$C^*$-algebras of anisotropic Schrödinger operators on trees

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Abstract

We study a $C^*$-algebra generated by differential operators on a tree. We give a complete description of its quotient with respect to the compact operators. This allows us to compute the essential spectrum of self-adjoint operators affiliated to this algebra. The results cover Schrödinger operators with highly anisotropic, possibly unbounded potentials.

1 Introduction

Given a $\nu$-fold tree $\Gamma$ of origin $e$ with its canonical metric $d$, we write $x \sim y$ when $x$ and $y$ are connected by an edge and we set $|x| = d(x, e)$. For each $x \in \Gamma \setminus \{e\}$, we denote by $x' \equiv x^{(1)}$ the unique element $y \sim x$ such that $|y| = |x| - 1$ and we set $x^{(p)} = (x^{(p-1)})'$ for $1 \leq p \leq |x|$. Let $x\Gamma = \{y \in \Gamma \mid |y| \geq |x| \text{ and } y^{(|y|-|x|)} = x\}$, where the convention $x^{(0)} = x$ has been used.

On $\ell^2(\Gamma)$ we define the bounded operator $\partial$ given by $(\partial f)(x) = \sum_{y' = x} f(y)$. Its adjoint is given by $(\partial^\ast f)(e) = 0$ and $(\partial^\ast f)(x) = f(x')$ for $|x| \geq 1$. Let $\mathcal{D}$ be the $C^*$-algebra generated by $\partial$.

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In order to obtain our algebra of potentials, we consider the “hyperbolic” compactification \( \hat{\Gamma} = \Gamma \cup \partial \Gamma \) of \( \Gamma \) constructed as follows. An element \( x \) of the boundary at infinity \( \partial \Gamma \) is a \( \Gamma \)-valued sequence \( x = (x_n)_{n \in \mathbb{N}} \) such that \( |x_n| = n \) and \( x_{n+1} \sim x_n \) for all \( n \in \mathbb{N} \). We set \( |x| = \infty \) for \( x \in \partial \Gamma \). The space \( \hat{\Gamma} \) is equipped with a natural ultrametric space structure.

For \( x \in \partial \Gamma \) and \( (y_n)_{n \in \mathbb{N}} \) a sequence in \( \Gamma \) we have \( \lim_{n \to \infty} y_n = x \) if for each \( m \in \mathbb{N} \) there is \( N \in \mathbb{N} \) such that for each \( n \geq N \) we have \( y_n \in x_m \Gamma \).

We denote by \( C(\hat{\Gamma}) \) the set of complex-valued continuous functions defined on \( \hat{\Gamma} \). Since \( \Gamma \) is dense in \( \hat{\Gamma} \), we can view \( C(\hat{\Gamma}) \) as a \( C^* \)-subalgebra of \( C_b(\Gamma) \), the algebra of bounded complex-valued functions defined on \( \Gamma \).

For \( V \in C(\hat{\Gamma}) \), we denote by \( V(Q) \) the operator of multiplication by \( V \) in \( \ell^2(\Gamma) \).

Let us now denote by \( \mathcal{C}(\hat{\Gamma}) \) the \( C^* \)-algebra generated by \( \mathcal{D} \) and \( C(\hat{\Gamma}) \). It contains the compact operators of \( \ell^2(\Gamma) \). Following the strategy exposed in [6], we shall first compute its quotient with respect to the ideal of compact operators. We stress that the crossed product technique introduced in [6] in order to compute quotients cannot be used in our case. Instead, we shall use the Theorem 4.5 in order to calculate the essential spectrum of self-adjoint operators related to \( \mathcal{C}(\hat{\Gamma}) \). In this introduction we consider only the most important case, when \( \nu > 1 \).

**Theorem 1.1** Let \( \nu > 1 \). There is a unique morphism \( \Phi : \mathcal{C}(\hat{\Gamma}) \to \mathcal{D} \otimes C(\partial \Gamma) \) such that \( \Phi(D) = D \otimes 1 \) for all \( D \in \mathcal{D} \) and \( \Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial \Gamma}) \).

This morphism is surjective and its kernel is \( \mathbb{K}(\Gamma) \).

The rest of this introduction is devoted to some applications of this theorem to spectral analysis. Let \( \nu > 1 \) and \( H = \sum_{\alpha,\beta} a_{\alpha,\beta}(Q) \partial^{\alpha} \partial^{\beta} + K \), where \( K \) is a compact operator, \( a_{\alpha,\beta} \in C(\hat{\Gamma}) \) and \( a_{\alpha,\beta} = 0 \) for all \( (\alpha, \beta) \in \mathbb{N}^2 \) but a finite number of pairs. Clearly \( H \in \mathcal{C}(\hat{\Gamma}) \). As a consequence of the Theorem 1.1, there is \( \Phi \) such that \( \Phi(H) = \sum_{\alpha,\beta} \partial^{\alpha} \partial^{\beta} \otimes (a_{\alpha,\beta}|_{\partial \Gamma}) \), and, if \( H \) self-adjoint, its essential spectrum is:

\[
\sigma_{\text{ess}}(H) = \bigcup_{\gamma \in \partial \Gamma} \sigma \left( \sum_{\alpha,\beta} a_{\alpha,\beta}(\gamma) \partial^{\alpha} \partial^{\beta} \right).
\]

This result can be made quite explicit in the particular case of a Schrödinger operator.
\( H = \Delta + V(Q) \) with potential \( V \) in \( C(\hat{\Gamma}) \). Since \( \Delta \) is a bounded operator on \( \ell^2(\Gamma) \) defined by \((\Delta f)(x) = \sum_{y \sim x} (f(y) - f(x))\), it belongs to \( \mathscr{C}(\hat{\Gamma}) \). We then set \( \Delta_0 = \partial + \partial^* - \nu \text{Id} \) (which belongs to \( \mathscr{D} \)) and notice that \( \Delta - \Delta_0 \) is compact. One then gets (see [1] for instance):

\[
\sigma_{\text{ess}}(\partial + \partial^*) = \sigma_{\text{ac}}(\partial + \partial^*) = \sigma(\partial + \partial^*) = [-2\sqrt{\nu}, 2\sqrt{\nu}],
\]

where \( \sigma_{\text{ac}}(T) \) denotes the absolute continuous part of the spectrum of a given self-adjoint operator \( T \). On the other hand, Theorem 1.1 gives us directly \( \sigma_{\text{ess}}(\partial^* + \partial) = \sigma(\partial^* + \partial) \). We thus get

\[
\sigma_{\text{ess}}(\Delta + V(Q)) = \sigma(\Delta_0) + V(\partial \Gamma) = [-\nu - 2\sqrt{\nu}, -\nu + 2\sqrt{\nu}] + V(\partial \Gamma).
\]

In fact this result holds (and is trivial) in the case of \( \nu = 1 \), i.e. when \( \Gamma = \mathbb{N} \).

Given a continuous function on \( \partial \Gamma \), the Tietze theorem allows us to extend it to a continuous function on \( \hat{\Gamma} \), so one may construct a large class of Hamiltonians with given essential spectra. Nevertheless, we are able to point out a concrete class of non-trivial potentials \( V \in C(\hat{\Gamma}) \) with uniform behaviour at infinity which form a dense family of \( C(\hat{\Gamma}) \). Namely, for each bounded function \( f : \Gamma \to \mathbb{R} \) and each real \( \alpha > 1 \) let

\[
V(x) = \sum_{k=1}^{\lvert x \rvert} \frac{f(x_k)}{k^\alpha}, \tag{1.1}
\]

where \( x_k = x^{\lvert x \rvert - k} \) for \( x \in \Gamma \) (\( V \) belongs to \( C(\hat{\Gamma}) \) because of Proposition 2.3).

Concerning finer spectral features, based mainly on the Mourre estimate, we mention that in the case \( H = \Delta + V(Q) \), with \( V \) as in (1.1) where \( \alpha \geq 3 \) and such that \( V(\partial \Gamma) = 0 \), the results of [1] can be applied (the hypotheses of the Lemmas 6 and 7 from [1] are verified since \( V(x) = O(|x|^{-\alpha+1}) \) when \( |x| \to \infty \)). The aim of our work in preparation [8] is to prove that the Mourre estimate holds for more general classes of Hamiltonians affiliated to \( \mathscr{C}(\hat{\Gamma}) \) and to develop a scattering theory for them. Theorem 1.1 remains the key technical point for these purposes.

The preceding results on trees allow us to treat more general graphs. We recall that a graph is said to be connected if two of its elements can
be joined by a sequence of neighbours. Let \( G = \bigcup_{i=1}^{n} \Gamma_i \cup G_0 \) be a finite disjoint union of \( \Gamma_i \), each \( \Gamma_i \) being a \( \nu_i \)-fold branching tree with \( \nu_i \geq 1 \) and of \( G_0 \), a compact connected graph. We endow \( G \) with a connected graph structure that respects the graph structure of each \( \Gamma_i \) and the one of \( G_0 \), such that \( \Gamma_i \) is connected to \( \Gamma_j \) \((i \neq j)\) only through \( G_0 \) and such that \( \Gamma_i \) is connected to \( G_0 \) only through \( e_i \), the origin of \( \Gamma_i \). The graph \( G \) is hyperbolic and its boundary at infinity \( \partial G \) is the disjoint union \( \bigcup_{i=1}^{n} \partial \Gamma_i \).

We now choose \( V \in C(G \cup \partial G) \). One has \( V|_{\Gamma_i} \in C(\hat{\Gamma}_i) \) for all \( i = 1, \ldots, n \) and we easily obtain:

\[
\sigma_{ess}(\Delta + V(Q)) = \bigcup_{i=1}^{n} \left( [-\nu_i - 2\sqrt{\nu_i}, -\nu_i + 2\sqrt{\nu_i}] + V(\partial \Gamma_i) \right).
\]

This covers in particular the case of the Cayley graph of a free group with finite system of generators. We recall that the Cayley graph of a group \( G \) with a system of generators \( S \) is the graph defined on the set \( G \) with the relation \( x \sim y \) if \( xy^{-1} \in S \) or \( yx^{-1} \in S \). Let \( G \) be a free group with a system of generators \( S \) such that \( S = S^{-1} \). We denote by \( e \) its neutral element and we set \( |S| = \nu + 1 \). One may associate the restriction of the Cayley graph to the set of words starting with a given generator with a \( \nu \)-fold branching tree having as origin the generator. Hence, the Cayley graph of \( G \) will be \( \bigcup_{i=1}^{\nu} \Gamma_i \cup \{e\} \) where \( \Gamma_i \) is a \( \nu \)-fold branching tree with the above graph structure.

