HARDY INEQUALITY AND ASYMPTOTIC EIGENVALUE DISTRIBUTION FOR DISCRETE LAPLACIANS

SYLVAIN GOLÉNIA

Abstract. In this paper we study in detail some spectral properties of the magnetic discrete Laplacian. We identify its form-domain, characterize the absence of essential spectrum and provide the asymptotic eigenvalue distribution.

Contents

1. Introduction

The uncertainty principle is a central point in quantum physics. It can be expressed by the following Hardy inequality:

$$\left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \left|\frac{1}{|x|} f(x)\right|^2 dx \leq \int_{\mathbb{R}^n} |\nabla f|^2 \, dx = \langle f, -\Delta_{\mathbb{R}^n} f\rangle, \text{ where } n \geq 3, \tag{1.1}$$

and $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Roughly speaking, the Laplacian controls some local singularities of a potential. In this paper, we investigate which potentials a discrete Laplacian is able to control. Obviously, since the value of a potential on a vertex has to be finite, we will not focus on local singularities. However, unlike in the continuous case, we will control potentials that explode at infinity.

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We start with some definitions and fix our notation for graphs. We refer to [CdV, Chu, MW] for surveys on the matter. Let $\mathscr{V}$ be a countable set. Let $\mathscr{E} := \mathscr{V} \times \mathscr{V} \to [0, \infty)$ and assume that

$$\mathscr{E}(x,y) = \mathscr{E}(y,x), \quad \text{for all } x, y \in \mathscr{V}.$$

We say that $G := (\mathscr{E}, \mathscr{V})$ is an unoriented weighted graph with *vertices* $\mathscr{V}$ and *weighted edges* $\mathscr{E}$. In the setting of electrical networks, the weights correspond to the conductances. We say that $x, y \in \mathscr{V}$ are *neighbors* if $\mathscr{E}(x,y) \neq 0$ and denote it by $x \sim y$. We say that there is a *loop* in $x \in \mathscr{V}$ if $\mathscr{E}(x,x) \neq 0$. The set of *neighbors* of $x \in \mathscr{E}$ is denoted by

$$\mathscr{N}_G(x) := \{y \in \mathscr{E}, x \sim y\}.$$

The *degree* of $x \in V$ is by definition $|\mathscr{N}_G(x)|$, the number of neighbors of x. A graph is *locally finite* if $|\mathscr{N}_G(x)|$ is finite for all $x \in V$. We also need a weight on the vertices

$$m : \mathscr{V} \to (0, \infty).$$

Finally, as we are dealing with magnetic fields, we fix a phase

$$\theta : \mathscr{V} \times \mathscr{V} \to [-\pi, \pi], \text{ such that } \theta(x,y) = -\theta(y,x).$$

We set $\theta_{x,y} := \theta(x,y)$. A graph is *connected*, if for all $x, y \in V$, there exists an *x-y-path*, i.e., there is a finite sequence

$$(x_1, \ldots, x_{N+1}) \in \mathscr{V}^{N+1} \text{ such that } x_1 = x,\ x_{N+1} = y \text{ and } x_n \sim x_{n+1},$$

for all $n \in \{1, \ldots, N\}$. The minimal possible N is called the (unweighted) *distance* between x and y.

We recall that a graph G is *simple* if $\mathscr{E}$ has values in $\{0,1\}$, $m = 1$, $\theta = 0$, and has no loop. A *bi-partite* graph is a graph whose vertex set can be partitioned into two subsets in such a way that no two points in the same subset are neighbors. Trees are bi-partite graphs.

In the sequel, we shall always consider (magnetic) graphs $G = (\mathscr{V}, \mathscr{E}, m, \theta)$, which are locally finite, connected and have no loop. We also fix $\omega \in \mathscr{V}$ and denote by $|x|$ the distance between x and ω.

We now associate a certain Hilbert space and some operators on it to a given graph $G = (\mathscr{V}, \mathscr{E}, m, \theta)$. Let $\ell^2(G, m^2) := \ell^2(\mathscr{V}, m^2; \mathbb{C})$ be the set of functions $f : \mathscr{V} \to \mathbb{C}$, such that $\|f\|^2 := \sum_{x\in\mathscr{V}} m(x)^2 |f(x)|^2$ is finite. The associated scalar product is given by $\langle f, g\rangle = \sum_{x\in\mathscr{V}} m^2(x)\overline{f(x)}g(x)$, for $f, g \in \ell^2(\mathscr{V}, m^2)$. We also denote by $\mathcal{C}_c(\mathscr{V})$ the set of functions $f : \mathscr{V} \to \mathbb{C}$, which have finite support. We define the quadratic form:

$$\mathscr{Q}(f,f) := \mathscr{Q}_{\mathscr{E},\theta}(f,f) := \frac{1}{2} \sum_{x,y\in\mathscr{V}} \mathscr{E}(x,y)|f(x) - e^{\mathrm{i}\theta_{x,y}} f(y)|^2 \geq 0, \text{ for } f \in \mathcal{C}_c(V). \tag{1.2}$$

It is closable and there exists a unique self-adjoint operator $\Delta_{\mathscr{E},\theta}$, such that

$$\mathscr{Q}_{\mathscr{E},\theta}(f,f) = \langle f, \Delta_{\mathscr{E},\theta} f\rangle, \text{ for } f \in \mathcal{C}_c(V)$$

and $\mathcal{D}(\Delta^{1/2}_{\mathscr{E},\theta}) = \mathcal{D}(\mathscr{Q}_{\mathscr{E},\theta})$, where the latter is the completion of $\mathcal{C}_c(\mathscr{V})$ under $\|\cdot\|^2 + \mathscr{Q}_{\mathscr{E},\theta}(\cdot,\cdot)$. This operator is the *Friedrichs extension* associated to the form $\mathscr{Q}_{\mathscr{E},\theta}$ (see Section 2.1 for its construction). It acts as follows:

$$\Delta_{\mathscr{E},\theta} f(x) := \frac{1}{m^2(x)} \sum_{y\in\mathscr{V}} \mathscr{E}(x,y)(f(x) - e^{\mathrm{i}\theta_{x,y}} f(y)), \text{ for } f \in \mathcal{C}_c(\mathscr{V}). \tag{1.3}$$

When $m = 1$, it is essentially self-adjoint on $\mathcal{C}_c(V)$ (see Section 2.2 for further discussion). If G is simple, we shall simply write Δ_G. There exist other definitions for the discrete Laplacian, e.g., [CdV, Chu, MW], the one we study here is sometimes called the "physical Laplacian".

In $\ell^2(G, m^2)$, we define the *weighted degree* by

$$d_G(x) := \frac{1}{m^2(x)} \sum_{y\in\mathscr{V}} \mathscr{E}(x,y).$$

Given a function $V : \mathscr{V} \to \mathbb{C}$, we denote by $V(Q)$ the operator of multiplication by V. It is elementary that $\mathcal{D}(d_G^{1/2}(Q)) \subset \mathcal{D}(\Delta_{\mathscr{E},\theta}^{1/2})$. Indeed, one has:

$$\begin{aligned}\langle f, \Delta_{\mathscr{E},\theta} f\rangle &= \frac{1}{2}\sum_{x\in\mathscr{V}}\sum_{y\sim x} \mathscr{E}(x,y)|f(x) - e^{\mathrm{i}\theta_{x,y}} f(y)|^2 \\ &\leq \sum_{x\in\mathscr{V}}\sum_{y\sim x} \mathscr{E}(x,y)(|f(x)|^2 + |f(y)|^2) = 2\langle f, d_G(Q) f\rangle,\end{aligned} \tag{1.4}$$

for $f \in \mathcal{C}_c(\mathscr{V})$. This inequality also gives a necessary condition for the absence of essential spectrum for $\Delta_{\mathscr{E},\theta}$ (see Corollary 2.9). In Proposition 4.11, we also prove that, in general, the constant 2 cannot be improved. It is also easy to see that $\Delta_{\mathscr{E},\theta}$ is bounded if and only if $d_G(Q)$ is (see Proposition 2.1).

In this paper we are interested in minorating the Laplacian with the help of the weighted degree. In the case of non-magnetic Laplacians, one standard approach is to use isoperimetric inequalities. The classical version gives estimates on the bottom of the spectrum . This is not adapted to our situation. We rely on a modified version. We define the following isoperimetric constant associated to (the weighted degree of) G by

$$\alpha(G) := \inf_{W\subset\mathscr{V},\, \sharp W<\infty} \frac{\langle \mathbf{1}_W, \Delta_{\varepsilon,0}\mathbf{1}_W\rangle}{\langle \mathbf{1}_W, d_G(Q)\mathbf{1}_W\rangle},$$

where $\mathbf{1}_X$ denotes the characteristic function of X. By [KL, page 14, line -4] (see also [Do, DK, Kel] and references therein), where α reads $\alpha_{b,c,n}$, we obtain:

$$\left(1 - \sqrt{1-\alpha^2(G)}\right)\langle f, d_G(Q) f\rangle \leq \langle f, \Delta_{\mathscr{E},0}\, f\rangle \leq \left(1 + \sqrt{1-\alpha^2(G)}\right)\langle f, d_G(Q) f\rangle, \tag{1.5}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. So, if $\alpha(G) > 0$, (1.4) is improved and we have: $\mathcal{D}(\Delta_{\mathscr{E},0}^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$. In particular, by Proposition 2.7, one sees that the essential spectrum $\sigma_{\mathrm{ess}}(d_G(Q))$ is empty if and only if $\sigma_{\mathrm{ess}}(\Delta_{\mathscr{E},0})$ is. We refer to Theorem 4.2 for the equality of the domains and to Proposition 5.2 for the stability of the essential spectrum.

We point out that a converse is also true. Namely, Proposition 3.4 ensures that if there is $a > 0$ so that $a\langle f, \Delta_{\mathscr{E},0}\, f\rangle \geq \langle f, d_G(Q) f\rangle$, for all $f \in \mathcal{C}_c(\mathscr{V})$, then $\alpha(G) > 0$.

Assume that $\alpha(G) > 0$. Supposing that $\sigma_{\mathrm{ess}}(\Delta_{\mathscr{E},0}) = \emptyset$ (or equivalently that $\lim_{|x|\to\infty} d_G(x) = +\infty$), the inequality (1.5) and the min-max principle, see Proposition 2.7, provide the bound

$$\left(1 - \sqrt{1-\alpha^2(G)}\right) \leq \liminf_{\lambda\to\infty} \frac{N_\lambda(\Delta_{\mathscr{E},0})}{N_\lambda(d_G(Q))} \leq \limsup_{\lambda\to\infty} \frac{N_\lambda(\Delta_{\mathscr{E},0})}{N_\lambda(d_G(Q))} \leq \left(1 + \sqrt{1-\alpha^2(G)}\right),$$

where

$$N_\lambda(A) := \dim \operatorname{Ran} \mathbf{1}_{(-\infty,\lambda]}(A),$$

for a self-adjoint operator A. This estimate has to be refined so as to give the asymptotic of eigenvalues and to deal with magnetic fields. Moreover, (1.5) is not stable by small perturbation for the question of the equality of the form-domain. For instance, take a simple graph G_1, such that $\alpha(G_1) > 0$ and the (simple) half-line graph G_2. Note that $\alpha(G_2) = 0$. Now connect the disjoint union of G_1 and G_2 by one edge to obtain a new graph G. One sees easily that $\alpha(G) = 0$ and $\mathcal{D}(\Delta_G^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$. This is why we seek a minoration by $ad_G(Q) - b$ for some $a, b > 0$.

