EIGENVALUE ASYMPTOTICS FOR SCHRÖDINGER OPERATORS ON SPARSE GRAPHS

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Abstract. We consider Schrödinger operators on sparse graphs. The geometric definition of sparseness turn out to be equivalent to a functional inequality for the Laplacian. In consequence, sparseness has in turn strong spectral and functional analytic consequences. Specifically, one consequence is that it allows to completely describe the form domain. Moreover, as another consequence it leads to a characterization for discreteness of the spectrum. In this case we determine the first order of the corresponding eigenvalue asymptotics.

1. Introduction

The spectral theory of discrete Laplacians on finite or infinite graphs has drawn a lot of attention for decades. One important aspect is to understand the relations between the geometry of the graph and the spectrum of the Laplacian. Often a particular focus lies on the study of the bottom of the spectrum and the eigenvalues below the essential spectrum.

Certainly the most well-known estimates for the bottom of the spectrum of Laplacians on infinite graphs are so called isoperimetric estimates or Cheeger inequalities. Starting with [D1] in the case of infinite graphs, these inequalities were intensively studied and resulted in huge body of literature, where we here mention only [BHJ, BKW, D2, DK, F, M1, M2, K1, KL2, Woj1]. In certain more specific geometric situations the bottom of the spectrum might be estimated in terms of curvature, see [BJL, H, JL, K1, K2, KP, LY, Woe]. There are various other more recent approaches such as Hardy inequalities in [G] and summability criteria involving the boundary and volume of balls in [KLW].

In this work we focus on sparse graphs to study discreteness of spectrum and eigenvalue asymptotics. In a moral sense, the term sparse means that there are not 'too many' edges, however, throughout the years various different definitions were investigated. We mention here [EGS, L] as seminal works which are closely related to our definitions. As it is impossible to give a complete discussion of the development, we refer to some selected more recent works such as [AABL, B, LS, M2] and references therein which also illustrates the great variety of possible definitions. Here, we discuss three notions of sparseness that result in a hierarchy of very general classes of graphs.

Let us highlight the work of Mohar [M3], where large eigenvalues of the adjacency matrix on finite graphs are studied. Although our situation of infinite graphs

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with unbounded geometry requires fundamentally different techniques – functional analytic rather than combinatorial – in spirit our work is certainly closely related.

The techniques used in this paper owe on the one hand to considerations of isoperimetric estimates as well as a scheme developed in [G] for the special case of trees. In particular, we show that a notion of sparseness is a geometric characterization for an inequality of the type

$$(1-a)\deg -k \leq \Delta \leq (1+a)\deg +k$$

for some $a \in (0,1)$, $k \geq 0$ which holds in the form sense (precise definitions and details will be given below). The moral of this inequality is that the asymptotic behavior of the Laplacian Δ is controlled by the vertex degree deg (the smaller a the better the control).

Furthermore, such an inequality has very strong consequences which follow from well-known functional analytic principles. These consequences include an explicit description of the form domain, characterization for discreteness of spectrum and eigenvalue asymptotics.

Let us set up the framework. Here, a graph $\mathscr{G}$ is a pair $(\mathscr{V},\mathscr{E})$, where $\mathscr{V}$ denotes a countable set of vertices of $\mathscr{G}$ and $\mathscr{E}:\mathscr{V}\times\mathscr{V}\to\{0,1\}$ is a symmetric function with zero diagonal determining the edges. We say two vertices x and y are *adjacent* or *neighbors* whenever $\mathscr{E}(x,y)=\mathscr{E}(y,x)>0$. In this case, we write $x\sim y$ and we call (x,y) and (x,y) the *(directed) edges* connecting x and y. We assume that $\mathscr{G}$ is *locally finite* that is each vertex has only finitely many neighbors. For any finite set $\mathscr{W}\subseteq\mathscr{V}$, the *induced subgraph* $\mathscr{G}_{\mathscr{W}}=(\mathscr{W},\mathscr{E}_{\mathscr{W}})$ is defined by setting $\mathscr{E}_{\mathscr{W}}:=\mathscr{E}|_{\mathscr{W}\times\mathscr{W}}$, i.e., an edge is contained in $\mathscr{G}_{\mathscr{W}}$ if and only if both of its vertices are in $\mathscr{W}$.

We consider the complex Hilbert space $\ell^2(\mathscr{V}):=\{\varphi:\mathscr{V}\to\mathbb{C}$ such that $\sum_{x\in\mathscr{V}}|\varphi(x)|^2<\infty\}$ endowed with the scalar product $\langle\varphi,\psi\rangle:=\sum_{x\in\mathscr{V}}\overline{\varphi(x)}\psi(x)$, $\varphi,\psi\in\ell^2(\mathscr{V})$.

For a function $g:\mathscr{V}\to\mathbb{C}$, we denote the operator of multiplication by g on $\ell^2(\mathscr{V})$ given by $\varphi\mapsto g\varphi$ and domain $\mathcal{D}(g):=\{\varphi\in\ell^2(\mathscr{V})\mid g\varphi\in\ell^2(\mathscr{V})\}$ with slight abuse of notation also by g.

Let $q:\mathscr{V}\to[0,\infty)$. We consider the Schrödinger operator $\Delta+q$ defined as

$$\mathcal{D}(\Delta+q):=\Big\{\varphi\in\ell^2(\mathscr{V})\mid\Big(v\mapsto\sum_{w\sim v}(\varphi(v)-\varphi(w))+q(v)\varphi(v)\Big)\in\ell^2(\mathscr{V})\Big\}$$

$$(\Delta+q)\varphi(v):=\sum_{w\sim v}(\varphi(v)-\varphi(w))+q(v)\varphi(v).$$

The operator is non-negative and selfadjoint as it is essentially selfadjoint on $\mathcal{C}_c(\mathscr{V})$, the set of finitely supported functions $\mathscr{V}\to\mathbb{R}$, (confer [Woj1, Theorem 1.3.1], [KL1, Theorem 6]). In Section 2 we will allow for potentials whose negative part is form bounded with bound strictly less than one. Moreover, in Section 4 we consider also magnetic Schrödinger operators.

As mentioned above sparse graphs have already been introduced in various contexts with slightly different definitions. In this article we also treat various natural generalizations of the concept. In this introduction we stick to an intermediate situation.

Definition. A graph $\mathscr{G}=(\mathscr{V},\mathscr{E})$ is called *k-sparse* if for any finite set $\mathscr{W}\subseteq\mathscr{V}$ the induced subgraph $\mathscr{G}_{\mathscr{W}}=(\mathscr{W},\mathscr{E}_{\mathscr{W}})$ satisfies

$$2|\mathscr{E}_{\mathscr{W}}|\leq k|\mathscr{W}|,$$

where $|A|$ denotes the cardinality of a finite set A and we set

$$|\mathscr{E}_{\mathscr{W}}| := \frac{1}{2}|\{(x,y), \mathscr{E}_{\mathscr{W}}(x,y) = 1\}|,$$

that is we count the non-oriented edges in $\mathscr{G}_{\mathscr{W}}$.

Examples of sparse graphs are planar graphs and, in particular, trees. We refer to Section 6 for more examples.

For a function $g : \mathscr{V} \to \mathbb{R}$ and a finite set $\mathscr{W} \subseteq \mathscr{V}$, we denote

$$g(\mathscr{W}) := \sum_{x\in\mathscr{W}} g(x).$$

Moreover, we define

$$\liminf_{|x|\to\infty} g(x) := \sup_{\mathscr{W}\subset\mathscr{V} \text{ finite}} \inf_{x\in\mathscr{V}\setminus\mathscr{W}} g(x), \qquad \limsup_{|x|\to\infty} g(x) := \inf_{\mathscr{W}\subset\mathscr{V} \text{ finite}} \sup_{x\in\mathscr{V}\setminus\mathscr{W}} g(x).$$

For two selfadjoint operators T_1, T_2 on a Hilbert space and a subspace $\mathcal{D}_0 \subseteq \mathcal{D}(T_1) \cap \mathcal{D}(T_2)$ we write $T_1 \leq T_2$ on $\mathcal{D}_0$ if $\langle T_1\varphi, \varphi\rangle \leq \langle T_2\varphi, \varphi\rangle$ for all $\varphi \in \mathcal{D}_0$. Moreover, for a selfadjoint semi-bounded operator T on a Hilbert space, we denote the eigenvalues below the essential spectrum by $\lambda_n(T)$, $n \geq 0$, with increasing order counted with multiplicity.

The next theorem is a special case of the more general Theorem 2.2 in Section 2. It illustrates our results in the case of sparse graphs introduced above and includes the case of trees, [G, Theorem 1.1], as a special case. While the proof in [G] uses a Hardy inequality, we rely on some new ideas which have their roots in isoperimetric techniques. The proof is given in Section 2.2.

Theorem 1.1. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a k-sparse graph and $q : \mathscr{V} \to [0,\infty)$. Then, we have the following:*

(a) *For all $0 < \varepsilon \leq 1$,*

$$(1-\varepsilon)(\deg + q) - \frac{k}{2}\left(\frac{1}{\varepsilon} - \varepsilon\right) \leq \Delta + q \leq (1+\varepsilon)(\deg + q) + \frac{k}{2}\left(\frac{1}{\varepsilon} - \varepsilon\right),$$

on $\mathcal{C}_c(\mathscr{V})$.

(b) $\mathcal{D}\left((\Delta + q)^{1/2}\right) = \mathcal{D}\left((\deg + q)^{1/2}\right)$.

(c) *The operator $\Delta + q$ has purely discrete spectrum if and only if*

$$\liminf_{|x|\to\infty} (\deg + q)(x) = \infty.$$

In this case, we obtain

$$\liminf_{\lambda\to\infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\deg + q)} = 1.$$

As a corollary, we obtain following estimate for the bottom and the top of the (essential) spectrum.

