Brownian motion and Ricci curvature in sub-Riemannian geometry

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Outline

1. Sub-Riemannian structures
2. Ricci curvature bounds and gradient estimates
3. Ricci curvature and analysis on path space
4. Analysis on path space over sub-Riemannian manifolds
5. Ricci curvature bounds in sub-Riemannian geometry

I. Sub-Riemannian structures

- (M, H, g_H) where
  - M smooth manifold, $\dim M = n$
  - $H \subsetneq TM$ subbundle ("horizontal directions"), $\operatorname{rank} H = m$
  - g_H fiberwise inner product on H

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- Let

$$d_H(x,y) = \inf_{\gamma} \left\{ \int_0^1 |\dot{\gamma}(t)|\, dt : \ \gamma(0) = x,\ \gamma(1) = y,\ \dot{\gamma}(t) \in H_{\gamma(t)}\ \forall\, t \right\}$$

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- H bracket generating (i.e. $\mathrm{Lie}(H)(x) = T_x M$ for each $x \in M$)
  $\Longrightarrow$ (M, d_H) metric space

- Canonical sub-Riemannian Laplacian?

$$\Delta^H = \sum_{i=1}^{m} A_i^2 + Z \quad \text{(locally)}$$

$A_1, \ldots, A_m$ local orthonormal frame of H,
Z first order term (horizontal vector field)

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- Some notation:
  1. Consider

$$\sharp^H \colon T^*M \to H \subset TM, \quad \langle \sharp^H \alpha, v \rangle_{g_H} := \alpha(v),$$

for $\alpha \in T_x^* M,\ v \in H_x,\ x \in M$.
Note that $\ker \sharp^H = \mathrm{Ann}\, H$.

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Note that $\ker \sharp^H = \mathrm{Ann}\, H$.

  2. The map $\sharp^H$ induces a (degenerate) co-metric g_H^* on T^*M via

$$\langle \alpha, \beta \rangle_{g_H^*} = \langle \sharp^H \alpha, \sharp^H \beta \rangle_{g_H}.$$

- Let L be a second order partial differential operator on M. Its symbol $\sigma(L)$ is the symmetric, bilinear 2-tensor on T^*M determined by the relation

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- A second order PDO L (without constant term) is called sub-Laplacian with respect to (M, H, g_H) if

$$\sigma(L) = g_H^*.$$

We write $L = \Delta^H$.

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Define

$$\nabla^H f = \mathrm{pr}_H \nabla f \equiv \sharp^H df$$

and let Δ^H be the generator of the Dirichlet form

$$\mathcal{E}(f,h) := -\int_M \langle \nabla^H f, \nabla^H h \rangle_H \, d\mathrm{vol}_g.$$

Then $\Delta^H := -(\nabla^H)^* \nabla^H = \mathrm{trace}_H \nabla^2$ is a sub-Laplacian.

In the situation of the last example:

- Canonical variation of the metric

$$\varepsilon > 0: \quad g_\varepsilon := g_H \oplus \frac{1}{\varepsilon} g_V$$

$\varepsilon \downarrow 0$: sub-Riemannian limit

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In the limit only horizontal curves have finite length.

- Observation

$$\mathrm{Ric}^{g_\varepsilon}(u,u) \xrightarrow{\varepsilon \downarrow 0} -\infty \quad \text{for any horizontal unit vector } u$$

Natural connections on a sub-Riemannian manifold (M, H, g_H)

- Would like to have a connection ∇ on M which is horizontally compatible with (H, g_H) in the sense that the horizontal subbundle H is preserved under parallel transport, as well as its metric g_H

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- In terms of the corresponding *horizontal Hessian*,

$$\nabla^2 f \equiv \mathrm{Hess}\, f \in \Gamma(H^* \otimes H^*), \quad (\nabla^2 f)(A, B) = ABf - (\nabla_A B)f,$$

the associated sub-Laplacian Δ^H is given by

$$\Delta^H f = \mathrm{trace}_H \nabla^2 f, \quad f \in C^\infty(M)$$

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- Note that horizontally compatible connections ∇ will always have torsion $\mathbf{T}$:

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- A horizontally compatible connection ∇ is uniquely determined by its torsion $\mathbf{T}$.
- Let V be a choice of complement to H. There exists a unique horizontally compatible partial connection ∇ with

$$\mathbf{T}(H,H) \subseteq V$$

Example Let again (M,g) and $g_H = g|H$. Then $TM = H \oplus_\perp V$ and

$$g = g_H \oplus g_V$$

Denote by ∇^g the Levi-Civita connection on M, g.