On the other hand, given $m_0 : \mathscr{V} \to (0,\infty)$, we know that the Laplacian acting in $\ell^2(\mathscr{V}, m^2)$ is unitarily equivalent to a Schrödinger operator acting in $\ell^2(\mathscr{V}, m_0^2)$ (see Proposition 3.2). This has already been noticed before, e.g., [CTT, HK]. By extracting some positivity, we obtain our analog of the Hardy inequality:

Proposition 1.1. *Let $G = (\mathscr{V}, \mathscr{E}, m_0, \theta)$ be a locally finite graph. Given $m : \mathscr{V} \to (0, \infty)$, one has*

$$\langle f, V_m(Q)f\rangle \leq \langle f, \Delta_{\mathscr{E},\theta} f\rangle, \text{ for } f \in \mathcal{C}_c(\mathscr{V}), \tag{1.6}$$

where

$$V_m(x) := d_G(x) - W_m(x), \text{ with } W_m(x) := \frac{1}{m_0^2(x)} \sum_{y\in\mathscr{V}} \mathscr{E}(x,y) \frac{m(y)}{m(x)} \frac{m_0(x)}{m_0(y)}. \tag{1.7}$$

Moreover, if G is bi-partite, we get:

$$\langle f, \Delta_{\mathscr{E},\theta} f\rangle \leq \langle f, (d_G(Q) + W_m(Q))f\rangle, \text{ for } f \in \mathcal{C}_c(\mathscr{V}).$$

Note that by choosing $m = m_0$, we recover that $\Delta_{\mathscr{E},\theta} \geq 0$. Moreover, V_m is independent of the magnetic field. The minoration is also different from the Kato's inequality of [DM]. We stress that the inequality (1.6) is in some cases trivial, e.g., Proposition 3.3. One has to find a favorable situation in order to exploit it. This is the case for some perturbations of weighted trees. We present our main result:

Theorem 1.2. *Let $G_\circ = (\mathscr{V}, \mathscr{E}_\circ, m, \theta)$ be a weighted tree. Assume that there is $\varepsilon_0 \in (0, 1)$, so that*

$$C_0 := \sup_{x\in\mathscr{V}} \max_{y\in\mathscr{V}} d_{G_\circ}^{\varepsilon_0 - 1}(x) \mathscr{E}_\circ(x,y) m^{-2}(x) < \infty \text{ and } C_1 := \inf_{x\in\mathscr{V}} d_{G_\circ}(x) > 0. \tag{1.8}$$

Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ be a perturbed graph and $V : \mathscr{V} \to \mathbb{R}$ be a potential, satisfying:

$$|V(x)| + \Lambda(x) = o(1 + d_{G_\circ}(x)), \text{ as } |x| \to \infty, \text{ where } \Lambda(x) := \frac{1}{m^2(x)} \sum_{y\sim x} |\mathscr{E}(x,y) - \mathscr{E}_\circ(x,y)|. \tag{1.9}$$

Then, one has that:

(a) *The quadratic form associated to $\Delta_{\mathscr{E},\theta} + V(Q)$ on $\mathcal{C}_c(\mathscr{V})$ is bounded from below by some constant $-C$. We denote by $\mathcal{H}_{\mathscr{F}}$ the associated Friedrichs extension.*

(b) *For all $\varepsilon > 0$, there is $c_\varepsilon \geq 0$, so that*

$$(1-\varepsilon)\langle f, d_{G_\circ}(Q)f\rangle - c_\varepsilon\|f\|^2 \leq \langle f, \mathcal{H}_{\mathscr{F}} f\rangle \leq (1+\varepsilon)\langle f, d_{G_\circ}(Q)f\rangle + c_\varepsilon\|f\|^2, \tag{1.10}$$

for $f \in \mathcal{C}_c(\mathscr{V})$. We have $\mathcal{D}(|\mathcal{H}_{\mathscr{F}}|^{1/2}) = \mathcal{D}((d_{G_\circ}(Q)^{1/2})$.

(c) *The essential spectrum of $\mathcal{H}_{\mathscr{F}}$ is equal to that of $\Delta_{\mathscr{E}_\circ,\theta_\circ}$.*

(d) *The essential spectrum of $\mathcal{H}_{\mathscr{F}}$ is empty if and only if $\lim_{|x|\to\infty} d_{G_\circ}(x) = +\infty$. In this case we obtain:*

$$\lim_{N\to\infty} \frac{\lambda_N(\mathcal{H}_{\mathscr{F}})}{\lambda_N(d_{G_\circ}(Q))} = 1, \tag{1.11}$$

where λ_N denotes the N-th eigenvalue counted with multiplicity.

The theorem will be proved in Section 3.2. In (1.9), we have use the Landau's notation for the small o. Namely, $f(x) = o(g(x))$, as $|x| \to \infty$ if $f/g(x)$ tends to 0 as $|x| \to \infty$. Notice that the convergence given by $|x| \to \infty$ corresponds that given by the filter generated by the complements of finite sets and is independent of the choice of ω.

Note that we improve on the bound (1.4). We point out that Hypothesis (1.8) is fulfilled by simple trees. We improve this condition in (3.9) and discuss it Remark 3.5. Since G_0 is a tree, we recall that $\Delta_{\mathscr{E}_\circ,\theta}$ is unitarily equivalent to $\Delta_{\mathscr{E}_\circ,0}$. However, G is a priori not a tree (recall that Zorn's Lemma ensures that every simple graph has a maximal subtree). Therefore it is interesting to observe that there is no hypothesis on θ. We indicate that the inequality (1.10) is valid for a larger class of perturbations (see Proposition 5.1).

We point out that the first part of d), namely the absence of the essential spectrum, has been studied in many works, e.g., [Kel, KL, KL2, KLW]. They generalize some ideas of [DK, Fuj]. Their approach is based on some isoperimetric estimates and on the Persson's Lemma. The latter characterizes the infimum of the essential spectrum.

The asymptotic of eigenvalues is a novelty and was not considered in the literature before. Here one should keep in mind that our approach is different from the one used in the continuous setting. Whereas one usually relies on the Dirichlet-Neumann bracketing technique, by cutting the space into boxes, it is hard to believe that such an approach would be efficient here. Indeed, cutting the graph gives a perturbation which is of the same size as the operator.

We stress that one can prescribe any asymptotic of eigenvalues by choosing a proper tree G (and in fact d_G). We mention that the spectral asymptotic estimates obtained in [DM] are for some operators with non-empty essential spectrum. They study graphs which are equipped with a free action of a discrete group and establish a bound on the $\operatorname{tr} e^{-t\Delta_{\mathscr{E},\theta}}$, where the trace is adapted to a fundamental domain.

We turn to the question of the form-domain. We stress that we do not suppose that the isoperimetric constant is non-zero. To our knowledge, this is the first time that the form-domain of the unbounded discrete Laplacian on a simple tree is identified. It is remarkable that the form-domain coincides with that of $d_G(Q)$, a multiplication operator. A useful consequence is the stability of the essential spectrum, obtained in c). This is also new. On the other hand, we stress that there are simple bi-partite graphs, such that the form-domain of the Laplacian is different from that of $d_G(Q)$ (see Proposition 4.11). In this case, (1.10) is not fulfilled.

Having the same form-domain does not necessarily ensure that the domains are also equal. In Proposition 4.8, we construct a simple tree which is such an example. However, under some further hypotheses on the graph, Theorem 4.2 ensures that the domain of the magnetic Laplacian is equal to that of $d_G(Q)$. In Proposition 4.6, we give an example of a simple tree T, which has 0 as associated isoperimetric constant and such that the domain of the Laplacian is the same as that of the weighted degree. Moreover, one obtains that $\sigma(\Delta_T) = \sigma_{\mathrm{ac}}(\Delta_T) = [0, \infty)$.

Finally we present the organization of this paper. In section 2.2 we provide a new criterion of essential self-adjointness. Next, in section 2.3, we recall some well-known facts about the min-max principle, its relation to the bottom of the essential spectrum and compactness. Then, in Section 3.1 we prove the Hardy inequality and discuss its triviality. In Section 3.2, we prove Theorem 1.2 in the context of trees. Next, in Section 4.1 we discuss the question of the domain of the Laplacian on a general graph and that of form-domain on bi-partite graphs in Section 4.2. Perturbation theory is developed in Section 5. Finally we provide two appendices, one concerning the $\mathcal{C}^1$ regularity and another one concerning the Helffer-Sjöstrand's formula.

Notation: We denote by $\mathbb{N}$ the non-negative integers. In particular, $0 \in \mathbb{N}$. We set $\langle x\rangle := (1 + x^2)^{1/2}$. Given a set X and $Y \subseteq X$ let $\mathbf{1}_Y : X \to \{0, 1\}$ be the characteristic function of Y. We denote also by Y^c the complement set of Y in X. We consider only separable complex Hilbert space. We denote by $\mathcal{B}(\mathscr{H}, \mathscr{K})$, the space of bounded operators between the Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$.

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2. General properties

2.1. **A few words about the Friedrichs extension.** Given a dense subspace $\mathscr{D}$ of a Hilbert space $\mathscr{H}$ and a non-negative symmetric operator H on $\mathscr{D}$. We define the quadratic from $\mathscr{Q}(f, g) := \langle f, Hg\rangle + \langle f, g\rangle$ on $\mathscr{D} \times \mathscr{D}$. Let $\mathscr{H}_1$ be the completion of $\mathscr{D}$ under the norm associated to $\mathscr{Q}$, namely by the norm $\|\cdot\|_{\mathscr{Q}}$

given by $\|\varphi\|^2_{\mathscr{Q}} := \mathscr{Q}(\varphi)^2 = \langle H\varphi, \varphi\rangle + \|\varphi\|^2$. The domain of the Friedrichs extension of H is given by

$$\begin{aligned}\mathcal{D}(H_{\mathscr{F}}) &= \{f \in \mathscr{H}_1, \mathscr{D} \ni g \mapsto \langle Hg, f\rangle + \langle g, f\rangle \text{ extends to a norm continuous function on } \mathscr{H}\}\\ &= \mathscr{H}_1 \cap \mathcal{D}(H^*).\end{aligned}$$

For each $f \in \mathcal{D}(H_{\mathscr{F}})$, there is a unique u_f such that $\langle Hg, f\rangle + \langle g, f\rangle = \langle g, u_f\rangle$, by Riesz' Theorem. The Friedrichs extension of H, is given by $H_{\mathscr{F}} f = u_f - f$. It is a self-adjoint extension of H, e.g., [RS, Theorem X.23]. Moreover $\mathcal{D}((H_{\mathscr{F}})^{1/2}) = \mathscr{H}_1$. In the sequel we drop the notation with $\mathscr{F}$ when we refer to the Friedrichs extension of the Laplacian, i.e., $\Delta_{\mathscr{E},\theta} = (\Delta_{\mathscr{E},\theta})_{\mathscr{F}}$.

It remains to describe the domain of the adjoint of a discrete Schrödinger operator. This is well-known, e.g., [CTT, KL2]. Let $G = (\mathscr{E}, \mathscr{V}, m, \theta)$ be a weighted graph and $V : \mathscr{V} \to \mathbb{R}$ be a potential. We set the Schrödinger operator $\mathcal{H} := \mathcal{H}|_{\mathcal{C}_c(\mathscr{V})} := (\Delta_{\mathscr{E},\theta} + V(Q))|_{\mathcal{C}_c(\mathscr{V})}$. The domain of its adjoint is given by

$$\mathcal{D}(\mathcal{H}^*) = \Big\{f \in \ell^2(\mathscr{V}, m^2), x \mapsto \frac{1}{m^2(x)} \sum_{y\in\mathscr{V}} \mathscr{E}(x, y)(f(x) - e^{\mathrm{i}\theta_{x,y}} f(y)) + V(x)f(x) \in \ell^2(\mathscr{V}, m^2)\Big\}.$$

Then, given $f \in \mathcal{D}(\mathcal{H}^*)$, one has:

$$(\mathcal{H}^* f)(x) = \frac{1}{m^2(x)} \sum_{y\sim x} \mathscr{E}(x, y)(f(x) - e^{\mathrm{i}\theta_{x,y}} f(y)) + V(x)f(x), \text{ for all } x \in V.$$

By definition, the operator $\mathcal{H}$ is *essentially self-adjoint* if its closure is equal to its adjoint. Recall that a symmetric operator is always closable since its adjoint is densely defined.

2.2. **Essential self-adjointness.** Before talking about essential self-adjointness, we deal with the trivial case, the boundedness of the Laplacian and refer to [KL2] for more discussions in the setting of Dirichlet forms and ℓ^p spaces:

Proposition 2.1. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ be a weighted graph. One has that $\Delta_{\mathscr{E},\theta}$ is bounded if and only if $d_G(Q)$ is bounded.*

Proof. First, (1.4) gives one direction. On the other hand, $\langle \mathbf{1}_{\{x\}}, \Delta_{\mathscr{E},\theta} \mathbf{1}_{\{x\}}\rangle = d_G(x)$, for all $x \in \mathscr{V}$. □

Essential self-adjointness of the discrete Laplacian was proved in many situations by Jørgensen (see [Jor] and references therein). In [Woj] Wojciechowski proves that every discrete Laplacian is essentially self-adjoint on simple graphs. This result was independently proved in [Jor] but the proof was incomplete (see [JP]). An alternative proof of this result can be found in [Web] where one uses the maximum principle. Similar ideas are found in [KL2], where one generalizes this fact to some weighted graphs by studying Dirichlet forms. Then come the works of [Tor, Ma] for weighted graphs which are metrically complete, see also [CTT] for the non-metrically complete case. Some other criteria, based on commutators, are given in [GS] (see also [Gol]). Finally for the magnetic case, we mention the works [CTT2, Mil, Mil2, MiTr]. We point out that in the older work of [Aom] ones gives some characterization of possible self-adjoint extensions of a weighted discrete Laplacian in the limit point/circle spirit in the case of trees. More recently, in [GS], the question of the deficiency indices is discussed. We point out that, in the latter, one can consider potentials that tend to $-\infty$.

