POSITIVE COMMUTATORS, FERMI GOLDEN RULE AND THE SPECTRUM OF 0 TEMPERATURE PAULI-FIERZ HAMILTONIANS.

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Abstract. We perform the spectral analysis of a zero temperature Pauli-Fierz system for small coupling constants. Under the hypothesis of Fermi golden rule, we show that the embedded eigenvalues of the uncoupled system disappear and establish a limiting absorption principle above this level of energy. We rely on a positive commutator approach introduced by Skibsted and pursued by Georgescu-Gérard-Møller. We complete some results obtained so far by Dereziński-Jakšić on one side and by Bach-Fröhlich-Segal-Soffer on the other side.

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1. Introduction

Pauli-Fierz operators are often used in quantum physics as generator of approximate dynamics of a (small) quantum system interacting with a free Bose gas. They describe typically a non-relativistic atom interacting with a field of massless scalar bosons. Pauli-Fierz operators appear also in solid state physics. They are used to describe the interaction of phonons with a quantum system with finitely many degrees of freedom. This paper is devoted to the justification of the second-order perturbation theory for a large class of perturbation. For positive temperature system, this property is related to the return to equilibrium, see for instance [DJ2] and reference therein.

This question has been studied in many places, see for instance [BFS, BFSS, DJ, FMS, FP] for zero temperature systems and [DJ, JP, M] for positive temperature. We mention also [FGS, GGM, HSp, S] who studied certain spectral properties using positive commutator techniques. Here, we focus on the zero temperature setting. In [BFS], one uses some analytic deformation techniques. In [BFSS] and in [DJ], one uses some kind of Mourre estimate approach. In the former, one enlarges the class of perturbation studied in [BFS] and in the latter, one introduces another class. These two classes do not fully overlap. This is due to the choice of the conjugate operator. In this paper, we enlarge the class of perturbations used in [DJ] for the question of the Virial theorem (one-commutator theory) and also for the limiting absorption principal (two-commutator theory).

Now, we present the model. For the sake of simplicity and as in [DJ], we start with a n-level atom. It is described by a self-adjoint matrix K acting on a finite dimensional Hilbert space $\mathscr{K}$. Let $(k_i)_{i=0,\dots,n}$ be its eigenvalues, with $k_i < k_{i+1}$. On the other hand, we have the Bosonic field $\Gamma_s(\mathfrak{h})$ with the 1-particle Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^d, dk)$. The Hamiltonian is given by the second quantization $\mathrm{d}\Gamma(\omega)$ of ω, where $\omega(k) = |k|$, see Section 2.1. This is a massless and zero temperature system.

The free operator is given by $H_0 = K \otimes 1_{\Gamma(\omega)} + 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(\omega)$ on $\mathscr{K} \otimes \Gamma(\mathfrak{h})$. Its spectrum is $[k_0, \infty)$. It has no singularly continuous spectrum. Its point spectrum is the same as K, with the same multiplicity. Let $\alpha \in \mathcal{B}(\mathscr{K}, \mathscr{K} \otimes \mathfrak{h})$ be a form-factor and $\phi(\alpha)$ the field operator associated to it, see Section 2.2. Under the condition

(I0) $(1 \otimes \omega^{-1/2})\alpha \in \mathcal{B}(\mathscr{K}, \mathscr{K} \otimes \mathfrak{h})$,

we define the interacting Hamiltonian on $\mathscr{K} \otimes \Gamma(\mathfrak{h})$ by

$$H_\lambda := K \otimes 1_{\Gamma(\omega)} + 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(\omega) + \lambda\phi(\alpha), \text{ where } \lambda \in \mathbb{R}. \tag{1.1}$$

The operator is self-adjoint with domain $\mathscr{K} \otimes \mathcal{D}\big(\mathrm{d}\Gamma(\omega)\big)$.

We now focus on a selected eigenvalue k_{i_0}, with $i_0 > 0$. The aim of this paper is to give hypotheses on the form factor α to ensure that H_λ has no eigenvalue in a neighborhood of k_{i_0} for λ small enough (and non-zero). First, we have to ensure that the perturbation given by the field operator will really couple the system at energy k_{i_0}; we have to avoid form factors like $\alpha(x) = 1 \otimes b$ for all $x \in \mathscr{K}$ and some $b \in \mathfrak{h}$, see Section 6. Here comes the second-order perturbation theory, namely the hypothesis of *Fermi golden rule* for the couple (H_0, α) at energy k_{i_0}:

$$\mathrm{w}-\lim_{\varepsilon \to 0^+} P\phi(\alpha)\overline{P}\,\mathrm{Im}(H_0 - k + i\varepsilon)^{-1}\overline{P}\phi(\alpha)P > 0, \text{ on } P\mathscr{H}, \tag{1.2}$$

where $P := P_{k_{i_0}} \otimes P_\Omega$ and $\overline{P} := 1 - P$. At first sight, this is pretty implicit. We make it explicit in Appendix A. This condition involves the form factor, the eigenvalues of H_0 lower than k_{i_0} and its eigenfunctions. Therein, we also explain why the ground state energy is tacitly excluded.

In this paper, we are establishing an extended Mourre estimate, in the spirit of [GGM2, S]; this is an extended version of the positive commutator technique initiated by E. Mourre, see [ABG, M] and [G2, GJe] for recent developments. Due to the method, we make further hypotheses on the form-factor. To formulate them, we shall take advantage of the *polar coordinates* and of the unitary map:

$$T := \left\{ \begin{array}{ccc} L^2(\mathbb{R}^d, dk) & \longrightarrow & L^2(\mathbb{R}^+, dr) \otimes L^2(S^{d-1}, d\theta) := \tilde{\mathfrak{h}} \\ u & \longmapsto & Tu := (r, \theta) \mapsto r^{(d-1)/2} u(r\theta). \end{array} \right. \tag{1.3}$$

We identify $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ through this transformation. We write ∂_r for $\partial_r \otimes 1$. We first give meaning to the commutator via:

(I1a) $\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K} \otimes \dot{\mathscr{H}}^1(\mathbb{R}^+) \otimes L^2(S^{d-1})\big)$, $1 \otimes \omega^{-1/2}\partial_r\alpha \in \mathcal{B}(\mathscr{K}, \mathscr{K} \otimes \mathfrak{h})$.

Here, the dot means the completion of $\mathcal{C}_c^\infty(\mathbb{R}^+)$ under the norm given by the space. We denote by $\|\cdot\|_2$ the L^2 norm. Recall the norm of $\mathscr{H}^1$ is given by $\|\cdot\|_2 + \|\partial_r \cdot\|_2$.

We explain the method on a formal level. We start by choosing a conjugate operator so as to obtain some positivity of the commutator. We choose $A := 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(i\partial_r)$. Note this operator is not self-adjoint and only maximal symmetric. We set $N := 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(\mathrm{Id})$, the *number operator*. Thanks to **(I1a)**, one obtains

$$[H_\lambda, iA] = \underbrace{N + 1_{\mathscr{K}} \otimes P_\Omega}_{\geq 1} + \underbrace{\lambda\phi(\partial_r\alpha) - 1_{\mathscr{K}} \otimes P_\Omega}_{H_\lambda\text{-bounded}} =: M + S.$$

Consider a compact interval $\mathcal{J}$. Since $\mathrm{d}\Gamma(\omega)$ is non-negative, we have:

$$E_{\mathcal{J}}(H_0) = \sum_{0 \leq i \leq \sup(\mathcal{J})} P_{k_i} \otimes E_{\mathcal{J}-k_i}\big(\mathrm{d}\Gamma(\omega)\big). \tag{1.4}$$

We infer $(1_{\mathscr{K}} \otimes P_\Omega)E_{\mathcal{J}}(H_0) = 0$ if and only if $\mathcal{J}$ contains no eigenvalues of K. We evaluate the commutator at an energy $\mathcal{J}$ which contains k_{i_0} and no other k_i. Thus,

$$M + E_{\mathcal{J}}(H_0)SE_{\mathcal{J}}(H_0) \geq 1 + \big(-1 + O(\lambda)\big)E_{\mathcal{J}}(H_0) \geq O(\lambda)E_{\mathcal{J}}(H_0), \tag{1.5}$$

since $\phi(i\partial_r\alpha)$ is H_0-bounded. We keep M outside the spectral measure as it is not H_λ-bounded. Note we have no control on the sign of $O(\lambda)$ so far. We have not yet used the Fermi golden rule assumption. We follow an idea of [BFSS] and set

$$B_\varepsilon := \mathrm{Im}\Big(\big((H_0 - k_{i_0})^2 + \varepsilon^2\big)^{-1}\overline{P}\phi(\alpha)P\Big)$$

Observe that (1.2) implies there exists $c > 0$ such that

$$P[H_\lambda, i\lambda B_\varepsilon]P = \frac{\lambda^2}{\varepsilon}P\phi(\alpha)\overline{P}\,\mathrm{Im}(H_0 - k_{i_0} + i\varepsilon)^{-1}\,\overline{P}\phi(\alpha)P \geq \frac{c\lambda^2}{\varepsilon}P,$$

holds true for ε small enough. Let $\hat{A} := A + \lambda B_\varepsilon$ and $\hat{S} := S + \lambda[H_\lambda, iB_\varepsilon]$. We have $[H_\lambda, i\hat{A}] = M + \hat{S}$. We go back to (1.5) and infer:

$$M + E_{\mathcal{J}}(H_0)\hat{S}E_{\mathcal{J}}(H_0) \geq \big(c\lambda^2/\varepsilon + O(\lambda)\big)E_{\mathcal{J}}(H_0) + \text{ error terms.} \tag{1.6}$$

By taking $\varepsilon := \varepsilon(\lambda)$, one hopes to obtain the positivity of the constant in front of $E_{\mathcal{J}}(H_0)$, to control the errors terms and to replace the spectral measure by the one of H_λ. Using the Feshbach method and with a more involved choice of conjugate operator, we show in Section 6 that there are $\lambda_0, c', \eta > 0$ so that

$$M + E_{\mathcal{J}}(H_\lambda)\hat{S}E_{\mathcal{J}}(H_\lambda) \geq c'|\lambda|^{1+\eta}E_{\mathcal{J}}(H_\lambda), \text{ for all } |\lambda| \leq \lambda_0, \tag{1.7}$$

on the sense of forms on $\mathcal{D}(N^{1/2})$.