We now go further by taking \( V \in C(\hat{\Gamma}, \mathbb{R}) \) such that \( V(\Gamma) \subset \mathbb{R} \) (here \( \mathbb{R} = \mathbb{R} \cup \{\infty\} \) is the Alexandrov compactification of \( \mathbb{R} \)). More precisely, \( V \in C(\hat{\Gamma}, \mathbb{R}) \) if and only if for each \( \gamma \in \partial \Gamma \) we have either \( \lim_{x \to \gamma} V(x) = l \) where \( l \in \mathbb{R} \) or for each \( M \geq 0 \) there is \( N \in \mathbb{N} \) such that \( |V(x)| \geq M \) for all \( n \geq N \) and \( x \in \gamma_n \Gamma \) (see Proposition 2.3). We set

\[
D(V) = \{ f \in \ell^2(\Gamma) \mid \|V(Q)f\|^2 < \infty \}.
\]

Let \( T \in \mathcal{D} \) and \( T_0 = \Phi(T) \). Since \( T \) is bounded, the operator \( H = T + V(Q) \) with domain \( D(V) \) is self-adjoint and it is affiliated to \( \mathcal{C}(\hat{\Gamma}) \) (i.e. its resolvent belongs to \( \mathcal{C}(\hat{\Gamma}) \)). Indeed, we have \((V(Q) + z)^{-1} \in C(\hat{\Gamma})\) for each \( z \in \mathbb{C} \setminus \mathbb{R} \), and for large such \( z \),

\[
(H + z)^{-1} = (V(Q) + z)^{-1} \sum_{n \geq 0} \langle T(V(Q) + z)^{-1} \rangle^n,
\]
where the series is norm convergent. Now, with the same $z$, we use Theorem 1.1 and the fact that $\mathcal{D} \otimes C(\partial \Gamma) \simeq C(\partial \Gamma, \mathcal{D})$ to obtain

$$
\Phi_\gamma((H+z)^{-1}) \equiv \Phi((H+z)^{-1})(\gamma) = (V(\gamma)+z)^{-1} \sum_{n \geq 0} (T_0(V(\gamma)+z)^{-1})^n.
$$

Note that $(V(\gamma)+z)^{-1} = 0$ if $V(\gamma) = \infty$. By analytic continuation we get

$$
\Phi_\gamma((T + V(Q) + z)^{-1}) = (T_0 + V(\gamma) + z)^{-1}, \text{ for all } z \in \mathbb{C} \setminus \mathbb{R}. \text{ We used the convention } (T_0 + V(\gamma) + z)^{-1} = 0 \text{ if } V(\gamma) = \infty.
$$

We now compute the essential spectrum of $H$. If $V(\gamma) = \infty$ then $\sigma(\Phi_\gamma(H)) = \emptyset$. Otherwise, one has $\sigma(\Phi_\gamma(H)) = \sigma(T_0 + V(\gamma)) = \sigma(T_0) + V(\gamma)$. Hence we obtain:

$$
\sigma_{\text{ess}}(T + V(Q)) = \sigma(T_0) + V(\partial \Gamma_0),
$$

where $\partial \Gamma_0$ is the set of $\gamma \in \partial \Gamma$ such that $V(\gamma) \in \mathbb{R}$.

**Remark:** We mention an interesting question which has not been studied in this paper. In fact, one could replace the algebra $\mathcal{D}$ by the (much bigger) $C^*$-algebra generated by all the right translations $\rho_a$ (see Subsection 3.4 for notations) and consider the corresponding algebra $C(\hat{\Gamma})$. This is a natural object, since it contains all the “right-differential” operators acting on the tree (not only polynomials in $\partial$ and $\partial^*$. A combination of the techniques that we use and that of [9, 10] could allow one to compute the quotient in this case too. We also note that in [9, 10] a certain connection with the notion of crossed-product is pointed out, and this could be useful in further investigations. I would like to thank the referee for bringing to my attention the two papers of A. Nica quoted above.

## 2 Trees and related objects

### 2.1 The free monoid $\Gamma$

Let $\mathcal{A}$ be a finite set consisting of $\nu$ objects. Let $\Gamma$ be the free monoid over $\mathcal{A}$; its elements are *words* and those of $\mathcal{A}$ *letters*. We refer to [3, Chapter I, §7] for a detailed discussion of these notions, but we recall that a word $x$ is an $\mathcal{A}$-valued map defined on a set of the form\(^1\) $[1, n]$ with $n \in \mathbb{N}$, $x(i)$

\(^1\)We use the notation $[1, n] = [1, n] \cap \mathbb{N}$ where $\mathbb{N}$ is the set of integers $\geq 0$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
being the \( i \)-th letter of the word \( x \). The integer \( n \) (the number of letters of \( x \)) is the length of the word and will be denoted \( |x| \). There is a unique word \( e \) of length 0, its domain being the empty set. This is the neutral element of \( \Gamma \). We will also identify \( A \) with the set of words of length 1.

The monoid \( \Gamma \) will be endowed with the discrete topology. If \( x \in \Gamma \), we denote \( x\Gamma \) and \( \Gamma x \) the right and left ideals generated by \( x \). We have on \( \Gamma \) a canonical order relation which is by definition:

\[ x \leq y \iff y \in x\Gamma. \]

We recall some terminology from the theory of ordered sets. If \( \Gamma \) is an arbitrary ordered set and \( x, y \in \Gamma \), then one says that \( y \) covers \( x \) if \( x < y \) and if \( x \leq z \leq y \Rightarrow z = x \) or \( z = y \). If \( x \in \Gamma \), we denote \( \tilde{x} = \{ y \in \Gamma \mid y \text{ covers } x \} \)

In our case, \( y \) covers \( x \) if \( x \leq y \) and \( |y| = |x| + 1 \). Notice that each element \( x \in \Gamma \setminus \{ e \} \) covers a unique element \( x' \), its father, and each element \( x \in \Gamma \) is covered by \( \nu \) elements, its sons. The set of sons of \( x \) clearly is \( \tilde{x} = \{ x \varepsilon \mid \varepsilon \in A \} \). Hence:

\[ y \text{ covers } x \iff y' = x \iff y \in \tilde{x}. \]

For \( |x| \geq n \), we define \( x^{(n)} \) inductively by setting \( x^{(0)} = x \) and \( x^{(m+1)} = (x^{(m)})' \) for \( m \leq n - 1 \). One may also notice that: \( |x^{(\alpha)}| = |x| - \alpha \), if \( \alpha \leq |x| \), and for \( \alpha \leq |ab| \):

\[ (ab)^{(\alpha)} = \begin{cases} 
ab^{(\alpha)}, & \text{if } \alpha \leq |b| \\
(a^{(\alpha-|b|)}, & \text{if } \alpha \geq |b|. 
\end{cases} \]

We remark that if \( \nu = 1 \) then \( \Gamma = \mathbb{N} \) and if \( \nu > 1 \) then \( \Gamma \) is the set of monoms of \( \nu \) non-commutative variables.

### 2.2 The tree \( \Gamma \) and the extended tree associated to \( A \)

Recall that a graph is a couple \( G = (V, E) \), where \( V \) is a set (of vertices) and \( E \) is a set of pairs of elements of \( V \) (the edges). If \( x \) and \( y \) are joined by an edge, one says that they are neighbours and one abbreviates \( x \sim y \). The graph structure allows one to endow \( V \) with a canonical metric \( d \), where \( d(x, y) \) is the length of the shortest path in \( G \) joining \( x \) to \( y \).
The graph $G$ associated to the free monoid $\Gamma$ is defined as follows: $V = \Gamma$ and $x \sim y$ if $x$ covers $y$ or $y$ covers $x$. It is usual to identify $\Gamma$ and $G$, the so-called $\nu$-fold branching tree. For all $x \in \Gamma$, we have $|x| = d(e, x)$. We set $B(x, r) = \{y \in \Gamma \mid d(x, y) < r\}$ and $S^n = \{x \in \Gamma \mid |x| = n\}$.

We shall now define an extended tree by mimicking the definition of a free monoid over $A$. We choose $o \in A$; this element will be fixed from now on. For each integer $r$, we set $Z_r = \{i \in \Z \mid i \leq r\}$. The extended tree $\tilde{\Gamma}$ associated to $A$ is the set of $A$-valued maps $x$ defined on sets of the form $Z_r$ such that $\{i \mid x(i) \neq o\}$ is finite. For $x \in \tilde{\Gamma}$, the unique $r \in \Z$ such that $x$ is a map $Z_r \to A$ will be denoted $|x|$ and will be called length of $x$.

We shall identify $\Gamma$ with the set $\{x \mid |x| \geq 0 \text{ and } x(i) = o \text{ if } i \leq 0\}$ as follows: if $x \in \Gamma$ then we associate to it the element of $\tilde{\Gamma}$ defined on $\Z_{|x|}$ by extending $x$ with $x(i) = o$ if $i \leq 0$. The element $e$ will be identified with the map $e \in \tilde{\Gamma}$ such that $|e| = 0$ and $e(i) = o, \forall i \leq 0$. Notice that the two notions of length are consistent on $\Gamma$.

There is a natural right action of $\Gamma$ on $\tilde{\Gamma}$ by concatenation, i.e. for $x \in \tilde{\Gamma}$ and $y \in \Gamma$, $xy$ will be the function $z$ defined on $\Z_{|x|+|y|}$ such that $z(i) = x(i)$, for $i \in \Z_{|x|}$ and $z(|x| + i) = y(i)$ for $i \in \Z_{|y|}$. Then we equip $\tilde{\Gamma}$ with an order relation by setting:

$$x \leq y \iff y \in x\Gamma.$$

As before, $y$ covers $x$ if and only if $x \leq y$ and $|y| = |x| + 1$. Now, each $x \in \tilde{\Gamma}$ covers a unique $x' \in \tilde{\Gamma}$ and each $x \in \tilde{\Gamma}$ is covered by $\nu$ elements, namely those of $\tilde{x} = \{x \epsilon \mid \epsilon \in A\}$. We still have: $y$ covers $x$ $\iff$ $y' = x \Leftrightarrow y \in \tilde{x}$. Observe that $x' = x|_{\Z_{|x|-1}}$. We will set $x(\alpha) = x|_{\Z_{|x|-\alpha}}$ for all $\alpha \in \Z$. As we did it for $\Gamma$, we shall indentify the graph $G_\tilde{\Gamma}$ with $\tilde{\Gamma}$. This justifies the notion of extended tree used for $\tilde{\Gamma}$.