We now improve a self-adjointness criteria given in [KL2] and extend it in two directions: we allow magnetic operators and potentials that are unbounded from below.

Proposition 2.2. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ be a weighted graph, $V : \mathscr{V} \to \mathbb{R}$ and $\gamma > 0$. Take $\lambda \in \mathbb{R}$ so that*

$$\{x \in \mathscr{V}, \lambda + d_G(x) + V(x) = 0\} = \emptyset. \tag{2.1}$$

Suppose that, for any $(x_n)_{n\in\mathbb{N}} \in \mathscr{V}^{\mathbb{N}}$, such that the weight $\mathscr{E}(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$, the property

$$\sum_{n\in\mathbb{N}} m^2(x_n)a_n = \infty, \text{ where } a_n := \prod_{i=0}^{n-1} \left(\left(\frac{\gamma}{d_G(x_i)}\right)^2 + \left(1 + \frac{\lambda + V(x_i)}{d_G(x_i)}\right)^2\right) \tag{2.2}$$

holds true. Then, the operator $\mathcal{H} := (\Delta_{\mathscr{E},\theta} + V(Q))|_{\mathcal{C}_c(\mathscr{V})}$ *is essentially self-adjoint.*

First, note it is always possible to find a λ fulfilling (2.1), as $\mathscr{V}$ is countable. Our technique relies on an improvement of [Woj, Theorem 1.3.1].

Proof. Let $f \in \mathcal{D}(\mathcal{H}^*) \setminus \{0\}$ such that $\mathcal{H}^* f + (\gamma \mathrm{i} + \lambda) f = 0$ or $\mathcal{H}^* f + (-\gamma \mathrm{i} + \lambda) f = 0$. We get easily:

$$|f(x)| \leq \frac{1}{m^2(x)} \sum_{y \in \mathscr{V}} \frac{\mathscr{E}(x,y)}{\sqrt{\gamma^2 + (\lambda + d_G(x) + V(x))^2}} |f(y)|.$$

We derive:

$$\max_{y \sim x} |f(y)|^2 \geq \left(\left(\frac{\gamma}{d_G(x)} \right)^2 + \left(1 + \frac{\lambda + V(x)}{d_G(x)} \right)^2 \right) |f(x)|^2, \text{ for all } x, y \in \mathscr{V}, \text{ so that } \mathscr{E}(x,y) \neq 0. \tag{2.3}$$

Now, since $f \neq 0$, there is $x_0 \in \mathscr{V}$ such that $f(x_0) \neq 0$. Therefore, inductively, we obtain a sequence $(x_n)_{n \in \mathbb{N}} \in \mathscr{V}^{\mathbb{N}}$ such that $\mathscr{E}(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$, and so that (2.3) holds for $y = x_{n+1}$ and $x = x_n$. Hence, we get

$$\sum_{n=0}^{N} m^2(x_n) |f(x_n)|^2 \geq \sum_{n=0}^{N} m^2(x_n) \prod_{i=0}^{n-1} \left(\left(\frac{\gamma}{d_G(x_i)} \right)^2 + \left(1 + \frac{\lambda + V(x_i)}{d_G(x_i)} \right)^2 \right) |f(x_0)|^2.$$

By letting N go to infinity and remembering (2.2), we obtain a contradiction of the fact that $f \in \ell^2(\mathscr{V}, m^2)$. We conclude with the help of [RS, Theorem X.1]. □

In [KL2], the hypothesis is stronger, i.e., they take $a_n = 1$, do not consider magnetic fields, and consider potentials that are bounded from below. We provide two examples which were not covered.

Corollary 2.3. *Let* $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ *be a weighted graph,* $\varepsilon > 0$, $C > 0$, *and* $V : \mathscr{V} \to \mathbb{R}$ *such that*

$$V(x) \geq -(1 - \varepsilon) d_G(x) - C,$$

for all $x \in \mathscr{V}$. *Suppose that, for any* $(x_n)_{n \in \mathbb{N}} \in \mathscr{V}^{\mathbb{N}}$, *such that the weight* $\mathscr{E}(x_n, x_{n+1}) > 0$ *for all* $n \in \mathbb{N}$,

$$\sum_{n \in \mathbb{N}} m^2(x_n) = \infty \tag{2.4}$$

holds true. Then, the operator $\mathcal{H} := (\Delta_{\mathscr{E},\theta} + V(Q))|_{\mathcal{C}_c(\mathscr{V})}$ *is essentially self-adjoint.*

Note that the result is optimal as one cannot take $\varepsilon = 0$. Indeed there are simple graphs on which the adjacency matrix $\mathcal{A} := d_G(Q) - \Delta_{\mathscr{E},0}$ is not essentially self-adjoint on $\mathcal{C}_c(\mathscr{V})$, see [GS] and references therein.

Corollary 2.4. *Let* $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ *be a weighted graph,* $V : \mathscr{V} \to \mathbb{R}$, *and* $\gamma > 0$. *Suppose that, for any* $(x_n)_{n \in \mathbb{N}} \in \mathscr{V}^{\mathbb{N}}$, *such that the weight* $\mathscr{E}(x_n, x_{n+1}) > 0$ *for all* $n \in \mathbb{N}$,

$$\sum_{n \in \mathbb{N}} m^2(x_n) \prod_{i=0}^{n-1} \left(\frac{\gamma}{d_G(x_i)} \right) = \infty \tag{2.5}$$

holds true. Then, the operator $\mathcal{H} := (\Delta_{\mathscr{E},\theta} + V(Q))|_{\mathcal{C}_c(\mathscr{V})}$ *is essentially self-adjoint.*

We stress that in the latter, we make no hypothesis on growth of V.

2.3. **Min-max principle.** We recall some well-known results. We refer to [RS][Chapter XIII.1] for more details and to [RS][Chapter XIII.15] for more applications. We start with the form-version of the standard variational characterization of the n-th eigenvalue.

Theorem 2.5. *Let A be a non-negative self-adjoint operator with form-domain $\mathcal{D}(A^{1/2})$. For all $n \geq 1$, we define:*

$$\mu_n(A) := \sup_{\varphi_1, \dots, \varphi_{n-1}} \inf_{\psi \in [\varphi_1, \dots, \varphi_{n-1}]^\perp} \langle \psi, A\psi \rangle,$$

where $[\varphi_1, \dots, \varphi_{n-1}]^\perp = \{\psi \in \mathcal{D}(A^{1/2})$, so that $\|\psi\| = 1$ and $\langle \psi, \varphi_i \rangle = 0$, with $i = 1, \dots, n-1\}$. Note that φ_i are not required to be linearly independent.

If μ_n is (strictly) below the essential spectrum of A, it is the n-th eigenvalue, counted with multiplicity, and we have:

$$\dim \operatorname{Ran} \mathbf{1}_{[0, \mu_n(A)]}(A) = n.$$

Otherwise, μ_n is the infimum of the essential spectrum. Moreover, $\mu_j = \mu_n$, for all $j \geq n$ and there are at most $n-1$ eigenvalues, counted with multiplicity, below the essential spectrum. In that case,

$$\dim \operatorname{Ran} \mathbf{1}_{[0, \mu_n(A)+\varepsilon]}(A) = +\infty, \text{ for all } \varepsilon > 0.$$

Remark 2.6. *One has a priori no control on the multiplicity of the (possible) eigenvalue which is at the bottom of the essential spectrum.*

This ensures the following useful criteria.

Proposition 2.7. *Let A, B be two non-negative self-adjoint operators. Suppose that*

$$\mathcal{D}(A^{1/2}) \supset \mathcal{D}(B^{1/2}) \text{ and } 0 \leq \langle \psi, A\psi \rangle \leq \langle \psi, B\psi \rangle,$$

for all $\psi \in \mathcal{D}(B^{1/2})$. Then one has $\inf \sigma_{\mathrm{ess}}(A) \leq \inf \sigma_{\mathrm{ess}}(B)$,

$$\mu_n(A) \leq \mu_n(B), \text{ for all } n \geq 0,$$

and

$$N_\lambda(A) \geq N_\lambda(B), \text{ for } \lambda \in [0, \infty) \setminus \{\inf \sigma_{\mathrm{ess}}(B)\}, \tag{2.6}$$

where $N_\lambda(A) := \dim \operatorname{Ran} \mathbf{1}_{[0,\lambda]}(A)$.

In particular, if A and B have the same form-domain, then $\sigma_{\mathrm{ess}}(A) = \emptyset$ if and only if $\sigma_{\mathrm{ess}}(B) = \emptyset$.

We refer to [LSW] for some applications of the last statement, in the context of the absence of the essential spectrum of differential operators. Because of Remark 2.6, we stress that we have to remove "$\inf \sigma_{\mathrm{ess}}(B)$" in (2.6), as the min-max principle does not decide whether or not the B has an eigenvalue at this energy.

Proof. Theorem 2.5 permits us to conclude for the first part. Supposing now they have the same form-domain, by the uniform boundedness principle, there are $a, b > 0$ such that:

$$\langle \psi, A\psi \rangle \leq a \langle \psi, B\psi \rangle + b\|\psi\|^2 \text{ and } \langle \psi, B\psi \rangle \leq a \langle \psi, A\psi \rangle + b\|\psi\|^2$$

for all $\psi \in \mathcal{D}(A^{1/2}) = \mathcal{D}(B^{1/2})$. By using the previous statement twice we get the result. □

We now turn to a criteria of compactness. We recall that a compact and self-adjoint operator A is in the $p-$Schatten class, for $p \in [1, \infty)$, if $\operatorname{tr}(|A|^p) < \infty$.

Proposition 2.8. *Let $\mathscr{H}$ be a Hilbert space and A, B be two bounded self-adjoint operators. Suppose that B is compact and*

$$|\langle f, Af \rangle| \leq \langle f, Bf \rangle, \text{ for all } f \in \mathscr{H}.$$

Then A is also compact. Moreover, given $p \in [1, \infty)$, if B is in the $p-$Schatten class, so is A.

Proof. First note that $-B \leq A$ and that $-B \leq -A$ in the form sense. By Proposition 2.7, this implies that the essential spectrum of A is $\{0\}$. Finally, as A is self-adjoint, we infer that A is compact. Now assume that B is in the $p-$Schatten class. The min-max principle, applied to A and $-A$, guarantees that $\operatorname{tr}(|A|^p) \leq 2^p \operatorname{tr}(B^p)$. ☐

Using (1.4), we give a straightforward consequence of Propositions 2.7 and 2.8:

Corollary 2.9. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ be a weighted graph. One has:*

$$\inf \sigma_{\mathrm{ess}}(\Delta_{\mathscr{E},\theta}) \leq 2 \inf \sigma_{\mathrm{ess}}(d_G(Q)) \text{ and } N_\lambda(\Delta_{\mathscr{E},\theta}) \geq N_\lambda(2 d_G(Q)),$$

for all $\lambda \in [0,\infty) \backslash \{\inf \sigma_{\mathrm{ess}}(d_G(Q))\}$. In particular, if $0 \in \sigma_{\mathrm{ess}}(d_G(Q))$, then $0 \in \sigma_{\mathrm{ess}}(\Delta_{\mathscr{E},\theta})$ and if $\Delta_{\mathscr{E},\theta}$ has compact resolvent then, $d_G(Q)$ has also compact resolvent. In other words, one has that $\sigma_{\mathrm{ess}}(d_G(Q)) \neq \emptyset$, then $\sigma_{\mathrm{ess}}(\Delta_{\mathscr{E},\theta}) \neq \emptyset$.

Moreover, if $d_G(Q)$ is compact, then $\Delta_{\mathscr{E},\theta}$ is also compact.

We mention, that for a simple graph, $d_G(Q) \geq 1$ and it is not a compact operator.

3. Surrounding the Laplacian

3.1. **A Hardy inequality.**

We start with a remark about bi-partite graphs:

Proposition 3.1. *Given a bi-partite graph $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ and a function $V : \mathscr{V} \to [0,\infty)$, the following assertions are equivalent:*

$$\langle f, (d_G(Q) - V(Q)) f\rangle \leq \langle f, \Delta_{\mathscr{E},\theta} f\rangle, \quad \text{for } f \in \mathcal{C}_c(\mathscr{V}), \tag{3.1}$$

$$\langle f, \Delta_{\mathscr{E},\theta} f\rangle \leq \langle f, (d_G(Q) + V(Q)) f\rangle, \quad \text{for } f \in \mathcal{C}_c(\mathscr{V}), \tag{3.2}$$

$$|\langle f, \mathcal{A}_{\mathscr{E},\theta} f\rangle| \leq \langle f, V f\rangle, \quad \text{for } f \in \mathcal{C}_c(\mathscr{V}), \tag{3.3}$$

where $\mathcal{A}_{\mathscr{E},\theta}$ is the magnetic adjacency matrix defined by

$$(\mathcal{A}_{\mathscr{E},\theta} f)(x) := \frac{1}{m^2(x)} \sum_{y \in \mathscr{V}} \mathscr{E}(x,y) e^{\mathrm{i}\theta_{x,y}} f(y), \tag{3.4}$$

for $f \in \mathcal{C}_c(\mathscr{V})$ and $x \in \mathscr{V}$.