Corollary 1.2. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a k-sparse graph and $q : \mathscr{V} \to [0,\infty)$. Define $d := \inf_{x\in\mathscr{V}}(\deg + q)(x)$ and $D := \sup_{x\in\mathscr{V}}$. Assume $d < k \leq D < +\infty$, then*

$$d - 2\sqrt{\frac{k}{2}\left(d - \frac{k}{2}\right)} \leq \inf \sigma(\Delta + q) \leq \sup \sigma(\Delta + q) \leq D - 2\sqrt{\frac{k}{2}\left(D - \frac{k}{2}\right)}.$$

Define $d_{\mathrm{ess}} := \liminf_{|x|\to\infty}(\deg+q)(x)$ *and* $D_{\mathrm{ess}} := \limsup_{|x|\to\infty}(\deg+q)(x)$. *Assume* $d_{\mathrm{ess}} < k \leq D_{\mathrm{ess}} < +\infty$, *then*

$$d_{\mathrm{ess}}-2\sqrt{\frac{k}{2}\left(d_{\mathrm{ess}}-\frac{k}{2}\right)} \leq \inf \sigma_{\mathrm{ess}}(\Delta+q) \leq \sup \sigma_{\mathrm{ess}}(\Delta+q) \leq D_{\mathrm{ess}}-2\sqrt{\frac{k}{2}\left(D_{\mathrm{ess}}-\frac{k}{2}\right)}.$$

Proof of Corollary 1.2. The conclusion of Corollary 1.2 just follows by taking ε to be

$$\varepsilon := \sqrt{\frac{k}{2d-k}} \wedge 1$$

in (a) in of Theorem 1.1. □

Remark 1.3. The bounds in Corollary 1.2 are optimal for the bottom and the top of the (essential) spectrum in the case of regular trees.

The paper is structured as follows. In the next section an extension of the notion of sparseness is introduced which is shown to be equivalent to a functional inequality and equality of the form domains of Δ and deg. In Section 3 we consider almost sparse graphs for which we obtain precise eigenvalue asymptotics. Furthermore, in Section 4 we shortly discuss magnetic Schrödinger operators. Our notion of sparseness has very explicit but non-trivial connections to isoperimetric inequalities which are made precise in Section 5. Finally, in Section 6 we discuss some examples.

2. A geometric characterization of the form domain

In this section we will characterize equality of the form domains of $\Delta + q$ and $\deg + q$ by a geometric property. This geometric property is a generalization of the notion of sparseness from the introduction. Before we come to this definition, we introduce the class of potentials that is treated in this paper.

Let $\alpha \in (0, 1)$. We say a potential $q : \mathscr{V} \to \mathbb{R}$ is in the class $\mathscr{K}_\alpha$ if there is $C \geq 0$ such that

$$q_- \leq \alpha(\Delta + q_+) + C,$$

and $\Delta + q \geq 0$ on $\mathcal{C}_c(\mathscr{V})$, where $q_\pm = \max(\pm q, 0)$. The second assumption on non-negativity is for convenience only. For a potential q that satisfies the first assumption the second one can be always achieved by adding a finite non-negative constant to q.

For $q \in \mathscr{K}_\alpha$, we define $\Delta + q$ via the form sum of the operators $\Delta + q_+$ and $-q_-$. Note that by the assumptions on q we have

$$\mathcal{D}((\Delta + q)^{\frac{1}{2}}) = \mathcal{D}((\Delta + q_+)^{\frac{1}{2}}),$$

by Theorem A.1. Indeed, by the virtue of [GKS, Theorem 5.6] we actually have $\mathcal{D}((\Delta + q)^{\frac{1}{2}}) = \mathcal{D}(\Delta^{\frac{1}{2}}) \cap \mathcal{D}(q_+^{\frac{1}{2}})$. By plugging characteristic functions of singleton sets of vertices in the non-negativity assumption $\Delta + q \geq 0$ we get

$$\deg + q \geq 0,$$

both in the sense of functions and forms. The more important class of potentials for this paper is the following

$$\mathscr{K}_0 := \bigcap_{\alpha \in (0,1)} \mathscr{K}_\alpha.$$

Let us mention that if q_- is in the Kato class with respect to $\Delta + q_+$ that is $\limsup_{t\to 0} \|e^{-t(\Delta+q_+)}q_-\|_\infty = 0$, then $q \in \mathscr{K}_0$ by [SV, Theorem 3.1]. In our context of sparseness, we can actually characterize the class $\mathscr{K}_0$ to be the potentials whose negative part q_- is morally $o(\deg + q_+)$, see Corollary 2.8.

Next, we come to an extension of the notion of sparseness. For a set $\mathscr{W} \subseteq \mathscr{V}$, let the boundary $\partial\mathscr{W}$ of $\mathscr{W}$ be the set of edges emanating from $\mathscr{W}$

$$\partial\mathscr{W} := \{(x, y) \in \mathscr{W} \times \mathscr{V} \setminus \mathscr{W} \mid x \sim y\}.$$

Definition. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph and $q : \mathscr{V} \to \mathbb{R}$. For given $a \geq 0$ and $k \geq 0$, we say that $(\mathscr{G}, q)$ is (a, k)-*sparse* if for any finite set $\mathscr{W} \subseteq \mathscr{V}$ the induced subgraph $\mathscr{G}_\mathscr{W} = (\mathscr{W}, \mathscr{E}_\mathscr{W})$ satisfies

$$2|\mathscr{E}_\mathscr{W}| \leq k|\mathscr{W}| + a(|\partial\mathscr{W}| + q_+(\mathscr{W})).$$

Remark 2.1. (a) As mentioned above there is a great variety of definitions which were so far predominantly established for (families of) finite graphs. For example it is asked that $|\mathscr{E}| = C|\mathscr{V}|$ in [EGS], $|\mathscr{E}_\mathscr{W}| \leq k|\mathscr{W}| + l$ in [L, LS], $|\mathscr{E}| \in O(|\mathscr{V}|)$ in [AABL] and $\deg(\mathscr{W}) \leq k|\mathscr{W}|$ in [M3].
(b) Observe that the definition depends only on q_+. This will be compensated by considering potentials in $\mathscr{K}_\alpha$ or $\mathscr{K}_0$ only in our theorems.
(c) If $(\mathscr{G}, q)$ is (a, k)-sparse, then $(\mathscr{G}, q')$ is (a, k)-sparse for every $q' \geq q$.

We now characterize the equality of the form domains in geometric terms.

Theorem 2.2. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph and $q \in \mathscr{K}_0$. The following assertions are equivalent:*

(i) *There are $a, k \geq 0$ such that $(\mathscr{G}, q)$ is (a, k)-sparse.*

(ii) *There are $\tilde{a} \in (0, 1)$ and $\tilde{k} \geq 0$ such that on $\mathcal{C}_c(\mathscr{V})$*

$$(1 - \tilde{a})(\deg + q) - \tilde{k} \leq \Delta + q \leq (1 + \tilde{a})(\deg + q) + \tilde{k}.$$

(iii) *There are $\tilde{a} \in (0, 1)$ and $\tilde{k} \geq 0$ such that on $\mathcal{C}_c(\mathscr{V})$*

$$(1 - \tilde{a})(\deg + q) - \tilde{k} \leq \Delta + q.$$

(iv) $\mathcal{D}((\Delta + q)^{1/2}) = \mathcal{D}((\deg + q)^{1/2})$.

Furthermore, $\Delta + q$ has purely discrete spectrum if and only if $\liminf_{|x|\to\infty}(\deg + q)(x) = \infty$*. In this case, we obtain*

$$1 - \tilde{a} \leq \liminf_{n\to\infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\deg + q)} \leq \limsup_{n\to\infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\deg + q)} \leq 1 + \tilde{a}.$$

Remark 2.3. (a) Observe that in the context of Theorem 2.2 statement (iv) is equivalent to

(iv') $\mathcal{D}((\Delta + q)^{1/2}) = \mathcal{D}((\deg + q_+)^{1/2})$.

Indeed, (ii) implies the corresponding inequality for $q = q_+$. Thus, as $q \in \mathscr{K}_0$,

$$\mathcal{D}((\Delta + q)^{\frac{1}{2}}) = \mathcal{D}((\Delta + q_+)^{\frac{1}{2}}) = \mathcal{D}((\deg + q)^{\frac{1}{2}}).$$

(b) The equivalence in the theorem above allows only for potentials in $\mathscr{K}_0$. However, for $\mathscr{K}_\alpha$, $\alpha \in (0, 1)$, parts of the statement remain true as it will be shown in Theorem 2.7 at the end of this section. For potentials in $\mathscr{K}_\alpha$, $\alpha \in (0, 1)$, one still has the equivalence (i) and (iii). Moreover, for potentials in $\mathscr{K}_\alpha$, $\alpha \in (0, 1/2)$, it can be seen by the inequality $\Delta \leq 2\deg$ (and Lemma A.3) that the upper bound in (ii) holds after replacing the constant $(1+\tilde{a})$, $\tilde{a}\in(0,1)$ by a constant $(1+\tilde{a}')$, $\tilde{a}'\in(0,\infty)$ which we denote by (ii'). This way one figures that (i), (ii'), (iii), (iv) are still equivalent in the case $\alpha\in(0,1/2)$.

(c) The definition of the class $\mathscr{K}_0$ is rather abstract. Indeed, Theorem 2.2 yields a very concrete characterization of these potentials, see Corollary 2.8 below.

(d) Theorem 2.2 characterizes equality of the form domains. Another natural question is under which circumstances the operator domains agree. For a discussion on this matter we refer to [G, Section 4.1].

The rest of this section is devoted to the proof of the results which are divided into three parts. The following two lemmas essentially show the equivalences (i)⇔(ii)⇔(iii) providing the explicit dependence of (a,k) on $(\tilde{a},\tilde{k})$ and vice versa. The third part uses general functional analytic principles collected in the appendix.

The first lemma gives (i)⇒(ii) for $q\geq 0$. The general case $q\in\mathscr{K}_0$ follows then by Lemma A.3.