- (Bott connection) There is a canonical connection ∇ preserving the decomposition $TM = H \oplus V$:

$$\nabla_X Y = \begin{cases} \mathrm{pr}_H(\nabla^g_X Y), & X, Y \in \Gamma(H), \\ \mathrm{pr}_H([X,Y]), & X \in \Gamma(V),\ Y \in \Gamma(H), \\ \mathrm{pr}_V([X,Y]), & X \in \Gamma(H),\ Y \in \Gamma(V), \\ \mathrm{pr}_V(\nabla^g_X Y), & X, Y \in \Gamma(V), \end{cases}$$

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- $\nabla g = 0$
- its torsion $T^\nabla(X, Y)$ is vertical for X and Y horizontal, and zero if either X or Y is vertical

Assumptions

- Let V be a choice of a complement to H in (M, H, g_H).
  Let pr_H and pr_V be the corresponding projections. Write ∇ for the unique partial connection with $\mathbf{T}(H, H) \subseteq V$.

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- We shall extend
$$\nabla_X Y, \quad X, Y \in \Gamma(H),$$
to an affine connection on M as follows:
$$\nabla_X Y = \begin{cases} \mathrm{pr}_H[X, Y] & \text{if } X \in \Gamma(V), Y \in \Gamma(H) \\ \mathrm{pr}_V[X, Y] & \text{if } X \in \Gamma(H), Y \in \Gamma(V) \end{cases}$$
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while ∇ on V can be an arbitrary partial connection on V in the direction of V.
- Connections of this form satisfy the following properties:
  1. both H and V are parallel with respect to ∇
  2. $\mathbf{T}(H, H) \subseteq V$
  3. $\mathbf{T}(H, V) = 0$.

  Conversely, any connection satisfying (i)-(iii) is of this form.

- (Metric preserving complement V) For simplicity, assume that

$$(L_Z \,\mathrm{pr}_H^* g_H)(X, X) = 0 \quad \text{for all } Z \in \Gamma(V) \text{ and } X \in \Gamma(H)$$

where L_Z denotes the Lie derivative with respect to Z.

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- Let $\mathrm{Ric}\colon TM \to TM$ be the Ricci tensor with respect to ∇:

$$\mathrm{Ric}(v) = \mathrm{trace}_H R^\nabla(v, \times)\times$$

  The object of our interest is

$$\mathrm{Ric}^H \in \Gamma(H^* \otimes H), \quad \mathrm{Ric}^H := \mathrm{Ric}|H \quad \text{(horizontal Ricci)}$$

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- We have

$$\operatorname{Ric}(v) = \operatorname{pr}_H \operatorname{Ric}^H \operatorname{pr}_H v, \quad v \in TM,$$

  where $\operatorname{pr}_H\colon TM \to H$ is the projection with kernel V.

- **Example**

  Let (M,g) be a Riemannian manifold and $g_H = g|H$ such that $TM = H \oplus V$, and

$$g = g_H \oplus g_V \quad \text{and} \quad g_\varepsilon = g_H \oplus \frac{1}{\varepsilon} g_V, \quad \varepsilon > 0.$$

  Then

$$\mathrm{Ric}_{g_\varepsilon}(X,X) = \mathrm{Ric}^H(X,X) + \frac{1}{2\varepsilon}\langle J^2 X, X\rangle_H, \quad X \in \Gamma(H),$$

  where for $Z \in \Gamma(V)$, $J_Z \in \Gamma(\mathrm{End}\, TM)$ is defined by

$$\langle J_Z X, Y\rangle_{g_H} = \langle Z, T^\nabla(X,Y)\rangle_{g_V},$$

  and, for $Z_1, \ldots, Z_r$ any local vertical frame,

$$J^2 := \sum_{i=1}^r J_{Z_i} J_{Z_i}.$$

- *(Laplacian)* For a compatible connection ∇ as above let

$$\Delta^H = \mathrm{trace}_H \nabla^2_{\times,\times}$$

be the subelliptic Laplacian (the trace of the Hessian ∇^2 is taken over H with respect to the inner product g_H)

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- *(Sub-Riemannian Brownian)* A sub-Riemannian Brownian motion is a diffusion process X_t with generator Δ^H
- *(Stochastic development)* Let $X_0 = x$ then

$$dX_t = //_{0,t} \circ dB_t \quad \text{or} \quad dB_t = //_{0,t}^{-1} \circ dX_t$$

where B_t is a (classical) Brownian motion in H_x and

$$//_{0,t} := U_t \circ U_0^{-1} : H_x M \to H_{X_t} M$$

is *stochastic parallel transport along* of horizontal vectors along X (by construction isometries with respect to g_H).
Here U_t is the horizontal lift of X_t to the orthonormal frame bundle $\mathrm{O}(H)$ over M.