Proof. Set $Uf(x) := (-1)^{|x|} f(x)$. Note that $U^2 = \mathrm{Id}$ and $U^{-1} = U^* = U$. Notice now that on $\mathcal{C}_c(\mathscr{V})$

$$U^{-1} \mathcal{A}_{\mathscr{E},\theta} U = -\mathcal{A}_{\mathscr{E},\theta} \text{ and } \Delta_{\mathscr{E},\theta} = d_G(Q) - \mathcal{A}_{\mathscr{E},\theta}.$$

We start with (3.1) and rewrite it as follows: $\langle f, \mathcal{A}_{\mathscr{E},\theta} f\rangle \leq \langle f, V(Q) f\rangle$, for $f \in \mathcal{C}_c(\mathscr{V})$. Applying this to Uf, we infer immediately (3.3). We start now from (3.3). We get:

$$\langle f, \Delta_{\mathscr{E},\theta}\, f\rangle = \langle f, (d_G(Q) - \mathcal{A}_{\mathscr{E},\theta}) f\rangle \geq \langle f, (d_G(Q) - V(Q)) f\rangle$$

for $f \in \mathcal{C}_c(\mathscr{V})$. In the same way, we have: (3.2) is equivalent to (3.3). ☐

We now turn to the key estimate of this paper and prove Proposition 1.1. First, given $m_0 : \mathscr{V} \to (0,\infty)$, we mention that a Laplacian in a certain $\ell^2(\mathscr{V}, m^2)$ is unitarily equivalent to a Schrödinger operator in any other $\ell^2(\mathscr{V}, m_0^2)$. This has already been noticed before, e.g., [CTT, HK].

Proposition 3.2. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ and $G_0 = (\mathscr{V}, \mathscr{E}_0, m_0, \theta)$ be two weighted graphs. Then the Friedrichs extension of $\Delta_{\mathscr{E}_0,\theta}$, acting in $\ell^2(\mathscr{V}, m_0^2)$, is unitarily equivalent to that of $\Delta_{\mathscr{E},\theta} + V(Q)$, in $\ell^2(\mathscr{V}, m^2)$, where*

$$\mathscr{E}_0(x,y) := \mathscr{E}(x,y) \frac{m_0(x) m_0(y)}{m(x) m(y)} \quad \text{and} \quad V(x) := \frac{1}{m_0^2(x)} \sum_{y \in \mathscr{V}} \mathscr{E}(x,y) \left(1 - \frac{m(y)}{m(x)} \frac{m_0(x)}{m_0(y)}\right).$$

Proof. Consider the unitary map $U : \ell^2(G, m^2) \to \ell^2(G, m_0^2)$ given by $(Uf)(x) = (m(x)/m_0(x))f(x)$. Straightforwardly, using (1.3), one has

$$U^{-1}\Delta_{\mathscr{E}_0,\theta} Uf := (\Delta_{\mathscr{E},\theta} + V(Q))f \tag{3.5}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. Then it holds true for the closures of $\Delta_{\mathscr{E}_0,\theta}|_{\mathcal{C}_c}$ and $(\Delta_{\mathscr{E},\theta} + V(Q))|_{\mathcal{C}_c}$. Note now that (3.5) also holds for the adjoints of the last two operators. Therefore, the Friedrichs extensions $\Delta_{\mathscr{E}_0,\theta}$ and $\Delta_{\mathscr{E},\theta} + V(Q)$ are unitarily equivalent. □

The novelty in the Hardy inequality (1.6) is more in the point of view. Rather than studying $\Delta_{\mathscr{E},\theta}$ in $\ell^2(\mathscr{V}, m^2)$ with the help of a simpler $\ell^2(\mathscr{V}, m_0^2)$ (where typically $m_0 = 1$), we study it with the help of all other weighted spaces. The applications we consider in this paper are also new.

Proof of Proposition 1.1. On $\ell^2(G, m^2)$, we consider the quadratic form:

$$\mathscr{Q}_{\mathscr{E},\theta}(f,f) := \frac{1}{2}\sum_{x,y\in\mathscr{V}} \tilde{\mathscr{E}}(x,y)|f(x) - e^{\mathrm{i}\theta_{x,y}}f(y)|^2 \geq 0, \text{ for } f \in \mathcal{C}_c(V),$$

where

$$\tilde{\mathscr{E}}(x,y) := \mathscr{E}(x,y)\frac{m(x)m(y)}{m_0(x)m_0(y)}.$$

Note that $\mathscr{Q}(f,f) = \langle f, H_{m^2}f\rangle_{\ell^2(\mathscr{V},m^2)}$, where

$$(H_{m^2}f)(x) = \frac{1}{m^2(x)}\sum_{y\sim x}\tilde{\mathscr{E}}(x,y)(f(x) - e^{\mathrm{i}\theta_{x,y}}f(y)), \text{ for } f \in \mathcal{C}_c(V).$$

Consider now the unitary map $U : \ell^2(G, m^2) \to \ell^2(G, m_0^2)$ given by $(Uf)(x) = (m(x)/m_0(x))f(x)$. Set $H := UH_{m^2}U^{-1}$ We have:

$$\begin{aligned}(Hf)(x) &= \frac{1}{m^2(x)}\sum_{y\in\mathscr{V}}\tilde{\mathscr{E}}(x,y)f(x) - \frac{1}{m_0^2(x)}\sum_{y\in\mathscr{V}}\mathscr{E}(x,y)e^{\mathrm{i}\theta_{x,y}}f(y)\\ &= \frac{1}{m_0^2(x)}\sum_{y\in\mathscr{V}}\mathscr{E}(x,y)(f(x) - e^{\mathrm{i}\theta_{x,y}}f(y)) - V_m(x)f(x).\end{aligned}$$

Since $H \geq 0$, we obtain (1.6). The rest of the statement is given by Proposition 3.1. □

In this paper, we are interested in minorating the Laplacian with the help of the weighted degree. Before proving the main result for simple trees in the next section, we start by giving some negative answers. First, it is obvious that one cannot find a non-negative V_m in (1.6) if the graph is finite. This continues to hold true for some infinite graphs.

Proposition 3.3. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ be a weighted graph and take $V : \mathscr{V} \to \mathbb{R}$ so that we have: $0 \leq V(Q) \leq \Delta_{\mathscr{E},\theta}$, in the form sense on $\mathcal{C}_c(\mathscr{V})$. If the constant function 1 is in $\mathcal{D}((\Delta_{\mathscr{E},\theta})^{1/2})$ then $V = 0$.*

If one supposes that $1 \in \ell^2(\mathscr{V}, m^2)$ and $\Delta_{\mathscr{E},\theta}$ is essentially self-adjoint on $\mathcal{C}_c(\mathscr{V})$, then 1 is in the form domain $\mathcal{D}((\Delta_{\mathscr{E},\theta})^{1/2})$, as $1 \in \mathcal{D}((\Delta_{\mathscr{E},\theta}|_{\mathcal{C}_c})^*)$.

Proof. By construction of the Friedrichs extension, there is $f_n \in \mathcal{C}_c(\mathscr{V})$, so that f_n tends to 1 in the graph norm of $\Delta_{\mathscr{E},\theta}^{1/2}$. Moreover,

$$0 \leq \sum_{x\in\mathscr{V}} m^2(x)V(x)|f_n(x)|^2 = \langle f_n, Vf_n\rangle \leq \langle f_n, \Delta_{\mathscr{E},\theta}f_n\rangle \longrightarrow 0,$$

as n goes to ∞. Now since $V^{1/2}(Q)$ is closed, the closed graph Theorem gives that the l.h.s. tends to $\sum_{x\in\mathscr{V}} m^2(x)V(x)$. Therefore $V = 0$. □

We now treat the impossibility of minorating by using the weighted degree. We say that $\{K_n\}_{n\in\mathbb{N}}$ is a *filtration* of a graph $(\mathscr{V},\mathscr{E},m,\theta)$ if $K_n\subset\mathscr{V}$ is finite, $K_n\subset K_{n+1}$, for all $n\in\mathbb{N}$, and such that $\cup_{n\in\mathbb{N}}K_n=\mathscr{V}$. Recall that given a subset $K\in\mathscr{V}$, $\partial K:=\{x\in K$, there is $y\sim x$ such that $y\notin K\}$.

Proposition 3.4. *Let $G=(\mathscr{V},\mathscr{E},m,0)$ be a weighted graph.*

(a) *Assume that $\alpha(G)=0$. Then, there is no $a>0$ such that*

$$\langle f,d_G(Q)f\rangle\le a\langle f,\Delta_{\mathscr{E},0}\,f\rangle,\ \textit{for } f\in\mathcal{C}_c(\mathscr{V}). \tag{3.6}$$

(b) *Suppose that there is a filtration $\{K_n\}_{n\in\mathbb{N}}$ associated with G, such that:*

$$\limsup_{n\to+\infty}\frac{\sum_{x\in K_n\setminus\partial K_n}m^2(x)d_G(x)}{\sum_{x\in K_n}m^2(x)}=+\infty, \tag{3.7}$$

then there is no $a\in[0,1]$ and $b\ge0$ such that

$$\langle f,d_G(Q)f\rangle\le a\langle f,\Delta_{\mathscr{E},0}\,f\rangle+b\|f\|^2,\ \textit{for all } f\in\mathcal{C}_c(\mathscr{V}). \tag{3.8}$$

In particular, (3.7) *is fulfilled if $1\in\ell^1(\mathscr{V},m^2)$ and $\sup_{x\in\mathscr{V}}m^2(x)d_G(x)=+\infty$.*

Proof. Take $K\subset\mathscr{V}$ finite and use (3.6) with $f=\mathbf{1}_K$. The statement follows from the definition of $\alpha(G)$.

For the second assertion, consider $K=K_n$ and note that (3.8) implies that

$$\sum_{x\in K_n\setminus\partial K_n}m^2(x)d_G(x)\le b\sum_{x\in K_n}m^2(x).$$

By letting n go to infinity, we obtain a contradiction with (3.7). □

3.2. **The case of trees.** We now turn to a minoration of the magnetic Laplacian and present it in the context of weighted trees. Perturbation theory will be considered in Section 5. We fix some notation.

We first recall that a *tree* is a connected graph $G=(\mathscr{E},\mathscr{V},m,\theta)$ such that for each edge $e\in\mathscr{V}\times\mathscr{V}$ with $\mathscr{E}(e)\neq0$ the graph $(\tilde{\mathscr{E}},\mathscr{V},m,\theta|_{\tilde{\mathscr{E}}})$, with $\tilde{\mathscr{E}}:=\mathscr{E}\times1_{\{e\}^c}$, i.e., with the edge e removed, is disconnected. It is convenient to choose a root in the tree. Due to its structure, one can take any point of $\mathscr{V}$. As in the definition of $|x|$, we choose ω to be the root. We define the *sphere* S_n by

$$S_{-1}=\emptyset, S_0:=\{\omega\},\ \text{and}\ S_n:=\{x\in\mathscr{V},|x|=n\}.$$

Given $n\in\mathbb{N}$, $x\in S_n$, and $y\in\mathscr{N}_G(x)$, one sees that $y\in S_{n-1}\cup S_{n+1}$. We write $x\rightsquigarrow y$ and say that x is a *son* of y, if $y\in S_{n-1}$, while we write $x\leftsquigarrow y$ and say that x is a *father* of y, if $y\in S_{n+1}$. Notice that ω has no father. Given $x\neq\omega$, note that there is a unique $y\in V$ with $x\rightsquigarrow y$, i.e., everyone apart from ω has one and only one father. We denote the father of x by $\overleftarrow{x}$.

We turn to the proof of Theorem 1.2. We strengthen slightly the result by working under the hypothesis

$$C_0:=\sup_{x\in\mathscr{V}}d_{G_\circ}^{\varepsilon_0-1}(x)\mathscr{E}_\circ(x,\overleftarrow{x})m^{-2}(x)<\infty\ \text{and}\ C_1:=\inf_{x\in\mathscr{V}}d_{G_\circ}(x)>0. \tag{3.9}$$

instead of (1.8).