Lemma 2.4. *Let $\mathscr{G}=(\mathscr{V},\mathscr{E})$ be a graph and $q:\mathscr{V}\to[0,\infty)$. If there are $a,k\geq 0$ such that $(\mathscr{G},q)$ is (a,k)-sparse, then*

$$(1-\tilde{a})(\deg+q)-\tilde{k}\leq\Delta+q\leq(1+\tilde{a})(\deg+q)+\tilde{k}.$$

on $\mathcal{C}_c(\mathscr{V})$, where if $(\mathscr{G},q)$ is sparse, i.e. $a=0$, we may choose $\tilde{a}\in(0,1)$ arbitrary and

$$\tilde{k}=\frac{k}{2}\Big(\frac{1}{\tilde{a}}-\tilde{a}\Big),$$

otherwise, for the general case, i.e. $a>0$, we may choose

$$\tilde{a}:=\frac{\sqrt{\min\left(\frac{1}{4},a^2\right)+2a+a^2}}{(1+a)}\quad\text{and}\quad\tilde{k}:=\max\left(\frac{\max\left(\frac{3}{2},\frac{1}{a}-a\right)k}{2(1+a)},2k(1-\tilde{a})\right).$$

Proof. Let $f\in\mathcal{C}_c(\mathscr{V})$ be complex valued. Assume first that $\langle f,(\deg+q)f\rangle<k\|f\|^2$. In this case, remembering $\Delta\leq 2\deg$, we can choose $\tilde{a}\in(0,1)$ arbitrary and $\tilde{k}$ such that

$$\tilde{k}\geq 2(1-\tilde{a})k.$$

So, assume $\langle f,(\deg+q)f\rangle\geq k\|f\|^2$. Using an area and a co-area formula (cf. [KL2, Theorem 12 and Theorem 13]) with

$$\Omega_t:=\{x\in\mathscr{V}\mid |f(x)|^2>t\},$$

in the first step and the assumption of sparseness in the third step, we obtain

$$\begin{aligned}\langle f,&(\deg+q)f\rangle-k\|f\|^2=\int_0^\infty\Big(\deg(\Omega_t)+q(\Omega_t)-k|\Omega_t|\Big)dt\\&=\int_0^\infty\Big(2|\mathscr{E}_{\Omega_t}|+|\partial\Omega_t|+q(\Omega_t)-k|\Omega_t|\Big)dt\\&\leq(1+a)\int_0^\infty|\partial\Omega_t|+q(\Omega_t)dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1+a)}{2} \sum_{x,y,x\sim y} \big||f(x)|^2 - |f(y)|^2\big| + (1+a)\sum_x q(x)|f(x)|^2 \\
&\leq \frac{(1+a)}{2} \sum_{x,y,x\sim y} |(f(x)-f(y))(\overline{f(x)}+\overline{f(y)})| + (1+a)\sum_x q(x)|f(x)|^2 \\
&\leq \frac{(1+a)}{2} \left(\sum_{x,y,x\sim y} |f(x)-f(y)|^2 + 2\sum_x q(x)|f(x)|^2 \right)^{1/2} \\
&\qquad \times \left(\sum_{x,y,x\sim y} |f(x)+f(y)|^2 + 2\sum_x q(x)|f(x)|^2 \right)^{1/2} \\
&= (1+a)\langle f, (\Delta+q)f\rangle^{\frac{1}{2}} \big(2\langle f, (\deg +q)f\rangle - \langle f, (\Delta+q)f\rangle\big)^{\frac{1}{2}},
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality in the last inequality and basic algebraic manipulation in the last equality. Since the left hand side is non-negative by the assumption $\langle f, (\deg +q)f\rangle \geq k\|f\|^2$, we can take square roots on both sides. To shorten notation, we assume for the rest of the proof $q \equiv 0$ since the proof with $q \neq 0$ is completely analogous.

Reordering the terms, yields

$$
(1+a)^2\langle f, \Delta f\rangle^2 - 2(1+a)^2\langle f, \deg f\rangle\langle f, \Delta f\rangle + (\langle f, (\deg -k)f\rangle)^2 \leq 0.
$$

Resolving the quadratic expression above gives,

$$
\langle f, \deg f\rangle - \sqrt{\delta} \leq \langle f, \Delta f\rangle \leq \langle f, \deg f\rangle + \sqrt{\delta},
$$

with

$$
\delta := \langle f, \deg f\rangle^2 - (1+a)^{-2}(\langle f, (\deg -k)f\rangle)^2.
$$

Using $4\xi\zeta \leq (\xi+\zeta)^2$, $\xi, \zeta \geq 0$, for all $0 < \lambda < 1$, we estimate δ as follows

$$
\begin{aligned}
(1+a)^2\delta &= (2a+a^2)\langle f, \deg f\rangle^2 + k\|f\|^2\langle f, (2\deg -k)f\rangle \\
&\leq (2a+a^2)\langle f, \deg f\rangle^2 + \left(\lambda\langle f, \deg f\rangle + \frac{k}{2}\left(\frac{1}{\lambda} - \lambda\right)\|f\|^2\right)^2 \\
&\leq \left(\sqrt{\lambda^2 + 2a + a^2}\langle f, \deg f\rangle + \frac{k}{2}\left(\frac{1}{\lambda} - \lambda\right)\|f\|^2\right)^2.
\end{aligned}
$$

If $a = 0$, then this is the k-sparse case and we take $\lambda = \tilde{a}$ to get

$$
\delta \leq k\|f\|^2\langle f, 2\deg f\rangle \leq \left(\tilde{a}\langle f, \deg f\rangle + \frac{k}{2}\left(\frac{1}{\tilde{a}} - \tilde{a}\right)\|f\|^2\right)^2.
$$

As $k/2\tilde{a} \geq 2(1-\tilde{a})k$, this proves the desired inequality with $\tilde{k} := k/2\tilde{a}$.

If $a > 0$, we take $\lambda := \min\left(\frac{1}{2}, a\right)$ to get:

$$
(1+a)^2\delta = \left(\sqrt{\min\left(\frac{1}{4}, a^2\right) + 2a + a^2}\langle f, \deg f\rangle + \frac{k}{2}\max\left(\frac{3}{2}, \left(\frac{1}{a} - a\right)\right)\|f\|^2\right)^2.
$$

Keeping in mind the restriction $\tilde{k} \geq 2(1-\tilde{a})k$ for the case $\langle f, (\deg +q)f\rangle < k\|f\|^2$, this gives the statement with the choice of $(\tilde{a}, \tilde{k})$ in the statement of the lemma. □

The second lemma shows (iii)⇒(i).

Lemma 2.5. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph and $q \in \mathscr{K}_\alpha$ for some $\alpha \in (0,1)$. If there are $\tilde{a} \in (0,1)$ and $\tilde{k} \geq 0$ such that on $\mathcal{C}_c(\mathscr{V})$*

$$(1-\tilde{a})(\deg + q) - \tilde{k} \leq \Delta + q,$$

then $(\mathscr{G}, q)$ is (a,k)-sparse with

$$a = \frac{\tilde{a}}{1-\tilde{a}} \quad \textit{and} \quad k = \frac{\tilde{k}}{1-\tilde{a}}.$$

Proof. By adding q_- to the assumed inequality we obtain immediately

$$(1-\tilde{a})(\deg + q_+) - \tilde{k} \leq \Delta + q_+.$$

Let $\mathscr{W} \subseteq \mathscr{V}$ be a finite set and denote by $\mathbf{1}_{\mathscr{W}}$ the characteristic function of the set $\mathscr{W}$. We recall the basic equalities

$$\deg(\mathscr{W}) = 2|\mathscr{E}_{\mathscr{W}}| + |\partial \mathscr{W}| \quad \text{and} \quad \langle \mathbf{1}_{\mathscr{W}}, \Delta \mathbf{1}_{\mathscr{W}} \rangle = |\partial W|.$$

Therefore, applying the asserted inequality to $\mathbf{1}_{\mathscr{W}}$, we obtain

$$2|\mathscr{E}_{\mathscr{W}}| \leq \frac{\tilde{k}}{1-\tilde{a}}|W| + \frac{\tilde{a}}{1-\tilde{a}}\left(|\partial W| + q_+(\mathscr{W})\right).$$

This proves the statement. □

Before we come to the proof of Theorem 2.2, we summarize the relation between the sparseness parameters (a,k) and the constants $(\tilde{a}, \tilde{k})$ in the inequality in Theorem 2.2 (ii) for $q : \mathscr{V} \to [0, \infty)$. By Lemma A.3 the same asymptotics hold for $q \in \mathscr{K}_0$.

Remark 2.6. Roughly speaking a tends to ∞ as $\tilde{a}$ tends to 1^- and a tends to 0^+ as $\tilde{a}$ tends to 0^+ and vice-versa. More precisely, the expressions of a and k with respect with $\tilde{a}$ and $\tilde{k}$ are given in Lemma 2.5. Reciprocally by Lemma 2.4, for $a = 0$ we refer to $\tilde{k} = k/2\tilde{a}$ for arbitrary $\tilde{a}$ and for $a > 0$ we obtain for $a \to 0^+$

$$\tilde{a} \simeq \sqrt{2a} \quad \text{and} \quad \tilde{k} \simeq \frac{k}{2a},$$

and for $a \to \infty$

$$\tilde{a} \simeq 1 - \frac{3}{8a^2} \quad \text{and} \quad \tilde{k} = \frac{3k}{4a}.$$

Proof of Theorem 2.2. The implication (i)⇒(ii) follows from Lemma 2.4 for $q \geq 0$. From Lemma A.3 we deduce the statement for $q \in \mathscr{K}_0$ where it remains to note $\tilde{a} \in (0,1)$. The implication (ii)⇒(iii) is trivial and the implication (iii)⇒(i) follows from Lemma 2.5. The equivalence (ii)⇔(iv) follows from an application of the Closed Graph Theorem, Theorem A.1. Finally, the statements about discreteness of spectrum and eigenvalue asymptotics follow from an application of the Min-Max-Principle, Theorem A.2. □

Proof of Theorem 1.1. (a) follows from Lemma 2.4. The other statements follow directly from Theorem 2.2. □

As mentioned in Remark 2.3 (b) above parts of Theorem 2.2 remain true for potentials in $\mathscr{K}_\alpha$, $\alpha < 1$, as opposed to $\mathscr{K}_0$.

Theorem 2.7. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph and $q \in \mathscr{K}_\alpha$ for some $\alpha < 1$. The following assertions are equivalent:*

(i) *There are $a, k \geq 0$ such that $(\mathscr{G}, q)$ is (a,k)-sparse.*

(ii) *There are* $\tilde{a} \in (0,1)$ *and* $\tilde{k} \geq 0$ *such that on* $\mathcal{C}_c(\mathscr{V})$

$$(1-\tilde{a})(\deg + q) - \tilde{k} \leq \Delta + q.$$

(iii) $\mathcal{D}((\Delta+q)^{1/2}) \subseteq \mathcal{D}((\deg + q)^{1/2})$

Furthermore, $\Delta + q$ *has purely discrete spectrum if* $\liminf_{|x|\to\infty}(\deg + q)(x) = \infty$. *In this case, we obtain*

$$1 - \tilde{a} \leq \liminf_{n\to\infty} \frac{\lambda_n(\Delta+q)}{\lambda_n(\deg + q)}.$$

Proof. The implication (i)⇒(ii) follows from Lemma 2.4 for $q \geq 0$ and we conclude the general case $q \in \mathscr{K}_\alpha$, $\alpha < 1$, by Lemma A.3 (a). The implication (ii)⇒(i) follows from Lemma 2.5. The equivalence (ii)⇔(iii) follows from Closed Graph Theorem, Theorem A.1 and the spectral statements follow from the Min-Max-Principle, Theorem A.2. □

As a corollary we can now determine the potentials in the class $\mathcal{K}_0$ explicitly and give necessary and sufficient criteria for potentials being in $\mathcal{K}_\alpha$, $\alpha \in (0,1)$.