Functional inequalities

- Consider the semigroup generated by Δ^H:

$$P_t f = e^{t\Delta^H} f$$

We have

$$P_t f(x) = \mathbb{E}[f(X_t^x)\, \mathbb{1}_{\{t<\zeta(x)\}}], \quad x \in M.$$

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- Question: How is Ric^H related to functional inequalities for P_t?

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- (Bakry-Émery Ricci tensor)

$$\mathrm{Ric}^Z = \mathrm{Ric} - \nabla Z$$

where $\quad \mathrm{Ric}^Z(X, Y) := \mathrm{Ric}(X, Y) - \langle \nabla_X Z, Y \rangle$

Theorem (classical probabilistic representations)

Let $f \in \mathscr{B}_b(M)$ *and* $u(x,t) = P_t f(x)$ *be the (minimal) solution to*

$$\frac{\partial}{\partial t} u = Lu, \; u|_{t=0} = f.$$

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- *(Semigroup formula) Then* $P_t f(x) = \mathbb{E}[f(X_t^x)\,1\!\!1_{\{t<\zeta(x)\}}]$.
- *(Derivative formula) If $f \in C_b^1(M)$ and Ric^Z bounded below,*

$$(\nabla P_t f)(x) = \mathbb{E}\Big[Q_t //_t^{-1} \nabla f(X_t^x)\Big]$$

where the random transformations $Q_t \in \mathrm{Hom}(T_x M, T_x M)$ are defined as solution to the pathwise ODE

$$dQ_t = -Q_t \,\mathrm{Ric}^Z_{//_t}\, dt, \quad Q_0 = \mathrm{id}_{T_x M}.$$

Here

$$\mathrm{Ric}^Z_{//_t} := //_t^{-1} \circ \mathrm{Ric}^Z_{X_t} \circ //_t \in End(T_x M)$$

is the equivariant representation of Ric^Z.

- In particular, if

$$\mathrm{CD}(K,\infty) \qquad \mathrm{Ric}^Z(v,v) \geq K|v|^2, \quad v \in TM,$$

for some constant K, then

$$|Q_t| \leq e^{-Kt}$$

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$$\text{(gradient estimate)} \qquad |\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|, \quad f \in C_b^1(M).$$

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- Actually, for $K \in \mathbb{R}$ the following two conditions are equivalent:

  - $\mathrm{CD}(K,\infty) \qquad \mathrm{Ric}(v,v) \geq K|v|^2, \quad v \in TM.$

  - $\text{(gradient estimate)} \qquad |\nabla P_t f| \leq e^{-Kt} P_t |\nabla f|, \quad f \in C_b^1(M).$

Well-known and classical: Let K be a real constant.
The following conditions are equivalent:

- **(Bakry-Émery lower curvature bound)**

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- **(Poincaré inequality)** for $p \in (1,2]$ and all $f \in C_c^\infty(M)$,

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- **(log-Sobolev inequality)** for all $f \in C_c^\infty(M)$,

$$P_t(f^2 \log f^2) - (P_t f^2)\log(P_t f^2) \leq \frac{2(1-e^{-2Kt})}{K} P_t |\nabla f|^2.$$

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Many other equivalent statements, e.g., transportation-cost inequalities; convexity properties of the entropy; Wang's dimension-free Harnack inequalities; Wang's log-Harnack inequalities, ...

Comparison with the sub-Riemannian case

- Example (Heisenberg group $\mathbb{H}$)

  $X, Y, Z \in \Gamma(\mathbb{H})$, $\quad [X,Y] = Z$, $[X,Z] = [Y,Z] = 0$

  $\mathbb{H} = \mathrm{span}(X,Y)$, $\quad V = \mathbb{R} \cdot Z$

  Let

  $$\Delta^H := X^2 + Y^2 \quad \text{and} \quad P_t f = (e^{t\Delta^H}) f$$

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  $$\Delta^H := X^2 + Y^2 \quad \text{and} \quad P_t f = (e^{t\Delta^H})f$$

  Theorem (Hong-Quan Li, 2006)

  $$\exists\, C > 0, \quad |\nabla^H P_t f|_{g_H} \le C\, P_t |\nabla^H f|_{g_H}, \quad \forall f \in C_c^\infty(\mathbb{H}),$$

  where $\nabla^H f = \mathrm{pr}_H \nabla f$.