2.3 The boundary at infinity of $\Gamma$

We shall see in the ending remark of this subsection that the boundary at infinity of $\Gamma$ can be thought as the boundary of a 0-hyperbolic space in the sense of Gromov. We prefer, however, to give a simpler presentation that
is closer to the theory of $p$-adic numbers (see [11] for instance). In fact, if $\nu$ is prime the boundary will be the set of $\nu$-adic integers.

**Definition 2.1** The boundary at infinity of $\Gamma$ is the set $\partial \Gamma = \{ x : \mathbb{N}^* \to \mathbb{A} \}$. For $x \in \partial \Gamma$, we set $|x| = \infty$.

Let $\hat{\Gamma}$ be $\Gamma \cup \partial \Gamma$. For $x \in \hat{\Gamma}$, we define the sequence $(x_n)_{n \in [0,|x|]}$ with values in $\Gamma$ by setting $x_0 = e$ and $x_n = x|_{[1,n]}$ for $n \geq 1$. Observe that the map $x \mapsto (x_n)_{n \in [0,|x|]}$ is injective. There is a natural left action of $\Gamma$ on $\hat{\Gamma}$. For $x \in \Gamma$ and $y \in \hat{\Gamma}$, $xy$ will be defined on the set $[1,|x| + |y|]$ by $x(i)$ for $i \leq |x|$ and by $y(i - |x|)$ for $i > |x|$.

We will now equip $\hat{\Gamma}$ with a structure of ultrametric space. We define a kind of valuation $v$ on $\hat{\Gamma}$ by

$$v(x, y) = \begin{cases} \max \{ n \mid x_n = y_n \} & \text{if } x \neq y \\ \infty & \text{if } x = y. \end{cases} \quad (2.1)$$

If $x, y, z \in \hat{\Gamma}$ it is easy to see that:

$$v(x, y) \geq \min(v(x, z), v(z, y)). \quad (2.2)$$

Let us set on $\hat{\Gamma}$:

$$\hat{d}(x, y) = \exp(-v(x, y)).$$

The relation (2.2) clearly implies that $(\hat{\Gamma}, \hat{d})$ is an ultrametric space, i.e. a metric space such that $\hat{d}(x, y) \leq \max(\hat{d}(x, z), \hat{d}(z, y))$, for $x, y, z \in \hat{\Gamma}$. We will denote, for $r > 0$, $\hat{B}(x, r) = \{ y \in \hat{\Gamma} \mid \hat{d}(x, y) < r \}$. Notice that ultrametricity implies that $\hat{B}(x, r)$ is closed for all $x \in \hat{\Gamma}$ and $r > 0$.

The topology induced by $\hat{\Gamma}$ on $\Gamma$ coincides with the initial topology of $\Gamma$, the discrete one. For $x \in \partial \Gamma$ and $n \in \mathbb{N}$,

$$x_n = \{ y \in \hat{\Gamma} \mid v(x, y) \geq n \} = \hat{B}(x, \exp(-n + 1))$$

which is the closure of $x_n \Gamma$ in $\hat{\Gamma}$. Hence for each $x \in \partial \Gamma$, $\{ x_n \} \in \mathbb{N}$ is a basis of neighbourhoods of $x$ in $\hat{\Gamma}$. Observe that if $x \in \Gamma$ then $x \partial \Gamma = x\hat{\Gamma} \cap \partial \Gamma$.

\footnote{We use the convention $[1, \infty] = \mathbb{N}^* \cup \{ \infty \}$.}
Proposition 2.2 $\hat{\Gamma}$ and $\partial \Gamma$ are compact spaces. $\hat{\Gamma}$ is a compactification of $\Gamma$.

Proof: $\partial \Gamma = \mathcal{A}^N$, thus the set $\partial \Gamma$ endowed with the product topology is compact. This topology coincides with the one induced by the restriction of $\hat{d}$ on $\partial \Gamma$ (for $x \in \partial \Gamma$, the product topology gives us the same basis of neighbourhoods $\{x_n \partial \Gamma\}_{n \in \mathbb{N}}$ as $\hat{d}|_{\partial \Gamma}$).

Since $\partial \Gamma$ is compact, in order to show that $\hat{\Gamma}$ is compact, it suffices to remark that $\cup_{x \in \partial \Gamma} B(x, \exp(-k)) = \{y \hat{\Gamma} \mid |y| = k + 1\}$ has a finite complementary in $\hat{\Gamma}$, for all $k \in \mathbb{N}$. Since $\Gamma$ is dense in $\hat{\Gamma}$, $\hat{\Gamma}$ is a compactification of $\Gamma$. □

Notice also that if $\nu > 1$, the topological space $\partial \Gamma$ is perfect.

The $C^*$-algebra $C(\hat{\Gamma})$ of continuous complex-valued functions on $\hat{\Gamma}$ plays an important rôle. The dense embedding $\Gamma \subset \hat{\Gamma}$ gives a canonical inclusion $C(\hat{\Gamma}) \subset C_b(\Gamma)$ ($C_b(\Gamma)$ is the space of bounded complex-valued functions on $\Gamma$). Moreover, we have

$$C_0(\Gamma) = \{f \in C(\hat{\Gamma}) \mid f|_{\partial \Gamma} = 0\}, \quad \text{(2.3)}$$

where $C_0(\Gamma) = \{f : \Gamma \to \mathbb{C} \mid \forall \varepsilon > 0, \exists M > 0 \mid |x| > M \Rightarrow |f(x)| < \varepsilon\}$. We shall often abbreviate $C_0(\Gamma)$ by $C_0$.

The following proposition gives us a better understanding of the functions in $C(\hat{\Gamma})$.

Proposition 2.3 Let $E$ be a metrisable topological space. A function $V : \Gamma \to E$ extends to a continuous function $\hat{V} : \hat{\Gamma} \to E$ if and only if for each $x \in \partial \Gamma$ the limit of $V(y)$, when $y \in \Gamma$ converges to $x$, exists.

Proof: Let $x \in \partial \Gamma$ and $\hat{V}(x)$ be the above limit. Let $F$ be a closed neighbourhood of $\hat{V}(x)$ in $E$; there is $k$ such that $V(x_k \Gamma) \subset F$. Then $x_k \hat{\Gamma}$ is a neighbourhood of $x$ in $\hat{\Gamma}$ and, since $F$ is closed, we have $\hat{V}(x_k \hat{\Gamma}) \subset F$. □

Later on, we will need the next ultrametricity result. We will say that $\mathcal{U} = \{x_i \Gamma\}$ is a covering of $\partial \Gamma$ if $\mathcal{U} = \{x_i \hat{\Gamma}\}$ is a covering of $\partial \Gamma$.

Proposition 2.4 For each open covering $\{\mathcal{O}_i\}_{i \in I}$ of $\partial \Gamma$, there is a disjoint and finite covering $\{x_j \Gamma\}_{j \in J}$ of $\partial \Gamma$ such that for each $j \in J$ there is $i \in I$ such that $x_j \hat{\Gamma} \subset \mathcal{O}_i$. 
Proof: For each $x \in \partial \Gamma$ there is $i$ such that $x$ belongs to the open set $\mathcal{O}_i$ and there is $n = n(x, i)$ such that $x_n \Gamma \subset \mathcal{O}_i$. Since $\partial \Gamma$ is compact, there is a finite sub-covering of $\partial \Gamma$ made by sets $\{y_j \Gamma\}_{j \in [1,m]}$ such that each of its elements is a subset of some $\mathcal{O}_i$. But in ultrametric spaces two balls are either disjoint or one of them is included in the other one. Since $\{y_j \Gamma\}$ are balls, we get the result. One may also choose $\{y \Gamma \mid |y| = \max_{j \in [1,m]} |y_j|\}$ as the required covering. □

Remark: As we said previously, this section could be presented from the perspective of hyperbolicity in the sense of Gromov, see [2, Chapter V] (a deeper investigation can be found in [4] and [7]). Let $(M, d)$ be a metric space. For $x, y \in M$ and a given $O \in M$, we define the Gromov product as:

$$((x, y)O = \frac{1}{2}(d(O, x) + d(O, y) - d(x, y)).$$

The space $(M, d)$ is called $\delta$-hyperbolic if there is $\delta$ such that for all $x, y, z, O \in M$,

$$(x, y)O \geq \min((x, z)O, (z, y)O) - \delta.$$ \hfill (2.5)

A metric space is hyperbolic if it is $\delta$-hyperbolic for a certain $\delta$. In fact, if there is $\delta$ such that (2.5) holds for all $x, y, z \in M$ and a given $O$ then $(M, d)$ is $2\delta$-hyperbolic. Classical examples of 0-hyperbolic spaces are trees (connected graphs with no cycle) and real trees (see [7] for this notion). Cartan-Hadamard manifolds, the Poincaré half-plane and, more generally, complete simply connected manifolds with sectional curvature bounded by $\kappa < 0$ are $\delta$-hyperbolic spaces with $\delta > 0$.

We equip the set of sequences with values in $M$ with an equivalence relation between $(u_n)$ and $(v_n)$ defined by the condition

$$\lim_{(n,m) \to \infty} (u_n, v_m)_O = \infty.$$ The boundary at infinity $\partial M$ is the set of equivalence classes. A basis of open sets of $\partial M$ is given by

$$\tilde{\mathcal{O}} = \{\gamma \in \partial M \mid \gamma \text{ is not associated to any sequence of } M \setminus \mathcal{O}\},$$

where $\mathcal{O}$ is an open set of $M$. The boundary of a 0-hyperbolic space is ultrametric.