Corollary 2.8. *Let* $(\mathscr{G}, q)$ *be an* (a,k)*-sparse graph for some* $a, k \geq 0$.

(a) *The potential* q *is in* $\mathscr{K}_0$ *if and only if for all* $\alpha \in (0,1)$ *there is* $C \geq 0$ *such that*

$$q_- \leq \alpha(\deg + q_+) + C.$$

(b) *Let* $\alpha \in (0,1)$ *and* $\tilde{a} = \sqrt{\min(1/4, a^2) + 2a + a^2}/(1+a)$. *If there is* $C \geq 0$ *such that* $q_- \leq \alpha(\deg + q_+) + C$, *then* $q \in \mathscr{K}_{\alpha/(1-\tilde{a})}$. *On the other, hand if* $q \in \mathscr{K}_\alpha$, *then there is* $C \geq 0$ *such that* $q_- \leq \alpha(1+\tilde{a})(\deg + q_+) + C$.

Proof. Using the assumption $q_- \leq \alpha(\Delta + q_-) + C$ and the lower bound of Theorem 2.2 (ii), we infer

$$q_- \leq \alpha(\deg + q_+) + C \leq \frac{\alpha}{(1-\tilde{a})}(\Delta + q_+) + \frac{\alpha}{(1-\tilde{a})}\tilde{k} + C.$$

Conversely, $q \in \mathscr{K}_\alpha$ and the upper bound of Theorem 2.2 (ii) yields

$$q_- \leq \alpha(\Delta + q_+) + C \leq \alpha(1+\tilde{a})(\deg + q_+) + \alpha\tilde{k} + C.$$

Hence, (a) follows. For (b), notice that $\tilde{a} = \sqrt{\min(1/4, a^2) + 2a + a^2}/(1+a)$ by Lemma 2.4. □

3. Almost-sparseness and asymptotic of eigenvalues

In this section we prove better estimates on the eigenvalue asymptotics in a more specific situation. Looking at the inequality in Theorem 2.2 (ii) it seems desirable to have $\tilde{a} = 0$. As this is impossible, we shall take a sequence of $\tilde{a}$ that tends to 0. Keeping in mind Remark 2.6, this leads naturally to the following definition.

Definition. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph and $q : \mathscr{V} \to \mathbb{R}$. We say $(\mathscr{G}, q)$ is *almost sparse* if for all $\varepsilon > 0$ there is $k_\varepsilon \geq 0$ such that $(\mathscr{G}, q)$ is $(\varepsilon, k_\varepsilon)$-sparse, i.e., for any finite set $\mathscr{W} \subseteq \mathscr{V}$ the induced subgraph $\mathscr{G}_\mathscr{W} = (\mathscr{W}, \mathscr{E}_\mathscr{W})$ satisfies

$$2|\mathscr{E}_\mathscr{W}| \leq k_\varepsilon|\mathscr{W}| + \varepsilon(|\partial\mathscr{W}| + q_+(\mathscr{W})).$$

Remark 3.1. (a) Every sparse graph $\mathscr{G}$ is almost sparse.
(b) For an almost sparse graph $(\mathscr{G}, q)$, every graph $(\mathscr{G}, q')$ with $q' \geq q$ is almost sparse.

The main result of this section shows how the first order of the eigenvalue asymptotics in the case of discrete spectrum can be determined for almost sparse graphs.

Theorem 3.2. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph and $q \in \mathscr{K}_0$. The following assertions are equivalent:*

(i) *$(\mathscr{G}, q)$ is almost sparse.*

(ii) *For every $\varepsilon > 0$ there are $k_\varepsilon \geq 0$ such that on $\mathcal{C}_c(\mathscr{V})$*
$$(1-\varepsilon)(\deg + q) - k_\varepsilon \leq \Delta + q \leq (1+\varepsilon)(\deg + q) + k_\varepsilon.$$

(iii) *For every $\varepsilon > 0$ there are $k_\varepsilon \geq 0$ such that on $\mathcal{C}_c(\mathscr{V})$*
$$(1-\varepsilon)(\deg + q) - k_\varepsilon \leq \Delta + q.$$

Moreover, $\mathcal{D}((\Delta+q)^{1/2}) = \mathcal{D}((\deg + q)^{1/2})$ and the operator $\Delta+q$ has purely discrete spectrum if and only if $\liminf_{|x|\to\infty}(\deg + q)(x) = \infty$. In this case, we have
$$\lim_{n\to\infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\deg + q)} = 1.$$

Proof. The implication (i)⇒(ii) follows from Lemma 2.4 and Lemma A.3, (ii)⇒(iii) is trivial and (iii)⇒(i) follows from Lemma 2.5. The statement about equality of the form domains is an obvious consequence of (ii). Again, the statements about discreteness of spectrum and eigenvalue asymptotics follow from the Min-Max-Principle, Theorem A.2. □

4. Magnetic Laplacians

In this section, we consider magnetic Schrödinger operators. Clearly, every lower bound can be deduced from Kato's inequality. However, for the eigenvalue asymptotics we also need to prove an upper bound.

We fix a phase
$$\theta : \mathscr{V} \times \mathscr{V} \to \mathbb{R}/2\pi\mathbb{Z} \text{ such that } \theta(x,y) = -\theta(y,x).$$
For a potential $q : \mathscr{V} \to [0,\infty)$ we consider the magnetic Schrödinger operator $\Delta_\theta + q$ defined as
$$\mathcal{D}(\Delta_\theta + q) := \Big\{\varphi \in \ell^2(\mathscr{V}) \,|\Big(v \mapsto \sum_{x\sim y}(\varphi(x) - e^{\mathrm{i}\theta(x,y)}\varphi(x)) + q(x)\varphi(x)\Big) \in \ell^2(\mathscr{V})\Big\}$$
$$(\Delta_\theta + q)\varphi(x) := \sum_{x\sim y}(\varphi(x) - e^{\mathrm{i}\theta(x,y)}\varphi(y)) + q(x)\varphi(x).$$
A computation for $\varphi \in \mathcal{C}_c(\mathscr{V})$ gives
$$\langle \varphi, (\Delta_\theta + q)\varphi\rangle = \frac{1}{2}\sum_{x,y\in X, x\sim y}\Big|\varphi(x) - e^{\mathrm{i}\theta(x,y)}\varphi(y)\Big|^2 + \sum_{x\in X} q(x)|\varphi(x)|^2.$$

The operator is non-negative and selfadjoint as it is essentially selfadjoint on $\mathcal{C}_c(\mathscr{V})$ (confer e.g. [G]). For $\alpha \in (0,1)$, let $\mathscr{K}_\alpha^\theta$ be the class of potentials q such that $q_- \leq \alpha(\Delta_\theta + q_+) + C$ for some $C \geq 0$ and $\Delta_\theta + q \geq 0$. Denote $\mathscr{K}_0^\theta = \bigcap_{\alpha\in(0,1)} \mathscr{K}_\alpha^\theta$. Again, for $\alpha \in (0,1)$ and $q \in \mathscr{K}_\alpha^\theta$, we define $\Delta_\theta + q$ to be the form sum of $\Delta_\theta + q_+$ and $-q_-$.

Next, we present our result for magnetic Schrödinger operators which consists of one implication from the equivalences of Theorem 2.2 and Theorem 3.2.

Theorem 4.1. *Let* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *be a graph,* θ *be a phase and* $q \in \mathscr{K}_0^\theta$ *be a potential. Assume* $(\mathscr{G}, q)$ *is* (a, k)*-sparse for some* $a, k \geq 0$*. Then, we have the following:*

(a) *There are* $\tilde{a} \in (0, 1)$*,* $k \geq 0$ *such that on* $\mathcal{C}_c(\mathscr{V})$

$$(1 - \tilde{a})(\deg + q) - k \leq \Delta_\theta + q \leq (1 + \tilde{a})(\deg + q) + k.$$

(b) $\mathcal{D}\left((\Delta_\theta + q)^{1/2}\right) = \mathcal{D}\left((\deg + q)^{1/2}\right)$.

(c) *The operator* $\Delta_\theta + q$ *has purely discrete spectrum if and only if*

$$\liminf_{|x| \to \infty} (\deg + q)(x) = \infty.$$

In this case, if $(\mathscr{G}, q)$ *is additionally almost sparse, then*

$$\liminf_{\lambda \to \infty} \frac{\lambda_n(\Delta_\theta + q)}{\lambda_n(\deg + q)} = 1.$$

Proof of Theorem 4.1. By Kato's inequality, (see e.g. [DM, Lemma 2.1] or [GKS, Theorem 5.2.b]), and Theorem 2.2, we get for $f \in \mathcal{C}_c(\mathscr{V})$

$$\begin{aligned} \langle f, (\Delta_\theta + q) f \rangle &\geq \langle |f|, (\Delta + q)|f| \rangle \geq (1 - \tilde{a}) \langle |f|, (\deg + q)|f| \rangle - k\|f\|^2 \\ &= (1 - \tilde{a}) \langle f, (\deg + q) f \rangle - k\|f\|^2, \end{aligned}$$

which gives the lower bound for all phases θ. For a fixed phase θ, we observe

$$\Delta_{\theta + \pi} = 2\deg - \Delta_\theta.$$

Hence, by the lower bound proven above, we arrive at

$$\Delta_\theta + q = 2(\deg + q) - (\Delta_{\theta + \pi} + q) \leq (1 + \tilde{a})(\deg + q) + k,$$

which is the upper bound. Simultaneously, we prove

$$(1 - \varepsilon)(\deg + q) - k_\varepsilon \leq \Delta_\theta + q \leq (1 + \varepsilon)(\deg + q) + k_\varepsilon$$

for all $\varepsilon > 0$ in the almost sparse case. The statement about the form domain is now immediate and the statements about discreteness of spectrum and eigenvalue asymptotics follow from the Min-Max-Principle, Theorem A.2. □

Remark 4.2. (a) The constants $\tilde{a}$ and $\tilde{k}$ can chosen to be the same as the ones we obtained in the proof of Theorem 2.2, i.e., these constants are explicitly given combining Lemma 2.4 and Lemma A.3.
(b) Instead of using Kato's inequality one can also reproduce the proof of Lemma 2.4 using the following estimate for complex-valued $f \in \mathcal{C}_c(\mathscr{V})$

$$||f(x)|^2 - |f(y)|^2| \leq |(f(x) - e^{\mathrm{i}\theta(x,y)} f(y))(\overline{f(x)} + e^{-\mathrm{i}\theta(x,y)} \overline{f(y)})|.$$

It can be observed that unlike in Theorem 2.2 or Theorem 3.2 we do not have an equivalence in the theorem above. A reason for this seems to be that our definition of sparseness does not involve the magnetic potential. This direction shall be pursued in a future project. Here, we restrict ourselves to some remarks on the perturbation theory in the context of Theorem 4.1 above.