One would like to deduce there is no eigenvalue in $\mathcal{J}$ from (1.7). To apply a Virial theorem, one has at least to check that the eigenvalues of H_λ are in the domain of $N^{1/2}$. One may proceed like in [M]. In this article, we follow [GGM, S] and construct a sequence of approximated conjugate operators $\hat{A}_n$ such that $[H_\lambda, i\hat{A}_n]$ is H_λ-bounded, converges to $[H_\lambda, i\hat{A}]$ and such that one may apply the Virial theorem with A_n. To justify these steps, we make a new assumption:

(I1b) $1_{\mathscr{K}} \otimes \omega^{-a}\alpha \in \mathcal{B}(\mathscr{K}, \mathscr{K} \otimes \mathfrak{h})$, for some $a > 1$.

We now give our first result, based on the Virial theorem, see Proposition 4.11.

Theorem 1.1. *Let $\mathcal{I}$ be an open interval containing k_{i_0} and no other k_i. Assume the Fermi golden rule hypothesis* (1.2) *at energy k_{i_0}. Suppose that* **(I0)**, **(I1a)** *and* **(I1b)** *are satisfied. Then, there is $\lambda_0 > 0$ such that H_λ has no eigenvalue in $\mathcal{I}$, for all $|\lambda| \in (0, \lambda_0)$.*

We now give more information on the resolvent $R_\lambda(z) := (H_\lambda - z)^{-1}$ as the imaginary part of z tends to 0. We show it extends to an operator in some weighted spaces around the real axis. This is a standard result in the Mourre theory, when one supposes some 2-commutators-like hypothesis, see [ABG]. Here, as the commutator is not H_λ-bounded, one relies on an adapted theory. We use [GGM] which is a refined version of [S]. We check the hypotheses **(M1)**–**(M5)** given in Appendix C and deduce a limiting absorption principle, thanks to Theorem C.8. Using again (1.3), we state our class of form factors:

(I2) $\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K} \otimes \dot{\mathscr{B}}_2^{1,1}(\mathbb{R}^+) \otimes L^2(S^{d-1})\big)$.

Recall that the dot denotes the completion of $\mathcal{C}_c^\infty$. One choice of norm for $\mathscr{B}_2^{1,1}$ is:

$$\|f\|_{\mathscr{B}_2^{1,1}(\mathbb{R}^+)} = \|f\|_2 + \int_0^1 \big\|f(2t+\cdot) - 2f(t+\cdot) + f(\cdot)\big\|_2 \frac{dt}{t^2}.$$

We refer to [ABG, T] for Besov spaces and real interpolation. To express the weights, consider $\tilde{b}$ the square root of the Dirichlet Laplacian on $L^2(\mathbb{R}^+, dr)$. Using (1.3), we define $b := 1_{\mathscr{K}} \otimes T^{-1}\tilde{b}T$ in $\mathscr{H}$. Set $\mathscr{P}_s := 1_{\mathscr{K}} \otimes (\mathrm{d}\Gamma(b)+1)^{-s}(N+1)^{1/2}$.

Theorem 1.2. *Let $\mathcal{I}$ be an open interval containing k_{i_0} and no other k_i. Assume the Fermi golden rule hypothesis* (1.2) *at energy k_{i_0}. Suppose that* **(I0)**, **(I1a)** *and* **(I2)** *(and not necessarily* **(I1b)***), there is $\lambda_0 > 0$ such that H_λ has no eigenvalue in $\mathcal{I}$, for all $|\lambda| \in (0, \lambda_0)$. Moreover, H_λ has no singularly continuous spectrum in $\mathcal{I}$. For each compact interval $\mathcal{J}$ included in $\mathcal{I}$, and for all $s \in (1/2, 1]$, the limits*

$$\mathscr{P}_s^* R_\lambda(x \pm i0)\, \mathscr{P}_s := \lim_{y\to 0^+} \mathscr{P}_s^* R_\lambda(x \pm iy)\mathscr{P}_s$$

exist in norm uniformly in $x \in \mathcal{J}$. Moreover the maps:

$$\mathcal{J} \ni x \mapsto \mathscr{P}_s^* R_\lambda(x \pm i0)\mathscr{P}_s$$

are Hölder continuous of order $s - 1/2$ for the norm topology of $\mathcal{B}(\mathscr{H})$

To our knowledge, the condition **(I2)** is new, even for the question far from the thresholds. We believe it to be optimal for limiting absorption principle.

We now compare our result with the literature. In [BFSS], they use a different conjugate operator, the second quantization of the generator of dilatation. With this choice they have $[H_0, iA] = 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(\omega)$. The commutator is H_λ-bounded. They modify the conjugate operator in the same way but the choice of parameters is more involved. The class of perturbations is thus different from ours.

In [DJ][Theorem 6.3], one shows the absence of embedded eigenvalues by proving a limiting absorption principal with the weights $1_{\mathscr{K}} \otimes (\mathrm{d}\Gamma(b)+1)^{-s}$, for $s > 1/2$, without any contribution in N. They suppose essentially **(I0)** and that $\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K} \otimes \dot{\mathscr{H}}^s(\mathbb{R}^+) \otimes L^2(S^{d-1})\big)$, for $s > 1$. The class of perturbations is chosen in relation with the weights. Their strategy is to take advantage the Fermi golden rule at the level of the limiting absorption principle, with the help of the Feshbach method. The drawback is that they are limited by the relation weights/class of form-factors and they cannot give a Virial-type theorem. On the other hand, their method allows to cover some positive temperature systems and we do not deal with this question. Their method leads to fewer problems with domains questions. We mention that they do not suppose the second condition of **(I1a)**.

Therefore, concerning merely the disappearance of the eigenvalues, the conditions **(I1a)** and **(I1b)** do not imply α to be better than $\dot{\mathscr{H}}^1(\mathbb{R}^+)$, in the Sobolev scale. Hence, Theorem 1.1 is a new result. We point out that the condition **(I2)** is weaker than the one used in [DJ]. The weights obtained in the limiting absorption principle are also better than the ones given in [DJ]. We mention that one could improve them by using some Besov spaces, see [GGM]. To simplify the presentation, we do not present them here. We believe they could hardly be reached by the method exposed in [DJ] due to the interplay between weights and form-factors.

In [GGM2] and in [S], one cares about showing that the point spectrum is locally finite, i.e. without clusters and of finite multiplicity. Here, they use a Virial theorem. Between the eigenvalues, one shows a limiting absorption principle, and uses a hypothesis on the second commutator, something stronger than **(I2)**, see Section 4.5. In our approach, we use the Virial theorem and the limiting absorption principle in an independent way. In particular, if one is interested only in the limiting absorption principle, one does not need to suppose the more restrictive condition **(I1b)** but only **(I0)**, **(I1a)** and **(I2)**. This is due to the fact that we are showing a strict Mourre estimate, i.e. without compact contribution.

We now give the plan of the paper. In Section 2, we recall some definitions and properties of Pauli-Fierz models. In Section 3, we construct the conjugate operators. In Section 4, we prove the regularity properties so that one may apply the Mourre theory. The Virial theorem is discussed in Section 4.4. In Section 5, we establish the extended Mourre estimate far from the thresholds for small coupling constants, we explain in Remark 5.3 why the method should be improved to obtain the result above a threshold. In Section 6, we settle the extended Mourre estimate above the thresholds under the hypotheses of a Fermi golden rule. In Appendix A, we explain how to check the Fermi golden rule and why this hypothesis is compatible with the hypothesis **(I0)**, **(I1a)**, **(I1b)** and **(I2)**. In Appendix B, we gather some properties of C_0-semigroups and in Appendix C we recall the properties of the $\mathcal{C}^1$ class in this setting and the hypotheses so as to apply the extended Mourre theory.

Notation: Given a borelian set $\mathcal{J}$, we denote by $E_{\mathcal{J}}(A)$ the spectral measure associated to a self-adjoint operator A at energy $\mathcal{J}$. Given Hilbert spaces $\mathscr{H}, \mathscr{K}$, we denote by $\mathcal{B}(\mathscr{H}, \mathscr{K})$ the set of bounded operator from $\mathscr{H}$ to $\mathscr{K}$. We simply write $\mathcal{B}(\mathscr{H})$, when $\mathscr{H} = \mathscr{K}$. We denote by $\sigma(H)$ the spectrum of H. We set $\langle x \rangle := (1+x^2)^{1/2}$. We denote by $\| \cdot \|_{\mathscr{H}}$ and by $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ the norm and the scalar product of $\mathscr{H}$, respectively. We omit the indices when no confusion arises. We denote by w$-$lim and s$-$lim the weak and strong limit, respectively. A dot over a Besov or a Sobolev space denotes the closure of the set $\mathcal{C}_c^\infty$ of smooth functions with compact support, with respect to the norm of the space.

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2. The Pauli-Fierz model

Pauli-Fierz operators are often used in quantum physics as generator of approximate dynamics of a (small) quantum system interacting with a free Bose gas. They describe typically a non-relativistic atom interacting with a field of massless scalar bosons. The quantum system is given by a (separable) complex Hilbert space $\mathscr{K}$. The Hamiltonian describing the system is denoted by a self-adjoint operator K, which is bounded from below. We will suppose that K has some discrete spectrum. One may consider purely discrete spectrum, like [GGM2], or not, like in [S]. To do not mutter the presentation, we will take $\mathscr{K} = \operatorname{Ran} E_{\mathcal{I}}(K)$, where $\mathcal{I}$ contains a finite number of eigenvalues and consider the restriction of K to this space. Hence, we restrict the analysis to a self-adjoint matrix K acting in a Hilbert space $\mathscr{K}$ of finite dimension. This corresponds to analyze n level atoms. Doing so, we avoid some light problems of domains, which are already discussed in details in [GGM2, S] and gain in clarity of presentation.