  The constant C must be strictly larger than 1!

Riemannian geometry

Boundedness of Ric

The problem of characterizing boundedness of Ric in Riemannian geometry has been solved by A. Naber via analysis on path space:

$|\mathrm{Ric}| \le K$ (i.e. $-K \le \mathrm{Ric} \le K$ for some constant $K \ge 0$)
$\iff$ certain functional inequalities on path space

III. Ricci curvature and analysis on path space

- For fixed $T > 0$, let $W^T = C([0,T]; M)$ and

$$\mathscr{F}C_{0,T}^{\infty} = \Big\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : \\ 0 < t_1 < \ldots < t_n \leq T,\ f \in C_c^{\infty}(M^n) \Big\}.$$

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- For $F \in \mathscr{F}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, the intrinsic gradient is defined as

$$D_t^{//} F(X_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t<t_i\}} //_{t,t_i}^{-1} \nabla^i f(X_{t_1}, \ldots, X_{t_n}), \quad t \in [0,T],$$

  where ∇^i denotes the gradient with respect to the i-th component.

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$$\left|\nabla_x \mathbb{E}[F(X^x_{[0,T]})]\right| \leq \mathbb{E}^x\left[|D_0^{//}F| + K\int_0^T e^{Kr}|D_r^{//}F|\,dr\right].$$

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- (L^2 gradient inequality on path space) for $F \in \mathcal{F}C_0^\infty$,

$$\left|\nabla_x \mathbb{E}[F(X^x_{[0,T]})]\right|^2 \leq e^{KT}\mathbb{E}^x\left[|D_0^{//}F|^2 + K\int_0^T e^{K(r-T)}|D_r^{//}F|^2\,dr\right].$$

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Important observation It is sufficient to check the estimates for very special $F \in \mathcal{F}C_0^\infty$. Namely:

(i) for $F(X^x_{[0,T]}) = f(X^x_t)$, and

(ii) for 2-point cylindrical functions of the form

$$F(X^x_{[0,T]}) = f(x) - \frac{1}{2}f(X^x_t)$$

From this observation, equivalence of the following two items follows:

(i) $|\mathrm{Ric}^Z| \le K$ for $K \ge 0$;

(ii) for $f \in C_c^\infty(M)$ and $t > 0$,

$$|\nabla P_t f|^2 \le e^{2Kt} P_t |\nabla f|^2 \quad \text{and}$$

$$\left|\nabla f - \frac{1}{2}\nabla P_t f\right|^2 \le e^{Kt}\,\mathbb{E}\left[\left|\nabla f - \frac{1}{2}//_{0,t}^{-1}\nabla f(X_t)\right|^2 + \frac{1}{4}\left(e^{Kt}-1\right)|\nabla f|^2(X_t)\right].$$

Path space characterization of pinched curvature

Let $F \in \mathscr{F}C^{\infty}_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$. Consider the gradients:

- *(intrinsic gradient)*

$$D_t^{//} F(X^x_{[0,T]}) = \sum_{i=1}^{n} \mathbb{1}_{\{t<t_i\}} //_{t,t_i}^{-1} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n});$$

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$$D_t F(X^x_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t<t_i\}} Q_{t,t_i} //_{t,t_i}^{-1} \, \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n})$$

  where $Q_{t,r}$ takes values in the linear automorphisms of $T_{X^x_t} M$ satisfying for fixed $t \geq 0$:

$$\frac{dQ_{t,r}}{dr} = -Q_{t,r} \, \mathrm{Ric}^Z_{//_{t,r}}, \quad Q_{t,t} = \mathrm{id}; \quad r \geq t$$

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- *(balanced gradient)* For constants $k_1 \leq k_2$ let

$$\bar{D}_t^{//} F(X^x_{[0,T]}) = \sum_{i=1}^n \mathbb{1}_{\{t\leq t_i\}} e^{-\frac{k_1+k_2}{2}(t_i - t)} //_{t,t_i}^{-1} \nabla_i f(X^x_{t_1}, \ldots, X^x_{t_n}).$$

Theorem (Path space characterization of pinched curvature)

The following conditions are equivalent:

(i) $k_1 \le \mathrm{Ric}^Z \le k_2$;

(ii) (Gradient estimate) *for any* $F \in \mathscr{F}C_{0,T}^{\infty}$,

$$\left|\nabla_x \mathbb{E} F(X_{[0,T]}^x)\right| \le \mathbb{E}|\bar{D}_0^{//} F| + \frac{k_2-k_1}{2} \int_0^T e^{-k_1 s}\, \mathbb{E}|\bar{D}_s^{//} F|\, ds;$$