Remark 4.3. (a) If the inequality Theorem 4.1 (a) holds for some θ, then the inequality holds with the same constants for $-\theta$ and $\theta \pm \pi$. This can be seen by the fact $\Delta_{\theta+\pi} = 2\deg - \Delta$ and $\langle f, \Delta_\theta f\rangle = \langle \overline{f}, \Delta_{-\theta}\overline{f}\rangle$ while $\langle f, \deg f\rangle = \langle \overline{f}, \deg \overline{f}\rangle$ for $f \in \mathcal{C}_c(\mathscr{V})$.

(b) The set of θ such that Theorem 4.1 (a) holds true for some fixed $\tilde{a}$ and $\tilde{k}$ is closed in the product topology, i.e., with respect to pointwise convergence. This follows as $\langle f, \Delta_{\theta_n} f\rangle \to \langle f, \Delta_\theta f\rangle$ if $\theta_n \to \theta$, $n \to \infty$, for fixed $f \in \mathcal{C}_c(\mathscr{V})$.

(c) For two phases θ and θ' let $h(x) := \max_{y\sim x} |\theta(x, y) - \theta'(x, y)|$. By a straight forward estimate $\limsup_{|x|\to\infty} h(x) = 0$ implies that for every $\varepsilon > 0$ there is $C \geq 0$ such that

$$-\varepsilon \deg - C \leq \Delta_\theta - \Delta_{\theta'} \leq \varepsilon \deg + C$$

on $\mathcal{C}_c(\mathscr{V})$. We discuss three consequences of this inequality:

First of all, this inequality immediately yields that if $\mathcal{D}(\Delta_\theta^{1/2}) = \mathcal{D}(\deg^{1/2})$ then $\mathcal{D}(\Delta_{\theta'}^{1/2}) = \mathcal{D}(\deg^{1/2})$ (by the KLMN Theorem, see e.g., [RS, Theorem X.17]) which in turn yields equality of the form domains of Δ_θ and $\Delta_{\theta'}$.

Secondly, combining this inequality with Theorem 3.2 we obtain the following: If $\limsup_{|x|\to\infty} \max_{y\sim x} |\theta(x, y)| = 0$ and for every $\varepsilon > 0$ there is $k_\varepsilon \geq 0$ such that

$$(1 - \varepsilon) \deg - k_\varepsilon \leq \Delta_\theta \leq (1 + \varepsilon) \deg + k_\varepsilon$$

then the graph is almost sparse and in consequence the inequality in Theorem 4.1 (a) holds for any phase.

Thirdly, using the techniques in the proof of [G, Proposition 5.2] one shows that the essential spectra of Δ_θ and $\Delta_{\theta'}$ coincide. With slightly more effort and the help of the Kuroda-Birman Theorem, [RS, Theorem XI.9] one can show that if $h \in \ell^1(\mathscr{V})$, then even the absolutely continuous spectra of Δ_θ and $\Delta_{\theta'}$ coincide.

5. Isoperimetric estimates and sparseness

In this section we relate the concept of sparseness with the concept of isoperimetric estimates. First, we present a result which should be viewed in the light of Theorem 2.2 as it points out in which sense isoperimetric estimates are stronger than our notions of sparseness. In the second subsection, we present a result related to Theorem 3.2. Finally, we present a concrete comparison of sparseness and isoperimetric estimates. As this section is of a more geometric flavor we restrict ourselves to the case of potentials $q : \mathscr{V} \to [0, \infty)$.

5.1. Isoperimetric estimates.

Let $\mathscr{U} \subseteq \mathscr{V}$ and define the *Cheeger constant* of $\mathscr{U}$ by

$$\alpha_{\mathscr{U}} := \inf_{\mathscr{W}\subset\mathscr{U} \text{ finite}} \frac{|\partial\mathscr{W}| + q(\mathscr{W})}{\deg(\mathscr{W}) + q(\mathscr{W})}.$$

Note that $\alpha_{\mathscr{U}} \in [0, 1)$.

The following theorem illustrates in which sense positivity of the Cheeger constant is a stronger assumption than (a, k)-sparseness.

Theorem 5.1. *Given $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ a graph and $q : \mathscr{V} \to [0, \infty)$. The following assertions are equivalent*

(i) $\alpha_{\mathscr{V}} > 0$.

(ii) *There is* $a \in (0,1)$

$$(1-a)(\deg + q) \leq \Delta + q \leq (1+a)(\deg + q).$$

(iii) *There is* $a \in (0,1)$ *such that*

$$(1-a)(\deg + q) \leq \Delta + q.$$

(iv) *There is* $a \in (0,1)$ *such that* $(\mathscr{G}, q)$ *is* $(a,0)$*-sparse*

The implication (iii)⇒(i) is already found in [G, Proposition 3.4]. The implication (i)⇒(ii) is a consequence from standard isoperimetric estimates which can be extracted from the proof of [KL2, Proposition 15].

Proposition 5.2 ([KL2])**.** *Let* $\mathscr{G} = (\mathscr{E}, \mathscr{V})$ *be a graph and* $q : \mathscr{V} \to [0,\infty)$*. Then, for all* $\mathscr{U} \subseteq \mathscr{V}$ *we have on* $\mathcal{C}_c(\mathscr{U})$*.*

$$\big(1 - \sqrt{1-\alpha_{\mathscr{U}}^2}\big)(\deg + q) \leq \Delta + q \leq \big(1 + \sqrt{1-\alpha_{\mathscr{U}}^2}\big)(\deg + q).$$

With the proposition above the proof now follows rather directly.

Proof of Theorem 5.1. (i)⇒(ii) follows from the proposition above and (ii)⇒(iii) is trivial. For (iii)⇒(iv) we refer to Lemma 2.5. For (iv)⇒(i) note that $(a,0)$-sparseness implies

$$\frac{a(|\partial \mathscr{W}| + q(\mathscr{W}))}{\deg(\mathscr{W}) + q(\mathscr{W})} \geq \frac{2|\mathscr{E}_{\mathscr{W}}|}{\deg(\mathscr{W}) + q(\mathscr{W})} = 1 - \frac{|\partial \mathscr{W}| + q(\mathscr{W})}{\deg(\mathscr{W}) + q(\mathscr{W})}.$$

Hence, $\alpha_{\mathscr{V}} \geq 1/(1+a)$ which shows (i). □

5.2. **Isoperimetric estimates at infinity.** Let the *Cheeger constant at infinity* be defined as

$$\alpha_\infty = \sup_{\mathcal{K} \subseteq \mathscr{V} \text{ finite}} \alpha_{\mathscr{V} \setminus \mathcal{K}}.$$

Clearly $0 \leq \alpha_{\mathscr{V}} \leq \alpha_{\mathscr{U}} \leq \alpha_\infty \leq 1$ for any $\mathscr{U} \subseteq \mathscr{V}$.

As a consequence of Proposition 5.2, we get the following theorem.

Theorem 5.3. *Let* $\mathscr{G} = (\mathscr{E}, \mathscr{V})$ *be a graph and* $q : \mathscr{V} \to [0,\infty)$ *be a potential. Assume* $\alpha_\infty > 0$*. Then we have the following:*

(a) *For every* $\varepsilon > 0$ *there is* $k_\varepsilon \geq 0$ *such that on* $\mathcal{C}_c(\mathscr{V})$

$$(1-\varepsilon)\big(1 - \sqrt{1-\alpha_\infty^2}\big)(\deg + q) - k_\varepsilon \leq \Delta + q$$
$$\leq (1+\varepsilon)\big(1 + \sqrt{1-\alpha_\infty^2}\big)(\deg + q) + k_\varepsilon.$$

(b) $\mathcal{D}((\Delta + q)^{1/2}) = \mathcal{D}((\deg + q)^{1/2})$.

(c) *The operator* $\Delta + q$ *has purely discrete spectrum if and only if we have* $\liminf_{|x|\to\infty}(\deg + q)(x) = \infty$*. In this case, if additionally* $\alpha_\infty = 1$*, we get*

$$\liminf_{\lambda \to \infty} \frac{\lambda_n(\Delta + q)}{\lambda_n(\deg + q)} = 1.$$

Proof. (a) Let $\varepsilon > 0$ and $\mathcal{K} \subseteq \mathscr{V}$ be finite and large enough such that

$$(1-\varepsilon)\big(1 - \sqrt{1-\alpha_\infty^2}\big) \leq \big(1 - \sqrt{1-\alpha_{\mathscr{V}\setminus\mathcal{K}}^2}\big)$$
$$\big(1 + \sqrt{1-\alpha_{\mathscr{V}\setminus\mathcal{K}}^2}\big) \leq (1+\varepsilon)\big(1 + \sqrt{1-\alpha_\infty^2}\big).$$

From Proposition 5.2 we conclude on $\mathcal{C}_c(\mathscr{V}\setminus\mathcal{K})$

$$\begin{aligned}(1-\varepsilon)\big(1-\sqrt{1-\alpha_\infty^2}\big)(\deg+q) &\le \big(1-\sqrt{1-\alpha_{\mathscr{V}\setminus\mathcal{K}}^2}\big)(\deg+q)\\ &\le \Delta+q\le \big(1+\sqrt{1-\alpha_{\mathscr{V}\setminus\mathcal{K}}^2}\big)(\deg+q)\\ &\le (1+\varepsilon)\big(1+\sqrt{1-\alpha_\infty^2}\big)(\deg+q)\end{aligned}$$

By local finiteness the operators $\mathbf{1}_{\mathscr{V}\setminus\mathcal{K}}(\Delta+q)\mathbf{1}_{\mathscr{V}\setminus\mathcal{K}}$ and $\mathbf{1}_{\mathscr{V}\setminus\mathcal{K}}(\deg+q)\mathbf{1}_{\mathscr{V}\setminus\mathcal{K}}$ are bounded (indeed, finite rank) perturbations of $\Delta+q$ and $\deg+q$. This gives rise to the constants k_ε and the inequality of (a) follows on $\mathcal{C}_c(\mathscr{V})$. Now, (b) is an immediate consequence of (a), and (c) follows by the Min-Max-Principle, Theorem A.2. □

5.3. **Relating sparseness and isoperimetric estimates.**

We now explain how the notions of sparseness and isoperimetric estimates are exactly related.

First, we consider classical isoperimetric estimates.

