound:

$$|g_t(x,y)| \leq \frac{C}{v(x,\sqrt{t})} e^{-c\frac{d^2(x,y)}{t}}.$$

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Note that the heat kernel $p_t(x,y)$ satisfies the above Gaussian upper bound.

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$$L\varphi_k = \lambda_k^2 \varphi_k.$$

$\varphi_k \in L^2(M)$, $\|\varphi_k\|_2 = 1$. *Then for every* $p \in [1, 2]$, *there exists a constant C such that for every N and* $\alpha_k \in \mathbb{C}$

$$\|\nabla\Big(\sum_{k=0}^{N} \alpha_k \varphi_k\Big)\|_p + \|V^{1/2}\Big(\sum_{k=0}^{N} \alpha_k \varphi_k\Big)\|_p \leq C\lambda_N \|\Big(\sum_{k=0}^{N} \alpha_k \varphi_k\Big)\|_p \qquad (B_p)$$

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$\varphi_k \in L^2(M)$, $\|\varphi_k\|_2 = 1$. *Suppose in addition that*

$$\|\nabla e^{-tL}\|_{p\to p} + \|V^{1/2} e^{-tL}\|_{p\to p} \le \frac{C}{\sqrt{t}}. \tag{R_p}$$

Then for every $p \in (2, \infty]$, there exists a constant C such that for every N and $\alpha_k \in \mathbb{C}$

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$$L = -div\,(A(x)\nabla\cdot)$$

an elliptic operator on $L^2(\Omega)$, subject to Dirichlet boundary conditions, $A = (a_{kl}), a_{kl} \in L^\infty(\Omega, \mathbb{R})$, Ω is a bounded domain of $\mathbb{R}^n$.

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For $p \in [2, \infty]$, (B_p) holds as well under the assumption that Ω is $C^{1+\epsilon}$ and the coefficients are C^ϵ for some $\epsilon > 0$. In this case, the regularity property

$$\|\nabla e^{-tL}\|_{\infty\to\infty} \leq \frac{C}{\sqrt{t}}$$

holds since the gradient of the heat kernel has a Gaussian upper bound (with $\frac{1}{\sqrt{t}}$). This holds even for complex coefficients, see

A.F.M. ter Elst and E.M. O.: Dirichlet-to-Neumann and elliptic operators on $C^{1+\kappa}$ domains: Poisson and Gaussian bounds (J. Diff Eqs 2019).

Main ideas of proof

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In order to prove the previous Bernstein type inequality (B_p) we introduce the *semi-classical Bernstein* inequality (for a given $\phi \in C_c^\infty([0,\infty))$):

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For the direct implication, suppose $\sigma(L) = (\lambda_k^2)_k$ and set

$$\phi(\lambda) = \begin{cases} 1 & \text{for } \lambda \in [0,1] \\ 0 & \text{for } \lambda \text{ large} \end{cases}$$

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Then, for $h = \frac{1}{\lambda_N^2}$

$$\phi(hL)\Big(\sum_{k=0}^{N} \alpha_k \varphi_k\Big) = \sum_{k=0}^{N} \alpha_k \phi\Big(\frac{\lambda_k^2}{\lambda_N^2}\Big)\varphi_k = \sum_{k=0}^{N} \alpha_k \varphi_k.$$

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$$\phi(hL) = \int_{\mathbb{R}} e^{-(2-i\xi)hL}\hat{\phi}_e(\xi)d\xi = \int_{\mathbb{R}} e^{-hL}e^{-(1-i\xi)hL}\hat{\phi}_e(\xi)d\xi \qquad (E)$$

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I) $\nabla e^{-hL}, V^{1/2}e^{-hL} : L^1(M) \to L^1(M)$ with norm $\leq \frac{C}{\sqrt{h}}$.

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$$\int_M |\nabla_x p_h(x,y)|^2 e^{\epsilon\frac{d^2(x,y)}{h}}dx \leq \frac{C}{h}v(y,\sqrt{h})^{-1}.$$

This is due to A. Grigor'yan, 1995 when $L = -\Delta$ (the proof is the same for $L = -\Delta + V$).

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Similarly,

$$\int_M |\sqrt{V(x)} p_h(x,y)|^2 e^{\epsilon \frac{d^2(x,y)}{h}} dx \leq \frac{C}{h} v(y,\sqrt{h})^{-1}.$$

Thus (for $\Gamma = \nabla_x$ or $\sqrt{V(x)}$),

$$\begin{aligned}\int_M |\Gamma p_h(x,y)| dx &\leq \left(\int_M |\Gamma p_h(x,y)|^2 e^{\epsilon \frac{d^2(x,y)}{h}} dx\right)^{1/2} \cdot \left(\int_M e^{-\epsilon \frac{d^2(x,y)}{h}} dx\right)^{1/2} \\ &= \frac{C}{\sqrt{h}} v(y,\sqrt{h})^{-1/2} v(y,\sqrt{h})^{1/2} = \frac{C}{\sqrt{h}}.\end{aligned}$$

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$$|p_z(x,y)| \leq C \frac{(|z|/\Re(z))^{-n}}{\sqrt{v\left(x, \frac{|z|}{\sqrt{\Re(z)}}\right) v\left(y, \frac{|z|}{\sqrt{\Re(z)}}\right)}} \exp\left(-c\frac{\Re(z) d(x,y)^2}{|z|^2}\right).$$

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We insert I) and II) into (E) and we obtain

$$\|\nabla \phi(hL)\|_{1\to 1} + \|V^{1/2}\phi(hL)\|_{1\to 1} \leq \frac{C}{\sqrt{h}} \|\phi\|_{H^{n/2+1}}$$

which is (scB_1). This proves the first theorem (after interpolation for $p \in (1,2]$).