2.1. **The bosonic field.** We refer to [BR, BSZ, RS] for a more thorough discussion of these matters. The bosonic field is described by the Hilbert space $\Gamma(\mathfrak{h})$, where $\mathfrak{h}$ is a Hilbert space. We recall its construction. Set $\mathfrak{h}^{0\otimes} = \mathbb{C}$ and $\mathfrak{h}^{n\otimes} = \mathfrak{h} \otimes \cdots \otimes \mathfrak{h}$. Given a closed operator A, we define the closed operator $A^{n\otimes}$ defined on $\mathfrak{h}^{n\otimes}$ by $A^{0\otimes} = 1$ if $n = 0$ and by $A \otimes \ldots \otimes A$ otherwise. Let S_n be the group of permutation of n elements. For each $\sigma \in S_n$, one defines the action on $\mathfrak{h}^{n\otimes}$ by $\sigma(f_{i_1} \otimes \ldots \otimes f_{i_n}) = f_{\sigma^{-1}(i_1)} \otimes \ldots \otimes f_{\sigma^{-1}(i_n)}$, where (f_i) is a basis of $\mathfrak{h}$. The action extends to $\mathfrak{h}^{n\otimes}$ by linearity to a unitary operator. The definition is independent of the choice of the basis. On $\mathfrak{h}^{n\otimes}$, we set

$$\Pi_n := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \text{ and } \Gamma_n(\mathfrak{h}) := \Pi_n(\mathfrak{h}^{n\otimes}). \tag{2.1}$$

Note that Π_n is an orthogonal projection. We call $\Gamma_n(\mathfrak{h})$ the n-particle bosonic space. The bosonic space is defined by

$$\Gamma(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \Gamma_n(\mathfrak{h}).$$

We denote by Ω the *vacuum*, the element $(1, 0, 0, \ldots)$ and by $P_\Omega := \Gamma(\mathfrak{h}) \to \Gamma_0(\mathfrak{h})$ the projection associated to it. We define $\Gamma_{\text{fin}}(\mathfrak{h})$ the set of finite particle vectors, i.e. $\Psi = (\Psi_1, \Psi_2, \ldots)$ such that $\Psi_n = 0$ for n big enough.

We now define the second quantized operators. We recall that a densely defined operator A is closable if and only if its adjoint A^* is densely defined. Given a closable operator q in $\mathfrak{h}$. We define $\Gamma_{\text{fin}}(q)$ acting from $\Gamma_{\text{fin}}(\mathcal{D}(q))$ into $\Gamma_{\text{fin}}(\mathfrak{h})$ by

$$\Gamma_{\text{fin}}(q)|_{\Pi_n(\mathcal{D}(q)^{n\otimes})} := q \otimes \ldots \otimes q.$$

Since q is closable, q^* is densely defined. Using that $\Gamma_{\text{fin}}(q^*) \subset \Gamma_{\text{fin}}(q)^*$, we see that $\Gamma_{\text{fin}}(q)$ is closable and we denote by $\Gamma(q)$ its closure. Note that $\Gamma(q)$ is bounded if and only if $\|q\| \le 1$.

Let b be a closable operator on $\mathfrak{h}$. We define $\mathrm{d}\Gamma_{\text{fin}}(b) : \Gamma_{\text{fin}}\big(\mathcal{D}(b)\big) \longrightarrow \Gamma_{\text{fin}}(\mathfrak{h})$ by

$$\mathrm{d}\Gamma_{\text{fin}}(b)|_{\Pi_n(\mathcal{D}(b)^{n\otimes})} := \sum_{j=1}^{n} 1 \otimes \ldots \otimes 1 \otimes \underbrace{b}_{j^{\text{th}}} \otimes 1 \otimes \ldots \otimes 1.$$

As above, $\mathrm{d}\Gamma_{\text{fin}}(b)$ is closable and $\mathrm{d}\Gamma(b)$ denotes also its closure. We link the objects.

Lemma 2.1. *Let $\mathbb{R}^+ \ni t \mapsto w_t \in \mathcal{B}(\mathfrak{h})$ be a C_0-semigroup of contractions (resp. of isometries), with generator a. Then $\mathbb{R}^+ \ni t \mapsto \Gamma(w_t) \in \mathcal{B}\big(\Gamma(\mathfrak{h})\big)$ is a C_0-semigroup of contractions (resp. of isometries) whose generator is $\mathrm{d}\Gamma(a)$.*

Proof. It is easy to see that $W_t := \Gamma(w_t)$ is a C_0-semigroup of contractions (resp. of isometries). Let A be its generator. Immediately, one gets $\mathrm{d}\Gamma_{\text{fin}}(a) \subset A$. Since $\Gamma_{\text{fin}}\big(\mathcal{D}(a)\big)$ is dense in $\mathrm{d}\Gamma(\mathfrak{h})$ and invariant under W_t, the Nelson lemma gives that $\Gamma_{\text{fin}}\big(\mathcal{D}(a)\big)$ is dense in $\mathcal{D}(A)$ for the graph norm and also that $\mathrm{d}\Gamma(a) = A$. □

2.2. **The interacting system.** Given a self-adjoint operator ω in $\mathfrak{h}$ and a finite dimensional Hilbert space $\mathscr{K}$. One defines the free Hamiltonian H_0 acting on the Hilbert space $\mathscr{H} := \mathscr{K} \otimes \Gamma(\mathfrak{h})$ by

$$H_0 := K \otimes 1_{\Gamma(\mathfrak{h})} + 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(\omega). \tag{2.2}$$

We recall also the definition of the *number operator* $N := 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(\mathrm{Id})$.

We now define the interaction. Let α be an element $\mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\mathfrak{h})$. This is a *form-factor*. We define $b(\alpha)$ on $\mathscr{H}$ by $b(\alpha) := \mathscr{K}\otimes h^{n\otimes} \to \mathscr{K}\otimes h^{(n-1)\otimes}$, where

$$b(\alpha)(\Psi\otimes\phi_1\otimes\ldots\phi_n) := \alpha^*(\Psi\otimes\phi_1)\otimes\phi_2\otimes\ldots\phi_n,$$

for $n\geq 1$ and by 0 otherwise. This operator is bounded and its norm is given by $\|\alpha\|_{\mathcal{B}(\mathscr{K},\mathscr{K}\otimes\mathfrak{h})}$. We define the *annihilation operator* on $\mathscr{K}\otimes\Gamma(\mathfrak{h})$ with domain $\mathscr{K}\otimes\Gamma_{\rm fin}(\mathfrak{h})$ by

$$a(\alpha) := (N+1)^{1/2} b(\alpha)\big(1\otimes\Pi\big),$$

where $\Pi := \sum_n \Pi_n$, see (2.1). As above, it is closable and its closure is denoted by $a(\alpha)$. Its adjoint is the *creation operator*. It acts as $a^*(\alpha) = b^*(\alpha)(N+1)^{1/2}$ on $\mathscr{H}$. Note that $b^*(\alpha)(\psi\otimes\phi_1\otimes\ldots\otimes\phi_n) = (\alpha\phi)\otimes\phi_1\otimes\ldots\otimes\phi_n$.

The *(Segal) Field operator* is defined by

$$\phi(\alpha) := \frac{1}{\sqrt{2}}\big(a(\alpha)+a^*(\alpha)\big).$$

We consider its closure on $\mathscr{K}\otimes\mathcal{D}(N^{1/2})$. We have the two elementary estimates:

$$\|(N+1)^{-1/2}a^{(*)}(\alpha)\|\leq\|\alpha\|,\quad \|(N+1)^{-1/2}\phi(\alpha)\|\leq\sqrt{2}\|\alpha\|. \tag{2.3}$$

An assertion containing $(*)$ holds with and without $*$.

We give the following N_τ-estimate and refer to [DJ, Proposition 4.1] for a proof of i). The point ii) is a direct consequence of the Kato-Rellich Lemma. This kind of estimates comes back to [GJ]. See also [BFS]. We refer to [G1][Appendix A] and [GGM2][Proposition 3.7] for unbounded K.

Proposition 2.2. *Let ω be a non-negative, injective, self-adjoint operator on $\mathfrak{h}$. Let $\beta\in\mathcal{B}\big(\mathscr{K},\mathscr{K}\otimes\mathcal{D}(\omega^{-1/2})\big)$.*

i) *Then $\phi(\beta)\in\mathcal{B}\big(\mathscr{K}\otimes\mathcal{D}(\mathrm{d}\Gamma(\omega)^{1/2}),\mathscr{H}\big)$ and for any $\Phi\in\mathcal{D}(\mathrm{d}\Gamma(\omega)^{1/2})$,*

$$\begin{aligned}|\phi(\beta)\Phi\|^2 \leq & \|\beta\|_{\mathcal{B}(\mathscr{K},\mathscr{K}\otimes\mathfrak{h})}\,\|\Phi\|^2\\ & + 2\|\omega^{-1/2}\beta\|_{\mathcal{B}(\mathscr{K},\mathscr{K}\otimes\mathfrak{h})}\,\langle\Phi, 1_{\mathscr{K}}\otimes\mathrm{d}\Gamma(\omega)\Phi\rangle.\end{aligned}\tag{2.4}$$

ii) *The field operator $\phi(\alpha)$ is H_0-operator bounded with relative bound ε, for all $\varepsilon>0$. Hence, $H_\lambda := H_0+\lambda\phi(\alpha)$, for $\lambda\in\mathbb{R}$, defines a self-adjoint operator with domain $\mathcal{D}(H_\lambda) = \mathscr{K}\otimes\mathrm{d}\Gamma(\omega)$ and is essentially self-adjoint on any core of H_0.*

2.3. **The zero-temperature Pauli-Fierz Model.** We now precise our model to the zero-temperature physical setting. The one particle space is given by $\mathfrak{h} := L^2(\mathbb{R}^d, dk)$, where k is the boson momentum. The *one particle kinetic energy* is the operator of multiplication by $\omega(k) := |k|$. Consider a self-adjoint matrix K on a finite dimensional Hilbert space $\mathscr{K}$ and denote by $(k_i)_{i=0,\ldots,n}$, with $k_i<k_{i+1}$ its eigenvalues. We denote by P_{k_i} the projection onto the i-th eigenspace.

The spectrum of $\mathrm{d}\Gamma(\omega)$ in $\Gamma(\mathfrak{h})$ is $[0,\infty)$ and due the vacuum part, 0 is the only eigenvalue. Its multiplicity is one. The spectrum of H_0 given by (2.2) is $[k_0,\infty)$. The eigenvalues are given by $(k_i)_{i=0,\ldots,n}$ and have the same multiplicity as those of $\mathscr{K}$. The singularly continuous component of the spectrum is empty. Here, $(k_i)_{i=0,\ldots,n}$ play also the rôle of *thresholds*.

We consider a form-factor α satisfying hypothesis **(I0)**. By applying Proposition 2.2, the operator H_λ, given by (1.1), is self-adjoint and $\mathcal{D}(H_\lambda) = \mathscr{K}\otimes\mathcal{D}\big(\mathrm{d}\Gamma(\omega)\big)$.