In our context, if we drop the convention $v(x, x) = \infty$, our valuation (2.1) is exactly (2.4). Hence (2.2) implies that $\Gamma$ is 0-hyperbolic. We define a geodesic ray as being $\gamma : \mathbb{N} \to \Gamma$ such that $|\gamma(n)| = n$ and $\gamma(n + 1) \sim$
\( \gamma(n) \). Geodesic rays are representative elements of the above equivalence classes. The two notions of boundary at infinity are identified by setting \( x_n = \gamma(n) \).

3 Operators in \( \ell^2(\Gamma) \)

3.1 Bounded and compact operators

We are interested in operators acting on the Hilbert space \( \ell^2(\Gamma) = \{ f : \Gamma \to \mathbb{C} \mid \sum_{x \in \Gamma} |f(x)|^2 < \infty \} \) endowed with the inner product:
\[
\langle f, g \rangle = \sum_{x \in \Gamma} f(x)g(x).
\]
We embed \( \Gamma \subset \ell^2(\Gamma) \) by identifying \( x \) with \( \chi_{\{x\}} \), where \( \chi_A \) is the characteristic function of the set \( A \). Observe that \( \Gamma \) is the canonical orthonormal basis in \( \ell^2(\Gamma) \) and each \( f \in \ell^2(\Gamma) \) writes as
\[
f = \sum_{x \in \Gamma} f(x)x.
\]
We denote by \( \mathbb{B}(\Gamma), \mathbb{K}(\Gamma) \) the sets of bounded, respectively compact operators in \( \ell^2(\Gamma) \). For \( T \in \mathbb{B}(\Gamma) \), we will denote by \( T^* \) its adjoint. Given \( A \subset \Gamma \) we denote by \( 1_A \) the operator of multiplication by \( \chi_A \) in \( \ell^2(\Gamma) \).

The orthogonal projection associated to \( \{ x \in \Gamma \mid |x| \geq r \} \) is denoted by \( 1_{\geq r} \). For \( T \in \Gamma \), we have the following compacity criterion for bounded operators \( T \) in \( \ell^2(\Gamma) \):

**Proposition 3.1** \( T \in \mathbb{K}(\Gamma) \iff \|1_{\geq r}T\| \rightarrow_{r \to \infty} 0 \iff \|T 1_{\geq r}\| \rightarrow_{r \to \infty} 0 \).

**Proof:** If one has for example \( \|1_{\geq r}T\| \rightarrow 0 \), then \( T \) is the norm limit of the sequence of finite rank operators \( 1_{B(\varepsilon,r)}T \), hence is compact. \( \square \)

3.2 The operator \( \partial \)

We now extend \( x \mapsto x' \) to a map \( \ell^2(\Gamma) \to \ell^2(\Gamma) \). We set \( e' = 0 \) and define the derivative of any \( f \in \ell^2(\Gamma) \) as:
\[
(\partial f)(x) \equiv f'(x) = \sum_{y' = x} f(y)y'(x) = \sum_{y' \in \Gamma} f(y) = \sum_{y \in \Gamma} f(y).
\]
Thus \( \partial \in \mathbb{B}(\Gamma) \). Indeed, \( \|f'\|^2 = \sum_{x \in \Gamma} |f'(x)|^2 \leq \nu \sum_{x \in \Gamma} \sum_{y \in \Gamma} |f(y)|^2 \leq \nu \|f\|^2 \). The adjoint \( \partial^* \) acts on each \( f \in \ell^2(\Gamma) \) as follows:
\[
\partial^* f(x) = \chi_{\Gamma \setminus \{e\}}(x)f(x').
\]
Indeed, \( \langle \partial f, f \rangle = \sum_{x \in \Gamma} \sum_{y \in \bar{x}} f(y)f(x) = \sum_{x \in \Gamma} \tilde{f}(x) \chi_{\Gamma \setminus \{e\}}(x)f(x') = \langle f, \partial^* f \rangle \). Moreover, \( \|\partial^* f\|^2 = \sum_{x \in \Gamma \setminus \{e\}} |f(x')|^2 = \nu \sum_{x \in \Gamma} |f(x)|^2 = \nu \|f\|^2 \) shows that \( \partial \partial^* = \nu \text{Id.} \) (3.1)

Thus \( \partial^*/\sqrt{\nu} \) is isometric on \( \ell^2(\Gamma) \) and \( \|\partial\| = \|\partial^*\| = \sqrt{\nu} \).

For \( \alpha \in \mathbb{N} \) we set \( f^{(\alpha)} = \partial^\alpha f \). Thus for each \( x \in \Gamma \), \( x^{(\alpha)} \) is well defined in \( \ell^2(\Gamma) \) and \( x^{(\alpha)} = 0 \Leftrightarrow \alpha > |x| \). For \( |x| \geq \alpha \) the notation is consistent with our old definition.

### 3.3 \( C^* \)-algebras of energy observables related to \( \Gamma \)

We first summarize the method used in [6] to study the essential spectrum of large families of operators. Let \( \mathcal{H} \) be a Hilbert space and \( H \) a bounded self-adjoint operator on \( \mathcal{H} \). If \( C(\mathcal{H}) = B(\mathcal{H})/K(\mathcal{H}) \) is the Calkin \( C^* \)-algebra, we denote by \( S \mapsto \hat{S} \) the canonical surjection of \( B(\mathcal{H}) \) onto \( C(\mathcal{H}) \) and we recall that \( \sigma_{\text{ess}}(H) = \sigma(\hat{H}) \) (this is a version of Weyl’s Theorem). If \( \mathcal{C} \) is a \( C^* \)-subalgebra of \( B(\mathcal{H}) \) which contains the compact operators, then one has a canonical embedding \( \mathcal{C}/K(\mathcal{H}) \subset C(\mathcal{H}) \). Thus, in order to determine the essential spectrum of an operator \( H \in \mathcal{C} \) it suffices to give a good description of the quotient \( \mathcal{C}/K(\mathcal{H}) \) and to compute \( \hat{H} \) as element of it. As explained in [6], we can actually go further by taking \( H \) as an unbounded operator over \( \mathcal{H} \) such that \( (H + i)^{-1} \in \mathcal{C} \). We shall apply this strategy in our context.

Let \( \mathcal{D}_{\text{alg}} \) be the \( \ast \)-algebra of operators in \( \ell^2(\Gamma) \) generated by \( \partial \) and \( \mathcal{D} \) the \( C^* \)-algebra of operators in \( \ell^2(\Gamma) \) generated by \( \partial \). Because of (3.1), \( \mathcal{D}_{\text{alg}} \) is unital. We denote by \( \varphi(Q) \) the operator of multiplication by \( \varphi \) on \( \ell^2(\Gamma) \). If \( C \) is a \( C^* \)-subalgebra of \( \ell^\infty(\Gamma) \) then we embed \( C \) in \( B(\Gamma) \) by \( \varphi \mapsto \varphi(Q) \). Let \( (\mathcal{D}, C) \) be the \( C^* \)-algebra generated by \( \mathcal{D} \cup C \). In this paper we shall take \( \mathcal{C} = (\mathcal{D}, C) \). This algebra contains many Hamiltonians of physical interest, for instance Schrödinger operators with potentials in \( C \). We recall that given a graph \( G \) the Laplace operator acts on \( \ell^2(G) \) as follows:

\[ (\Delta f)(x) = \sum_{y \sim x} (f(y) - f(x)). \]
With our definitions $\Delta = \partial + \partial^* - \nu \Id + \chi_\{e\}$. Notice that if $\nu > 1$ then $\mathcal{D}$ does not contain compact operators (see below), so $\Delta \notin \mathcal{D}$. On the other hand, if $C \supseteq C_0$ and $V \in C$ then the Schrödinger operator $\Delta + V(Q)$ clearly belongs to $\langle \mathcal{D}, C \rangle$.

We now give a new description of $\mathbb{K}(\Gamma)$.

**Proposition 3.2** If $\mathcal{C}_0$ be the $C^*$-algebra generated by $\mathcal{D} \cdot C_0$ then $\mathcal{C}_0 = \mathbb{K}(\Gamma)$.

**Proof:** For each $\varphi \in C_0$, Proposition 3.1 shows $\varphi(Q) \in \mathbb{K}(\Gamma)$. Hence $\mathcal{C}_0 \subset \mathbb{K}(\Gamma)$. For the opposite inclusion, let $T \in \mathbb{K}(\Gamma)$ and fix $\varepsilon > 0$. Proposition 3.1, shows that there is an operator $T'$ with compactly supported kernel such that $\|T - T'\| \leq \varepsilon$. Define $\delta_{x,y} \in \mathbb{K}(\Gamma)$ by $\langle \delta_{x,y}, f \rangle(z) = f(y)$ if $z = x$ and 0 elsewhere. We have $\delta_{x,x} = \chi_x(Q) \in C_0$. As $T'$ is a linear combination of $\delta_{x,y}$, it suffices to show that $\delta_{x,y}$ is in $\mathcal{C}_0$. But this follows from $\delta_{x,y} = \delta_{x,x}(\partial^*|\partial|\nu|\delta_{y,y})$. □

If $C$ is a $C^*$-subalgebra of $\ell^\infty(\Gamma)$ that contains $C_0$, then $\mathbb{K}(\Gamma) \subset \langle \mathcal{D}, C \rangle$. Hence, in order to apply the technique described above, we have to give a sufficiently explicit description of the quotient $\langle \mathcal{D}, C \rangle / \mathbb{K}(\Gamma)$. In this paper we concentrate on the case $\mathcal{C} \equiv C(\hat{\Gamma})$ which is, geometrically speaking, the most interesting one (see the last Remark in §2.3). The $C^*$-algebra generated by $\partial$ and $C(\hat{\Gamma})$ will be denoted by $\mathcal{C}(\hat{\Gamma})$ and the $*$-subalgebra generated by $\partial$ and $C(\hat{\Gamma})$ will be denoted by $\mathcal{C}(\hat{\Gamma})_{\text{alg}}$. We will need the next fundamental property.