Theorem 5.4. *Let $\mathscr{G}=(\mathscr{V},\mathscr{E})$ be a graph and let $q:\mathscr{V}\to[0,\infty)$ be a potential.*

(a) *If $(\mathscr{G},q)$ is (a,k)-sparse, then*

$$\alpha_{\mathscr{V}}\ge\frac{d-k}{d(1+a)},$$

where $d=\inf_{x\in\mathscr{V}}(\deg+q)(x)$. In particular, $\alpha_{\mathscr{V}}>0$ if $d>k$.

(b) *If $\mathscr{G}$ is a d-regular k-sparse graph which is not k'-sparse for all $k'<k$, then $\alpha_{\mathscr{V}}=\frac{d-k}{d}$.*

Proof. Recalling the identity $\deg(\mathscr{W})=2|\mathscr{E}_{\mathscr{W}}|+|\partial\mathscr{W}|$ for finite $\mathscr{W}\subset\mathscr{V}$, we obtain

$$\frac{|\partial\mathscr{W}|+q(\mathscr{W})}{(\deg+q)(\mathscr{W})}=1-\frac{2|\mathscr{E}_{\mathscr{W}}|}{(\deg+q)(\mathscr{W})}.$$

Employing (a,k)-sparseness for (a) and regularity and sparseness for (b), the statements follow. □

We next come to almost sparseness and show two 'almost equivalences'.

Theorem 5.5. *Let $\mathscr{G}=(\mathscr{V},\mathscr{E})$ be a graph and let $q:\mathscr{V}\to[0,\infty)$ be a potential.*

(a) *If $\alpha_\infty>0$, then $(\mathscr{G},q)$ is (a,k)-sparse for some $a>0$, $k\ge0$. On the other hand, if $(\mathscr{G},q)$ is (a,k)-sparse for some $a>0$, $k\ge0$ and*

$$l:=\liminf_{|x|\to\infty}(\deg+q)(x)>k,$$

then

$$\alpha_\infty\ge\frac{l-k}{l(a+1)}>0,$$

if l is finite and $\alpha_\infty\ge1/(1+a)$ otherwise.

(b) *If $\alpha_\infty=1$, then $(\mathscr{G},q)$ is almost sparse. On the other hand, if $(\mathscr{G},q)$ is almost sparse and $\liminf_{|x|\to\infty}(\deg+q)(x)=\infty$, then $\alpha_\infty=1$.*

Proof. The first implication of (a) follows from Theorem 5.3 (a) and Theorem 2.2 (ii)⇒(i). For the opposite direction let $\varepsilon>0$ and $\mathcal{K}\subseteq\mathscr{V}$ be finite such that

$\deg + q \geq l - \varepsilon$ on $\mathscr{V} \setminus \mathcal{K}$. Using the formula in the proof of Theorem 5.4, yields for $\mathscr{W} \subseteq \mathscr{V} \setminus \mathcal{K}$

$$\frac{|\partial \mathscr{W}| + q(\mathscr{W})}{(\deg + q)(\mathscr{W})} = 1 - \frac{2|\mathscr{E}_{\mathscr{W}}|}{(\deg + q)(\mathscr{W})} \geq 1 - \frac{k|\mathscr{W}| + a(|\partial \mathscr{W}| + q(\mathscr{W}))}{(\deg + q)(\mathscr{W})}$$
$$\geq 1 - \frac{k}{(l - \varepsilon)} - \frac{a(|\partial \mathscr{W}| + q(\mathscr{W}))}{(\deg + q)(\mathscr{W})}.$$

This proves (a).
The first implication of (b) follows from Theorem 5.3 (a) and Theorem 3.2 (ii)⇒(i). The other implication follows from (a) employing the definition of almost sparseness.
□

Remark 5.6. (a) We point out that the backwards implications without the assumptions on $(\deg + q)$ do not hold. For example the Cayley graph of $\mathbb{Z}$ is 2-sparse (cf. Lemma 6.2), but has $\alpha_\infty = 0$.
(b) Observe that $\alpha_\infty = 1$ implies $\liminf_{|x|\to\infty}(\deg + q)(x) = \infty$. Hence, (b) can be rephrased as the following equivalence: $\alpha_\infty = 1$ is equivalent to $(\mathscr{G}, q)$ almost sparse and $\liminf_{|x|\to\infty}(\deg + q)(x) = \infty$.
(c) In the situation of Theorem 5.4 and Theorem 5.5, using Proposition 5.2, one can recover exactly the same bounds for the bottom and the top of the (essential) spectrum as in Corollary 1.2.

The previous theorems provides a slightly simplified proof of [K1] which also appeared morally in somewhat different forms in [D1, Woe].

Corollary 5.7. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a planar graph.*
(a) *If for all vertices* $\deg \geq 7$*, then* $\alpha_{\mathscr{V}} > 0$.
(b) *If for all vertices away from a finite set* $\deg \geq 7$*, then* $\alpha_\infty > 0$.

Proof. Combine Theorem 5.4 and Theorem 5.5 with Lemma 6.2. □

6. Examples

6.1. **Examples of sparse graphs.** To start off, we exhibit two classes of sparse graphs. First we consider the case of graphs with bounded degree.

Lemma 6.1. *Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. Assume $D := \sup_{x\in\mathscr{V}} \deg(x) < +\infty$, then $\mathscr{G}$ is D-sparse.*

Proof. Let $\mathscr{W}$ be a finite subset of $\mathscr{V}$. Then, $2|\mathcal{E}_{\mathscr{W}}| \leq \deg(\mathscr{W}) \leq D|\mathscr{W}|$. □

We turn to graphs which admit a 2-cell embedding into S_g, where S_g denotes a compact orientable topological surface of genus g. (The surface S_g might be pictured as a sphere with g handles.) Admitting a 2-cell embedding means that the graphs can be embedded into S_g without self-intersection. By definition we say that a graph is planar when $g = 0$. Note that unlike other possible definitions of planarity, we do not impose any local compactness on the embedding.

Lemma 6.2. *(a) Trees are 2-sparse.*
(b) Planar graphs are 6-sparse.
(c) Graphs admitting a 2-cell embedding into S_g with $g \geq 1$ are $4g + 2$-sparse.

Proof. (a) Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a tree and $\mathscr{G}_{\mathscr{W}} = (\mathscr{W}, \mathscr{E}_{\mathscr{W}})$ be a finite induced subgraph of $\mathscr{G}$. Clearly $|\mathscr{E}_{\mathscr{W}}| \leq |\mathscr{W}| - 1$. Therefore, every tree is 2-sparse.

We treat the cases (b) and (c) simultaneously. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph which is connected 2-cell embedded in S_g with $g \geq 0$ (as remarked above planar graphs correspond to $g = 0$). Let $\mathscr{G}_{\mathscr{W}} = (\mathscr{W}, \mathscr{E}_{\mathscr{W}})$ be a finite induced subgraph of $\mathscr{G}$ which, clearly, also admits a 2-cell embedding into S_g. The statement is clear for $|\mathscr{W}| \leq 2$. Assume $|\mathscr{W}| \geq 3$. Let $\mathscr{F}_{\mathscr{W}}$ be the faces induced by $\mathscr{G}_{\mathscr{W}} = (\mathscr{W}, \mathscr{E}_{\mathscr{W}})$ in S_g. Here, all faces (even the outer one) contain at least 3 edges, each edge belongs only to 2 faces, thus,

$$2|\mathscr{E}_{\mathscr{W}}| \geq 3|\mathscr{F}_{\mathscr{W}}|.$$

Euler's formula, $|\mathscr{W}| - |\mathscr{E}_{\mathscr{W}}| + |\mathscr{F}_{\mathscr{W}}| = 2 - 2g$, gives then

$$2 - 2g + |\mathscr{E}_{\mathscr{W}}| = |\mathscr{W}| + |\mathscr{F}_{\mathscr{W}}| \leq |\mathscr{W}| + \frac{2}{3}|\mathscr{E}_{\mathscr{W}}|$$

that is

$$|\mathscr{E}_{\mathscr{W}}| \leq 3|\mathscr{W}| + 6(g-1) \leq \max(2g+1, 3)|\mathscr{W}|.$$

This concludes the proof. □

Next, we explain how to construct sparse graphs from existing sparse graphs.

Lemma 6.3. *Let $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$ be two graphs.*

(a) *Assume $\mathscr{V}_1 = \mathscr{V}_2$, $\mathscr{G}_1$ is k_1-sparse and $\mathscr{G}_2$ is k_2-sparse. Then, $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with $\mathscr{E} := \max(\mathscr{E}_1, \mathscr{E}_2)$ is $(k_1 + k_2)$-sparse.*

(b) *Assume $\mathscr{G}_1$ is k_1-sparse and $\mathscr{G}_2$ is k_2-sparse. Then $\mathcal{G}_1 \oplus \mathcal{G}_2 = (\mathscr{V}, \mathscr{E})$ with where $\mathscr{V} := \mathscr{V}_1 \times \mathscr{V}_2$ and*

$$\mathscr{E}\left((x_1, x_2), (y_1, y_2)\right) := \delta_{\{x_1\}}(y_1) \cdot \mathscr{E}_2(x_2, y_2) + \delta_{\{x_2\}}(y_2) \cdot \mathscr{E}_1(x_1, y_1),$$

is $(k_1 + k_2)$-sparse.

(c) *Assume $\mathscr{V}_1 = \mathscr{V}_2$, $\mathscr{G}_1$ is k-sparse and $\mathscr{E}_2 \leq \mathscr{E}_1$. Then, $\mathscr{G}_2$ is k-sparse.*

Proof. For (a) let $\mathscr{W} \subseteq \mathscr{V}$ be finite and note that $|\mathscr{E}_{\mathscr{W}}| \leq |\mathscr{E}_{1,\mathscr{W}}| + |\mathscr{E}_{2,\mathscr{W}}|$. For (b) let p_1, p_2 the canonical projections from $\mathscr{V}$ to $\mathscr{V}_1$ and $\mathscr{V}_2$. For finite $\mathscr{W} \subseteq \mathscr{V}$ we observe

$$|\mathscr{E}_{\mathscr{W}}| = |\mathscr{E}_{1,p_1(\mathscr{W})}| + |\mathscr{E}_{2,p_2(\mathscr{W})}| \leq k_1|p_2(\mathscr{W})| + k_2|p_1(\mathscr{W})| \leq (k_1 + k_2)|\mathscr{W}|.$$

For (c) and $\mathscr{W} \subseteq \mathscr{V}$ finite, we have $|\mathscr{E}_{2,\mathscr{W}}| \leq |\mathscr{E}_{1,\mathscr{W}}|$ which yields the statement. □

Remark 6.4. We point out that there are bi-partite graphs which are not sparse. See for example [G, Proposition 4.11] or take an antitree, confer [KLW, Section 6], where the number of vertices in the spheres grows monotonously to ∞.