Since we study form factors in $\mathcal{B}(\mathscr{K},\mathscr{K}\otimes\mathfrak{h})$, we forbid some eventual singularities of the form-factor from the very beginning. However, if the atomic part has a particular shape, one may use some gauge transformations and gains in singularity, see for instance [GGM2][Section 2.4] and [DJ][Section 1.6]. Nevertheless, it is an open question if there exists some gauge transformation that allows one to cover the physical form factor studied in [BFS, BFSS], from our conditions. Conversely, the classes of perturbations studied in the latter does not fully cover ours.

3. The conjugate operators

In this paper, we analyze the spectrum of the Pauli-Fierz Hamiltonian H_λ described in Section 2.3 using some commutator techniques. We study the behavior of the embedded eigenvalues of H_λ under small coupling constants and establish some refined spectral properties. To do so, we establish a version of the Mourre estimate, see Appendix C.2. Hence, we start by constructing the conjugate operator. We follow similar ideas as in [GGM2, HSp, S]. Later, we modify it by a finite rank perturbation, in the spirit of [BFSS]. Unlike in the standard Mourre theory, the conjugate operator is not self-adjoint and only maximal-symmetric. We refer to Appendix C.1 for discussions about 1-commutators properties in this setting. We point out that one may avoid to work with maximal-symmetric operator by symmetrizing the space and thus gluing non-physical free bosons, see [DJ][Section 5.2]. This trick leads to some problems of domains with our method and would be treated elsewhere.

We point out that the real drawback of this choice of conjugate operator comes from the fact that the commutator is not H_λ bounded, like in the standard Mourre theory and [BFS, BFSS, FGS, FP]. Some difficulties appear to apply the Virial theorem. To overcome them, we follow ideas of [S, GGM2] and construct a series of approximate conjugate operators. One may also proceed like in [M].

3.1. **The semigroup on the 1-particle space.** Fix $\chi \in \mathcal{C}_c^\infty\big(\mathbb{R}^+;[0,1]\big)$ decreasing such that $\chi(x) = 1$ for $x \leq 1$ and 0 for $x \geq 2$. Set $\tilde\chi := 1 - \chi$. We consider the following vector fields on $\mathbb{R}^+$:

$$m_n(t) := \begin{cases} \tilde\chi\,(nt)\,, & \text{for } n \in \mathbb{N}, \\ 1, & \text{for } n = \infty, \end{cases} \text{ and } s_n(t) = \frac{m_n(t)}{t}. \tag{3.1}$$

Note that m_n converges increasingly to m_∞, almost everywhere, as n goes to infinity. As in [S] and in [GGM2], the rôle of m_∞ would be to ensure the positivity of the commutator and the one of m_n would be to guarantee of the Virial theorem.

We define the associated vector fields in $\mathbb{R}^d$ as follows:

$$\overrightarrow{s_n}(k) := s_n\big(|k|\big)k, \text{ for } k \in \mathbb{R}^d \text{ and } n \in \mathbb{N}^* \cup \{\infty\}. \tag{3.2}$$

We shall construct the C_0-semigroup of isometries associated to the vector fields $\overrightarrow{s_n}$ on $\mathfrak{h} = L^2(\mathbb{R}^d)$ and identify the generators. We define

$$a_n := -\frac{1}{2}\Big(\overrightarrow{s_n}\cdot D_k + D_k \cdot \overrightarrow{s_n}\Big) \tag{3.3}$$

on $\mathcal{C}_c^\infty(\mathbb{R}^d \setminus \{0\})$ for all $n \in \mathbb{N}^* \cup \{\infty\}$ and where $D_k = i\nabla$. These operators are closable as the domains of their adjoints are dense. In the sequel, we denote by the same symbol their closure.

We work in polar coordinates. We identify $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ through the transformation (1.3). Given an operator B in $\mathfrak{h}$, we denote by $\tilde B$ the corresponding operator acting in the $\tilde{\mathfrak{h}}$ and given by $\tilde B := TBT^{-1}$. We have:

Proposition 3.1. *For n finite, a_n is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d \setminus \{0\})$ and a_∞ is maximal symmetric with deficiency indices $(N, 0)$. Here, $N = \infty$ for $d \geq 2$ and $N = 2$ for $d = 1$. The operator a_n generates a C_0-semigroup of isometries denoted by $\{w_{n,t}\}_{t \in \mathbb{R}^+}$. In polar coordinates, the domains are given by*

$$\mathcal{D}(\tilde{a}_n) \supset \mathcal{D}(\tilde{a}_\infty) = \dot{\mathscr{H}}^1(\mathbb{R}^+) \otimes L^2(S^{d-1}),\ \textit{for all } n \in \mathbb{N}^*,$$
$$\mathcal{D}(\tilde{a}_\infty^*) = \mathscr{H}^1(\mathbb{R}^+) \otimes L^2(S^{d-1}),$$

where $\dot{\mathscr{H}}^1(\mathbb{R}^+)$ is the closure of $\mathcal{C}_c^\infty(\mathbb{R}^+)$ under the norm $\|\cdot\| + \|\partial_r \cdot\|$ and where $\mathscr{H}^1(\mathbb{R}^+)$ is the Sobolev space of first order.

See Section B for an overview on C_0-semigroups. For n finite, the C_0-semigroup extends to a C_0-group since a_n is self-adjoint.

Proof. When n is finite, it is well known that a_n is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ and follows by studying C_0-group associated to the flow defined by the smooth vector field $\overrightarrow{s_n}$. The density follows by the Nelson lemma. See for instance [ABG][Proposition 4.2.3]. Hence, for n finite, it remains to show that $\mathcal{C}_c^\infty(\mathbb{R}^d \setminus \{0\})$ is a core for a_n.

Straightforwardly, for $n \in \mathbb{N}^* \cup \{\infty\}$, one gets

$$\tilde{a}_n := T a_n T^{-1} = i\big(m_n(\cdot)\partial_r + \frac{1}{2}(m_n)'(\cdot)\big) \otimes 1,\ \text{where } m_n(r) := r s_n(r). \tag{3.4}$$

We extend m_n on $\mathbb{R}$ by setting $m_n(-r) := m_n(r)$ for $r > 0$ and prolongate it by continuity in 0. Let $\phi_{n,t}$ be the flow generated by the smooth vector field m_n on $\mathbb{R}$. In other words, $\phi_{n,t} := \phi_n(t, \cdot)$ is the unique solution of $(\partial_t \phi_n)(t, r) = m_n\big(\phi_n(t, r)\big)$, where $\phi_n(0, r) = r$. Since m_n is globally Lipschitz, $\phi_{n,t}$ exists for all time t. Moreover, $\phi_{n,t}$ is a smooth diffeomorphism of $\mathbb{R}$ with inverse $\phi_{n,-t}$ for all $t \in \mathbb{R}$. Let $\tilde{\phi}_{n,t}$ be the restriction of $\phi_{n,t}$ from $\mathbb{R}^{+*}$ to $\mathbb{R}^{+*}$. Let $\Omega_{n,t}$ be the domain of this restriction, i.e. the set of $r > 0$ such that $\phi_{n,t}(r) > 0$. One has $\Omega_{n,t} = \mathbb{R}^{+*}$ for $t \geq 0$ as $m_n(r)$ is positive. For the same reason, $t \mapsto \Omega_{n,t}$ is increasing. Note also that we have $\Omega_{n,-t} = \phi_{n,t}(\mathbb{R}^{+*})$ for $t \geq 0$. For $u \in \tilde{\mathfrak{h}}$, we set:

$$(\tilde{w}_{n,t} u)(r, \theta) := 1_{\Omega_{n,-t}}(r) \sqrt{\phi'_{n,-t}(r)}\, u(\phi_{n,-t}(r), \theta),\ \text{ for } t \geq 0. \tag{3.5}$$

A change of variable gives that $\tilde{w}_{n,t}$ is an isometry of $L^2(\mathbb{R}^+)$ with range $L^2(\Omega_{n,-t})$ for all $t \geq 0$. Since $\phi_{n,t}$ is a smooth flow, $\{\tilde{w}_{n,t}\}_{t \geq 0}$ is a C_0-semigroup of isometries. The adjoint C_0-semigroup is given by

$$(\tilde{w}^*_{n,t} u)(r, \theta) := 1_{\mathbb{R}^{+*}}(r) \sqrt{\phi'_{n,t}(r)}\, u(\phi_{n,t}(r), \theta),\ \text{ for } t \geq 0. \tag{3.6}$$

This is *not* a semigroup of isometries when $n = \infty$.

We compute the generator of the semigroup $\{\tilde{w}_{n,t}\}_{t \geq 0}$. Take $u \in \mathcal{C}_c^\infty\big(\tilde{\mathfrak{h}}\big)$. We have $\tilde{w}_{n,t} u \in \mathcal{C}_c^\infty(\Omega_{n,-t} \times S^{d-1})$. Let $r \in \Omega_{n,-t}$, we get

$$-\left(\frac{d}{dt} \tilde{w}_{n,t} u\right)(r, \theta) = \left(\tilde{w}_{n,t}\big(m_n(\cdot)\partial_r + \frac{1}{2}(m_n)'(\cdot)\big) u\right)(r, \theta).$$

Denoting by the same symbol the closure of $\tilde{a}_n$ on $\mathcal{C}_c^\infty(\mathbb{R}^{+*} \times S^{d-1})$, we obtain

$$-i \frac{d}{dt} \tilde{w}_{n,t} u = \tilde{w}_{n,t} \tilde{a}_n u.$$

The closed operator is *a priori* only a restriction of the generator of the semigroup (in the sense of the inclusion of graph of operators). Now, since $\tilde{w}_{n,t}$ stabilizes $\mathcal{C}_c^\infty(\tilde{\mathfrak{h}})$ for all $t \geq 0$, the Nelson lemma gives that this space is a core for generator of the C_0-semigroup $\{\tilde{w}_{n,t}\}_{t\geq 0}$. Since this one is an extension of $\tilde{a}_n$, we have shown that $\tilde{a}_n$ is really the generator. One may denote formally $\tilde{w}_{n,t} = e^{it\tilde{a}_n}$. The domain of $\tilde{a}_n$ contains $\dot{\mathscr{H}}^1(\mathbb{R}^+) \otimes L^2(S^{d-1})$. Easily, this is an equality for $n = \infty$.

