THE PROBLEM OF DEFICIENCY INDICES ADJACENCY MATRICES ON LOCALLY FINITE GRAPHS

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ABSTRACT. In this note we answer negatively to our conjecture concerning the deficiency indices. More precisely, given any non-negative integer n, there is locally finite graph on which the adjency matrix has deficiency indices (n, n).

Let V be a countable set. Let $E := V \times V \to \{0, 1\}$ and assume that $E(x, y) = E(y, x)$, for all $x, y \in V$. Set that $E(x, x) = 0$, for all $x \in V$. We say that $x, y \in V$ are *neighbors* if $E(x, y) \neq 0$ and denote it by $x \sim y$. We say that $G := (E, V)$ is an unoriented simple graph with *vertices* V, *weights* E, and no loop. We associate to G the complex Hilbert space $\ell^2(V)$. We denote by $\langle \cdot, \cdot \rangle$ and by $\|\cdot\|$ the scalar product and the associated norm, respectively. The set of complex functions with compact support in V is denoted by $\mathcal{C}_c(G)$. The *adjacency matrix* is defined by:

$$(\mathcal{A}_{G,\circ} f)(x) := \sum_{y \sim x} f(y), \text{ with } f \in \mathcal{C}_c(G). \tag{1}$$

The operator is symmetric and thus closable. We denote the closures by $\mathcal{A}_G$. We denote the domains by $\mathcal{D}(\mathcal{A}_G)$, and its adjoint by $(\mathcal{A}_G)^*$. Even in the case of a locally finite tree G, $\mathcal{A}_G$ may have many self-adjoint extensions. We investigate the number of possible self-adjoint extensions of the adjacency matrix by computing its deficiency indices. Given a closed and densely defined symmetric operator T acting on a complex Hilbert space, the deficiency indices of T are defined by $\eta_\pm(T) := \dim \ker(T^* \mp \mathrm{i}) \in \mathbb{N} \cup \{+\infty\}$. The operator T possesses a self-adjoint extension if and only if $\eta_+(T) = \eta_-(T)$. If this is the case, we denote the common value by $\eta(T)$. T is self-adjoint if and only if $\eta(T) = 0$. Since the operator $\mathcal{A}_G$ commutes with the complex conjugation, its deficiency indices are equal. This means that $\mathcal{A}_G$ possesses a self-adjoint extension. Note that $\eta(\mathcal{A}_G) = 0$ if and only if $\mathcal{A}_G$ is essentially self-adjoint on $\mathcal{C}_c(G)$.

In [MO, Mü], one constructs adjacency matrices for simple trees with positive deficiency indices. In fact, it follows from the proof that the deficiency indices are infinite in both references. As a general result, a special case of [GS, Theorem 1.1] gives that, given a locally finite simple tree G, one has the following alternative:

$$\eta(\mathcal{A}_G) \in \{0, +\infty\}. \tag{2}$$

The value of $\eta(\mathcal{A}_G)$ is discussed in [GS] and is linked with the growth of the tree.

In [MW, Section 3], one finds:

Theorem 1. *For all $n \in \mathbb{N} \cup \{\infty\}$, there is a simple graph G, such that $\eta(\mathcal{A}_G) = n$.*

Their proof is unfortunately incomplete. However, the statement is correct, this is aim of this note. Using standard perturbation theory, e.g., [GS, Proposition A.1], one sees that the validity of Theorem 1 is equivalent to the existence of a simple graph G for which

$$\eta(\mathcal{A}_G) = 1. \tag{3}$$

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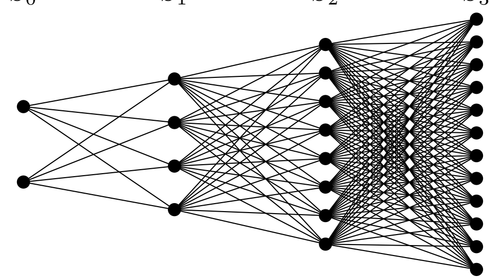

FIGURE 1. An antitree with $s_0 = 2, s_1 = 4, s_2 = 8$, and $s_4 = 12$

In [MW], instead of pointing a simple graph such that (3) holds, they relied on a tree. More precisely, they refer to the works of [MO, Mü].

Keeping that in mind and strongly motivated by some other examples, we had proposed a drastically different senario and had conjectured in [GS] that that for any simple graph, one has (2).

We now turn to the proof of Theorem 1 and therefore disprove our conjecture. We rely on the decomposition of anti-trees, see [BK, Theorem 4.1].

Let S_n, $n \in \mathbb{N}$, be nonempty, finite and pairwise disjoint sets. We set $s_n := |S_n|$, $V := \cup_{n\in\mathbb{N}} S_n$ and $|x| := n$ for $x \in S_n$. Set also $E(x, y) = 1$, if $\big||x| - |y|\big| = 1$, and $E(x, y) = 0$ otherwise. We define:

$$P := \bigoplus_{n\in\mathbb{N}} P_n, \text{ where } P_n f(x) := \frac{1}{s_n}\mathbf{1}_{S_n}(x) \sum_{y\in S_n} f(y),$$

for all $f \in \ell^2(V)$. Note that $P = P^2 = P^*$ and $\operatorname{rank} P\mathbf{1}_{S_n} = 1$ for all $n \in \mathbb{N}$. Given $f \in \ell^2(V)$ such that $f = Pf$, i.e., f is radially symmetric, we set $\tilde{f}(|x|) := f(x)$, for all $x \in V$. Note that

$$P\ell^2(V) = \{f \in \ell^2(V), \sum_{n\in\mathbb{N}} s_n|\tilde{f}(n)|^2 < \infty\} \simeq \ell^2(\mathbb{N}, (s_n)_{n\in\mathbb{N}}),$$

where $(s_n)_{n\in\mathbb{N}}$ is now a sequence of weights. The key remark of [BK, Theorem 4.1] is that

$$\mathcal{A}_G = P\mathcal{A}_G P \text{ and } \widetilde{\mathcal{A}_G Pf}(|x|) = s_{|x|-1}\widetilde{Pf}(|x| - 1) + s_{|x|+1}\widetilde{Pf}(|x| + 1),$$

for all $f \in \mathcal{C}_c(V)$ with the convention that $s_{-1} = 0$. Using the unitary transformation $U\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}, s_n)$ given by $U\tilde{f}(n) = \sqrt{s_n}\tilde{f}(n)$, we see that $\mathcal{A}_G$ is unitarily equivalent to the direct sum of 0 and of a Jacobi matrix acting on $\ell^2(\mathbb{N})$, with 0 on the diagonal and the sequence $(\sqrt{s_n}\sqrt{s_{n+1}})_{n\in\mathbb{N}}$ on the off-diagonal. Finally using [Ber, Page 504 and 507], we derive that: Given $\alpha > 0$ and $s_n := \lfloor n^\alpha \rfloor$,

$$\eta(\mathcal{A}_G) = \begin{cases} 0, & \text{if } \alpha \le 1, \\ 1, & \text{if } \alpha > 1. \end{cases}$$

Notation: The set of nonpositive integers is denoted by $\mathbb{N}$, note that $0 \in \mathbb{N}$. Given a set X and $Y \subseteq X$ let $\mathbf{1}_Y$ be the multiplication operator by the characteristic function of Y.

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