6.2. **Examples of almost-sparse and (a, k)-sparse graph.** We construct a series of examples which are perturbations of a radial tree. They illustrate that sparseness, almost sparseness and (a, k)-sparseness are indeed different concepts.

Let $\beta = (\beta_n)$, $\gamma = (\gamma_n)$ be two sequences of natural numbers. Let $\mathscr{T} = \mathscr{T}(\beta)$ with $\mathscr{T} = (\mathscr{V}, \mathscr{E}^{\mathscr{T}})$ be a radial tree with root o and vertex degree β_n at the n-th sphere, that is every vertex which has natural graph distance n to o has $(\beta_n - 1)$ forward neighbors. We denote the distance spheres by S_n. We let $\mathscr{G}(\beta, \gamma)$ be the set of graphs $\mathscr{G} = (\mathscr{V}, \mathscr{E}^{\mathscr{G}})$ that are super graphs of $\mathscr{T}$ such that the induced subgraphs $\mathscr{G}_{S_n}$ are γ_n-regular and $\mathscr{E}^{\mathscr{G}}(x, y) = \mathscr{E}^{\mathscr{T}}(x, y)$ for $x \in S_n, y \in S_m$, $m \neq n$. Observe that $\mathscr{G}(\beta, \gamma)$ is non empty if and only if $\gamma_n \prod_{j=0}^{n}(\beta_j - 1)$ is even and $\gamma_n < |S_n| = \prod_{j=0}^{n}(\beta_j - 1)$ for all $n \geq 0$.

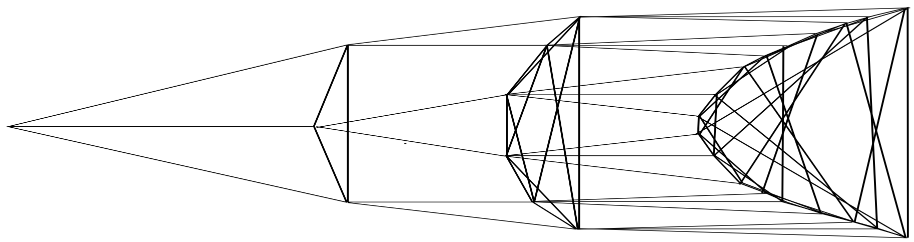

FIGURE 1. $\mathscr{G}$ with $\beta = (3, 3, 4, \ldots)$ and $\gamma = (0, 2, 4, 5, \ldots)$.

Proposition 6.5. *Let $\beta, \gamma \in \mathbb{N}_0^{\mathbb{N}_0}$, $a = \limsup_{n\to\infty} \gamma_n/\beta_n$ and $\mathscr{G} \in \mathscr{G}(\beta, \gamma)$.*

(a) *If $a = 0$, then $\mathscr{G}$ is almost sparse. The graph $\mathscr{G}$ is sparse if and only if $\limsup_{n\to\infty} \gamma_n < \infty$.*
(b) *If $a > 0$, then $\mathscr{G}$ is (a', k)-sparse for some $k \geq 0$ if $a' > a$. Conversely, if $\mathscr{G}$ is (a', k)-sparse for some $k \geq 0$, then $a' \geq a$.*

Proof. Let $\varepsilon > 0$ and let $N \geq 0$ be so large that

$$\gamma_n \leq (a + \varepsilon)\beta_n, \quad n \geq N.$$

Set $C_\varepsilon := \sum_{n=0}^{N-1} \deg^{\mathscr{G}}(S_n)$. Let $\mathscr{W}$ be a non-empty finite subset of $\mathscr{V}$. We calculate

$$\begin{aligned}
2|\mathscr{E}_{\mathscr{W}}^{\mathscr{G}}| + |\partial^{\mathscr{G}}\mathscr{W}| = \deg^{\mathscr{G}}(\mathscr{W}) &= \deg^{\mathscr{T}}(\mathscr{W}) + \sum_{n\geq 0} |\mathscr{W} \cap S_n|\gamma_n \\
&\leq \deg^{\mathscr{T}}(\mathscr{W}) + (a + \varepsilon)\sum_{n\geq 0} |\mathscr{W} \cap S_n|\beta_n + \sum_{n=0}^{N-1} |\mathscr{W} \cap S_n|\gamma_n \\
&\leq (1 + a + \varepsilon)\deg^{\mathscr{T}}(\mathscr{W}) + C_\varepsilon|\mathscr{W}| \\
&= 2(1 + a + \varepsilon)|\mathscr{E}_{\mathscr{W}}^{\mathscr{T}}| + (1 + a + \varepsilon)|\partial^{\mathscr{T}}\mathscr{W}| + C_\varepsilon|\mathscr{W}| \\
&\leq (2(1 + a + \varepsilon) + C_\varepsilon)|\mathscr{W}| + (1 + a + \varepsilon)|\partial^{\mathscr{T}}\mathscr{W}|,
\end{aligned}$$

where we used that trees are 2-sparse in the last inequality. Finally, since $|\partial^{\mathscr{G}}\mathscr{W}| \geq |\partial^{\mathscr{T}}\mathscr{W}|$, we conclude

$$2|\mathscr{E}_{\mathscr{W}}^{\mathscr{G}}| \leq (2(1 + a + \varepsilon) + C_\varepsilon)\,|\mathscr{W}| + (a + \varepsilon)|\partial^{\mathscr{G}}\mathscr{W}|.$$

This shows that the graph in (a) with $a = 0$ is almost sparse and that the graph in (b) with $a > 0$ is $(a + \varepsilon, k_\varepsilon)$-sparse for $\varepsilon > 0$ and $k_\varepsilon = 2(1 + a + \varepsilon) + C_\varepsilon$. Moreover, for the other statement of (a) let $k_0 = \limsup_{n\to\infty} \gamma_n$ and note that for $\mathscr{G}_{S_n}$

$$2|\mathscr{E}_{S_n}| = \gamma_n|S_n|.$$

Hence, if $k_0 = \infty$, then $\mathscr{G}$ is not sparse. On the other hand, if $k_0 < \infty$, then $\mathscr{G}$ is $(k_0 + 2)$-sparse by Lemma 6.3 as $\mathscr{T}$ is 2-sparse by Lemma 6.2. This finishes the proof of (a). Finally, assume that $\mathscr{G}$ is (a', k)-sparse with $k \geq 0$. Then, for $\mathscr{W} = S_n$

$$\gamma_n|S_n| = 2|\mathscr{E}_{S_n}| \leq k|S_n| + a'|\partial^{\mathscr{G}} S_n| = k|S_n| + a'\beta_n|S_n|$$

Dividing by $\beta_n|S_n|$ and taking the limit yields $a \leq a'$. This proves (b). □

Remark 6.6. In (a), we may suppose alternatively that we have the complete graph on S_n and the following exponential growth $\lim_{n\to\infty} \frac{|S_n|}{|S_{n+1}|} = 0$

Appendix A. Some general operator theory

We collect some consequences of standard results from functional analysis that are used in the paper. Let H be a Hilbert space with norm $\|\cdot\|$. For a quadratic form Q, denote the form norm by $\|\cdot\|_Q = \sqrt{Q(\cdot)+\|\cdot\|^2}$. The following is a direct consequence of the Closed Graph Theorem, (confer e.g. [We, Satz 4.7]).

Theorem A.1. *Let $(Q_1, \mathcal{D}(Q_1))$ and $(Q_2, \mathcal{D}(Q_2))$ be closed non-negative quadratic forms with a common form core $\mathcal{D}_0$. Then, the following are equivalent.*

(i) $\mathcal{D}(Q_1) \leq \mathcal{D}(Q_2)$.

(ii) *There are constants $c_1 > 0$, $c_2 \geq 0$ such that $c_1 Q_2 - c_2 \leq Q_1$ on D_0.*

Proof. If (ii) holds, then any $\|\cdot\|_{Q_1}$-Cauchy sequence is a $\|\cdot\|_{Q_2}$-Cauchy sequence. Thus, (ii) implies (i). On the other hand, consider the identity map $j : (\mathcal{D}(Q_1), \|\cdot\|_{Q_1}) \to (\mathcal{D}(Q_2), \|\cdot\|_{Q_2})$. The map j is closed as it is defined on the whole Hilbert space $(\mathcal{D}(Q_1), \|\cdot\|_{Q_1})$ and, thus, bounded by the Closed Graph Theorem [RS, Theorem III.12] which implies (i). □

For a selfadjoint operator T which is bounded from below, we denote the bottom of the spectrum by $\lambda_0(T)$ and the bottom of the essential spectrum by $\lambda_0^{\mathrm{ess}}(T)$. Let $n(T) \in \mathbb{N}_0 \cup \{\infty\}$ be the dimension of the range of the spectral projection of $(-\infty, \lambda_0^{\mathrm{ess}}(T))$. For $\lambda_0(T) < \lambda_0^{\mathrm{ess}}(T)$ we denote the eigenvalues below $\lambda_0^{\mathrm{ess}}(T)$ by $\lambda_n(T)$, for $0 \leq n \leq n(T)$, in increasing order counted with multiplicity.