Considering the spectrum of a_n, we derive the deficiency indices of the closure of a_n on $\mathcal{C}_c^\infty(\mathbb{R}^d \setminus \{0\})$ are of the form $(N, 0)$. For n finite these indices are equal, we infer the essential self-adjointness of a_n on $\mathcal{C}_c^\infty(\mathbb{R}^d \setminus \{0\})$.

At this point, one may feel the real difference between the case n finite and infinite. On one hand $m_\infty \geq 1$ and on the other hand, for finite n, $m_n(r)$ tends to 0 as r tends to 0. The domain of the adjoint of $\tilde{a}_\infty$ would be different. Indeed,

$$\big(\tilde{a}_\infty^* u\big)(r,\theta) = i\big(m_\infty(r)(\partial_r u)(r,\theta) + \frac{1}{2}(m_\infty)'(r)u(r,\theta)\big), \tag{3.7}$$

where $u \in \mathcal{D}(\tilde{a}_\infty^*) = \mathscr{H}^1(\mathbb{R}^+) \otimes L^2(S^{d-1})$. Moreover, when $n = \infty$, the deficiency indices are then $(\infty, 0)$, as the dimension of $L^2(S^{d-1})$ is infinite. □

3.2. **The C_0-semigroup on the Fock space.** Thanks to Proposition 3.1 and Lemma 2.1, we define the C_0-semigroups on the whole Hilbert space. We set:

$$W_{n,t} := 1_{\mathscr{K}} \otimes \Gamma(w_{n,t}) \text{ and } W_{n,t}^* = 1_{\mathscr{K}} \otimes \Gamma(w_t^*), \text{ for } t \geq 0. \tag{3.8}$$

Clearly, $\{W_{n,t}\}_{t\geq 0}$ is a C_0-semigroup of isometries. Let A_∞ be its generator. In the same way, for n finite, we set

$$A_n := 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(a_n). \tag{3.9}$$

This is the generator of the C_0-group $1_{\mathscr{K}} \otimes \Gamma(e^{ita_n})$ by Lemma 2.1. Recall the rôle of the A_n is to ensure a Virial theorem, see Proposition 4.11.

In Section 5, we see that the operator A_∞ alone is not enough to deal with threshold energy as the system could be uncoupled. One needs to take in account the Fermi golden rule. One way is to follow [DJ] and to take advantage of it in the limiting absorption principle. Another way is to modify the conjugate operator with a *finite rank perturbation* so as to obtain more positivity above the thresholds, by letting appearing the Fermi golden rule in the commutator, see Section 6. This idea comes from [BFSS]. We follow it.

Choose k_{i_0} an eigenvalue of K and assume that (6.1) holds true at energy k_{i_0} for the couple (H_0, α). Let P be the projector $P_{k_{i_0}} \otimes P_\Omega$. For $\varepsilon < \varepsilon_0$, we define

$$\hat{A}_n := A_n + \lambda\theta B_\varepsilon, \text{ for } n \in \mathbb{N}^* \cup \{\infty\},$$

where $B_\varepsilon := \mathrm{Im}(\overline{R_\varepsilon}^2\phi(\alpha)P)$, $R_\varepsilon := \big((H_0 - k_{i_0})^2 + \varepsilon^2\big)^{-1/2}$ and $\overline{R_\varepsilon} := \overline{P}R_\varepsilon$. Note that the conjugate operator depends on the two parameters $\lambda \in \mathbb{R}$ from the coupling constant, $\varepsilon > 0$ from the Fermi golden rule hypothesis and on an extra technical $\theta > 0$. For the sake of clarity, we do not write these extra dependences.

Using Proposition B.5 and the fact that B_ε is bounded, one gets $\hat{A}_\infty$ is the generator of a C_0-semigroup. A bit more is true.

Lemma 3.2. *The operator $\hat{A}_\infty$ is maximal symmetric on $\mathcal{D}(A_\infty)$ and is the generator of C_0-semigroup of isometries, denoted by $\{\widehat{W}_{n,t}\}_{t\geq 0}$. For n finite, the operator $\hat{A}_n$ is self-adjoint on the domain of $\mathcal{D}(A_n)$.*

Proof. The second point is obvious. We concentrate on the first one. By Proposition 3.1, A_∞ is maximal symmetric with deficiency indices $(N,0)$ for some $N \neq 0$. Since B_ε is bounded, there is $c < 0$ such that $\|B_\varepsilon(A_\infty - z)^{-1}\| < 1$, for all $z \in \mathbb{C}$ where $\mathrm{Im}(z) \leq c$. Since $(I + B_\varepsilon(A_\infty - z)^{-1})(A_\infty - z) = A_\infty + B_\varepsilon - z$ on the domain of A_∞, we get the spectrum of $\hat{A}_\infty$ is contained in an upper half plane $\mathbb{R} + i[c, \infty)$. Now, since B_ε is symmetric, so is $\hat{A}_\infty$. If the indices of $\hat{A}_\infty$ would be both non-zero then its spectrum would be $\mathbb{C}$. Therefore, the deficiency indices of $\hat{A}_\infty$ are $(N', 0)$ for some non-negative N'. Note that $N' \neq 0$ by the Kato-Rellich theorem applied on $\hat{A}_\infty$, since B_ε is bounded. Hence, $\hat{A}_\infty$ is maximal symmetric on $\mathcal{D}(A_\infty)$ and its spectrum is $\mathbb{R} + i[0, \infty)$. It is automatically a C_0-semigroup of isometries. □

4. Smoothness with respect to the C_0-semigroup

In Section 4.1, we recall a general result. In Section 4.2, we give some 1-commutator properties for A_n. We check the hypothesis **(M1)**–**(M4)** of Appendix C.2. We identify the spaces and operators appearing therein in Lemma 4.3. In Section 4.3, we extend these properties to $\hat{A}_n$, using Proposition B.5 and Lemma 4.5. The Virial theorem is discussed in Section 4.4. At last, second commutator assumptions and the hypothesis **(M5)** are discussed in Section 4.5.

4.1. **A general result.** In order to check the C^1 properties, the b-stability, see Definition B.3, and to be able to deduce hypothesis **(M1)-(M5)** of Appendix C.2, we recall [GGM2][Proposition 4.10]. We formulate it for bounded K. Set first a C_0-semigroup of isometries $\mathbb{R}^+ \ni t \to v_t \in \mathcal{B}(\mathfrak{h})$ with generator a. By Lemma 2.1, $V_t := 1_{\mathscr{K}} \otimes \Gamma(v_t)$ is a C_0-semigroup of isometries with generator $A = 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(a)$. Let $b \geq 0$ be a self-adjoint operator on $\mathfrak{h}$, and K as in (2.2). Set

$$B := K \otimes 1_{\Gamma(\mathfrak{h})} + 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma(b), \quad \mathscr{G}_B := \mathcal{D}(B^{1/2}) = 1_{\mathscr{K}} \otimes \mathcal{D}\big(\mathrm{d}\Gamma(b)^{1/2}\big).$$

Proposition 4.1. *Let ω and $b \geq 0$ acting in $\mathfrak{h}$. Then,*

i) *The space $\mathscr{G}_B$ is b-stable under $\{V_t\}_{t\in\mathbb{R}^+}$ (resp. $\{V_t^*\}_{t\in\mathbb{R}^+}$), if*

$$v_t^* b v_t \leq C_t b, (\text{resp. } v_t b v_t^* \leq C_t b) \text{ with } \sup_{0<t<1} C_t < \infty. \tag{4.1}$$

ii) *Assuming* (4.1) *and that there is a constant C such that for all $u_i \in \mathcal{D}(b^{1/2})$*

$$\omega \leq Cb, \ |\langle u_2, (\omega v_t - v_t\omega)u_1\rangle| \leq Ct\|b^{1/2}u_1\| \cdot \|b^{1/2}u_2\|, \text{ for } 0 < t < 1. \tag{4.2}$$

Then $H_0 \in \mathcal{C}^1(A; \mathscr{G}_B, \mathscr{G}_B^*)$. Besides, in the sense of forms on $\mathscr{G}_B$, one has

$$[H_0, iA]^\circ = 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma([\omega, ia]^\circ).$$

iii) *Assume* (4.1) *and that α is a form-factor satisfying*

$$\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K} \otimes \mathcal{D}(a)\big), \ a\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K} \otimes \mathcal{D}(b^{-1/2})\big). \tag{4.3}$$

Then $\phi(\alpha) \in \mathcal{C}^1(A; \mathscr{G}_B, \mathscr{G}_B^*)$ and in the sense of forms on $\mathscr{G}_B$, we get

$$[\phi(\alpha), iA]^\circ = -\phi(ia\alpha).$$

Here $[\cdot,\cdot]^\circ$ denotes the closure of the form defined by $[\cdot,\cdot]$, H_0 is defined in (2.2) and $a\alpha$ is a short for $(1 \otimes a)\alpha$. If **(I0)** and (4.1)–(4.3) hold true, then H_λ, defined in (1.1), is self-adjoint with the same domain as H_0 and lies in $\mathcal{C}^1(A; \mathscr{G}_B, \mathscr{G}_B^*)$.

4.2. **Estimation on the first commutator.** In this section, we compute the first commutator with respect to the conjugate operator A_∞ and check the hypotheses **(M1)-(M4)** discussed in Appendix C.2. We follow [GGM] and use only the hypotheses **(I0)** and **(I1a)**. We start with a direct consequence of Proposition 3.1.

Lemma 4.2. *We assume* **(I0)** *and* **(I1a)**. *Then,* $\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K}\otimes\mathcal{D}(a_n)\big)$ *and* $a_n\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K}\otimes\mathcal{D}(\omega^{-1/2})\big)$, *for all* $n \in \mathbb{N}^*\cup\{\infty\}$.

We formally decompose the commutator $[H_\lambda, iA_n]$ into two parts. We set:

$$\begin{cases} M_n := 1_{\mathscr{K}}\otimes \mathrm{d}\Gamma(m_n) + 1\otimes P_\Omega, \\ S_n := -\phi(ia_n\alpha) - 1_{\mathscr{K}}\otimes P_\Omega, \end{cases} \qquad \text{for all } n\in\mathbb{N}^*\cup\{\infty\}. \tag{4.4}$$

Here, we add $1_{\mathscr{K}}\otimes P_\Omega$ to obtain $M_\infty \geq 1$. We stress that, for finite n, M_n has a different domain as M_∞. Indeed, $\mathcal{D}(M_n)\subset\mathcal{D}(H_0)$ when n is finite and $\mathcal{D}(M_\infty) = \mathcal{D}(N)$, since $M_\infty = N + 1_{\mathscr{K}}\otimes P_\Omega$.