Remark 3.5. *We mention that because of part (b) of Proposition 3.4, one cannot solely suppose that G is a weighted tree. This is why a relation between m and $\mathcal{E}$ has to be assumed. Whereas the condition on C_1 is easy to check, the condition on C_0 seems more technical. In order to understand it better, we point out that it would be automatically fulfilled if we could take $\varepsilon_0=0$. The optimality of* (3.9) *remains open.*

Proof of Theorem 1.2. Before going into perturbation theory, we focus on the left hand side of (1.10) for $G_\circ$ and $V=0$, i.e., for $\Delta_{\mathscr{E}_\circ,\theta}$ instead of $\mathcal{H}_{\mathscr{F}}$. Take $\eta>0$. We define:

$$\tilde m(\omega):=1\ \text{and}\ \tilde m(x):=\eta\,\tilde m(\overleftarrow{x})\frac{m(x)}{m(\overleftarrow{x})}d_{G_\circ}^{-\varepsilon_0/2}(x),\ \text{for all}\ x\in\mathscr{V}\setminus\{\omega\}.$$

With this definition and $V_{\tilde m}$ as in (1.7), we obtain:

$$\begin{aligned}\frac{V_{\tilde m}(x)}{d_{G_\circ}(x)} &= 1 - \frac{1}{d_{G_\circ}(x)m^2(x)}\left(\mathscr{E}_\circ(\overleftarrow{x},x)\frac{\tilde m(\overleftarrow{x})}{\tilde m(x)}\frac{m(x)}{m(\overleftarrow{x})} + \sum_{y\rightsquigarrow x}\mathscr{E}_\circ(y,x)\frac{\tilde m(y)}{\tilde m(x)}\frac{m(x)}{m(y)}\right)\\ &= 1 - \frac{1}{\eta}\frac{1}{d_{G_\circ}^{(1-\varepsilon_0/2)}(x)m^2(x)}\mathscr{E}_\circ(\overleftarrow{x},x) - \frac{\eta}{d_{G_\circ}(x)m^2(x)}\sum_{y\rightsquigarrow x}\mathscr{E}_\circ(y,x)d_{G_\circ}^{-\varepsilon_0/2}(y)\\ &\geq 1 - \eta C_1^{-\varepsilon_0/2} - \frac{1}{\eta}d_{G_\circ}^{-\varepsilon_0/2}(x)C_0, \end{aligned}\tag{3.10}$$

for all $x \in \mathscr{V}$ and where C_0 and C_1 are given by (3.9). Now note that for all $\tilde\eta > 0$ there is $c_{\tilde\eta} > 0$ such that $t^{1-\varepsilon_0/2} \leq \tilde\eta t + c_{\tilde\eta}$, for all $t \in [0,\infty)$. Therefore by Proposition 1.1 applied to $G_\circ$, for all $\varepsilon > 0$, there is $c_\varepsilon > 0$ such that:

$$\begin{aligned}\langle f, \Delta_{\mathscr{E}_\circ,\theta} f\rangle &\geq \langle d_{G_\circ}^{1/2}(Q)f, \left(1 - \eta C_1^{-\varepsilon_0/2} - C_0/d_{G_\circ}^{\varepsilon_0/2}(Q)\eta\right) d_{G_\circ}^{1/2}(Q)f\rangle\\ &\geq (1-\varepsilon)\langle f, d_{G_\circ}(Q)f\rangle - c_\varepsilon\|f\|^2,\end{aligned}\tag{3.11}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. This gives (1.10) for $G_\circ$ and $V = 0$, where the second inequality is obtained by applying Proposition 3.1.

Next, the equality of the domains of the forms follows immediately and so the essential spectrum of $\Delta_{\mathscr{E}_\circ,\theta}$ is empty if and only if $\lim_{|x|\to\infty} d_{G_\circ}(x) = +\infty$. Finally, we use twice Proposition 2.7 with the double inequality (1.10). This yields:

$$1 - \varepsilon \leq \liminf_{N\to\infty}\frac{\lambda_N(\Delta_{\mathscr{E},\theta})}{\lambda_N(d_{G_\circ}(Q))} \leq \limsup_{N\to\infty}\frac{\lambda_N(\Delta_{\mathscr{E},\theta})}{\lambda_N(d_{G_\circ}(Q))} \leq 1 + \varepsilon.$$

By letting ε go to zero we obtain the asymptotic (1.11) for $\Delta_{\mathscr{E}_\circ,\theta}$.

We finish with $\mathcal{H}_{\mathscr{F}} = \Delta_{\mathscr{E},\theta} + V(Q)$ by perturbing $\Delta_{\mathscr{E}_\circ,\theta}$. Thanks to (1.9), Propositions 5.1 and 5.2 end the proof. ☐

Remark 3.6. *It is important to note that ε has to be positive in* (1.10). *For instance, on a simple tree, if ε was equal to* 0*, this would imply that the magnetic adjacency matrix was bounded (see Proposition 3.1). Finally note that, by considering delta functions as test functions, the magnetic adjacency matrix is bounded if and only if the weighted degree is bounded.*

Remark 3.7. *We could not provide an example of a Schrödinger operator for which we can compute the asymptotic of eigenvalues and which is not essentially self-adjoint on $\mathcal{C}_c(\mathscr{V})$.*

4. Comparison of domains

4.1. **From form-domain to domain.** Once one knows that the form-domains are equal, the next question is to guarantee that the domains are equal. We present an approach by commutators. Some subtleties arise since we have to deal with the square root of the Laplacian.

We first prove the invariance of the domain of $\Delta_{\mathscr{E},\theta}$ under the C_0−group $\{e^{\mathrm{i}td_G(Q)}\}_{t\in\mathbb{R}}$ and that $\Delta_{\mathscr{E},\theta} \in \mathcal{C}^1(d_G(Q))$, i.e., the map $t \mapsto e^{-\mathrm{i}td_G(Q)}(\Delta_{\mathscr{E},\theta}+1)^{-1}e^{\mathrm{i}td_G(Q)}$ is strongly $\mathcal{C}^1$. We refer to Appendix A for a discussion about the $\mathcal{C}^1$ regularity.

Proposition 4.1. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ be a weighted graph, such that $\Delta_{\mathscr{E},\theta}$ is essentially self-adjoint on $\mathcal{C}_c(\mathscr{V})$, $\mathcal{D}(\Delta_{\mathscr{E},\theta}^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$, and*

$$\sup_{x\in\mathscr{V}}\left(\frac{1}{m^2(x)}\sum_{y\in\mathscr{V}}\frac{\mathscr{E}(x,y)}{\min(\langle d_G(x)\rangle^{1/2}, \langle d_G(y)\rangle^{1/2})}\,|d_G(x) - d_G(y)|\right) < \infty.\tag{4.1}$$

Then, one has the invariance $e^{\mathrm{i}td_G(Q)}\mathcal{D}(\Delta_{\mathscr{E},\theta}) \subset \mathcal{D}(\Delta_{\mathscr{E},\theta})$*, for all* $t \in \mathbb{R}$*. Moreover, we have that* $\Delta_{\mathscr{E},\theta} \in \mathcal{C}^1(d_G(Q))$ *and that* $[\Delta_{\mathscr{E},\theta}, d_G(Q)]|_{\mathcal{C}_c(\mathscr{V})}$ *extends to* $[\Delta_{\mathscr{E},\theta}, d_G(Q)]_\circ \in \mathcal{B}\left(\mathcal{D}(\Delta_{\mathscr{E},\theta}^{1/2}), \ell^2(\mathscr{V}, m^2)\right)$*.*

Proof. We denote by $[\Delta_{\mathscr{E},\theta}, e^{\mathrm{i}td_G(Q)}]_\circ$ the closure of the commutator $[\Delta_{\mathscr{E},\theta}, e^{\mathrm{i}td_G(Q)}]|_{\mathcal{C}_c(\mathscr{V})}$ and by c_0 the constant appearing in (4.1). We have:

$$\begin{aligned}
|\langle f, [\Delta_{\mathscr{E},\theta}, e^{\mathrm{i}td_G(Q)}]_\circ\, g\rangle| &= \sum_{x\in\mathscr{V}}\sum_{y\in\mathscr{V}} \overline{f(x)}\mathscr{E}(x,y)e^{\mathrm{i}\theta_{x,y}}\left(e^{\mathrm{i}td_G(x)} - e^{\mathrm{i}td_G(y)}\right)g(y)\\
&\leq \frac{|t|}{2}\sum_{x\in\mathscr{V}}\sum_{y\in\mathscr{V}} \frac{\mathscr{E}(x,y)}{\langle d_G(y)\rangle^{1/2}}\,|d_G(x)-d_G(y)|\left(|f(x)|^2 + |\langle d_G(y)\rangle^{1/2}g(y)|^2\right)\\
&\leq c_0|t|\left(\|f\|^2 + \left\|\langle d_G(Q)\rangle^{1/2}g\right\|^2\right), \qquad (4.2)
\end{aligned}$$

for all $f, g \in \mathcal{C}_c(\mathscr{V})$. Therefore, $[\Delta_{\mathscr{E},\theta}, e^{\mathrm{i}td_G(Q)}]_\circ$ is bounded from $\mathcal{D}(\Delta_{\mathscr{E},\theta}^{1/2})$ to $\ell^2(\mathscr{V}, m^2)$. On the other hand, we get:

$$\begin{aligned}
\|\Delta_{\mathscr{E},\theta}\, e^{\mathrm{i}td_G(Q)}f\| &\leq \|e^{\mathrm{i}td_G(Q)}\Delta_{\mathscr{E},\theta}\, f\| + \|[\Delta_{\mathscr{E},\theta}, e^{\mathrm{i}td_G(Q)}]\langle d_G(Q)\rangle^{-1/2}\langle d_G(Q)\rangle^{1/2}f\|\\
&\leq C\left(\|f\| + \|\Delta_{\mathscr{E},\theta}\, f\|\right),
\end{aligned}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. By density we obtain the invariance of the domain.

Next, thanks to (4.2), we obtain that

$$\liminf_{t\to 0^+}\left\|[\Delta_{\mathscr{E},\theta}, e^{\mathrm{i}td_G(Q)}/t]\right\|_{\mathcal{B}(\mathcal{D}(\Delta_{\mathscr{E},\theta}),\ell^2(\mathscr{V},m^2))} < \infty.$$

Therefore, Theorem A.2 yields that $\Delta_{\mathscr{E},\theta} \in \mathcal{C}^1(d_G(Q))$. Finally, by estimating as in (4.2), we obtain that $[\Delta_{\mathscr{E},\theta}, d_G(Q)]_\circ$, belongs to $\mathcal{B}\left(\mathcal{D}(\Delta_{\mathscr{E},\theta}^{1/2}), \ell^2(\mathscr{V}, m^2)\right)$. □

We turn to the central result of this section.

Theorem 4.2. *Let* $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ *be a weighted graph, such that* $\mathcal{D}(\Delta_{\mathscr{E},\theta}^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$*,* $\Delta_{\mathscr{E},\theta}$ *is essentially self-adjoint on* $\mathcal{C}_c(\mathscr{V})$*, and there is* $\varepsilon > 0$ *such that*

$$\sup_{x\in\mathscr{V}}\left(\frac{1}{m^2(x)}\sum_{y\in\mathscr{V}}\frac{\mathscr{E}(x,y)}{\min(\langle d_G(x)\rangle^{1/2-\varepsilon}, \langle d_G(y)\rangle^{1/2-\varepsilon})}\,|d_G(x)-d_G(y)|\right) < \infty. \tag{4.3}$$

Take also a potential $V : \mathscr{V} \to \mathbb{R}$*, such that:*

$$\lim_{|x|\to+\infty}\frac{V(x)}{d_G(x)+1} = 0. \tag{4.4}$$

Then, the operator $\mathcal{H} := (\Delta_{\mathscr{E},\theta} + V(Q))|_{\mathcal{C}_c(\mathscr{V})}$ *is bounded from below. We denote by* $\mathcal{H}_{\mathscr{F}}$ *its Friedrichs extension. We have:* $\mathcal{D}(\mathcal{H}_{\mathscr{F}}) = \mathcal{D}(d_G(Q))$ *and* $\sigma_{\mathrm{ess}}(\mathcal{H}_{Fr}) = \sigma_{\mathrm{ess}}(\Delta_{\mathscr{E},\theta})$*.*

Remark 4.3. *We point out that the hypothesis on the essential self-adjointness is also necessary, as* $d_G(Q)$ *is essentially self-adjoint on* $\mathcal{C}_c(\mathscr{V})$*.*

Remark 4.4. *By taking* $\varepsilon = 0$ *in* (4.3)*, i.e., under the same hypothesis as Proposition 4.1, the proof ensures only that* $\mathcal{D}(d_G(Q)) \subset \mathcal{D}(\Delta_{\mathscr{E},\theta})$*.*