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(iii) (Log-Sobolev inequality) *for any* $F \in \mathscr{F}C_{0,T}^{\infty}$ *and* $t_1 < t_2$ *in* $[0,T]$,

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\log \mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_2}]\Big]\\ &\qquad -\mathbb{E}\Big[\mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\log \mathbb{E}[F^2(X_{[0,T]})|\mathscr{F}_{t_1}]\Big]\\ &\le 2\int_{t_1}^{t_2}\left(1+\frac{k_2-k_1}{2}\int_t^T e^{-k_1(s-t)}\,ds\right)\\ &\qquad \times\left(\mathbb{E}|\bar{D}_t^{//} F|^2 + \frac{k_2-k_1}{2}\int_t^T e^{-k_1(s-t)}\mathbb{E}|\bar{D}_s^{//} F|^2 ds\right)dt.\end{aligned}$$

Theorem (continuation)

(iv) (Poincaré type inequality) *for* $F \in \mathscr{F}C^{\infty}_{0,T}$ *and* $t_1 < t_2$ *in* $[0,T]$,

$$\mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_2}]^2\Big] - \mathbb{E}\Big[\mathbb{E}[F(X_{[0,T]})|\mathscr{F}_{t_1}]^2\Big]$$

$$\leq \int_{t_1}^{t_2} \left(1 + \frac{k_2 - k_1}{2} \int_t^T e^{-k_1(s-t)}\, ds\right)$$

$$\times \left(\mathbb{E}|\bar{D}_t^{//} F|^2 + \frac{k_2 - k_1}{2} \int_t^T e^{-k_1(s-t)}\, \mathbb{E}|\bar{D}_s^{//} F|^2\, ds\right) dt.$$

The theorem allows to characterize

- Einstein manifolds (Ric is a multiple of the metric g)
- Ricci solitons ($\mathrm{Ric} + \mathrm{Hess} f = c\, g$)
- manifolds such that $\mathrm{Ric} = \nabla Z$
- etc

- Let $\mathcal{L}$ be the Ornstein-Uhlenbeck operator defined as generator associated to the Dirichlet form

$$\mathcal{E}(F,F) = \mathbb{E}\left[\int_0^T |D_t^{//}F|^2(X_{[0,T]})\,dt\right].$$

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- It is well-known that a log-Sobolev inequality

$$\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2\, H(T,k_1,k_2) \int_0^T |D_t^{//} F|^2(X_{[0,T]})\,dt$$

or a Poincaré inequality

$$\mathbb{E}\Big[(F - \mathbb{E}[F])^2\Big] \leq H(T,k_1,k_2) \int_0^T |D_t^{//} F|^2(X_{[0,T]})\,dt$$

for some explicit bound $H(T,k_1,k_2)$, give the spectral gap lower bound $H(T,k_1,k_2)^{-1}$ for the operator $\mathcal{L}$.

IV. Analysis on path space over sub-Riemannian manifolds

(cf. also F. Baudoin, Qi Feng, M. Gordina, J. Funct. Anal. **277** (2019))

Let again ∇ be a partial connection on H, extended as above to a compatible connection on M.

Weitzenböck formula

- Consider the corresponding rough sub-Laplacian

$$L(\nabla) := \mathrm{trace}_H \nabla^2$$

(on functions and 1-forms).

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- Consider the corresponding rough sub-Laplacian

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  (on functions and 1-forms).
- Would like to have a Weitzenböck type commutation formula of the form:

$$dLf = (L - \mathscr{R})df, \quad L = L(\nabla),$$

  where $\mathscr{R} \in \Gamma(\mathrm{End}(T^*M))$.

- Let $\hat{\nabla}$ be the adjoint connection to ∇, i.e.

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- **Proposition** Let L be a rough sub-Laplacian of a connection on M. There exists a vector bundle endomorphism

$$\mathscr{R} : T^*M \to T^*M$$

such that

$$(L-\mathscr{R})df = dLf, \quad f \in C^\infty(M),$$

if and only if $L = L(\hat{\nabla})$ for some adjoint $\hat{\nabla}$ of a connection ∇ that is compatible with (H, g_H).