**Proposition 3.3** $[\partial, C(\hat{\Gamma})] \subset \mathbb{K}(\Gamma)$.

**Proof:** For each $\varphi \in C(\hat{\Gamma})$ one has $[\partial, \varphi(Q)]f(x) = \sum_{y' = x} (\varphi(y) - \varphi(x)) f(y) = (\partial \circ \psi(Q))f(x)$, where $\psi$ belongs to $C(\hat{\Gamma})$ and is defined by $\psi(y) = \varphi(y) - \varphi(y')$ when $|y| \geq 1$ and $\psi(e) = 0$. Observe that for $\gamma \in \partial \Gamma$ we have $\psi(\gamma) = \varphi(\gamma) - \varphi(\gamma) = 0$. Hence by (2.3), $\psi \in C_0$. Proposition 3.2 implies $\psi(Q) \in \mathbb{K}(\Gamma)$. □

**Remark:** The algebra $\mathcal{D}$ is the tree analogous of the algebra generated by the momentum operator on the real line. However, these algebras are rather different: $\mathcal{D}$ is not commutative and the spectrum and the essential spectrum of the operators from $\mathcal{D}$ are not connected sets in general. For instance, one has $\sigma(\partial^* \partial) = \sigma_{\text{ess}}(\partial^* \partial) = \{0, \nu\}$ if $\nu > 1$. Indeed, we remind
that if \( A, B \) are elements of a Banach algebra we have \( \sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\} \) and, as noticed below, \( \dim \ker \partial \) is infinite for \( \nu > 1 \).

### 3.4 Translations in \( \ell^2(\Gamma) \)

\( \Gamma \) acts on itself to the left and to the right: for each \( a \in \Gamma \) we may define \( \lambda_a, \rho_a : \Gamma \to \Gamma \) by \( \lambda_a(x) = ax \) and \( \rho_a(x) = xa \) respectively. Clearly, for \( a, b \in \Gamma \), \( \lambda_a \rho_b = \rho_b \lambda_a \) and for any \( x \in a \Gamma \) we define \( a^{-1}x \) as being the \( y \) for which \( x = ay \). For each \( x \in \Gamma a = \{ y \in \Gamma \mid \exists z \in \Gamma \text{ s.t. } y = za \} \), we define \( y = xa^{-1} \) by \( x = ya \). We extend now these translations to \( \ell^2(\Gamma) \). The translation \( \lambda_a \) acts on each \( f \in \ell^2(\Gamma) \) as \( \sum_{x \in \Gamma} f(x)ax \), i.e. \( (\lambda_a f)(x) = \chi_{a \Gamma}(x)f(a^{-1}x) \). In the same manner, we define \( (\rho_a f)(x) = \chi_{\Gamma a}(x)f(xa^{-1}) \).

The operators \( \lambda_a \) and \( \rho_a \) are isometries:

\[
\lambda_a^* \lambda_a = \text{Id} \quad \text{and} \quad \rho_a^* \rho_a = \text{Id}.
\]

It is easy to check that the adjoints act on any \( f \in \ell^2(\Gamma) \) as \( (\lambda_a^* f)(x) = f(ax) \) and \( (\rho_a^* f)(x) = f(xa) \). Moreover,

\[
\lambda_a \lambda_a^* = 1_{a \Gamma} \quad \text{and} \quad \rho_a \rho_a^* = 1_{\Gamma a}.
\]

Note also that \( \partial^* = \sum_{|a|=1} \rho_a \) and \( \partial = \sum_{|a|=1} \rho_a^* \).

### 3.5 Localizations at infinity

In order to study \( \mathcal{C}(\widehat{\Gamma})/\mathbb{K}(\Gamma) \) we have to define the localizations at infinity of \( T \in \mathcal{C}(\widehat{\Gamma}) \) by looking at the behavior of the translated operator \( \lambda_a^* T \lambda_a \) as \( a \) converges to \( \gamma \) in \( \widehat{\Gamma} \) (abbreviated \( a \to \gamma \)), for each \( \gamma \in \partial \Gamma \).

If \( T \in \mathbb{K}(\Gamma) \) then \( u\lim_{a \to \gamma} \lambda_a^* T \lambda_a = 0 \), where \( u\lim \) means convergence in norm. Indeed, by (3.2), (3.3) and Proposition 3.1 we get

\[
\|\lambda_a^* T \lambda_a\| = \|1_{a \Gamma} T 1_{a \Gamma}\| \to 0, \quad \text{as} \quad a \to \gamma.
\]

Now, we compute the uniform limit of \( \lambda_a^* T \lambda_a \) when \( T \in \mathcal{C}(\widehat{\Gamma})_{\text{alg}} \). There is \( P \), a non-commutative complex polynomial in \( m + 2 \) variables, and functions \( \varphi_i \in C(\widehat{\Gamma}) \) for \( i = [1, m] \), such that \( T = P(\varphi_1, \varphi_2, \ldots, \varphi_m, \partial, \partial^*) \). We set \( T(\gamma) = P(\varphi_1(\gamma), \varphi_2(\gamma), \ldots, \varphi_m(\gamma), \partial, \partial^*) \).
Lemma 3.4 There is \( a_0 \in \Gamma \) such that \( u \)-\( \lim_{u \to \gamma} \lambda_a^* T \lambda_a = \lambda_{a_0}^* T(\gamma) \lambda_{a_0} \).

**Proof:** The Proposition 3.3 and (3.1) give some \( \phi_k \in C(\tilde{\Gamma}) \), \( K \in \mathbb{K}(\Gamma) \) and \( \alpha_k, \beta_k \in \mathbb{N} \) such that \( T = \sum_{k=1}^n \phi_k(Q) \partial^{\alpha_k} \partial^{\beta_k} + K \) and \( T(\gamma) = \sum_{k=1}^n \phi_k(\gamma) \partial^{\alpha_k} \partial^{\beta_k} \). Thus, it suffices to compute a limit of the form \( u \)-\( \lim_{u \to \gamma} \lambda_a^* \varphi(Q) \partial^{\alpha} \partial^{\beta} \lambda_a \) with \( \varphi \in C(\tilde{\Gamma}) \). We suppose \( |a| \geq \alpha \) and take \( f \in \ell^2(\Gamma) \). We first show the result for \( \varphi = 1 \). Since

\[
(\lambda_a^* \partial^{\alpha} \partial^{\beta} \lambda_a f)(x) = \sum_{y | (ay)^{(\beta)} = (ax)^{(\alpha)}} (\lambda_a f)(y) = \sum_{y | (ay)^{(\beta)} = (ax)^{(\alpha)}} f(y), \tag{3.4}
\]

it suffices to show that the set \( \{ y | (ay)^{(\beta)} = (ax)^{(\alpha)} \} \) is independent of \( a \) if \( |a| \geq \alpha \). But this is precisely what asserts the Lemma 3.5 below.

We now treat the general case \( \varphi \in C(\tilde{\Gamma}) \). The identity \( (\lambda_a^* \varphi(Q) \partial^{\alpha} \partial^{\beta} \lambda_a f)(x) = \varphi(ax)(\lambda_a^* \partial^{\alpha} \partial^{\beta} \lambda_a f)(x) \) gives us that \( \| \lambda_a^* \varphi(Q) \partial^{\alpha} \partial^{\beta} \lambda_a - \varphi(\gamma) \lambda_a^* \partial^{\alpha} \partial^{\beta} \lambda_a \| \leq \| \varphi(aQ) - \varphi(\gamma) \| \cdot \| \partial^{\alpha} \partial^{\beta} \| \to 0 \) as \( a \to \gamma \). On the other hand, by the Lemma 3.5, \( \varphi(\gamma) \lambda_a^* \partial^{\alpha} \partial^{\beta} \lambda_a \) is constant for \( |a| \geq \alpha \). Thus, it suffices to choose \( |a_0| \geq \max \{ \alpha_k | k = 1, \ldots, n \} \) in the statement of the lemma to end the proof. \( \square \)

Lemma 3.5 For \( |a| \geq \alpha \) we have:

\[
\{ y | (ay)^{(\beta)} = (ax)^{(\alpha)} \} = \begin{cases} \emptyset & \text{for } |x| + \beta - \alpha < 0, \\ S^{[x]} - \alpha - \beta & \text{for } |x| < \alpha \text{ and } |x| + \beta - \alpha \geq 0, \\ x^{(\alpha)} S^{\beta} & \text{for } |x| \geq \alpha \text{ and } |x| + \beta - \alpha \geq 0. \end{cases} \tag{3.5}
\]

**Proof:** Let \( J_x = \{ y | (ay)^{(\beta)} = (ax)^{(\alpha)} \} \). Then

\[
aJ_x = \{ ay | (ay)^{(\beta)} = (ax)^{(\alpha)} \} = \{ y | y^{(\beta)} = (ax)^{(\alpha)} \} \cap a\Gamma
= ((\alpha x)^{(\alpha)} S^{\beta}(\Gamma)) \cap a\Gamma.
\]

We first notice that \((\alpha x)^{(\alpha)} S^{\beta} \subset S^{[x]+|x| - \alpha, \beta} \). If \( |x| - \alpha + \beta < 0 \) then \((\alpha x)^{(\alpha)} S^{\beta} \cap a\Gamma = \emptyset \), so \( aJ_x = \emptyset \). This implies \( J_x = \emptyset \). If \( |x| - \alpha + \beta \geq 0 \) then \((\alpha x)^{(\alpha)} S^{\beta} \cap a\Gamma \neq \emptyset \). If we suppose that \( |x| < \alpha \), i.e. \( |(ax)^{(\alpha)}| < |a| \), we have \( a \in (ax)^{(\alpha)} \Gamma \). Let \( b \) such that \( a = (ax)^{(\alpha)} b \). Thus

\[
((ax)^{(\alpha)} S^{\beta}) \cap a\Gamma
= ((ax)^{(\alpha)} S^{\beta}) \cap (ax)^{(\alpha)} b\Gamma
= (ax)^{(\alpha)} (S^{\beta} \cap b\Gamma)
= (ax)^{(\alpha)} b S^{\beta - |b|}
= a S^{\beta - |b|}
= a S^{\beta + |x| - \alpha},
\]

15
so we have $aJ_x = aS^{\beta+|x|^{-\alpha}}$, hence $J_x = S^{\beta+|x|^{-\alpha}}$.

Finally, if $|x| \geq \alpha$, i.e. $|(ax)^{(\alpha)}| \geq |a|$, one has $(ax)^{(\alpha)} \in a\Gamma$. Thus we obtain $aJ_x = (ax)^{(\alpha)}S^{\beta} = ax^{(\alpha)}S^{\beta}$, hence $J_x = x^{(\alpha)}S^{\beta}$.