Proof. The stability of the essential spectrum is ensured by Proposition 5.2. We focus on the domain. By Kato-Rellich's Theorem, e.g., [RS, Theorem X.12], it is enough to consider $V = 0$. We assume for the moment that

$$M := (\Delta_{\mathscr{E},\theta}+1)^{1/2}[d_G(Q), (\Delta_{\mathscr{E},\theta}+1)^{1/2}]_\circ + [\Delta_{\mathscr{E},\theta}, (d_G(Q)+1)^{1/2}]_\circ(d_G(Q)+1)^{1/2}.$$

is a bounded operator from $\mathcal{D}(d_G^{1-\varepsilon}(Q))$ to $\mathcal{D}(d_G^{1-\varepsilon}(Q))^*$. The 1 is here to make the square root smooth over the spectrum. Since the form-domain of $\Delta_{\mathscr{E},\theta}$ and $d_G(Q)$ are equal, by the uniform boundedness principle, we have: there are $a, b > 0$ so that (3.8) holds true. By using twice (3.8) and working in the form sense on $\mathcal{C}_c(\mathscr{V})$, we infer:

$$\begin{aligned}(\Delta_{\mathscr{E},\theta}+1)^2 &\leq a(\Delta_{\mathscr{E},\theta}+1)^{1/2}(d_G(Q)+1)(\Delta_{\mathscr{E},\theta}+1)^{1/2}+(b+1)(\Delta_{\mathscr{E},\theta}+1)\\ &= a\,(d_G(Q)+1)^{1/2}\Delta_{\mathscr{E},\theta}\,(d_G(Q)+1)^{1/2}+aM+(b+1)(\Delta_{\mathscr{E},\theta}+1)\\ &\leq a' d_G^2(Q)+b',\end{aligned}\tag{4.5}$$

for some $a', b' > 0$. By using the fact that $\varepsilon > 0$, the reverse inequality holds true for the same reasons for some $a' \in (0, 1)$ and $b' > 0$. Therefore, the domains are equal. It remains to prove the boundedness of M. We start with the r.h.s. term. Let c_1 be the constant in (4.3). We estimate as above:

$$\begin{aligned}&|\langle f, [\Delta_{\mathscr{E},\theta}, (d_G(Q)+1)^{1/2}]_\circ \langle d_G(Q)\rangle^{1/2} g\rangle|\\ &\qquad= \sum_{x\in\mathscr{V}}\sum_{y\in\mathscr{V}} \overline{f(x)}\mathscr{E}(x,y)e^{\mathrm{i}\theta_{x,y}}((d_G(x)+1)^{1/2}-(d_G(y)+1)^{1/2})\langle d_G(y)\rangle^{1/2}g(y)\\ &\qquad\leq \frac{1}{2}\sum_{x\in\mathscr{V}}\sum_{y\in\mathscr{V}} \frac{\mathscr{E}(x,y)}{\langle d_G(y)\rangle^{1/2-\varepsilon}}\,|d_G(x)-d_G(y)|\left(|f(x)|^2+|\langle d_G(y)\rangle^{1-\varepsilon}g(y)|^2\right)\\ &\qquad\leq c_1\left(\|f\|^2+\left\|\langle d_G(Q)\rangle^{1-\varepsilon}g\right\|^2\right),\end{aligned}$$

for all $f, g \in \mathcal{C}_c(\mathscr{V})$. In particular, $[\Delta_{\mathscr{E},\theta}, (d_G(Q)+1)^{1/2}]_\circ\langle d_G(Q)\rangle^{1/2}$ is bounded from $\mathcal{D}(d_G^{1-\varepsilon}(Q))$ to $\ell^2(\mathscr{V}, m^2)$.

We turn to the second part of M and use Appendix B. Let φ be in $\mathcal{S}^{1/2}$ such that $\varphi(x) = \sqrt{x}$, for all $x \geq 1$. Since $\Delta_{\mathscr{E},\theta}$ is non-negative, $\varphi(\Delta_{\mathscr{E},\theta}+1) = (\Delta_{\mathscr{E},\theta}+1)^{1/2}$. We cannot use the (B.5) directly with φ as the integral does not seem to exist. We proceed as in [GJ]. Take $\chi_1 \in \mathcal{C}_c^\infty(\mathbb{R};\mathbb{R})$ with values in $[0, 1]$ and being 1 on $[-1, 1]$. Set $\chi_R := \chi(\cdot/R)$. As R goes to infinity, χ_R converges pointwise to 1. Moreover, $\{\chi_R\}_{R\in[1,\infty)}$ is bounded in $\mathcal{S}^0$. We infer $\varphi_R := \varphi\chi_R$ tends pointwise to φ and that $\{\varphi_R\}_{R\in[1,\infty)}$ is bounded in $\mathcal{S}^{1/2}$. Now, recalling $\Delta_{\mathscr{E},\theta} \in \mathcal{C}^1(d_G(Q))$ and (A.3), we obtain

$$\begin{aligned}&[\varphi_R(\Delta_{\mathscr{E},\theta}), d_G(Q)]_\circ\langle\Delta_{\mathscr{E},\theta}\rangle^{-1/2} =\\ &\qquad= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\frac{\partial\varphi_R^{\mathbb{C}}}{\partial\overline{z}}(z-\Delta_{\mathscr{E},\theta})^{-1}[d_G(Q),\Delta_{\mathscr{E},\theta}]_\circ(z-\Delta_{\mathscr{E},\theta})^{-1}\langle\Delta_{\mathscr{E},\theta}\rangle^{-1/2}dz\wedge d\overline{z}.\\ &\qquad= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{C}}\frac{\partial\varphi_R^{\mathbb{C}}}{\partial\overline{z}}(z-\Delta_{\mathscr{E},\theta})^{-1}[d_G(Q),\Delta_{\mathscr{E},\theta}]_\circ\,\langle\Delta_{\mathscr{E},\theta}\rangle^{-1/2}(z-\Delta_{\mathscr{E},\theta})^{-1}dz\wedge d\overline{z}.\end{aligned}\tag{4.6}$$

By Proposition 4.1, $[d_G(Q), \Delta_{\mathscr{E},\theta}]_\circ\langle\Delta_{\mathscr{E},\theta}\rangle^{-1/2}$ is bounded. Moreover, using (B.2) with $l = 2$, we bound the integrand, uniformly in R, by $C\langle x\rangle^{-1+1/2-2}|y|^2|y|^{-1}|y|^{-1}$, for some constant C. It is integrable on the domain given by (B.4). By Lebesgue domination, the r.h.s. of (4.6) has a limit in norm. Note now that the l.h.s., as form on $\mathcal{C}_c(\mathscr{V})$, tends to the operator $[(\Delta_{\mathscr{E},\theta}+1)^{1/2}, d_G(Q)]\langle\Delta_{\mathscr{E},\theta}\rangle^{-1/2}$. This gives that

$$-C(d_G(Q)+1) \leq \langle\Delta_{\mathscr{E},\theta}\rangle^{1/2}[(\Delta_{\mathscr{E},\theta}+1)^{1/2}, d_G(Q)]_\circ \leq C(d_G(Q)+1)$$

in the form sense on $\mathcal{C}_c(\mathscr{V})$ for some constant C. This ensures the announced boundedness of M. □

Remark 4.5. *Assuming also that for all $\varepsilon > 0$, there is $c_\varepsilon \geq 0$ so that* (1.10) *holds true, one observes that a' can be arbitrary close to* 1 *in* (4.5). *Therefore, using again the Kato-Rellich's Theorem, one can* *weaken* (4.4) *in Theorem 4.2 and replace it by*

$$\limsup_{|x|\to+\infty} \frac{|V(x)|}{d_G(x)+1} < 1. \tag{4.7}$$

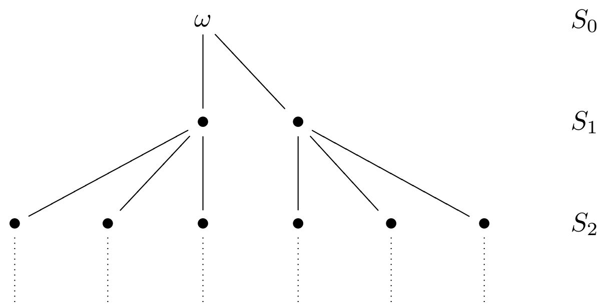
to obtain $\mathcal{D}(\mathcal{H}_{\mathscr{F}}) = \mathcal{D}(d_G(Q))$.

Finally, we examine the question of the equality of the domains in the context of simple trees. We provide a positive result.

Proposition 4.6. *There is a simple tree T, such that $\sigma(\Delta_T) = \sigma_{\mathrm{ac}}(\Delta_T) = [0,\infty)$ and such that $\mathcal{D}(\Delta_T) = \mathcal{D}(d_T(Q))$. In particular, $\alpha(T) = 0$.*

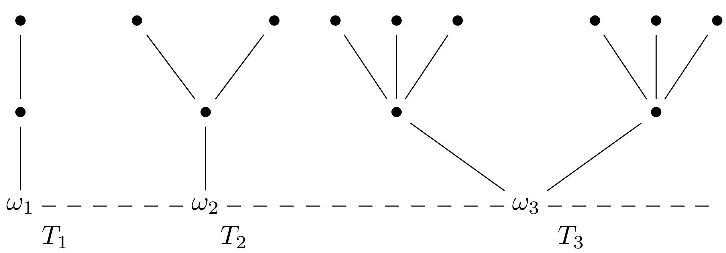
Here ac stands for absolutely continuous.

Proof. We start by fixing some notation. Given an *offspring sequence* $(b_n)_{n\in\mathbb{N}}$, with $b_n \in \mathbb{N}^*$, we associate a simple tree with root ε such that, for all $x \in S_n$, $b_n = \sharp\{y, \overleftarrow{y} = x\}$.



Example of a tree with $b_0 = 2$ and $b_1 = 3$.

We turn to our example and construct some trees $T_n = (\mathscr{E}_n, \mathscr{V}_n)$. For $n = 1$, we take $\mathscr{V}_1 := \mathbb{N}$, with $\omega_1 := 0$ and with $\mathscr{E}_1(p,q) = 1$ if and only if $|p-q| = 1$. For $n \geq 2$, we take trees that are n-ary after the first generation. For each $n \in \mathbb{N}\setminus\{0,1\}$, let ω_n be the root and set that the offspring $b(n)_k := n$, for all $k \in \mathbb{N}\setminus\{0\}$ and $b(n)_0 := n-1$. Now take $T := (\mathscr{E}, \mathscr{V})$, where $\mathscr{V} := \cup_{n\in\mathbb{N}\setminus\{0,1\}}\mathscr{V}_n$ and $\mathscr{E}(x,y) := \mathscr{E}_n(x,y)$, if $x, y \in \mathscr{V}_n$, $\mathscr{E}(\omega_n, \omega_{n+1}) := 1$, and $\mathscr{E}(x,y) := 0$ otherwise.



Graph of T

Note that $x \mapsto \sum_{y\sim x} |d_T(x) - d_T(y)|$ has support contained in $\cup_n\{\omega_n\}$ and takes values in $\{0,1\}$. Hence, (4.3) is fulfilled. Since the graph is simple, Δ_T is essentially self-adjoint on $\mathcal{C}_c(\mathscr{V})$ (see Section 2.2). Therefore, we derive from Theorem 4.2 that the $\mathcal{D}(\Delta_T) = \mathcal{D}(d_T(Q))$.

We turn to the spectrum. First, $\sigma(\Delta_{G_1}) = \sigma_{\mathrm{ac}}(\Delta_{G_1}) = [0,4]$, where $G_1 := T_1$. This is easy to see by discrete Fourier transformation, e.g., [AF]. For each $n \geq 2$, T contains a subtree G_n which is $n-$ary and which is connected to the rest of T by only one edge. It is well-known that $\sigma_{\mathrm{ac}}(\Delta_{G_n}) = [n+1-2\sqrt{n}, n+1+2\sqrt{n}]$, e.g., [AF]. Now, we denote by $\tilde{G}_i := T \setminus G_i$, the graph obtained by removing the only edge that is connecting G_i to the rest of the graph. Note that $\Delta_{\tilde G_i} - \Delta_{G_i}$ is a rank one operator, for all $i \geq 1$. Therefore, $\sigma_{\rm ac}(\Delta_{G_i}) \subset \sigma_{\rm ac}(\Delta_T)$ for all $i \geq 1$. Hence, $\sigma(\Delta_T) = \sigma_{\rm ac}(\Delta_T) = [0, \infty)$. Finally note that if $\alpha(T) > 0$, (1.5) ensures that $\inf \sigma(\Delta_T) > 0$. This is a contradiction. $\square$

Remark 4.7. *This construction also provides an example of a graph on which the adjacency matrix $\mathcal{A}_{\mathscr{E},0}$ (see* (3.4)*) is essentially self-adjoint on $\mathcal{C}_c(\mathscr{V})$ (see* [GS, Lemma 2.1]*) and has absolutely continuous spectrum equal to $\mathbb{R}$.*

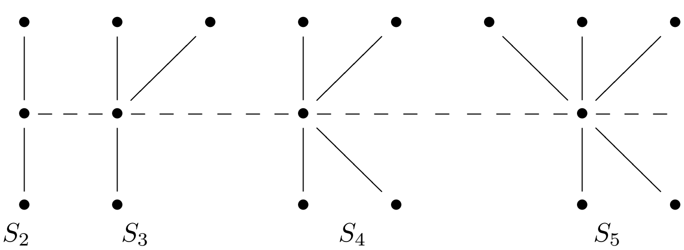
Finally, we give a negative example.