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- In this case,

$$\mathscr{R} = \mathrm{Ric}^\nabla$$

where for $(\alpha, v) \in T^*M \oplus TM$,

$$\mathrm{Ric}^\nabla(\alpha)(v) = \mathrm{trace}_H R^\nabla(\cdot, v)\alpha(\cdot)$$

- **Proposition** (Weitzenböck formula)
  Then, for all $f \in C^\infty(M)$,

$$\left(L(\hat{\nabla}) - \mathscr{R}\right) df = dL(\hat{\nabla}) f = dL(\nabla) f = d\Delta^H f$$

Derivative formula

- Define $\hat{Q}_t = \hat{Q}_t(x) \in \mathrm{End}(T_xM)$ by

$$\frac{d}{dt}\hat{Q}_t = -\mathscr{R}_{\hat{//}_t}\hat{Q}_t\,, \quad \hat{Q}_0 = \mathrm{id}_{T_xM},$$

where $\mathscr{R} = \mathrm{Ric}^\nabla$ and $\mathscr{R}_{\hat{//}_t} = \hat{//}_t^{-1}\mathscr{R}\hat{//}_t$.

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- (Derivative formula) For $P_t = e^{t\Delta_H}$ and $f \in C^\infty(M)$, we have

$$dP_tf(x) = \mathbb{E}[\hat{Q}_t^*\hat{//}_t^{-1}df_{X_t(x)}]$$

Integration by parts on path space over a sub-Riemannian manifold

- Let (M, H, g_H) be a sub-Riemannian manifold equipped with a compatible connection ∇ and let

$$L = \mathrm{trace}_H \nabla^2_{\times,\times}$$

be defined as the trace of the Hessian ∇^2 over H with respect to the inner product g_H.

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- Assume that there is a decomposition $TM = H \oplus V$ such that
  a. both H and V are parallel with respect to ∇
  b. $\mathbf{T}(H, H) \subseteq V$
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  a. both H and V are parallel with respect to ∇
  b. $\mathbf{T}(H, H) \subseteq V$
  c. $\mathbf{T}(H, V) = 0$.

  No choice of a Riemannian metric g on M satisfying $g|_H = g_H$ is required.

  Assume again that the complement V metric preserving.

- Let $X_t(x) \equiv X_t^x$ be the sub-Riemannian Brownian motion with generator L such that $X_0(x) = x$ and

$$dB_t^x = //_t^{-1} \circ dX_t(x), \quad B_0 = 0 \in H_x$$

Recall that B_t^x is a standard Brownian motion in H_x.

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- (Cameron-Martin space) Let

$$\mathbb{H} = \left\{ h\colon [0,T] \to H_x \text{ abs. cont.} \;\middle|\; \int_0^T |\dot{h}(t)|^2_{g_H} dt < \infty \right\}$$

which becomes a Hilbert space with inner product

$$\langle h_1, h_2 \rangle_{\mathbb{H}} = \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t) \rangle_{g_H} dt.$$

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As usual, we write $\langle h, B^x \rangle_{\mathbb{H}} = \int_0^t \langle \dot{h}_s, dB_s^x \rangle_{g_H}$.

- For fixed $T > 0$, let $W^T = C([0,T]; M)$ and

$$\mathscr{F}C^\infty_{0,T} = \Big\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : \\ 0 < t_1 < \ldots < t_n \le T,\ f \in C^\infty_c(M^n) \Big\}$$

be the class of smooth cylindrical functions on W^T.

Derivatives on path space of sub-Riemannian manifolds

- For fixed $T > 0$, let $W^T = C([0,T]; M)$ and
$$\mathscr{F}C^\infty_{0,T} = \Big\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n}) : \\ 0 < t_1 < \ldots < t_n \le T,\ f \in C_c^\infty(M^n) \Big\}$$
be the class of smooth cylindrical functions on W^T.
- Let the operator $A_t : T_xM \to T_xM$ be given by
$$A_t = \int_0^t \mathbf{T}_{//_t}(\circ dB_t^x, \cdot)$$
(Note that $A_t(H_x) \subseteq V_x$ and $A_t(V_x) = 0$)
- For an adapted process h with paths in $\mathbb{H}$ let
$$\begin{aligned} S(h)_t &= h_t + \int_0^t \mathbf{T}_{//_s}(\circ dB_s^x, h_s) \\ &= h_t + \int_0^t dA_s h_s = \int_0^t (\mathrm{id} + A_t + A_s) dh_s. \end{aligned}$$

- (Derivative operator on path space) For $F \in \mathscr{F}C_{0,T}^{\infty}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$ and $h \in \mathbb{H}$, let

$$D_h F(\gamma) = \sum_{i=1}^{n} \left\langle //_{t_i}^{-1} d_i f(\gamma_{t_1}, \ldots, \gamma_{t_n}), S(h)_{t_i} \right\rangle$$

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- Motivation For any horizontal curve γ on M (starting from x) with anti-development $u = \mathrm{Dev}^{-1}(\gamma)$ in H_x, we have that

$$\left\{ \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathrm{Dev}(u + \varepsilon k) : \ k \in \mathbb{H} \right\} = \left\{ D_h|_{\gamma} : \ h \in \mathbb{H} \right\}$$

  where the vector field D_h on path space is defined by

$$D_h|_{\gamma} = //_t^{-1} \left(h_t + \int_0^t \mathbf{T}_{//_s}(du_s, h_s) \right) = //_t^{-1} \left(h_t + \int_0^t dA_s h_s \right)$$

  with $A_t = \int_0^t \mathbf{T}_{//_s}(du_s, h_s)$.

