ON THE UNIQUENESS OF SOLUTIONS FOR THE FUBUKI GAME

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ABSTRACT. We discuss the problem of uniqueness of solutions for the Fubuki game when the diagonal is prescribed.

1. INTRODUCTION

The Fubuki game is a generalization of the magical square. The rules are simple: Fill a 3 by 3 grid with the numbers 1 to 9 so that each line and column adds up to a given sum. For instance,

1			=10
	2		=15
		3	=20
= 16	=15	=14	

leads to the two solutions:

1	4	5	=10
7	**2**	6	=15
8	9	**3**	=20
= 16	=15	=14	

1	5	4	=10
6	**2**	7	=15
9	8	**3**	=20
= 16	=15	=14	

whereas

1			=11
	2		=14
		3	=20
= 14	=15	=16	

leads to the unique solution:

1	4	6	=11
5	**2**	7	=14
8	9	**3**	=20
= 14	=15	=16	

In these two examples we have prescribed the diagonal. It is not necessary in the general case but it will be the setting of our study. It is the first interesting case since, as a linear system, i.e, solving it with real numbers, the affine subspace of solutions is of dimension 1, when the solution exists.

The first problem is to ensure the uniqueness of the solution. This can be done solely by choosing the diagonal.

Theorem 1. *a) Assume that the diagonal belongs to the following set:*

$$\mathscr{D} := \big\{\{1,3,4\}, \{1,3,6\}, \{1,4,6\}, \{1,5,7\}, \{1,5,8\}, \{1,7,8\}, \{2,3,4\}, \{2,3,5\},$$
$$\{2,3,6\}, \{2,3,8\}, \{2,3,9\}, \{2,4,6\}, \{2,4,7\}, \{2,4,8\}, \{2,5,6\}, \{2,5,9\},$$
$$\{2,6,7\}, \{2,6,8\}, \{2,7,8\}, \{3,4,6\}, \{3,4,8\}, \{3,5,6\}, \{3,5,7\}, \{3,5,9\},$$
$$\{3,6,8\}, \{4,5,7\}, \{4,5,8\}, \{4,6,7\}, \{4,6,8\}, \{4,6,9\}, \{4,7,8\}, \{4,7,9\},$$
$$\{5,7,8\}, \{6,7,8\}, \{6,7,9\}\big\}.$$

Then if the solution of the Fubuki game with prescribed diagonal exists, it is unique.
b) If the diagonal does not belong to $\mathscr{D}$, there are at most 2 solutions.

We stress that the result is independent of the choice of the sum of the lines and that of the column.

Next we count the number of grids that lead to a unique solution.

Theorem 2. *Among the Fubuki grids with prescribed diagonal which have a solution, there are 351432 of them which have a unique solution.*

Recalling that there are 362880 possible grids we have therefore around 96% chance to have a unique solution. The novelty in this note is not really about the statement of the two theorems, as it could have been guessed and done pretty easily with Matlab for instance, but relies in the fact that it can be humanly proven.

To conclude we mention that brute force computations give that there are: $281304 (\simeq 77\%)$ unique solutions if we prescribe the two first squares on the diagonal, $163387 (\simeq 45\%)$ ones if we prescribe the upper left square, and $46147 (\simeq 12\%)$ ones if we prescribe no square at all.

2. Proofs

Suppose that we have a solution of a Fubuki problem with prescribed diagonal given by

(2.1)
$$S_0 := \begin{array}{|c|c|c|} \hline \mathbf{s_{1,1}} & s_{1,2} & s_{1,3} \\ \hline s_{2,1} & \mathbf{s_{2,2}} & s_{2,3} \\ \hline s_{3,1} & s_{3,2} & \mathbf{s_{3,3}} \\ \hline \end{array}$$

We stress that we have prescribed the diagonal by using a bold font.