□

Remark: As seen in the proof of lemma 3.4, one may choose any $a_0$ such that $|a_0| \geq \deg(P)$. On the other hand, we stress that the limit is not a multiplicative function of $T$. Indeed,

$$u \lim_{a \to \gamma} \lambda_\alpha^* \partial^* \partial \lambda_a \neq (u \lim_{a \to \gamma} \lambda_\alpha^* \partial^* \lambda_a) \cdot (u \lim_{a \to \gamma} \lambda_\alpha^* \partial \lambda_a).$$

Therefore, in order to describe the morphism of the algebra $\mathcal{C}(\tilde{\Gamma})$ onto its quotient $\mathcal{C}(\tilde{\Gamma})/K(\Gamma)$ we have to improve our definition of the localizations at infinity.

3.6 Extensions to $\tilde{\Gamma}$

The space $\ell^2(\tilde{\Gamma})$ is defined similarly to $\ell^2(\Gamma)$. Since $\Gamma \subset \tilde{\Gamma}$, we have $\ell^2(\Gamma) \hookrightarrow \ell^2(\tilde{\Gamma})$. As before, we embed $\tilde{\Gamma}$ in $\ell^2(\tilde{\Gamma})$ by sending $x$ on $\chi_{(x)}$ and we notice that $\tilde{\Gamma}$ is an orthonormal basis of $\ell^2(\tilde{\Gamma})$. We define $\tilde{\partial} : \ell^2(\tilde{\Gamma}) \rightarrow \ell^2(\tilde{\Gamma})$ by

$$(\tilde{\partial} f)(x) = f'(x) = \sum_{y' = x} f(y).$$

For $\alpha \in \mathbb{N}$, we set $f^{(\alpha)} = \tilde{\alpha}^* f$, notation which is consistent with our old definition of $x^{(\alpha)}$ as the restriction of $x$ to $\mathbb{Z}_{|x|^{-\alpha}}$. Obviously $\tilde{\partial} \in \mathcal{B}(\Gamma)$, its adjoint $\tilde{\partial}^*$ acts as $(\tilde{\partial}^* f)(x) = f(x')$, $\tilde{\partial}^*/\sqrt{\nu}$ is an isometry on $\ell^2(\tilde{\Gamma})$:

$$\tilde{\partial} \tilde{\partial}^* = \nu 1\Gamma,$$

thus $||\tilde{\partial}|| = ||\tilde{\partial}^*|| = \nu$. We denote by $\tilde{\mathcal{D}}$ the $C^*$-algebra generated by $\tilde{\partial}$ and by $\tilde{\mathcal{D}}_{\text{alg}}$ the $*$-algebra generated by $\tilde{\partial}$. Both of them are unital.

We now make the connection between $\mathcal{D}_{\text{alg}}$ and $\tilde{\mathcal{D}}_{\text{alg}}$.

**Lemma 3.6** For $|a| \geq \alpha$, one has: $\lambda_\alpha^* \partial^* \partial \lambda_a = 1\Gamma \tilde{\partial}^* \alpha \tilde{\partial} \beta 1\Gamma$.

**Proof:** For any $f \in \ell^2(\tilde{\Gamma})$, one has $(1\Gamma \tilde{\partial}^* \alpha \tilde{\partial} \beta 1\Gamma f)(x) = 1\Gamma(x) \sum_{|y| = (\beta)} 1\Gamma(y) f(y)$. Using the same arguments as in the proof
of the Lemma 3.5, one shows that for each \( x \in \Gamma \) the set \( \{ y \in \Gamma \mid y^{(j)} = x^{(\alpha)} \} \) equals the r.h.s. of (3.5). Thus the above sum is the same as that of the r.h.s. of (3.4).\( \square \)

We will also need a result concerning the localization of the norm on \( \mathcal{D}_{alg} \).

**Lemma 3.7** If \( \tilde{T} \in \mathcal{D}_{alg} \), then \( \| \tilde{T} \| = \| 1_\Gamma \tilde{T} 1_\Gamma \| . \)

**Proof:** Because of (3.6), we can suppose that \( \tilde{T} = \sum_{k=1}^{n} c_k \partial^{\alpha_k} \tilde{x}^{\beta_k} \). We denote by \( \beta \) the integer \( \max \{ \beta_k \mid k \in \{1, n\} \} \). For each \( \varepsilon > 0 \), there is some \( g \in \ell^2(\Gamma) \) with compact support such that \( \|g\| = 1 \) and \( \| \tilde{T} g \| \geq \| \tilde{T} \| - \varepsilon \). Note that if \( y_1, y_2, \ldots, y_m \) are distinct points of \( \Gamma \), \( a_1, a_2, \ldots, a_m \) are complex numbers and \( x_1, x_2 \in \Gamma \), we have

\[
\| \sum_{i=1}^{m} a_i x_i y_i \|^2 = \sum_{i=1}^{m} |a_i|^2 = \sum_{i=1}^{m} a_i x_i y_i \|^2. \tag{3.7}
\]

Thus, since \( g \) has compact support, there are \( x \in \Gamma \), \( m \in \mathbb{N}^* \) and \( y_i \in \Gamma \), \( |y_i| \geq \beta \), \( a_i \in \mathbb{C} \), for all \( i \in \{1, m\} \) such that \( g = \sum_{k=1}^{m} a_i x_i y_i \). We set \( f = \sum_{k=1}^{m} a_i e y_i \). Then (3.7) gives us \( \|f\| = \|g\| = 1 \). Using \( |y_i| \geq \beta \), we get \( f \in \ell^2(\Gamma) \) and \( \tilde{T} f \in \ell^2(\Gamma) \). Also with (3.7) we obtain for \( z \in \Gamma \),

\[
\| \tilde{T} g \| = \left\| \sum_{k=1}^{n} \sum_{i=1}^{m} c_k a_i \partial^{\alpha_k} \tilde{x}^{\beta_k} x_i y_i \right\| = \left\| \sum_{k=1}^{n} \left( \sum_{i=1}^{m} \left( \sum_{|z| = \alpha_k} c_k a_i (x(y_i))^{(\beta_k)} z \right) \right) \right\| \\
= \| \sum_{k=1}^{n} \sum_{i=1}^{m} \left( \sum_{|z| = \alpha_k} c_k a_i x(y_i)^{(\beta_k)} z \right) \| = \| \sum_{k=1}^{n} \sum_{i=1}^{m} \left( \sum_{|z| = \alpha_k} c_k a_i e(y_i)^{(\beta_k)} z \right) \| \\
= \| \sum_{k=1}^{n} \sum_{i=1}^{m} \left( \sum_{|z| = \alpha_k} c_k a_i (e(y_i))^{(\beta_k)} z \right) \| = \| \sum_{k=1}^{n} \sum_{i=1}^{m} c_k a_i \partial^{\alpha_k} \tilde{x}^{\beta_k} e y_i \| = \| \tilde{T} f \|. 
\]

Hence, there is \( f \in \ell^2(\Gamma) \) such that \( \| 1_\Gamma \tilde{T} 1_\Gamma f \| = \| \tilde{T} f \| = \| \tilde{T} g \| \geq \| \tilde{T} \| - \varepsilon. \square \)
4 The main results

4.1 The morphism

In the sequel, a morphism will be understood as a morphism of $C^*$-algebras. To describe the quotient $\mathcal{C}(\hat{\Gamma})/\mathbb{K}(\Gamma)$, we need to find an adapted morphism.

**Theorem 4.1** For each $\gamma \in \partial \Gamma$ there is a unique morphism $\Phi_\gamma : \mathcal{C}(\hat{\Gamma}) \to \tilde{\mathcal{D}}$ such that $\Phi_\gamma(\partial) = \partial$ and $\Phi_\gamma(\varphi(Q)) = \varphi(\gamma)$, for all $\varphi \in \mathcal{C}(\hat{\Gamma})$. One has $\mathbb{K}(\Gamma) \subset \text{Ker} \Phi_\gamma$.

**Proof:** We use the notations from §3.5. If $T \in \mathcal{C}(\hat{\Gamma})_{\text{alg}}$ then by Lemma 3.4 we have $\lim_{a \to \gamma} \lambda_a^* T \lambda_a = \lambda_{a_0}^* T(\gamma) \lambda_{a_0}$. Let $\tilde{T}(\gamma)$ be $P(\varphi_1(\gamma), \varphi_2(\gamma), \ldots, \varphi_m(\gamma), \partial, \tilde{\partial}^*)$. By Lemma 3.6 and (3.6) one can choose $a_0$ such that $\lambda_{a_0}^* T(\gamma) \lambda_{a_0} = 1_{\Gamma} \tilde{T}(\gamma) 1_{\Gamma}$. Lemma 3.7 implies

$$\|\tilde{T}(\gamma)\| = \|1_{\Gamma} \tilde{T}(\gamma) 1_{\Gamma}\| = \|\lambda_{a_0}^* T(\gamma) \lambda_{a_0}\| = \|\lim_{a \to \gamma} \lambda_a^* T \lambda_a\| \leq \|T\|.$$ 

Thus there is a linear multiplicative contraction $\Phi_0^\gamma : \mathcal{C}(\hat{\Gamma})_{\text{alg}} \to \tilde{\mathcal{D}}$, $\Phi_0^\gamma(T) = T(\gamma)$. The density of $\mathcal{C}(\hat{\Gamma})_{\text{alg}}$ in $\mathcal{C}(\hat{\Gamma})$ allows us to extend $\Phi_0^\gamma$ to a morphism $\Phi_\gamma : \mathcal{C}(\hat{\Gamma}) \to \tilde{\mathcal{D}}$ which clearly satisfies the conditions of the theorem. The uniqueness of $\Phi_\gamma$ is obvious and the last assertion of the theorem follows from the Proposition 3.2. □

4.2 The case $\nu > 1$

In this case, we are able to improve the Theorem 4.1. We recall first that an isometry is said to be *proper* if it is not unitary. The operators $\partial^*$ and $\tilde{\partial}^*$ are proper isometries and the dimensions of the kernels of $\partial$ and $\tilde{\partial}$ are infinite: in the case of $\partial$, if one lets $a, b$ be two different letters of $\mathcal{A}$, and one chooses $g \in \ell^2(\Gamma a)$ and $h \in \ell^2(\Gamma b)$ such that $h(xb) = g(xa)$ for all $x \in \Gamma$, then $g - h$ is in $\text{Ker} \partial$.