- Defining $D_t F \in H_x$ by

$$D_t F := \sum_{i=1}^{n} \mathbb{1}_{\{t \le t_i\}} \sharp^H (\mathrm{id} + A_{t_i} - A_t)^* //_{t_i}^{-1} d_i f(\gamma_{t_1}, \ldots, \gamma_{t_n}),$$

  we have

$$D_h F = \int_0^t \langle D_t F, \dot{h}_t \rangle_{g_H} dt.$$

- The *gradient* DF is then defined by the relation

$$\langle DF, h \rangle_{\mathbb{H}} = D_h F$$

Proposition (Integration by parts formula)

- For $F \in \mathscr{F}C^{\infty}_{0,T}$ and any adapted process h_t with paths in $\mathbb{H}$, we have

$$\mathbb{E}\left[\langle DF, h\rangle_{\mathbb{H}}\right] = \mathbb{E}\left[F \int_0^T \langle \dot{h}_t + \mathrm{Ric}_{//_t} h_t, dB_t\rangle_{g_H}\right].$$

- In particular, for $f \in C^{\infty}(M)$,

$$\mathbb{E}\left[\langle //_t^{-1} df_{X_t(x)}, S(h)_t\rangle\right] = \mathbb{E}\left[f(X_t(x)) \int_0^t \langle \dot{h}_s + \mathrm{Ric}_{//_s} h_s, dB_s\rangle_{g_H}\right].$$

Damped gradients

- For $F \in \mathscr{F}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_{t_1}, \ldots, \gamma_{t_n})$, define

$$\tilde{D}_t F(\gamma) := \sum_{i=1}^{n} 1_{\{t \le t_i\}} \sharp^H //_t^{-1} \hat{Q}^*_{t,t_i} \hat{//}^{-1}_{t,t_i} d_i f(\gamma_{t_1}, \ldots, \gamma_{t_n})$$

  and

$$\tilde{D}_h F = \langle \tilde{D} F, h \rangle_{\mathbb{H}} = \int_0^T \tilde{D}_t F \, dh_t$$

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- (Quasi-invariance) For adapted process h with paths in $\mathbb{H}$ one has

$$\mathbb{E}_x[\langle \tilde{D} F, h \rangle_{\mathbb{H}}] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{F(X^{\varepsilon}_{[0,T]}) - F(X_{[0,T]})}{\varepsilon}\right]$$

where

$$dX^{\varepsilon}_t = //^{\varepsilon}_t \circ dB_t + \varepsilon //^{\varepsilon}_t dh_t, \quad X^{\varepsilon}_0 = x$$

- Let $Q_t : T_xM \to T_xM$ be the solution of

$$Q_0 = \mathrm{id}_{T_xM}, \quad dQ_t = -\mathrm{Ric}_{//_t} Q_t dt$$

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- For any adapted process h_t with paths in $\mathbb{H}$, we then have

$$\langle \tilde{D}F, h \rangle_{\mathbb{H}} = \langle DF, k \rangle_{\mathbb{H}}, \quad k_t = Q_t \int_0^t Q_s^{-1} dh_s$$

and hence

$$\mathbb{E}[\langle \tilde{D}F, h \rangle_{\mathbb{H}}] = \mathbb{E}\left[F \int_0^T \langle h, B^x \rangle_{\mathbb{H}} \right]$$

V. Ricci curvature bounds in sub-Riemannian geometry

- (Derivative formula on path space)

  For $F \in \mathscr{F}C^{\infty}_{0,T}$ and $t > 0$, we have

$$D_t \mathbb{E}[F|\mathscr{F}_t] = \mathbb{E}[\tilde{D}_t F|\mathscr{F}_t]$$

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- (Semigroup derivative formula)

$$dP_t f(v) = \mathbb{E}\left[\left\langle //_t^{-1} df_{X_t(x)}, Q_t v + \int_0^t dA_r\, Q_r v \right\rangle\right], \quad v \in T_x M.$$