We start with the hypothesis **(M1)**. We need to precise the definition the commutator H'_λ given formally by $[H_\lambda, iA_\infty]$. Note that it does not extend to a H_λ-bounded operator, as in the standard Mourre theory. We follow [GGM] and define

$$B_\infty := K\otimes 1_{\Gamma(\mathfrak{h})} + 1_{\mathscr{K}}\otimes \mathrm{d}\Gamma\big((k^2+1)^{1/2}\big). \tag{4.5}$$

Let $\mathscr{D}_\infty := \mathcal{D}(B_\infty)$ and $\mathscr{G}_\infty := \mathcal{D}(B_\infty^{1/2})$. We would drop the subscripts after this lemma as no more confusion could arise with Appendix C.2.

Lemma 4.3. *Assume* **(I0)** *and* **(I1a)**. *Then:*

i) $H_\lambda\in\mathcal{C}^1(M_\infty)$, $\mathcal{D}(H_\lambda)\cap\mathcal{D}(M_\infty)$ *is a core for* M_∞, S_∞ *is symmetric and lies in* $\mathcal{B}(\mathcal{D}(H_0),\mathscr{H})$.

ii) *Let* H'_λ *be the closure of* $M_\infty + S_\infty$ *defined on* $\mathcal{D}(H_\lambda)\cap\mathcal{D}(M_\infty)$. *Therefore,* H_λ *and* H'_λ *satisfy* **(M1)**.

iii) $\mathscr{D}_\infty = \mathcal{D}(H'_\lambda)\cap\mathcal{D}(H_\lambda) = \mathcal{D}(M_\infty)\cap\mathcal{D}(H_\lambda)$ *and* $\mathscr{G}_\infty$ *is the same as in* (C.2).

Proof. We start with i). Take the C_0-group generated by m_∞ acting by $(v_t f)(x) = e^{it}f(x)$ for $f\in\mathfrak{h}$. We use Proposition 4.1, with $a = m_\infty$ and $b=\omega$. Conditions (4.1) and (4.2) are trivially satisfied. Condition (4.3) follows from Lemma 4.2. Therefore, $H_\lambda\in\mathcal{C}^1\big(M_\infty;\mathcal{D}(|H_\lambda|^{1/2}),\mathcal{D}(|H_\lambda|^{1/2})^*\big)$ and thus [ABG][Lemma 7.5.3] gives $H_\lambda\in\mathcal{C}^1(M_\infty)$. Therefore, Proposition 2.2 gives that $\mathcal{D}(H_\lambda)\cap\mathcal{D}(M_\infty) = \mathcal{D}(N)\cap 1_{\mathscr{K}}\otimes\mathcal{D}\big(\mathrm{d}\Gamma(\omega)\big) = \mathcal{D}(B_\infty)$. This is an obvious core for M_∞.

Now, Lemma C.7 implies point ii) and also gives the statements on $\mathscr{D}=\mathscr{D}_\infty$ in iii). By Proposition 2.2 and **(I1a)**, we have that S_∞ is H_0-form bounded. Then, the norm $\|\cdot\|_{\mathscr{G}}$, given by (C.2), is equivalent to $\sqrt{\langle\cdot,(M_\infty+H_0+1)\cdot\rangle}$ on $\mathscr{D}$. Since $\mathscr{D}$ is a form core for B_∞, we infer $\mathscr{G}=\mathscr{G}_\infty$. □

From now on, we drop the subscripts for $\mathscr{D}$ and $\mathscr{G}$. We clarify the $\mathcal{C}^1$ property. The hypothesis **(M2)** is checked in Theorem 5.1.

Lemma 4.4. *Assume* **(I0)** *and* **(I1a)**. *Then,*

i) $\{W_{\infty,t}\}_{t\geq 0}$ *b-stabilizes* $\mathscr{G}$ *and* $\mathscr{G}^*$.

ii) $H_\lambda\in\mathcal{C}^1(A_\infty;\mathscr{G},\mathscr{G}^*)$ *and* $[H_\lambda, iA] = H'_\lambda$ *on* $\mathscr{D}$.

Therefore, hypotheses **(M3)** *and* **(M4)** *are fulfilled.*

Proof. We apply Proposition 4.1. As in the proof of Proposition 3.1, we work in polar coordinate through the isomorphism (1.3). In this representation, the operator b acts by $\tilde{b} = (r^2+1)^{1/2} \otimes 1$ in $\tilde{\mathfrak{h}}$. Using (3.5) and (3.6), we obtain

$$\tilde{w}^*_{\infty,t}\tilde{b}\tilde{w}_{\infty,t} = b\big(\phi_{\infty,t}(\cdot)\big) \text{ and } \tilde{w}_{\infty,t}\tilde{b}\tilde{w}^*_{\infty,t} = 1_{\mathbb{R}^+}\big(\phi_{\infty,-t}(\cdot)\big)b\big(\phi_{\infty,-t}(\cdot)\big).$$

Therein, the flow $\phi_{\infty,t}$ was extended in $\mathbb{R}$. We have,

$$|b\big(\phi_{\infty,t}(r)\big) - b(r)| \leq \|\nabla b\|_\infty |\phi_{\infty,t}(r) - r| \leq \|\nabla b\|_\infty |t|, \text{ for } 0 \leq |t| \leq 1. \tag{4.6}$$

We infer $1 \leq b\big(\phi_{\infty,t}(r)\big) \leq \|\nabla b\|_\infty \big(1+|t|\big)b(r)$, for $0 \leq |t| \leq 1$. Hence, the condition (4.1) is satisfied. The C_0-semigroup $\{W_{\infty,t}\}_{t\in\mathbb{R}^+}$ and $\{W^*_{\infty,t}\}_{t\in\mathbb{R}^+}$ b-stabilizes $\mathscr{G}$.

We prove the second point with the help of Proposition 4.1 ii) and iii). First, $\omega \leq b$. Now, $\omega w_{\infty,t} - w_{\infty,t}\omega = \big(\omega - \omega\big(\phi_{\infty,-t}(\cdot)\big)\big)w_t$. By (4.6), we obtain that $|\phi_{\infty,t}(r) - r| \leq C|t|b(r)$ and hence $|\omega - \omega\big(\phi_{\infty,-t}(\cdot)\big)| \leq C|t|b$, for all $t \in [0,1]$. Since $\{\omega_t\}_{t\in\mathbb{R}^+}$ b-stabilizes $\mathcal{D}(b^{1/2})$, we get (4.2). Now by Lemma 4.2, we check (4.3). We obtain $H_\lambda \in \mathcal{C}^1(A_\infty; \mathscr{G}, \mathscr{G}^*)$. □

4.3. **Estimation on the first perturbed commutator.** We now add the finite rank perturbation B_ε to the conjugate operator. We consider the conjugate operator $\hat{A}_n$, given by (6.6). We denote with a hat the perturbed operators. Set

$$\hat{S}_n := S_n + [H_\lambda, i\lambda\theta B_\varepsilon], \qquad \text{for all } n \in \mathbb{N}^* \cup \{\infty\}. \tag{4.7}$$

Note that the operator M_n, given in (4.4), is unaffected by B_ε.

Although B_ε is a finite rank perturbation, one needs to be careful, especially in the 2-commutators properties. We give the key-lemma which allows us to transfer safely properties of A_n to $\hat{A}_n$. We point out that Lemma 6.7 shows that $[H_\lambda, B_\varepsilon]$ is also a finite rank operator in $\mathscr{H}$. Recall that $\mathscr{G} = \mathscr{G}_\infty$ is given in (4.5).

Lemma 4.5. *Assume* **(I0)**. *We have:*

i) *B_ε is a finite rank self-adjoint operator.*

ii) *$B_\varepsilon \in \mathcal{B}(\mathscr{G})$.*

iii) *Assume also* **(I1a)**, *then B_ε is belonging to $\mathcal{C}^1(A_\infty; \mathscr{G}, \mathscr{G})$.*

Proof. Since P is of finite rank and B_ε is symmetric, we need only to show that B_ε is bounded. **(I0)** gives that $P\phi(\alpha)\overline{P} = P\alpha\overline{P}$ belongs to $\mathcal{B}(\mathscr{H}, \mathscr{K} \otimes \mathcal{D}(\omega^{-1/2}))$. Now, recall that $\varepsilon R^2_\varepsilon = \mathrm{Im}(H_0 - k + i\varepsilon)^{-1}$ and that $1 \otimes \omega^{1/2}(H_0 - k \pm i\varepsilon)^{-1}$ is bounded by functional calculus in $\mathscr{K} \otimes \mathfrak{h}$. This concludes i).

For point ii), note that $B_\varepsilon \in \mathcal{B}(\mathscr{G})$ is equivalent to $B_\varepsilon \in \mathcal{B}\big(\mathcal{D}(|H_0|^{1/2})\big)$, since $P\alpha\overline{P}$ is with image in the 1-particle space. Hence, the assertion follows by noticing that $1 \otimes \omega^{1/2}(1+\omega)^{1/2}(H_0 - k \pm i\varepsilon)^{-1}$ is bounded in $\mathscr{K} \otimes \mathfrak{h}$.

As in ii), it is enough to show that $T := P\phi(\alpha)\overline{P}(H_0 - z)^{-1}$ and its adjoint are in $\mathcal{C}^1\big(A_\infty; \mathcal{D}(|H_0|^{1/2}), \mathcal{D}(|H_0|^{1/2})\big)$, where $z \in \mathbb{C} \setminus \mathbb{R}$. We treat T. Note that $H_0|_{\mathscr{K}\otimes\mathfrak{h}} \in \mathcal{C}^1(A_\infty)$. Using (1.3), we have:

$$[T, iA_\infty] = P(1 \otimes \partial_r)\alpha\overline{P}(H_0 - z)^{-1} - P\alpha\overline{P}(H_0 - z)^{-2}.$$

Like in ii), the second term is easily bounded in $\mathcal{D}(|H_0|^{1/2})$. The boundedness of the first one is ensured by the second part of **(I1a)**. □

As an immediate corollary, we infer from Lemma 4.3 the following.