Let $T$ be the unit circle of $\mathbb{R}^2$ and $H^2$ the closure of the subspace spanned by $\{e^{inQ}, n \in \mathbb{N}\}$ in $\ell^2(T)$. For $g \in L^\infty(T)$, we define the *Toeplitz operator* $T_g$ on $H^2$ by $T_g h = P_{H^2} gh$, where $P_{H^2}$ is the projection on $H^2$. 

18
For each \( z \in \mathbb{C} \setminus \{0\} \), we denote by \(\mathcal{F}\) the \( C^*\)-algebra generated by \( T_z \). The next theorem is due to Coburn (see [5] for a proof).

**Theorem 4.2** If \( S \) is a proper isometry, then there is a unique isomorphism \(\mathcal{F}\) of \(\mathcal{F}\) onto \(\mathcal{F}\), the \( C^*\)-algebra generated by \( S \), such that \(\mathcal{F}(T_z) = S \).

Thus there is a unique isomorphism \(\mathcal{F}\) of \(\mathcal{D}\) onto \(\mathcal{D}\) such that \(\mathcal{F}(\partial) = \mathcal{F}(\partial)\), so in the case \( \nu > 1 \) we can rewrite our Theorem 4.1 as follows.

**Theorem 4.3** Let \( \gamma \in \partial \Gamma \). There is a unique morphism \(\Phi_\gamma : \mathcal{C}(\hat{\Gamma}) \to \mathcal{D}\) such that \(\Phi_\gamma(\varphi(Q)) = \varphi(\gamma)\) for all \( \varphi \in \mathcal{C}(\hat{\Gamma}) \) and \(\Phi_\gamma(D) = D \) for all \( D \in \mathcal{D} \).

**Remark:** When \( \nu = 1 \), there is no isomorphism \(\mathcal{F} : \mathcal{D} \to \mathcal{D}\) such that \(\mathcal{F}(\partial) = \partial\) because \(\mathcal{D}\) is commutative. Thus, in this case, one cannot hope in a result as above. There is another way of proving Theorem 4.3 which uses the next proposition.

**Proposition 4.4** If \( \nu \geq 1 \) then \(\{\partial^\alpha \partial^\beta\}_{\alpha, \beta \in \mathbb{N}}\) is a basis of the vector space \(\mathcal{D}_{\text{alg}}\). One has \(\nu > 1\) if and only if \(\{\tilde{\partial}^\alpha \tilde{\partial}^\beta\}_{\alpha, \beta \in \mathbb{N}}\) is a basis of space \(\mathcal{D}_{\text{alg}}\).

**Proof:** Let \( \lambda_i \neq 0 \) for all \( i \in [1, n] \). Assume that \(\sum_{i=1}^{n} \lambda_i \partial^\alpha_i \partial^\beta_i = 0\), where \((\alpha_i, \beta_i)\) are distinct couples. We set \( \alpha = \min\{\alpha_i \mid i \in [1, n]\} \) and \( I = \{ i \mid \alpha_i = \alpha \} \). We take \( x \in \Gamma \) such that \( |x| = \alpha \) and we obtain \(\sum_{i \in I} \lambda_i(\partial^\beta_i f)(e) = 0\). Notice that \(\{\beta_i\}_{i \in I}\) are pairwise distinct by hypothesis. Now, by taking \( i_0 \in I \) and \( f \) the characteristic function of \( S_{\beta_{i_0}} \), we get that \(\lambda_{i_0} = 0\) which is a contradiction. Hence \(\sum_{i=1}^{n} \lambda_i \partial^\alpha_i \partial^\beta_i \neq 0\), i.e. the family is free. Let now \( \nu > 1 \) and \( \lambda_i \neq 0 \) for all \( i \in [1, n] \). We suppose \(\sum_{i=1}^{n} \lambda_i \partial^\alpha_i \partial^\beta_i = 0\), with \((\alpha_i, \beta_i)\) pairwise distinct. We fix \( x \in \hat{\Gamma} \) and set \(\alpha = \max\{\alpha_i, i \in [1, n]\}\). One has \(\sum_{i=1}^{n} \lambda_i \partial^\alpha_i \partial^\beta_i f(x) = \sum_{i=1}^{n} \lambda_i \sum_{y \in x(\alpha_i) S_{\beta_i}} f(y) = 0\). Notice that \( x(\alpha) S_{\beta} \cap x(\alpha') S_{\beta'} = \emptyset \) if and only if \(\alpha' - \alpha \neq \beta' - \beta\). Taking \( f \in L^2(S_{x-\alpha+\beta}) \), we see that one can reduce oneself to the case when there is some \( k \) such that \(\alpha_i - \beta_i = k\) for all \( i \in [1, n] \). Since \( x(\alpha-l) S_{\alpha-k-l} \subset x(\alpha-l) S_{\alpha-k-l} \subset x(\alpha) S_{\alpha-k} \) for all \( l \in [1, (\alpha - k)] \), there is some \( y_0 \in x(\alpha) S_{\alpha-k} \setminus \cup_{\alpha_i \neq \alpha} x(\alpha_i) S_{\beta_i} \). Then, taking \( f = \chi_{(y_0)} \), we get some \( i_0 \) such that \(\lambda_{i_0} = 0\), which is a contradiction. Hence \(\sum_{i=1}^{n} \lambda_i \partial^\alpha_i \partial^\beta_i \neq 0\). Finally, since when \( \nu = 1 \) one has \(\tilde{\partial} \partial^* = \tilde{\partial} \tilde{\partial} = \text{Id}\), \(\{\tilde{\partial}^\alpha \tilde{\partial}^\beta\}_{\alpha, \beta \in \mathbb{N}}\) is obviously not a basis. □
4.3 Description of $\mathcal{C}(\hat{\Gamma})/\mathbb{K}(\Gamma)$

**Theorem 4.5**

i) For any $\nu \geq 1$, there is a unique morphism $\Phi : \mathcal{C}(\hat{\Gamma}) \to \mathcal{T}$ such that $\Phi(\partial) = \mathcal{T}$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$. This morphism is surjective and its kernel is $\mathbb{K}(\Gamma)$.

ii) For $\nu > 1$, there is a unique surjective morphism $\Phi : \mathcal{C}(\hat{\Gamma}) \to \mathcal{T}$ such that $\Phi(\partial) = \mathcal{T}$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$ and $\text{Ker} \Phi = \mathbb{K}(\Gamma)$.

Once again, as in Remark 4.2, the statement (ii) of the theorem is false if $\nu = 1$. As a corollary of Theorem 4.5 we obtain the following result.

**Proposition 4.6** If $\nu > 1$ then $\mathcal{T} \cap \mathbb{K}(\Gamma) = \{0\}$ and if $\nu = 1$ one has $\mathbb{K}(\Gamma) \subset \mathcal{T}$.

**Proof:** Let $\nu > 1$ and $T \in \mathcal{T} \cap \mathbb{K}(\Gamma)$. Theorem 4.5 gives us both $\Phi(T) = T$ and $\Phi(T) = 0$ (since $T$ is compact). For $\nu = 1$, as in the proof of Proposition 3.2, it suffices to prove that $\delta_{x,x}$ is in $\mathcal{T}$. But this is clear since $\delta_{x,x} = \partial_{x}[x+1]\partial_{x}[x+1] - \partial_{x}[x]\partial_{x}[x]$.

We devote the rest of the section to the proof of the Theorem 4.5.

**Proof:** By Theorem 4.1 there is a morphism $\Phi : \mathcal{C}(\hat{\Gamma}) \to \mathcal{T}$ such that $(\Phi(\partial))(\gamma) = \mathcal{T}$ and $(\Phi(\varphi(Q)))(\gamma) = \varphi(\gamma)$, for all $\gamma \in \partial\Gamma$, $\varphi \in \mathcal{C}(\hat{\Gamma})$. Since the images of $\partial$ and $\varphi(Q)$ through $\Phi$ belong to the $C^*$-subalgebra $\mathcal{C}(\partial\Gamma, \mathcal{T})$, and since $\mathcal{C}(\hat{\Gamma})$ is generated by $\partial$ and such $\varphi(Q)$, it follows that the range of $\Phi$ is included in $\mathcal{C}(\partial\Gamma, \mathcal{T})$. We have $\mathcal{C}(\partial\Gamma, \mathcal{T}) \cong \mathcal{T} \otimes \mathcal{C}(\partial\Gamma)$, so we get the required morphism $\Phi : \mathcal{C}(\hat{\Gamma}) \to \mathcal{T} \otimes \mathcal{C}(\partial\Gamma)$. Now since $\Phi(\partial) = \mathcal{T} \otimes 1$ and $\Phi(\varphi(Q)) = 1 \otimes (\varphi|_{\partial\Gamma})$, and since any function in $\mathcal{C}(\partial\Gamma)$ is the restriction of some function from $\mathcal{C}(\hat{\Gamma})$, it follows that $\Phi$ is surjective. Its uniqueness is clear. It remains to compute the kernel.

As seen in the Theorem 4.1, $\mathbb{K}(\Gamma) \subset \text{Ker} \Phi$. In the remainder of this section we shall prove the reverse inclusion. For this we need some preliminary lemmas.

**Lemma 4.7** Let $R = \varphi(Q)\partial^\alpha \partial^\beta$ and $\mathcal{U} = \{a_i\}_{i \in [1,n]}$ be a disjoint covering of $\partial\Gamma$. For each $\varepsilon > 0$ there are $c_1, c_2, \ldots, c_m \in \text{Ran}(\varphi)$ and there is a disjoint covering $\mathcal{U}' = \{b_j\}_{j \in [1,m]}$ of $\partial\Gamma$ finer than $\mathcal{U}$ such that $\|1_{U'} R - R'\| \leq \varepsilon$, where $R' = \sum_{j=1}^m b_j c_j \partial^\alpha \partial^\beta$ and $U' = \bigcup_{j=1}^m b_j \Gamma$. 