Theorem A.2. *Let $(Q_1, \mathcal{D}(Q_1))$ and $(Q_2, \mathcal{D}(Q_2))$ be closed non-negative quadratic forms with a common form core $\mathcal{D}_0$ and let T_1 and T_2 be the corresponding selfadjoint operators. Assume there are constants $c_1 > 0$, $c_2 \in \mathbb{R}$ such that on $\mathcal{D}_0$*

$$c_1 Q_2 - c_2 \leq Q_1.$$

Then, $c_1 \lambda_n(T_2) - c_2 \leq \lambda_n(T_1)$, for $0 \leq n \leq \min(n(T_1), n(T_2))$. Moreover, $c_1 \lambda_0^{\mathrm{ess}}(T_2) - c_2 \leq \lambda_0^{\mathrm{ess}}(T_1)$, in particular, $\sigma_{\mathrm{ess}}(T_1) = \emptyset$ if $\sigma_{\mathrm{ess}}(T_2) = \emptyset$ and in this case

$$c_1 \leq \liminf_{n\to\infty} \frac{\lambda_n(T_1)}{\lambda_n(T_2)}.$$

Proof. Letting

$$\mu_n(T) = \sup_{\varphi_1,\ldots,\varphi_n \in H} \inf_{0\neq\psi\in\{\varphi_1,\ldots,\varphi_n\}^\perp \cap \mathcal{D}_0} \frac{\langle T\psi, \psi\rangle}{\langle \psi, \psi\rangle},$$

for a selfadjoint operator T, we know by the Min-Max-Principle [RS, Chapter XIII.1] $\mu_n(T) = \lambda_n(T)$ if $\lambda_n(T) < \lambda_0^{\mathrm{ess}}(T)$ and $\mu_n(T) = \lambda_0^{\mathrm{ess}}(T)$ otherwise, $n \geq 0$. Assume $n \leq \min\{n(T_1), n(T_2)\}$ and let $\varphi_0^{(j)}, \ldots, \varphi_n^{(j)}$ be the eigenfunctions of T_j to $\lambda_0(T_j), \ldots, \lambda_n(T_j)$ we get

$$\begin{aligned} c_1 \lambda_n(T_2) - c_3 &= \inf_{0\neq\psi\in\{\varphi_1^{(2)},\ldots,\varphi_n^{(2)}\}^\perp \cap \mathcal{D}_0} \Big(c_1 \frac{\langle T_2\psi, \psi\rangle}{\langle \psi, \psi\rangle} - c_3 \Big) \\ &\leq \inf_{0\neq\psi\in\{\varphi_1^{(2)},\ldots,\varphi_n^{(2)}\}^\perp \cap \mathcal{D}_0} \frac{\langle T_1\psi, \psi\rangle}{\langle \psi, \psi\rangle} \leq \mu_n(T_1) = \lambda_n(T_1) \end{aligned}$$

This directly implies the first statement. By a similar argument the statement about the bottom of the essential spectrum follows, in particular, $\lambda_0^{\mathrm{ess}}(T_2) = \infty$ implies $\lim_{n\to\infty} \mu_n(T_1) = \infty$ and, thus, $\lambda_0^{\mathrm{ess}}(T_1) = \infty$. In this case $\lambda_n(T_2) \to \infty$, $n \to \infty$, which implies the final statement. □

Finally, we give a lemma which helps us to transform inequalities under form perturbations.

Lemma A.3. *Let* $(Q_1, \mathcal{D}(Q_1))$, $(Q_2, \mathcal{D}(Q_2))$ *and* $(q, \mathcal{D}(q))$ *be closed symmetric non-negative quadratic forms with a common form core* $\mathcal{D}_0$ *such that there are* $\alpha \in (0,1)$, $C \geq 0$ *such that*

$$q \leq \alpha Q_1 + C \text{ and } q \leq Q_1$$

on $\mathcal{D}_0$. *Moreover, let* $a \in (0,1)$ *and* $k \geq 0$ *be given*

(a) *If* $(1-a)Q_2 - k \leq Q_1$ *on* $\mathcal{D}_0$, *then*

$$\frac{(1-\alpha)(1-a)}{(1-\alpha(1-a))}(Q_2 - q) - \frac{(1-\alpha)k + aC}{(1-\alpha(1-a))} \leq Q_1 - q \qquad \text{on } \mathcal{D}_0.$$

(b) *If* $Q_1 \leq (1+a)Q_2 + k$ *on* $\mathcal{D}_0$ *and* $\alpha < 1/(1+a)$, *then*

$$Q_1 - q \leq \frac{(1-\alpha)(1+a)}{(1-\alpha(1+a))}(Q_2 - q) + \frac{(1-\alpha)k + aC}{(1-\alpha(1+a))} \qquad \text{on } \mathcal{D}_0.$$

In particular, if $a \to 0$, *then* $(1-\alpha)(1 \pm a)/(1-\alpha(1 \pm a)) \to 1$ *and if* $\alpha \to 0$, *then* $(1-\alpha)(1 \pm a)/(1-\alpha(1 \pm a)) \to (1 \pm a)$.

Proof. The assumption on q implies

$$q \leq \frac{\alpha}{(1-\alpha)}(Q_1 - q) + \frac{C}{(1-\alpha)}.$$

(a) For the lower bound, we subtract $(1-a)q$ on each side of the lower bound in the assumed inequality. Then, we get

$$(1-a)(Q_2 - q) - k \leq (Q_1 - q) + aq \leq \frac{1-\alpha(1-a)}{(1-\alpha)}(Q_1 - q) + \frac{aC}{(1-\alpha)}$$

and, thus,

$$\frac{(1-\alpha)(1-a)}{(1-\alpha(1-a))}(Q_2 - q) - \frac{(1-\alpha)k + aC}{(1-\alpha(1-a))} \leq Q_1 - q.$$

For the upper bound we get after subtracting $(1+a)q$

$$(1+a)(Q_2 - q) + k \geq Q_1 - q - aq \geq \frac{1-\alpha(1+a)}{(1-\alpha)}(Q_1 - q) - \frac{aC}{(1-\alpha)}.$$

Observing that $1 - \alpha(1+a) > 0$ if and only if $\alpha < 1/(1+a)$ we deduce the statement. ☐

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References

[AABL] N. Alon, O. Angel, I. Benjamini, E. Lubetzky, *Sums and products along sparse graphs*, Israel J. Math. **188** (2012), 353–384.

[BHJ] F. Bauer, B. Hua, J. Jost, *The dual Cheeger constant and spectra of infinite graphs*, to appear in Adv. in Math., arXiv:1207.3410.

[BJL] F. Bauer, J. Jost, S. Liu, *Ollivier-Ricci curvature and the spectrum of the normalized graph Laplace operator*, Mathematical research letters, 19 (2012) 6, 1185–1205.

[BKW] F. Bauer, M. Keller, R.K. Wojciechowski, *Cheeger inequalities for unbounded graph Laplacians*, to appear in J. Eur. Math. Soc. (JEMS), arXiv:1209.4911, (2012).

[B] J. Breuer, *Singular continuous spectrum for the Laplacian on certain sparse trees*, Comm. Math. Phys. **269**, no. 3, (2007) 851–857.

[D1] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc. **284**, no. 2, (1984)787–794.

[D2] J. Dodziuk, *Elliptic operators on infinite graphs*, Analysis, geometry and topology of elliptic operators, World Sci. Publ., Hackensack, NJ, 2006, 353–368.

[DK] J. Dodziuk, W. S. Kendall, *Combinatorial Laplacians and isoperimetric inequality*, From Local Times to Global Geometry, Control and Physics, Pitman Res. Notes Math. Ser., 150, (1986) 68–74.

[DM] J. Dodziuk, V. Matthai, *Kato's inequality and asymptotic spectral properties for discrete magnetic Laplacians*, The ubiquitous heat kernel, Cont. Math. **398**, Am. Math. Soc. (2006), 69–81.

[EGS] P. Erdös, R.L. Graham, E. Szemerédi, *On sparse graphs with dense long paths*, Computers and Mathematics with Applications **1**, Issues 3-4, (1975) 365–369.

[F] K. Fujiwara, *Laplacians on rapidly branching trees*, Duke Math Jour. **83**, (1996) 191–202.

[G] S. Golénia, *Hardy inequality and eigenvalue asymptotic for discrete Laplacians*, to appear in J. Funct. Anal., arXiv:1106.0658.

[GG] V. Georgescu, S. Golénia, *Decay Preserving Operators and stability of the essential spectrum*, J. Operator Theory **59**, no. 1, (2008) 115–155.

[GKS] B. Güneysu, M. Keller, M. Schmidt, *A Feynman-Kac-Itō Formula for magnetic Schrödinger operators on graphs*, 2012, arXiv:1301.1304.

[H] Y. Higuchi, *Combinatorial Curvature for Planar Graphs*, Journal of Graph Theory **38**, Issue 4, (2001) 220-229.

[JL] J. Jost, S. Liu, *Ollivier's Ricci curvature, local clustering and curvature dimension inequalities on graphs*, preprint 2011, arXiv:1103.4037v2.

[K1] M. Keller, *Essential spectrum of the Laplacian on rapidly branching tessellations*, Math. Ann. **346**, (2010) 51–66.

[K2] M. Keller, *Curvature, geometry and spectral properties of planar graphs*, Discrete Comput. Geom., **46**, (2011), 500-525

[KL1] M. Keller, D. Lenz, *Dirichlet forms and stochastic completeness of graphs and subgraphs*, J. Reine Angew. Math. **666**, (2012), 189–223.

[KL2] M. Keller, D. Lenz, *Unbounded Laplacians on graphs: basic spectral properties and the heat equation*, Math. Model. Nat. Phenom. **5**, no. 4, (2010) 198–224.

[KLW] M. Keller, D. Lenz, R.K. Wojciechowski, *Volume Growth, Spectrum and Stochastic Completeness of Infinite Graphs*, Math. Z. **274**, Issue 3 (2013), 905–932.

[KP] M. Keller, N. Peyerimhoff, *Cheeger constants, growth and spectrum of locally tessellating planar graphs*, Math. Z., **268**, (2011), 871-886.

[LS] A. Lee, I. Streinu, *Pebble game algorithms and sparse graphs*, Discrete Math. **308** (2008), no. 8, 1425–1437.

[LY] Y. Lin, S.T. Yau, *Ricci curvature and eigenvalue estimate on locally finite graphs*, Math. Res. Lett. **17** (2010), 343–356.

[L] M. Loréa, *On matroidal families*, Discrete Math. **28**, (1979) 103–106.

[M1] B. Mohar, *Isoperimetics inequalities, growth and the spectrum of graphs*, Linear Algebra Appl. **103**, (1988), 119–131.

[M2] B. Mohar, *Some relations between analytic and geometric properties of infinite graphs,* Discrete Math. 95 (1991), no. 1–3, 193–219.

[M3] B. Mohar, *Many large eigenvalues in sparse graphs*, European J. Combin. **34**, no. 7, (2013) 1125–1129.

[RS] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I, II, IV: Functional analysis. Fourier analysis, Self-adjointness*, Academic Press, New York e.a., 1975.

[SV] P. Stollmann, J. Voigt, *Perturbation of Dirichlet forms by measures*, Potential Anal. **5** (1996), 109–138.

[We] J. Weidmann, *Lineare Operatoren in Hilberträumen 1*, B. G. Teubner, Stuttgart, 2000.

[Woe] W. Woess, *A note on tilings and strong isoperimetric inequality*, Math. Proc. Camb. Phil. Soc. **124**, (1998) 385-393.

[Woj1] R. K. Wojciechowski, *Stochastic completeness of graphs*, ProQuest LLC, Ann Arbor, MI, 2008, Thesis (Ph.D.)–City University of New York.

[Woj2] R. K. Wojciechowski,*Heat kernel and essential spectrum of infinite graphs*, Indiana Univ. Math. J. **58** (2009), 1419–1441.

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