Lemma 4.6. *Assume* **(I0)** *and* **(I1a)**. *Then:*

i) *$H_\lambda \in \mathcal{C}^1(M_\infty)$, $\mathcal{D}(H_\lambda) \cap \mathcal{D}(M_\infty)$ is a core for M_∞, $\hat{S}_\infty$ is symmetric and lies in $\mathcal{B}(\mathcal{D}(H_0), \mathscr{H})$.*

ii) *Let* $\hat{H}'_\lambda$ *be the closure of* $M_\infty + \hat{S}_\infty$ *defined on* $\mathcal{D}(H_\lambda) \cap \mathcal{D}(M_\infty)$. *Therefore,* H_λ *and* $\hat{H}'_\lambda$ *satisfy* **(M1)**.

iii) $\mathscr{D} = \mathcal{D}(\hat{H}'_\lambda) \cap \mathcal{D}(H_\lambda) = \mathcal{D}(M_\infty) \cap \mathcal{D}(H_\lambda)$ *and* $\mathscr{G}$ *is the same as in* (C.2).

We now strengthen Lemma 4.4 and check **(M3)** and **(M4)**. The hypothesis **(M2)** is checked in Theorem 6.2.

Lemma 4.7. *Assume* **(I0)** *and* **(I1a)**. *Then,*

i) $\{\widehat{W}_{\infty,t}\}_{t\geq 0}$ *b-stabilizes* $\mathscr{G}$ *and* $\mathscr{G}^*$.

ii) $\mathcal{C}^1(A_\infty, \mathscr{G}, \mathscr{G}^*) = \mathcal{C}^1(\hat{A}_\infty, \mathscr{G}, \mathscr{G}^*)$.

iii) $H_\lambda \in \mathcal{C}^1(\hat{A}_\infty; \mathscr{G}, \mathscr{G}^*)$ *and* $[H_\lambda, i\hat{A}] = \hat{H}'_\lambda$ *on* $\mathscr{D}$.

Therefore, hypotheses **(M3)** *and* **(M4)** *are fulfilled.*

Proof. We consider $\{\widehat{W}_{\infty,t}\}_{t\in\mathbb{R}}$. The argument is the same for the adjoint. Let A'_∞ be the generator of $\{W_{\infty,t}\}_{t\in\mathbb{R}}$ in $\mathscr{G}$. As in (6.6), set $\widetilde{A_\infty}' := A'_\infty + \lambda\theta B_\varepsilon$. Thanks to Proposition B.5, since $B_\varepsilon \in \mathcal{B}(\mathscr{G})$, $\widetilde{A_\infty}'$ is the generator of a C_0-semigroup in $\mathscr{G}$. We name it $\{\widehat{W}'_{\infty,t}\}_{t\in\mathbb{R}}$. By duality and interpolation, it extends to a C_0-semigroup in $\mathscr{H}$. Comparing the generators, we obtain that $\{\widehat{W}'_{\infty,t}\}_{t\in\mathbb{R}}$ is really the restriction of $\{\widehat{W}_{\infty,t}\}_{t\in\mathbb{R}}$ and it gives point i). By Lemma 4.4, it is enough to show ii) to get iii). Proposition C.6 and the boundedness of B_ε in $\mathscr{G}$ and $\mathscr{G}^*$ give the former. $\square$

4.4. **The Virial theorem.** In order to obtain a Virial theorem, we proceed like in [GGM2, S] by approximating the conjugate operator. Indeed, since $\hat{H}'_\lambda$ is not H_λ-bounded, one can not apply *a priori* H'_λ to an eigenfunction of H_λ even in the form sense. In this section, we use the hypotheses **(I0)** and **(I1)**. Here, **(I1)** means **(I1a)** and **(I1b)**. In a zero temperature setting, this method is less demanding in hypotheses than the one used in [M], see for instance [M][Proposition 6.1]. Note that we do not deal with the positive temperature Hamiltonians treated therein.

Lemma 4.8. *Assume* **(I0)** *and* **(I1)**. *Then* $\phi(ia_n\alpha)$ *tends to* $\phi(ia_\infty\alpha)$, *as quadratic forms on* $\mathcal{D}(|H_\lambda|^{1/2})$, *as* n *goes to infinity.*

Proof. Thanks to Proposition 2.2, it is enough to show that $\|(a_n - a_\infty)\alpha\|_{\mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\mathfrak{h})}$ and that $\|\omega^{-1/2}(a_n - a_\infty)\alpha\|_{\mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\mathfrak{h})}$ tend to 0 as n goes to infinity.

We start with the first point. Like in the proof of Proposition 3.1, we work in polar coordinates. We focus on the expression of $\tilde{a}_n$ obtained in (3.4). We have $\alpha \in \mathcal{B}(\mathscr{K}, \mathscr{K} \otimes \dot{\mathscr{H}}^1 \otimes L^2)$. Moreover, since $m_n(r) \leq m_\infty(r) = 1$ and m_n converges simply to 1 almost everywhere, by the Lebesgue dominated convergence theorem, we obtain $\|(m_n - m_\infty)\partial\alpha\|_{\mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\mathfrak{h})}$ tends to 0. We treat the term in $\big(m'_n(r) - m'_\infty(r)\big)\alpha = m'_n(r)\alpha = m'_n(r)r^a r^{-a}\alpha$. As $a > 1/2$, dominated convergence proves it tends to 0 in $\mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\mathfrak{h})$. The proof of the second point is the same but use the fact that $r^{-a}\alpha \in \mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\mathfrak{h})$ for the term in m'_n, for some $a > 1$. $\square$

We point out that if one knows that $\omega^{-1}\alpha \in \mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\mathcal{C}_0(\mathbb{R}^+)\otimes L^2(S^{d-1}))$, one may relax **(I1b)** and take $a = 1$. Here $\mathcal{C}_0(\mathbb{R}^+)$ denotes the continuous functions vanishing in 0 and in $+\infty$.

Lemma 4.9. *Assume* n *finite,* **(I0)** *and* **(I1a)**. *Then,* $\{\widehat{W}_{n,t}\}_{t\in\mathbb{R}}$ *b-stabilizes the form domain of* H_λ.

Proof. First we apply Proposition 4.1 i) with $v_t = w_{n,t}$ and $b = w$. As we have a C_0-group, by taking t negative we obtain the result for the adjoint. As in the proof of Proposition 3.1, we denote by $\phi_{n,t} : \mathbb{R}^d \to \mathbb{R}^d$ the flow generated by the smooth vector field $\overrightarrow{s_n}$. Since $m_n(0) = 0$, we have

$$\begin{aligned} |\phi_{n,t}(k) - k| = |\phi_{n,t}(k) - \phi_{n,0}(k)| &\leq \int_0^{|t|} |m_n(\phi_{n,s}(k)) - m_n(0)|\, ds \\ &\leq \|\nabla m_n\|_\infty \int_0^{|t|} |\phi_{n,s}(k)|\, ds, \text{ for all } t \in \mathbb{R}. \end{aligned} \tag{4.8}$$

By the Gronwall lemma, we infer there is C such that $|\phi_{n,t}(k)| \leq C|k|$, for all $t \in [1, 1]$. Plugging back into (4.8), we obtain $|\phi_{n,t}(k) - k| \leq C|tk|$, for all $t \in [1, 1]$. Now using (3.5) and (3.6), we infer $e^{-ita_n} w e^{ita_n} = w\big(\phi_{n,t}(\cdot)\big)$. Since m_n is globally Lipschitz, there is C' such that

$$|w(\phi_{n,t}(k)) - w(k)| \leq C'|t|w(k), \text{ for all } t \in [1, 1]. \tag{4.9}$$

Hence, we satisfy the hypothesis (4.1) and $\mathcal{D}(|H_\lambda|^{1/2})$ is b-stable under $\{W_{n,t}\}_{t\in\mathbb{R}}$.

We now take care about $\{\widehat{W}_{n,t}\}_{t\in\mathbb{R}}$. Let A_n' be the generator of $\{W_{n,t}\}_{t\in\mathbb{R}}$ in $\mathcal{D}(|H_\lambda|^{1/2})$. As in (6.6), set $\widehat{A_n}' := A_n' + \lambda\theta B_\varepsilon$. By Lemma 4.5 ii) and the fact that B_ε is with values in the 0 and 1 particles space, we get B_ε bounded in $\mathcal{D}(|H_\lambda|^{1/2})$. Thanks to Proposition B.5, $\widehat{A_n}'$ is the generator of a C_0-group in $\mathcal{D}(|H_\lambda|^{1/2})$. We name it $\{\widehat{W}'_{n,t}\}_{t\in\mathbb{R}}$. By duality and interpolation, it extends to a C_0-group in $\mathscr{H}$. Comparing the generators, we obtain that $\{\widehat{W}'_{n,t}\}_{t\in\mathbb{R}}$ is really the restriction of $\{\widehat{W}_{n,t}\}_{t\in\mathbb{R}}$ and this gives the result. □

Lemma 4.10. *Assume n finite,* **(I0)** *and* **(I1a)**. *Then $H_\lambda \in \mathcal{C}^1(\hat{A}_n)$. Moreover:*

$$\big[H_\lambda, i\hat{A}_n\big] = M_n + \hat{S}_n, \tag{4.10}$$

holds true in the sense of forms on $\mathcal{D}(|H_\lambda|^{1/2})$.

Proof. Using again (4.9), we check (4.2). We get $[H_0, iA_n]^\circ = 1_{\mathscr{K}} \otimes \mathrm{d}\Gamma([\omega, ia_n]^\circ)$ in the sense of form on $\mathcal{D}(|H_\lambda|^{1/2})$. By computing $[\omega, ia_n]^\circ$ on the core $\mathcal{C}_c^\infty(\mathbb{R}^d \setminus \{0\})$, we obtain $[\omega, ia_n]^\circ = m_n$. Now, by Lemma 4.2, we can use Proposition 4.1 iii) and deduce $\big[H_\lambda, iA_n\big] = M_n + S_n$ in the sense of forms on $\mathcal{D}(|H_\lambda|^{1/2})$. Finally, by Lemma 6.7, $[H_\lambda, B_\varepsilon]$ is of finite rank, we also obtain (4.10) on the same domain.