20
Proof: Let $\varepsilon > 0$ and denote $\varepsilon/\|\partial^x \partial^y\|$ by $\varepsilon'$. Since $\varphi(\partial \Gamma)$ is compact, there are $\gamma_1, \gamma_2, \ldots, \gamma_N \subset \partial \Gamma$ such that $\varphi(\partial \Gamma) \subset \bigcup_{k=1}^{N} D(\varphi(\gamma_k), \varepsilon')$, where $D(z, r)$ is the complex open disk of center $z$ and ray $r$. The open sets $\mathcal{O}_{i,k} = a_i \Gamma \cap \varphi^{-1}(D(\varphi(\gamma_k), \varepsilon'))$ cover $\partial \Gamma$. The Proposition 2.4 gives us a disjoint covering $\{b_j \Gamma\}_{j \in \{1, m\}}$ of $\partial \Gamma$ such that for each $j \in \{1, m\}$ there are $i$ and $k$ such that $b_j \Gamma \subset \mathcal{O}_{i,k}$. To simplify the notations, we will denote by $\gamma_j$ those $\gamma_k$ associated to $b_j \Gamma$. We set $\mathcal{U}' = \{b_j \Gamma\}_{j \in \{1, m\}}$ and $R' = \sum_{j=1}^{m} b_j \Gamma \varphi(\gamma_j) \partial^x \partial^y$. Recall that $\sup_{x \in b_j \Gamma} |\varphi(\gamma_j) - \varphi(x)| \leq \varepsilon'$, so

$$
\|(R' - 1_{\mathcal{U}'} R)f\|^2 = \sum_{x \in \Gamma} \left| \sum_{j=1}^{m} b_j \Gamma(x)(\varphi(\gamma_j) - \varphi(x))(\partial^x \partial^y f)(x) \right|^2
$$

$$
= \sum_{j=1}^{m} \sum_{x \in b_j \Gamma} \left| (\varphi(\gamma_j) - \varphi(x))(\partial^x \partial^y f)(x) \right|^2
$$

$$
\leq \sum_{j=1}^{m} \sup_{x \in b_j \Gamma} |\varphi(\gamma_j) - \varphi(x)|^2 \sum_{x \in b_j \Gamma} \left| (\partial^x \partial^y f)(x) \right|^2
$$

$$
\leq \varepsilon'^2 \sum_{j=1}^{m} \sum_{x \in b_j \Gamma} \left| (\partial^x \partial^y f)(x) \right|^2
$$

$$
\leq \varepsilon'^2 \|\partial^x \partial^y\|^2 \cdot \|\partial^x \partial^y\|^2 \cdot \|f\|^2 = \varepsilon'^2 \|f\|^2.
$$

Denoting $\varphi(\gamma_j)$ by $c_j$ we obtain the result. $\Box$

Lemma 4.8 Let $T = \sum_{k=1}^{n} \varphi_k(Q) \partial^x \partial^y a_k$ with $\varphi_k \in C(\hat{\Gamma})$ and let $\varepsilon > 0$. There are a compact operator $K$, a disjoint covering $\{a_j \Gamma\}_{j \in \{1, m\}}$ of $\partial \Gamma$ and $S = \sum_{k=1}^{n} \sum_{j=1}^{m} a_j \Gamma \varphi_k(\gamma_j, k) \partial^x \partial^y a_k$, with $\max_{j \in \{1, m\}} a_j \geq \max_{k \in \{1, n\}} (\alpha_k$ and $\gamma_j,k) \in \partial \Gamma$ such that $\|T - S - K\| \leq \varepsilon$.

Proof: We denote by $\alpha = \max\{\alpha_k \mid k \in \{1, n\}\}$. Let $T_k$ be $\varphi_k(Q) \partial^x \partial^y a_k$. Setting $\mathcal{U}_0 = \bigcup_{\{a_k \mid k \in \{1, n\}\}} \{a_i \Gamma\}$, we apply the Lemma 4.7 inductively for $k \in \{1, n\}$ with $\varepsilon/n$ instead of $\varepsilon$, $\mathcal{U} = \mathcal{U}_{k-1}$ and $R = T_k$, denoting $\mathcal{U}'$ by $\mathcal{U}_k$ and $R'$ by $S_k$. Then, for $k \in \{1, n\}$ we get $\|1_{U_k} T_k - S_k\| \leq \varepsilon/k$. Since $\mathcal{U}_{k+1}$ is finer than $\mathcal{U}_k$ for $k \in \{1, n - 1\}$, we obtain $\|1_{U_k} \sum_{k=1}^{n} (T_k - S_k)\| \leq \varepsilon$, hence $\|T - 1_{U_n} T - 1_{U_n} \sum_{k=1}^{n} S_k\| \leq \varepsilon$. To finish the proof, we denote the compact operator $1_{U_n} T$ by $\hat{K}$, $1_{U_n} \sum_{k=1}^{n} S_k$ by $S$ and $\mathcal{U}_n$ by $\{a_j \Gamma\}_{j \in \{1, m\}}$. $\Box$
We now go back to the proof of Theorem 4.5. Let $T \in \text{Ker } \Phi$. For each $\varepsilon > 0$ there is $T' \in \mathcal{C}(\hat{\Gamma})_{\text{alg}}$ such that $\|T - T'\| \leq \varepsilon/4$. By relation (3.1) and Proposition 3.3, we can write $T' = \sum_{k=1}^{n} \varphi_k(Q) \partial^{*\alpha_k} \partial^{\beta_k} + K$, where $K \in \mathbb{K}(\Gamma)$ and $\varphi_k \in C(\hat{\Gamma})$. Thus $\|\Phi(T')\| \leq \varepsilon/4$. Using Lemma 4.8, we get an operator $S$ and a compact operator $K_1$ such that $\|T' - S - K_1\| \leq \varepsilon/4$. This implies that $\|\Phi(S)\| \leq \varepsilon/2$.

**Lemma 4.9** There is $K_2 \in \mathbb{K}(\Gamma)$ such that $\|S - K_2\| \leq \|\Phi(S)\|$.

Before proving the lemma, let us remark that it implies

$$\|T - K_1 - K_2\| \leq \|T - T'\| + \|T' - S - K_1\| + \|S - K_2\| \leq \varepsilon.$$

Hence $T \in \mathbb{K}(\Gamma)$. Thus Theorem 4.5 is proved.$\square$

**Proof of Lemma 4.9.** First, we remark that for each $a \in \Gamma$ and $\alpha, \beta \geq 0$, the Proposition 3.3 gives us that $1_{a\Gamma} \partial^{*\alpha} \partial^{\beta} - 1_{a\Gamma} \partial^{*\alpha} \partial^{\beta} 1_{a\Gamma}$ is a compact operator. We define $S' = \sum_{j=1}^{m} \sum_{j=1}^{n} (1_{a_j \Gamma} \varphi_k(\gamma_{j,k}) \partial^{*\alpha_k} \partial^{\beta_k} 1_{a_j \Gamma} f)(x)^2$

$$= \sum_{j=1}^{m} \sum_{j' \in \Gamma} \sum_{k=1}^{n} (1_{a_{j'} \Gamma} \varphi_k(\gamma_{j,k}) \partial^{*\alpha_k} \partial^{\beta_k} 1_{a_{j'} \Gamma} f)(x)^2$$

$$\leq \sum_{j=1}^{m} \sum_{j' \in \Gamma} \sum_{k=1}^{n} \left\| 1_{a_{j'} \Gamma} \varphi_k(\gamma_{j,k}) \partial^{*\alpha_k} \partial^{\beta_k} 1_{a_{j'} \Gamma} \right\|^2 \cdot \left\| 1_{a_{j'} \Gamma} f \right\|^2.$$

Now we use (3.2) and (3.3) and get:

$$\left\| 1_{a_{j'} \Gamma} \left( \sum_{k=1}^{n} \varphi_k(\gamma_{j,k}) \partial^{*\alpha_k} \partial^{\beta_k} \right) 1_{a_{j'} \Gamma} \right\| = \left\| \lambda_{a_{j'}} \left( \sum_{k=1}^{n} \varphi_k(\gamma_{j,k}) \partial^{*\alpha_k} \partial^{\beta_k} \right) \lambda_{a_{j'}} \right\|.$$

Since $|a_{j'}| \geq \max\{a_k \mid k \in [1, n]\}$, the Lemmas 3.6 and 3.7 give us:

$$\left\| \lambda_{a_{j'}} \left( \sum_{k=1}^{n} \varphi_k(\gamma_{j,k}) \partial^{*\alpha_k} \partial^{\beta_k} \right) \lambda_{a_{j'}} \right\| = \left\| 1_{\Gamma} \left( \sum_{k=1}^{n} \varphi_k(\gamma_{j,k}) \partial^{*\alpha_k} \partial^{\beta_k} \right) 1_{\Gamma} \right\|$$

$$= \left\| \sum_{k=1}^{n} \varphi_k(\gamma_{j,k}) \partial^{*\alpha_k} \partial^{\beta_k} \right\|.$$
For each $j$ we choose $\gamma_j \in a_j \partial \Gamma$. The family $\{a_j \Gamma\}_{j \in \{1, m\}}$ is a disjoint covering of $\partial \Gamma$, so we have $\lim_{x \to \gamma_j} \chi_{a_j \Gamma}(x) = 1$ and $\lim_{x \to \gamma_i} \chi_{a_i \Gamma}(x) = 0$ for $i \neq j$. Hence $\Phi_{\gamma_j}(S') = \sum_{k=1}^{n} \varphi_k(\gamma_{j,k}) \overline{\partial}_{a_k} \overline{\partial}_{\beta_k}$. We obtain

$$\|S'f\|^2 \leq \sum_{j=1}^{m} \|\Phi_{\gamma_j}(S')\|^2 \cdot \|1_{a_j \Gamma}f\|^2 \leq \sup_{\gamma \in \partial \Gamma} \|\Phi_{\gamma}(S')\|^2 : \|f\|^2.$$  

Finally, since $\mathbb{K}(\Gamma) \subset \text{Ker } \Phi$, $\|\Phi(S)\| = \|\Phi'(S')\| = \sup_{\gamma \in \partial \Gamma} \|\Phi_{\gamma}(S')\|$. □

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References