If we have a second solution then there is $a \in \mathbb{Z} \setminus \{0\}$ such that $s_{1,2} + a$ is at the place of the $s_{1,2}$. Since the sum of the elements of the first line is the same for the two grids, we see that $s_{1,3}$ is hence replaced by $s_{1,3} - a$. We fill the rest of the grid in the same manner, we obtain:

(2.2)
$$S_a := \begin{array}{|c|c|c|} \hline \mathbf{s_{1,1}} & s_{1,2} + a & s_{1,3} - a \\ \hline s_{2,1} - a & \mathbf{s_{2,2}} & s_{2,3} + a \\ \hline s_{3,1} + a & s_{3,2} - a & \mathbf{s_{3,3}} \\ \hline \end{array}$$

Note that S_a is an other solution if and only if

$$\{s_{1,2} + a, s_{1,3} - a, s_{2,1} - a, s_{2,3} + a, s_{3,1} + a, s_{3,2} - a\}$$

(2.3)
$$= \{s_{1,2}, s_{1,3}, s_{2,1}, s_{2,3}, s_{3,1}, s_{3,2}\} := X.$$

We give a first remark on the structure of the solutions.

Remark 2.1. By considering the minimal element of the set X we see that: if there are $a, b \in \mathbb{Z} \setminus \{0\}$ such that S_a and S_b exist, then a and b are of the same sign.

2.1. **Triplet structure.** We turn to the structure of the set X with respect to a.

Lemma 2.2. *Given $a \in \mathbb{Z} \setminus \{0\}$ such that (2.3) holds true, then there is a unique triplet $y_1, y_2, y_3 \in X$ such that $y_1 < y_2 < y_3$ and*

$$\{y_1, y_2, y_3\} + \{0, a\} := \{y_1, y_1 + a, y_2, y_2 + a, y_3, y_3 + a\} = X.$$

Here the plus between ensembles denotes the Minkowski sum.

Proof. To fix ideas say that $a > 0$. Set $y_1 := \min(X)$. Therefore by (2.3) $y_1 + a$ or $y_1 - a$ belongs to X. By minimality, we have that $y_1 + a \in X$. To complete the proof, take $y_2 := \min(X \setminus \{y_1, y_1 + a\})$ and $y_3 := \min(X \setminus \{y_1, y_1 + a, y_2, y_2 + a\})$. The uniqueness follows by construction. $\qquad\square$

Definition. When it exists, we call $\mathcal{T}_a := \{y_1, y_2, y_3\}$ the *triplet* associated to a and to S_0.

Remark 2.3. Note that $\mathcal{T}_a$ exists if and only if $\mathcal{T}_{-a}$ exists (set $y_i(a) := y_i(-a) + a$). Therefore to prove the existence of $\mathcal{T}_a$ it is enough to do it for $a > 0$. However, recalling Remark 2.1, S_a cannot be a solution if S_{-a} is one.

Next we notice the incompatibility of the two different triplets.

Lemma 2.4. *Given a and b in $\mathbb{Z} \setminus \{0\}$ such that (2.2) and (2.3) hold true and that $a \neq b$. Then $\mathcal{T}_a \neq \mathcal{T}_b$.*

Proof. First note that by Remark 2.1 that a and b are of the same sign. To fix ideas say that they are both positive. Suppose that $\mathcal{T}_a = \mathcal{T}_b$. By maximality $y_3(a) + a = y_3(b) + b$. We infer that $a = b$. Contradiction. $\qquad\square$

Finally we rephrase the existence of the solution (2.2) with the help of the triplet $\mathcal{T}_a$ into a key-stone Proposition.

Proposition 2.5. *Let S_0 be the solution of (2.1) and $a \in \mathbb{Z}$.*
a) If the triplet $\mathcal{T}_a$ associated to S_0 and to $a \in \mathbb{Z} \setminus \{0\}$ does not exists, then S_a does not exist.
b) Assume that there exists the triplet $\mathcal{T}_a$ associated to S_0 and to $a \in \mathbb{Z} \setminus \{0\}$. Then the solution S_a exists if and only if:

$$(2.4) \qquad\qquad \mathcal{T}_a = \{s_{1,2}, s_{2,3}, s_{3,1}\}.$$

Proof. The point a) is clear from (2.3) and Lemma 2.2.

We turn to b). To fix idea say that $a > 0$. We denote by $S^+ := \{s_{1,2}, s_{2,3}, s_{3,1}\}$ and by $S^- := X \setminus S^+$. If $\mathcal{T}_a = S^+$ by Lemma 2.2 we see that $\mathcal{T}_a + \{a\} = S^-$. Consequently (2.2) and (2.3) are satisfied: S_a exists.