Theorem (Characterization of horizontal Ricci curvature)

Assume that V is metric preserving. For a non-negative constant K the following conditions are equivalent:

1. *(Bounded Ricci curvature) the horizontal Ricci curvature $Ric^H = \mathrm{Ric}|_H \in \mathrm{End}(H)$ is bounded by K, i.e.*

$$-K \le Ric^H \le K$$

2. *(Gradient estimate) for any $F \in \mathscr{F}C_0^\infty$,*

$$|D_0 \mathbb{E}_x[F]|_{g_H} \le \mathbb{E}_x\left[|D_0 F|_{g_H} + K\int_0^T e^{Ks}|D_s F|_{g_H}\,ds\right]$$

3. *(L^2 gradient estimate) for any $F \in \mathscr{F}C_0^\infty$,*

$$|D_0 \mathbb{E}_x[F]|_{g_H}^2 \le e^{-KT}\mathbb{E}_x\left[|D_0 F|_{g_H}^2 + K\int_0^T e^{Ks}|D_s F|_{g_H}^2\,ds\right]$$

Theorem (continuation)

- iii *(Log-Sobolev inequality) for any* $F \in \mathscr{F}C_0^\infty$ *and* $t > 0$ *in* $[0, T]$,

$$\mathbb{E}_x\Big[\mathbb{E}_x[F^2|\mathscr{F}_t]\log\mathbb{E}_x[F^2|\mathscr{F}_t]\Big] - \mathbb{E}_x[F^2]\log\mathbb{E}_x[F^2]$$
$$\leq 2\int_0^t e^{K(T-r)}\left(\mathbb{E}_x|D_rF|_{g_H}^2 + \frac{K}{2}\int_r^T e^{K(s-r)}\mathbb{E}_x|D_sF|_{g_H}^2\,ds\right)dr;$$

- iv *(Poincaré inequality) for any* $F \in \mathscr{F}C_0^\infty$ *and* $t > 0$ *in* $[0, T]$,

$$\mathbb{E}_x\Big[\mathbb{E}_x[F|\mathscr{F}_t]^2\Big] - \mathbb{E}_x[F]^2$$
$$\leq \int_0^t e^{K(T-r)}\left(\mathbb{E}_x|D_rF|_{g_H}^2 + \frac{K}{2}\int_r^T e^{K(s-r)}\mathbb{E}_x|D_sF|_{g_H}^2\,ds\right)dr.$$

- For non-symmetric bounds, i.e. $K_1 \le \mathrm{Ric}^H \le K_2$, one can give similar equivalent conditions redefining $\bar{D}_t F$ by

$$\bar{D}_t F = \sum_{i=1}^{n} 1\!\!1_{\{t \le t_i\}} \, e^{-\frac{K_1+K_2}{2}(t_i - t)} \sharp^H (\mathrm{id} + A_{t_i} - A_t)^* //_{t_i}^{-1} d_i F$$

- (Ornstein-Uhlenbeck operator)

  For $F, G \in \mathscr{F} C_{0,T}^{\infty}$ let

$$\mathscr{E}(F,G) = \mathbb{E}\langle DF, DG \rangle_{\mathbb{H}} = \mathbb{E}\left[\int_0^T \langle D_t F, D_t G \rangle_{g_H} \, dt \right].$$

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  Integration by parts formula implies the closability of the form.

- Let $\mathscr{L}$ be the generator of the the Dirichlet form

$$\mathscr{E}(F,F) = \mathbb{E}\left[\int_0^T |D_t F|^2_{g_H} \, dt\right].$$

  Let $\mathrm{gap}(\mathscr{L})$ denote its spectral gap.

Theorem Suppose there exists a constant $K \geq 0$ such that

$$\left|\mathrm{Ric}^H\right| \leq K.$$

Then

(i) (Poincaré inequality) for any $F \in \mathrm{dom}(\mathscr{E})$ with $\mathbb{E}[F] = 0$,

$$\mathbb{E}[F^2] \leq \frac{1}{2}(e^{KT} + 1)\mathscr{E}(F, F)$$

(ii) (Log-Sobolev inequality) for any $F \in \mathrm{dom}(\mathscr{E})$ with $\mathbb{E}[F^2] = 1$,

$$\mathbb{E}[F^2 \log F^2] \leq (e^{KT} + 1)\mathscr{E}(F, F)$$

(iii) (Spectral gap estimate) the following estimate holds:

$$\mathrm{gap}(\mathscr{L})^{-1} \leq \frac{1}{2}(e^{KT} + 1)$$