Proposition 4.8. *There is a simple tree T, such that $\mathcal{D}(\Delta_T) \neq \mathcal{D}(d_T(Q))$ and such that the form-domains $\mathcal{D}(\Delta_T^{1/2}) = \mathcal{D}(d_T^{1/2}(Q))$.*

Proof. The second point follows from Theorem 1.2. We start by constructing the star graph S_n. Let $S_n := (\mathscr{E}_n, \mathscr{V}_n)$ be defined as follows: $\mathscr{V}_n := \{1, \ldots, n+1\}$ and so that $\mathscr{E}_n((1, j)) := 1, \forall j \in \{2, \ldots, n+1\}$ and $\mathscr{E}_n((j, k)) := 0, \forall j, k \in \{2, \ldots, n+1\}$. Consider now $f_n(x) := 1$ on $\mathscr{V}_n$. One has:

$$\|\Delta_{S_n} f_n\|^2 = 0, \quad \|f_n\|^2 = n+1, \quad \text{and} \quad \|d_{S_n}(Q) f_n\|^2 = n(n+1). \tag{4.8}$$

Take $T := (\mathscr{E}, \mathscr{V})$, where $\mathscr{V} := \cup_{n \in \mathbb{N}\setminus\{0\}} \mathscr{V}_n$ and $\mathscr{E}(x, y) := \mathscr{E}_n(x, y)$, if $x, y \in \mathscr{V}_n$, $\mathscr{E}(x, y) := 1$, if $x = 1 \in \mathscr{V}_n$ and $y = 1 \in \mathscr{V}_{n+1}$, for all $n \geq 1$, and $\mathscr{E}(x, y) := 0$ otherwise.



Graph of T

Now if $\mathcal{D}(d_T(Q)) \supset \mathcal{D}(\Delta_T)$, by the uniform boundedness principle, there are constants $a, b > 0$, so that

$$\|d_T(Q) f\|^2 \leq a \|\Delta_T f\|^2 + b\|f\|^2, \quad \text{for all } f \in \mathcal{C}_c(T).$$

This leads to a contradiction with (4.8), as $\|\Delta_T f_n\|^2 = 4$ and $\|d_T(Q) f_n\|^2 = (n+2)^2 + n$ for $n \geq 3$. $\square$

4.2. **The form-domain for bi-partite graphs.** As (1.10) holds true for trees, it is natural to ask the question for bi-partite graphs. The answer is no. We start by relating the form-domain of the magnetic Laplacian with the inequality (4.9).

Proposition 4.9. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ be a weighted bi-partite graph. Then there is $a \in (0, 1]$ and $C_a > 0$ so that:*

$$(1-a)\langle f, d_G(Q) f\rangle - C_a \|f\|^2 \leq \langle f, \Delta_{\mathscr{E},\theta} f\rangle \leq (1+a)\langle f, d_G(Q) f\rangle + C_a\|f\|^2, \tag{4.9}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. Moreover, one can take some $a < 1$ in (4.9) *if and only if $\mathcal{D}((\Delta_{\mathscr{E},\theta})^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$.*

Suppose also that $\Delta_{\mathscr{E},\theta}$ has compact resolvent. Then, $d_G(Q)$ has also compact resolvent and, with the same a as in (4.9)*, one has:*

$$1 - a \leq \liminf_{\lambda\to\infty} \frac{N_\lambda(\Delta_{\mathscr{E},\theta})}{N_\lambda(d_G(Q))} \leq \limsup_{\lambda\to\infty} \frac{N_\lambda(\Delta_{\mathscr{E},\theta})}{N_\lambda(d_G(Q))} \leq 1 + a. \tag{4.10}$$

We recall that (1.4) ensures that $\mathcal{D}(d_G^{1/2}(Q)) \subset \mathcal{D}((\Delta_{\mathscr{E},\theta})^{1/2})$.

Proof. First, note that (4.9) follows from (1.4) and Proposition 3.1. Suppose that (4.9) holds true for some $a < 1$, this gives immediately that $\mathcal{D}((\Delta_{\mathscr{E},\theta})^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$. Reciprocally, suppose now that $\mathcal{D}((\Delta_{\mathscr{E},\theta})^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$. By the uniform boundedness principle, there is $a_0, b_0 > 0$ so that

$$a_0\langle f, d_G(Q)f\rangle - b_0\|f\|^2 \leq \langle f, \Delta_{\mathscr{E},\theta} f\rangle,$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. Using again Proposition 3.1, (4.9) holds true with $a = 1 - a_0 < 1$.

We turn to the second part and work under the hypothesis that $\Delta_{\mathscr{E},\theta}$ has compact resolvent. Corollary 2.9 ensures that $d_G(Q)$ has also compact resolvent. We conclude by using Proposition 2.7 twice. □

Remark 4.10. *For a bi-partite graph, the constant* 2 *in* (1.4) *can be improved, in the sense of* (4.9)*, if and only if* $\mathcal{D}((\Delta_{\mathscr{E},\theta})^{1/2}) = \mathcal{D}(d_G^{1/2}(Q))$.

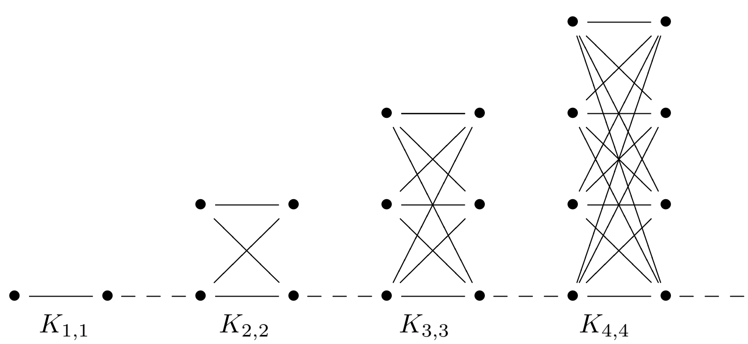
We finally provide an example.

Proposition 4.11. *There is a simple bi-partite graph K such that* $\mathcal{D}(d_K^{1/2}(Q)) \subsetneq \mathcal{D}(\Delta_K^{1/2})$*. In particular, the constant* 2 *in* (1.4) *is optimal in the sense of* (4.9).

Proof. We start by constructing a complete bi-partite graph. Let $K_{n,n} := (\mathscr{E}_n, \mathscr{V}_n)$ be defined as follows: $\mathscr{V}_n = \{1, \dots n\} \times \{1, \dots n\}$ and such that $\mathscr{E}_n((k,i),(j,i)) = 0, \forall j,k \in \{1, \dots n\}$ and $i = 1,2$ and $\mathscr{E}_n((k,1),(j,2)) = 0, \forall j,k \in \{1, \dots n\}$. Consider now $f_n(x) = 1$ on $K_{n,n}$. One has:

$$\langle f_n, \Delta_K f_n\rangle = 0, \quad \|f_n\|^2 = 2n, \quad \text{and} \quad \langle f_n, d_K(Q) f_n\rangle = 2n^2. \tag{4.11}$$

Now take $K := (\mathscr{E}, \mathscr{V})$, where $\mathscr{V} = \cup_{n \in \mathbb{N}\setminus\{0\}} \mathscr{V}_n$ and $\mathscr{E}(x,y) := \mathscr{E}_n(x,y)$, if $x, y \in \mathscr{V}_n$, $\mathscr{E}(x,y) = 1$, if $x = (1,2) \in \mathscr{V}_n$ and $y = (1,1) \in \mathscr{V}_{n+1}$, for all $n \geq 1$, and $\mathscr{E}(x,y) = 0$ otherwise.



Graph of K

On the other hand, if $\mathcal{D}(d_K^{1/2}(Q)) = \mathcal{D}(\Delta_K^{1/2})$, the uniform boundedness principle ensures that there are constants $a, b > 0$, so that

$$\langle f, d_K(Q) f\rangle \leq a\langle f, \Delta_K f\rangle + b\|f\|^2, \quad \text{for all } f \in \mathcal{C}_c(K).$$

This leads to a contradiction with (4.11), as $\langle f_n, \Delta_K f_n\rangle = 2$, for $n \geq 2$. Optimality follows by Proposition and 4.9 and Remark 4.10. □

5. Perturbation theory

We finally go into perturbation theory in order to obtain the stability of the essential spectrum, of the inequality (5.1), and of the asymptotic of eigenvalues.

Proposition 5.1. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ and $G_\circ = (\mathscr{V}, \mathscr{E}_\circ, m, \theta_\circ)$ be weighted graphs and $V : \mathscr{V} \to \mathbb{R}$. Suppose that for all $\varepsilon > 0$ there is $c_\varepsilon > 0$ so that* (1.10) *holds true for $\Delta_{\mathscr{E}_\circ,\theta_\circ}$. Suppose that there is $\eta \in (0, 1)$ and $\kappa_\eta \geq 0$, so that*

$$|\langle f, V(Q)f\rangle| + 2\langle f, \Lambda(Q)f\rangle \leq \eta\langle f, d_{G_\circ}(Q)f\rangle + \kappa_\eta\|f\|^2, \tag{5.1}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$, where

$$\Lambda(x) := \frac{1}{m^2(x)} \sum_{y\sim x} |\mathscr{E}(x, y) - \tilde{\mathscr{E}}(x, y)|. \tag{5.2}$$

Then, one has that:

(a) *The operator $\mathcal{H} := (\Delta_{\mathscr{E},\theta} + V(Q))|_{\mathcal{C}_c(\mathscr{V})}$ is bounded from below by some negative constant $-C$, in the form sense. We denote by $\mathcal{H}_\mathscr{F}$ its Friedrichs extension. We have the equality of the form domains: $\mathcal{D}(|\mathcal{H}_\mathscr{F}|^{1/2}) = \mathcal{D}((\Delta_{\mathscr{E}_\circ,\theta_\circ})^{1/2})$.*

(b) *The three following assertions are equivalent:*
   - i) *The essential spectrum of $\mathcal{H}_\mathscr{F}$ is empty,*
   - ii) *the essential spectrum of $\Delta_{\mathscr{E}_\circ,\theta_\circ}$ is empty,*
   - iii) *$\lim_{|x|\to\infty} d_{G_\circ}(x) = +\infty$.*

(c) *Supposing that the essential spectrum of $\mathcal{H}_\mathscr{F}$ is empty and that for all $\eta \in (0, 1)$ there is $\kappa_\eta \geq 0$, so that* (5.1) *holds true, then:*

$$\lim_{N\to\infty} \frac{\lambda_N(\mathcal{H}_\mathscr{F})}{\lambda_N(d_{G_\circ}(Q))} = 1.$$

Proof. We start by noticing that, as in (1.4), (5.1) ensures that:

$$|\langle f, (\Delta_{\mathscr{E},\theta} + V(Q) - \Delta_{\mathscr{E}_\circ,\theta_\circ})f\rangle| \leq |\langle f, V(Q)f\rangle| + 2\langle f, \Lambda(G)f\rangle \leq \eta\langle f, d_{G_\circ}(Q)f\rangle + \kappa_\eta\|f\|^2 \tag{5.3}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. Therefore, for all $\varepsilon > 0$ and all $\eta \in (0, 1)$, satisfying (5.1), there is $c_{\varepsilon,\eta}$ so that:

$$\begin{aligned}(1 - \eta - \varepsilon)\langle f, d_{G_0}(Q)f\rangle - c_{\varepsilon,\eta}\|f\|^2 \leq \langle f, (\Delta_{\mathscr{E},\theta} + V(Q))f\rangle \\ \leq (1 + \eta + \varepsilon)\langle f, d_{G_0}(Q)f\rangle + c_{\varepsilon,\eta}\|f\|^2,\end{aligned}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. This gives, directly, the first point. Moreover, as above, the second and third points follow by Proposition 2.7. □

We now turn to the stability of the essential spectrum.