Assume now that S_a exists: (2.2) and (2.3) are satisfied. If $y_3 + a \in S^+$ then we obtain a contradiction by maximality on X and by (2.3). Therefore $y_3 + a \in S^-$. Next since $y_2 < y_3$, (2.3) ensures that $y_3 \in S^+$. We repeat the proof for y_2 and then for y_1 to conclude that $\mathcal{T}_a = S^+$. $\qquad\square$

2.2. **List of possibilities.** As seen in Proposition 2.5, the number of solutions of a grid is linked with the existence of a Triplet. We classify them. Given $x_1, x_2, x_3 \in$

$\{1, \ldots, 9\}$, in the first column, we give the possible $c > 0$, in the second column, such that there exist y_1, y_2, and y_3 in $\{1, \ldots, 9\} \setminus \{x_1, x_2, x_3\}$ satisfying

(2.5) $\{y_1, y_1 + c, y_2, y_2 + c, y_3, y_3 + c\} = \{1, \ldots, 9\} \setminus \{x_1, x_2, x_3\}.$

This gives:

$\{1, 2, 3\}$	$\{1, 3\}$	$\{1, 2, 4\}$	$\{2\}$	$\{1, 2, 5\}$	$\{1\}$	$\{1, 2, 6\}$	$\{4\}$
$\{1, 2, 7\}$	$\{1\}$	$\{1, 2, 8\}$	$\{2\}$	$\{1, 2, 9\}$	$\{1, 3\}$	$\{1, 3, 4\}$	$\emptyset$
$\{1, 3, 5\}$	$\{2\}$	$\{1, 3, 6\}$	$\emptyset$	$\{1, 3, 7\}$	$\{4\}$	$\{1, 3, 8\}$	$\{3\}$
$\{1, 3, 9\}$	$\{2\}$	$\{1, 4, 5\}$	$\{1\}$	$\{1, 4, 6\}$	$\emptyset$	$\{1, 4, 7\}$	$\{1\}$
$\{1, 4, 8\}$	$\{4\}$	$\{1, 4, 9\}$	$\{1\}$	$\{1, 5, 6\}$	$\{5\}$	$\{1, 5, 7\}$	$\emptyset$
$\{1, 5, 8\}$	$\emptyset$	$\{1, 5, 9\}$	$\{4\}$	$\{1, 6, 7\}$	$\{1\}$	$\{1, 6, 8\}$	$\{2\}$
$\{1, 6, 9\}$	$\{1\}$	$\{1, 7, 8\}$	$\emptyset$	$\{1, 7, 9\}$	$\{2\}$	$\{1, 8, 9\}$	$\{1, 3\}$
$\{2, 3, 4\}$	$\emptyset$	$\{2, 3, 5\}$	$\emptyset$	$\{2, 3, 6\}$	$\emptyset$	$\{2, 3, 7\}$	$\{3\}$
$\{2, 3, 8\}$	$\emptyset$	$\{2, 3, 9\}$	$\emptyset$	$\{2, 4, 5\}$	$\{2\}$	$\{2, 4, 6\}$	$\emptyset$
$\{2, 4, 7\}$	$\emptyset$	$\{2, 4, 8\}$	$\emptyset$	$\{2, 4, 9\}$	$\{2\}$	$\{2, 5, 6\}$	$\emptyset$
$\{2, 5, 7\}$	$\{5\}$	$\{2, 5, 8\}$	$\{2\}$	$\{2, 5, 9\}$	$\emptyset$	$\{2, 6, 7\}$	$\emptyset$
$\{2, 6, 8\}$	$\emptyset$	$\{2, 6, 9\}$	$\{4\}$	$\{2, 7, 8\}$	$\emptyset$	$\{2, 7, 9\}$	$\{3\}$
$\{2, 8, 9\}$	$\{2\}$	$\{3, 4, 5\}$	$\{1\}$	$\{3, 4, 6\}$	$\emptyset$	$\{3, 4, 7\}$	$\{1\}$
$\{3, 4, 8\}$	$\emptyset$	$\{3, 4, 9\}$	$\{1\}$	$\{3, 5, 6\}$	$\emptyset$	$\{3, 5, 7\}$	$\emptyset$
$\{3, 5, 8\}$	$\{5\}$	$\{3, 5, 9\}$	$\emptyset$	$\{3, 6, 7\}$	$\{1\}$	$\{3, 6, 8\}$	$\emptyset$
$\{3, 6, 9\}$	$\{1\}$	$\{3, 7, 8\}$	$\{3\}$	$\{3, 7, 9\}$	$\{4\}$	$\{3, 8, 9\}$	$\{1\}$
$\{4, 5, 6\}$	$\{6\}$	$\{4, 5, 7\}$	$\emptyset$	$\{4, 5, 8\}$	$\emptyset$	$\{4, 5, 9\}$	$\{5\}$
$\{4, 6, 7\}$	$\emptyset$	$\{4, 6, 8\}$	$\emptyset$	$\{4, 6, 9\}$	$\emptyset$	$\{4, 7, 8\}$	$\emptyset$
$\{4, 7, 9\}$	$\emptyset$	$\{4, 8, 9\}$	$\{4\}$	$\{5, 6, 7\}$	$\{1\}$	$\{5, 6, 8\}$	$\{2\}$
$\{5, 6, 9\}$	$\{1\}$	$\{5, 7, 8\}$	$\emptyset$	$\{5, 7, 9\}$	$\{2\}$	$\{5, 8, 9\}$	$\{1\}$
$\{6, 7, 8\}$	$\emptyset$	$\{6, 7, 9\}$	$\emptyset$	$\{6, 8, 9\}$	$\{2\}$	$\{7, 8, 9\}$	$\{1, 3\}$