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Theorem

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i) *there exists a non-trivial function $\psi_0 \in C_c^\infty([0,\infty))$ for which the semi-classical Bernstein inequality (scB_p) holds.*

ii) *for every $\psi \in C_c^\infty([0,\infty))$, the semi-classical Bernstein inequality (scB_p) holds, i.e.,*

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$ii) \Longrightarrow iii)$ is based on the following facts:

- Set $\psi_0(\lambda) := (1-\lambda)_+^a$ and take $\phi \in C_c^\infty([0,\infty))$ such that $\psi_0 = \phi\psi_0$.

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Theorem (Hebisch if $M = \mathbb{R}^d$, X.T. Duong-EM. O.- A. Sikora, 2002 for a more general version)

Let $F : [0,\infty) \to \mathbb{C}$ bounded such that

$$\sup_{t>0} \|F(t.)\eta(.)\|_{W^{s,\infty}} < \infty$$

for some non-trivial $\eta \in C_c^\infty(0,\infty)$ and some $s > n/2$. Then:
- $\sup_{h>0} \|F(hL)\|_{p\to p} < \infty$ *for all $p \in [1,\infty]$ for F compactly supported.*
- *$F(L)$ is weak type $(1,1)$ and bounded on L^p for all $p \in (1,\infty)$.*

- Finally, use

$$e^{-x} = \Gamma(a)^{-1}\int_0^\infty e^{-s}(s-x)_+^a\,ds$$

for some $a > n/2$.

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$$\frac{1}{|B|}\int_B \left| f - \frac{1}{|B|}\int_B f \right|^2 dx \le Cr^2 \frac{1}{|B|}\int_B |\nabla f|^2 dx$$

for all ball B with radius r.

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for all ball B with radius r. Then by a theorem of P. Auscher-Th. Coulhon-X.T. Duong-S. Hofmann, 2004 (see also F. Bernicot-D. Frey, 2016):

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The previous theorem shows that the semi-classical Bernstein inequality (scB_p) for $p > 2$ is related to the boundedness of the Riesz transform. In particular, counter-examples for the boundedness of the Riesz transform on $L^p(M)$ for $p > 2$ are counter-examples to (scB_p).

A ”reverse” Bernstein inequality

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Y. Shi-B. Xu, 2010 proved on a compact manifold without boundary: if $\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$ with $\lambda \geq 1$ then

$$C_1\lambda \|\varphi_\lambda\|_\infty \leq \|\nabla\varphi_\lambda\|_\infty \leq C_2\lambda \|\varphi_\lambda\|_\infty$$

A "reverse" Bernstein inequality

Y. Shi-B. Xu, 2010 proved on a compact manifold without boundary: if $\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$ with $\lambda \geq 1$ then

$$C_1\lambda \, \|\varphi_\lambda\|_\infty \leq \|\nabla\varphi_\lambda\|_\infty \leq C_2\lambda \, \|\varphi_\lambda\|_\infty$$

Under the sole assumptions of doubling and Gaussian upper bound for the heat kernel of Δ we have

Theorem

Let $L = -\Delta + V$ and suppose $\sigma(L) = (\lambda_k^2)_k$, $L\varphi_k = \lambda_k^2\varphi_k$ with $\|\varphi_k\|_2 = 1$. Then for every $p \in [2, \infty]$, N and $\alpha_k \in \mathbb{C}$:

$$\|\nabla\Big(\sum_{k=N}^{\infty}\alpha_k\varphi_k\Big)\|_p + \|V^{1/2}\Big(\sum_{k=N}^{\infty}\alpha_k\varphi_k\Big)\|_p \geq C_p\lambda_N \, \|\Big(\sum_{k=N}^{\infty}\alpha_k\varphi_k\Big)\|_p \qquad (RB_p)$$

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- For non compactly supported ϕ we need a weak factorization for $f \in L^1 \cap C^\infty(\mathbb{R})$ function (based on J. Dixmier-P. Malliavin, 1978):

$$f = \psi_1 * g_1 + \psi_2 * g_2,\ \psi_i \in C^\infty_c(\mathbb{R}),\ g_i \in L^1$$

Proposition

Let $\phi \in C_c^\infty(0,\infty)$. Then $\phi(L) : L^1 \to H_L^1$ is bounded and

$$\sup_{h>0} \|\phi(hL)\|_{L^1 \to H_L^1} < \infty.$$

The Hardy space H_L^1 is defined by the square function S_L as follows. Set

$$S_L f(x) := \left(\int_0^\infty \int_{\rho(x,y)<t} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{v(x,t)} \frac{dt}{t} \right)^{\frac{1}{2}}, \tag{1}$$

and $D := \left\{ f \in \overline{R(L)} : \ S_L f \in L^1(X) \right\}$. Then H_L^1 is the completion of the space D with respect to the norm

$$\|f\|_{H_L^1} := \|S_L f\|_{L^1(M)}.$$

Molecular decomposition for $\phi(L)g$: there exist m and ϵ, a ball $B = B(x_B, r_B)$ and a function b such that $\phi(L)g = L^m b$ and

$$\|({r_B}^2 L)^k b\|_{L^2(U_j(B))} \leq {r_B}^{2m} 2^{-j\epsilon} Vol(2^j B)^{-1/2}$$

for $k = 0, 1, ..., m$ and $j = 0, 1, 2, ...$ Here $U_j(B) := 2^{j+1}B \setminus 2^j B$, $2^j B := B(x_B, 2^j r_B)$, $Vol(2^j B) := Vol(x_B, 2^j r_B)$ for $j \geq 1$ and $U_0(B) = B$.

Thank you for your attention

Place à l'HDR de Michel !