Proposition 5.2. *Let $G = (\mathscr{V}, \mathscr{E}, m, \theta)$ and $G_\circ = (\mathscr{V}, \mathscr{E}_\circ, m, \theta_\circ)$ be weighted graphs and $V : \mathscr{V} \to \mathbb{R}$. Suppose that $\mathcal{D}(\Delta^{1/2}_{\mathscr{E}_\circ,\theta_\circ}) = \mathcal{D}(d^{1/2}_{G_0}(Q))$ and that*

$$|V(x)| + \Lambda(x) = o(1 + d_{G_\circ}(x)), \text{ as } |x| \to \infty, \tag{5.4}$$

where Λ is defined in (5.2)*. Then $\mathcal{H} := \Delta_{\mathscr{E},\theta} + V(Q)|_{\mathcal{C}_c(\mathscr{V})}$ is bounded from below by some negative constant $-C$, in the form sense. We denote by $\mathcal{H}_\mathscr{F}$ its Friedrichs extension. We obtain that $\mathcal{D}(|\mathcal{H}_\mathscr{F}|^{1/2}) = \mathcal{D}(d^{1/2}_{G_0}(Q))$ and $\sigma_{\rm ess}(\Delta_{\mathscr{E}_\circ,\theta_\circ}) = \sigma_{\rm ess}(\mathcal{H}_\mathscr{F})$.*

Proof. The uniform boundedness theorem and (5.4) implies that $\mathcal{H}$ is bounded from below by some negative constant $-C$, in the form sense. We consider its Friedrichs extension. Next, KLMN's Theorem, e.g., [RS, Theorem X.17] ensures that $\mathcal{D}(|\mathcal{H}_\mathscr{F}|^{1/2}) = \mathcal{D}(d^{1/2}_G(Q))$.

By Weyl's Theorem, e.g., [RS, Theorem XII.1], it is enough to show that the difference of the resolvents is compact. As we have to work with forms, one should be careful with the resolvent equation. We give a complete proof and refer to [GGo] for more discussions of this matter. To lighten notation, we set $H_0 := \Delta_{\mathscr{E}_\circ,\theta_\circ}$ and $H := \Delta_{\mathscr{E},\theta} + V$. To start off, we give a rigorous meaning to

$$(H + \mathrm{i})^{-1} - (H_0 + \mathrm{i})^{-1} = (H + \mathrm{i})^{-1}(H_0 - H)(H_0 + \mathrm{i})^{-1}. \tag{5.5}$$

Since $\mathscr{G} := \mathcal{D}((H+C)^{1/2}) = \mathcal{D}((H_0)^{1/2})$, both operators extend to an element of $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$. Here we use the Riesz lemma to identify $\mathscr{H}$ with its anti-dual $\mathscr{H}^*$. We denote these extensions with a tilde.

We have $(H_0+\mathrm{i})^{-1*}\mathscr{H} \subset \mathscr{G}$. This allows one to deduce that $(H_0+\mathrm{i})^{-1}$ extends to a unique continuous operator $\mathscr{G}^* \to \mathscr{H}$. We denote it for the moment by R. From $R(H_0+\mathrm{i})u = u$ for $u \in \mathcal{D}(H_0)$ we get, by density of $\mathcal{D}(H_0)$ in $\mathscr{G}$ and continuity, $R(\widetilde{H_0}+\mathrm{i})u = u$ for $u \in \mathscr{G}$. In particular

$$(H+\mathrm{i})^{-1} = R(\widetilde{H_0}+\mathrm{i})(H+\mathrm{i})^{-1}.$$

Clearly,

$$(H_0+\mathrm{i})^{-1} = (H_0+\mathrm{i})^{-1}(H+\mathrm{i})(H+\mathrm{i})^{-1} = R(\widetilde{H}+\mathrm{i})(H+\mathrm{i})^{-1}.$$

We subtract the last two relations to obtain that

$$(H_0+\mathrm{i})^{-1} - (H+\mathrm{i})^{-1} = R(\widetilde{H} - \widetilde{H_0})(H+\mathrm{i})^{-1}.$$

Since R is uniquely determined as the extension of $(H_0+\mathrm{i})^{-1}$ to a continuous map $\mathscr{G}^* \to \mathscr{H}$, one may keep the notation $(H_0+\mathrm{i})^{-1}$ for it. With this convention, the rigorous version of (5.5) that we use is:

$$(H_0+\mathrm{i})^{-1} - (H+\mathrm{i})^{-1} = (H_0+\mathrm{i})^{-1}(\widetilde{H} - \widetilde{H}_0)(H+\mathrm{i})^{-1}. \tag{5.6}$$

Therefore, to prove the equality of the essential spectra, it is enough to show that $\widetilde{H} - \widetilde{H}_0$ is a compact operator from $\mathscr{G}$ to $\mathscr{G}^*$. By (5.3), one gets:

$$\begin{aligned}|\langle (1+d_{G_0})^{-1/2}(Q)f, (\Delta_{\mathscr{E},\theta} + V(Q) - \Delta_{\mathscr{E}_\circ,\theta_\circ})(1+d_{G_0})^{-1/2}(Q)f\rangle| \le \\ \langle (1+d_{G_0})^{-1/2}(Q)f, (V+2\Lambda)(Q)(1+d_{G_0})^{-1/2}(Q)f\rangle,\end{aligned}$$

for all $f \in \mathcal{C}_c(\mathscr{V})$. Now, by hypothesis (5.4), $(V+2\Lambda)(1+d_{G_0})^{-1}(Q)$ is compact in $\ell^2(\mathscr{V}, m^2)$. We conclude by using Proposition 2.8. □

Remark 5.3. *Note that* (5.1) *and* (5.4) *allow us to consider some potentials $V(Q)$ that are unbounded from below, whereas $\Delta_{\mathscr{E},\theta} + V(Q)$ is bounded from below. This is due to the fact that we know the form-domain explicitly.*

Appendix A. The $\mathcal{C}^1$–Regularity

We start with some generalities. Given a bounded operator B and a self-adjoint operator A acting in a Hilbert space $\mathscr{H}$, one says that $B \in \mathcal{C}^k(A)$ if $t \mapsto e^{-\mathrm{i}tA}Be^{\mathrm{i}tA}$ is strongly $\mathcal{C}^k$. Given a closed and densely defined operator B, one says that $B \in \mathcal{C}^k(A)$ if for some (hence any) $z \notin \sigma(B)$, $t \mapsto e^{-\mathrm{i}tA}(B-z)^{-1}e^{\mathrm{i}tA}$ is strongly $\mathcal{C}^k$. The two definitions coincide in the case of a bounded self-adjoint operator. We recall a result following from Lemma 6.2.9 and Theorem 6.2.10 of [ABG].

Theorem A.1. *Let A and B be two self-adjoint operators in the Hilbert space $\mathscr{H}$. For $z \notin \sigma(A)$, set $R(z) := (B-z)^{-1}$. The following points are equivalent to $B \in \mathcal{C}^1(A)$:*

(a) *For one (then for all) $z \notin \sigma(B)$, there is a finite c such that*

$$|\langle Af, R(z)f\rangle - \langle R(\overline{z})f, Af\rangle| \le c\|f\|^2, \text{ for all } f \in \mathcal{D}(A). \tag{A.1}$$

(b) i) *There is a finite c such that for all $f \in \mathcal{D}(A)\cap\mathcal{D}(B)$:*

$$|\langle Af, Bf\rangle - \langle Bf, Af\rangle| \le c\big(\|Bf\|^2 + \|f\|^2\big). \tag{A.2}$$

ii) *For some (then for all) $z \notin \sigma(B)$, the set $\{f \in \mathcal{D}(A), R(z)f \in \mathcal{D}(A)$ and $R(\overline{z})f \in \mathcal{D}(A)\}$ is a core for A.*

Note that the condition ii) could be delicate to check (see [GGé]). We mention [GM][Lemma A.2] and [GŁ][Lemma 3.2.2] to overcome this subtlety.

Note that (A.1) yields that the commutator $[A, R(z)]$ extends to a bounded operator, in the form sense. We shall denote the extension by $[A, R(z)]_\circ$. In the same way, since $\mathcal{D}(B) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(B)$, (A.2) ensures that the commutator $[B, A]$ extends to a unique element of $\mathcal{B}\big(\mathcal{D}(B), \mathcal{D}(B)^*\big)$ denoted by $[B, A]_\circ$. Moreover, when $B \in \mathcal{C}^1(A)$, one has:

$$\big[A, (B-z)^{-1}\big]_\circ = \underbrace{(B-z)^{-1}}_{\mathscr{H} \leftarrow \mathcal{D}(B)^*} \quad \underbrace{[B, A]_\circ}_{\mathcal{D}(B)^* \leftarrow \mathcal{D}(B)} \quad \underbrace{(B-z)^{-1}}_{\mathcal{D}(B) \leftarrow \mathscr{H}}. \tag{A.3}$$

Here we use the Riesz lemma to identify $\mathscr{H}$ with its anti-dual $\mathscr{H}^*$.

It turns out that an easier characterization is available if the domain of B is conserved under the action of the C_0-group generated by A.

Theorem A.2. *(*[ABG, p. 258]*) Let A and B be two self-adjoint operators in the Hilbert space $\mathscr{H}$ such that $e^{itA}\mathcal{D}(B) \subset \mathcal{D}(B)$, for all $t \in \mathbb{R}$. Then, $B \in \mathcal{C}^1(A)$ if and only if*

$$\liminf_{t \to 0^+} \big\| [B, e^{\mathrm{i}tB}/t] \big\|_{\mathcal{B}(\mathcal{D}(B), \mathcal{D}(B)^*)} < \infty.$$

Note that $e^{itA}\mathcal{D}(B)^* \subset \mathcal{D}(B)^*$ by duality.

Appendix B. The Helffer-Sjöstrand's formula

We present briefly the Helffer-Sjöstrand's formula. We refer to [GJ][Appendix B] and [BG][Appendix A] (see also [DG, HSS, M]) for commutator expansion. We first recall some well-known facts about almost analytic extensions. For $\rho \in \mathbb{R}$, let $\mathcal{S}^\rho$ be the class of function $\varphi \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{C})$ such that

$$\forall k \in \mathbb{N}, \quad C_k(\varphi) := \sup_{t \in \mathbb{R}} \langle t \rangle^{-\rho+k} |\varphi^{(k)}(t)| < \infty. \tag{B.1}$$

Equipped with the semi-norms defined by (B.1), $\mathcal{S}^\rho$ is a Fréchet space. Leibniz' formula implies the continuous embedding: $\mathcal{S}^\rho \cdot \mathcal{S}^{\rho'} \subset \mathcal{S}^{\rho+\rho'}$. We shall use the following result, e.g., [DG, Appendix C.2].

Lemma B.1. *Let $\varphi \in \mathcal{S}^\rho$ with $\rho \in \mathbb{R}$. For all $l \in \mathbb{N}$, there is a smooth function $\varphi^{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, such that:*

$$\varphi^{\mathbb{C}}|_{\mathbb{R}} = \varphi, \qquad \left| \frac{\partial \varphi^{\mathbb{C}}}{\partial \overline{z}}(z) \right| \leq c_1 \langle \Re(z) \rangle^{\rho-1-l} |\Im(z)|^l \tag{B.2}$$

$$\operatorname{supp} \varphi^{\mathbb{C}} \subset \{x + \mathrm{i}y, |y| \leq c_2 \langle x \rangle\}, \tag{B.3}$$

$$\varphi^{\mathbb{C}}(x + \mathrm{i}y) = 0, \ \text{if } x \notin \operatorname{supp} \varphi. \tag{B.4}$$

for some constants c_1, c_2 depending on the semi-norms (B.1) *of φ in $\mathcal{S}^\rho$ and not on φ.*

One calls $\varphi^{\mathbb{C}}$ an *almost analytic extension* of φ. Let A be a self-adjoint operator, $\rho < 0$ and $\varphi \in \mathcal{S}^\rho$. By functional calculus, one has $\varphi(A)$ bounded. The Helffer-Sjöstrand's formula, e.g., [HS] and [DG], gives that for all almost analytic extension of $\varphi \in \mathcal{S}^\rho$, with $\rho < 0$, we have:

$$\varphi(A) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{C}} \frac{\partial \varphi^{\mathbb{C}}}{\partial \overline{z}} (z - A)^{-1} dz \wedge d\overline{z}. \tag{B.5}$$

Note that the integral exists in the norm topology, by (B.2) with $l = 1$ and by taking in account the domain of integration given in (B.3).

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Institut de Mathématiques de Bordeaux, Université Bordeaux 1, 351, cours de la Libération, 33405 Talence cedex, France

E-mail address: sylvain.golenia@u-bordeaux1.fr