2.3. **Proof of the main result.** We are now in position to prove the first theorem.

Proof of Theorem 1. a) Assume that the elements of the diagonal belong to $\mathscr{D}$ then by Remark 2.3 and Section 2.2 we see that there exists no $a \in \mathbb{Z} \setminus \{0\}$ such that $\mathscr{T}_a$ exists. Therefore by Proposition 2.5 the solution is unique.
b) Let S_0 be a solution as in (2.1). Suppose now that the elements of the diagonal does not belong to $\mathscr{D}$. Then by Section 2.2 there is $c > 0$ such that $\mathscr{T}_c$ and $\mathscr{T}_{-c}$ exist. We first discuss the case where c is unique. Thanks to (2.4), we see that we will have at most one other solution. Suppose now that we have two different $c > 0$ such that $\mathscr{T}_c$ exists, namely c_1 and c_2. Lemma 2.4 ensures that $\mathscr{T}_{c_1} \neq \mathscr{T}_{c_2}$. Using again (2.4), we conclude we will have at most one other solution. $\square$

We conclude by counting the number of unique solutions.

Proof of Theorem 2. We split the proof according to the number of $c > 0$ that exists in (2.5).
a) no c: there are 35 different sets $\{x_1, x_2, x_3\}$ that give that $\mathscr{T}_c$ and $\mathscr{T}_{-c}$ do not exist (as seen before the solution is therefore unique). By placing x_1, x_2, and x_3 on the diagonal, we have 3! choices and 6! choices for the rest of the grid. We obtain

$$35 \times 3! \times 6! \text{ possibilities}$$

that lead to a unique solution.
b) A unique c: there are 45 different sets $\{x_1, x_2, x_3\}$ that give a unique c in (2.5). Then we place x_1, x_2, and x_3 on the diagonal. Suppose that we have two solutions

(recall that there can not be more than 2 by Theorem 1). Recalling Remark 2.1 we consider solutions either of the form S_c or S_{-c} (but not both of them). Finally using (2.4), we have $3! \times 3!$ grids with two solutions. We obtain

$$45 \times 3! \times (6! - 3! \times 3!) \text{ possibilities}$$

that lead to a unique solution.

c) Two different c: there are 4 different sets $\{x_1, x_2, x_3\}$ that give two different c in (2.5), namely c_1 and c_2. As before we start by fixing x_1, x_2, and x_3 on the diagonal. Suppose that we have two solutions. Then recalling Remark 2.1 the solutions are either $\{S_{c_1}, S_{c_2}\}$ or $\{S_{-c_1}, S_{-c_2}\}$ (but not the union). Finally by Lemma 2.4 and by using we have $2 \times 3! \times 3!$ grids with two solutions. We obtain

$$4 \times 3! \times (6! - 2 \times 3! \times 3!) \text{ possibilities}$$

that lead to a unique solution.

To conclude it remains to sum the three results. $\qquad\square$

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