Now, $H_\lambda \in \mathcal{C}^1\big(\hat{A}_n; \mathcal{D}\big(|H_\lambda|^{1/2}\big), \mathcal{D}\big(|H_\lambda|^{1/2}\big)^*\big)$ by Lemma 4.9 and Proposition C.6. We apply [GGM2][Lemma 6.3] to get $H_\lambda \in \mathcal{C}^1(\hat{A}_n)$. □

Therefore, the Virial theorem holds true when $\hat{A}_n$ is the conjugate operator and when n is finite. However, there is no Mourre estimate for $\hat{A}_n$ but only one for $\hat{A}_\infty$. To overcome this problem, we take advantage of the monotone convergence of $[H_0, iA_n]$ to $[H_0, iA_\infty]$ and of the uniformity given in Lemma 4.8 to prove:

Proposition 4.11 (Virial theorem)**.** *Assume* **(I0)** *and* **(I1)**. *Let u be an eigenfunction of H_λ then $u \in \mathcal{D}(N^{1/2})$ and $\langle u, (M_\infty + \hat{S}_\infty)u\rangle = 0$, as a quadratic form on $\mathcal{D}(N^{1/2}) \cap \mathcal{D}(H_\lambda)$.*

Proof. First, M_n is a bounded form for H_λ. Note that $0 \leq m_n \leq m$ implies $0 \leq \mathrm{d}\Gamma(m_n) \leq \mathrm{d}\Gamma(m)$ for all n. Now, since m_n is increasing and converges to m as n goes to infinity, monotone convergence gives

$$0 \leq \langle g, M_n g\rangle \leq \langle g, M_\infty g\rangle \text{ and } \langle g, M_n g\rangle \underset{n\to\infty}{\longrightarrow} \langle g, M_\infty g\rangle,$$

for all $g \in \mathcal{D}(M_\infty) \cap \mathcal{D}(H_\lambda)$. Using some Cauchy sequences, this holds true also in the sense of forms for $g \in \mathcal{D}(M_\infty^{1/2}) \cap \mathcal{D}(H_\lambda)$. By authorizing the value $+\infty$ on the two r.h.s. when $g \notin \mathcal{D}(M_\infty^{1/2})$, one allows $g \in \mathcal{D}(H_\lambda)$. On the other hand, Lemma 4.2 gives that $\hat{S}_n$ tends to $\hat{S}_\infty$ as a quadratic form on $\mathcal{D}(H)$.

Let $\dot{H}$ be the closure of quadratic form $\langle u, \hat{H}'_\lambda u\rangle$ defined on $\mathcal{D}(M_\infty) \cap \mathcal{D}(H)$. It is given by the quadratic form $\langle u, (M_\infty + \hat{S}_\infty) u\rangle$ defined on $\mathcal{D}(M_\infty^{1/2}) \cap \mathcal{D}(H)$. Take now an eigenfunction u of H_λ. By Lemma 4.10 and the Virial theorem, see [ABG][Proposition 7.2.10], we get $\langle u, (M_n + \hat{S}_n)u\rangle = 0$. By letting n go to infinity and noticing that $\mathcal{D}(M_\infty^{1/2}) = \mathscr{K} \otimes \mathcal{D}(N^{1/2})$, we get the result. $\square$

4.5. **Estimation on the second commutator.** In this section, we discuss the second commutator hypothesis **(I2)** so as to obtain a limiting absorption principle through the Theorem C.8. We stress we forgo the hypothesis **(I1b)** in this section. We start with the important remark.

Lemma 4.12. *We have* $\mathcal{C}^2(A_\infty, \mathscr{G}, \mathscr{G}^*) = \mathcal{C}^2(\hat{A}_\infty, \mathscr{G}, \mathscr{G}^*)$.

Proof. It is enough to show one inclusion. Using Proposition C.6 and the invariance of $\mathscr{G}$ and $\mathscr{G}^*$ given in Lemmata 4.4 and 4.7, one may work directly with A_∞ and $\hat{A}_\infty$. Let $H \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ be in $\mathcal{C}^2(A_\infty, \mathscr{G}, \mathscr{G}^*)$. One justifies the next expansion, by working in the form sense on $\mathcal{D}\big((A_\infty^*)^2|_{\mathscr{G}}\big) \times \mathcal{D}\big((A_\infty)^2|_{\mathscr{G}}\big)$. This is legal by using Lemma 4.5. We have:

$$\begin{aligned}[[H, \hat{A}_\infty], \hat{A}_\infty] =& [[H, A_\infty], A_\infty] + [[H, A_\infty], \lambda\theta B_\varepsilon] \\ & [[H, \lambda\theta B_\varepsilon], A_\infty] + [[H, \lambda\theta B_\varepsilon], \lambda\theta B_\varepsilon].\end{aligned}$$

The first term is in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ by hypothesis. For the second one, note that $[H, A_\infty] \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ since H is $\mathcal{C}^1(A_\infty, \mathscr{G}, \mathscr{G}^*)$. For the third one, we expand the commutator inside, use again that $H \in \mathcal{C}^1(A_\infty, \mathscr{G}, \mathscr{G}^*)$ and finish with Lemma 4.5 iii). For the last one, one expands it and use Lemma 4.5 ii). $\square$

We start by discussing the $\mathcal{C}^2$ theory used in [GGM2, S] and check the point **(M5')**. Through the isomorphism given by (1.3), we suppose the stronger

(I2') $\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K} \otimes \dot{\mathscr{H}}^2(\mathbb{R}^+) \otimes L^2(S^{d-1})\big)$.

This hypothesis is stronger than $\alpha \in \mathcal{B}\big(\mathscr{K}, \mathscr{K} \otimes \dot{\mathscr{H}}^s(\mathbb{R}^+) \otimes L^2(S^{d-1})\big)$ for $s > 1$, the one used in [DJ][Theorem 6.3].

Lemma 4.13. *Assume* **(I0)**, **(I1a)** *and* **(I2')**. *Then* $H_\lambda \in \mathcal{C}^2(\hat{A}_\infty, \mathscr{G}, \mathscr{G}^*)$ *and*

$$[\hat{H}'_\lambda, i\hat{A}_\infty] = \lambda\phi\big(a_\infty^2 \alpha\big) + \lambda\theta[[H_\lambda, B_\varepsilon], iA] + \lambda^2\theta^2[\hat{H}'_\lambda, B_\varepsilon].$$

Therefore, the hypothesis **(M5')** *is fulfilled.*

Proof. We use Proposition 4.1 ii) and iii) for the operator $H := N - \lambda\phi(ia_\infty\alpha)$. Point ii) is trivially satisfied. The hypothesis **(I2)** and Proposition 2.2 give (4.3). We obtain $H \in \mathcal{C}^1(A_\infty; \mathscr{G}, \mathscr{G}^*)$. $\square$

We now work with the hypothesis **(I2)** which is weaker than the one used in [DJ]. Thanks to Lemma 4.12, we have

$$\begin{aligned}\mathcal{C}^{1,1}(A_\infty, \mathscr{G}, \mathscr{G}^*) &:= \big(\mathcal{C}^2(A_\infty, \mathscr{G}, \mathscr{G}^*), \mathcal{B}(\mathscr{G}, \mathscr{G}^*)\big)_{1/2,1} \\ &= \big(\mathcal{C}^2(\hat{A}_\infty, \mathscr{G}, \mathscr{G}^*), \mathcal{B}(\mathscr{G}, \mathscr{G}^*)\big)_{1/2,1} =: \mathcal{C}^{1,1}(\hat{A}_\infty, \mathscr{G}, \mathscr{G}^*).\end{aligned}$$

We refer to [ABG, T] for real interpolation. We obtain:

Lemma 4.14. *Assume* **(I0)**, **(I1a)** *and* **(I2)**. *Then* $H_\lambda \in \mathcal{C}^{1,1}(\hat{A}_\infty, \mathscr{G}, \mathscr{G}^*)$ *and the hypothesis* **(M5)** *is fulfilled.*

Proof. By Lemma 4.13, we have $H_0 \in \mathcal{C}^2(\hat{A}_\infty, \mathscr{G}, \mathscr{G}^*)$. It is enough to show that $\phi(\alpha) \in \mathcal{C}^{1,1}(A_\infty, \mathscr{G}, \mathscr{H})$. By [DJ][Lemma 2.7], we have $W_{\infty,t}\phi(\alpha) = \phi(w_{\infty,t}\alpha)W_{\infty,t}$ for $t \geq 0$. By Proposition 2.2 and $b \geq 1$ and since $\{W_{\infty,t}\}$ b-preserves $\mathscr{G}$, we get

$$\begin{aligned}\int_0^1 \big\|[W_{\infty,t}[W_{\infty,t}, \phi(\alpha)]]\big\|_{\mathcal{B}(\mathscr{G},\mathscr{H})} \frac{dt}{t^2} &\leq \int_0^1 \big\|\phi([w_{\infty,t}[w_{\infty,t}, \alpha]])W_{\infty,2t}\big\|_{\mathcal{B}(\mathscr{G},\mathscr{H})} \frac{dt}{t^2} \\ &\leq C \int_0^1 \big\|[w_{\infty,t}[w_{\infty,t}, \alpha]]\big\|_{\mathcal{B}(\mathscr{K},\mathscr{K}\otimes\mathfrak{h})} \frac{dt}{t^2}.\end{aligned}$$

The latter is finite if and only if α belongs to $\big(\mathcal{B}\big(\mathscr{K}, \mathcal{D}(a_\infty^2)\big), \mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\mathfrak{h})\big)_{1/2,1}$. On the other hand, using the isomorphism (1.3) and Proposition 3.1, this space is the same as $\big(\mathcal{B}\big(\mathscr{K}, \mathscr{K}\otimes\dot{\mathscr{H}}^2(\mathbb{R}^+)\otimes L^2(S^{d-1})\big), \mathcal{B}(\mathscr{K}, \mathscr{K}\otimes\tilde{\mathfrak{h}})\big)_{1/2,1}$. Finally, using [T][Section 2.10.4], this is equivalent to the fact that α satisfies **(I2)**. $\square$

5. A Mourre estimate far from the thresholds

5.1. **The result.** The aim in this part is to show a Mourre estimate far from thresholds for small coupling constants. This is a well-known result, see [BFS, DJ] for instance. For the sake of completeness, we give a proof of the estimate. Doing so, we point out, in Remark 5.3, where the lack of positivity occurs above the thresholds. We use the approach based on the theory described in Appendix C. To obtain information just above the thresholds and without supposing the Fermi golden rule, one should add a compact term in (5.1), see [GGM, S].

Theorem 5.1. *Let $\mathcal{I}_0$ be a compact interval containing no element of $\sigma(K)$. Suppose also that* **(I0)** *and* **(I1a)** *are satisfied. Then, for all open interval $\mathcal{I} \subset \mathcal{I}_0$:*

i) *There are $M_\infty \geq 1$ and S_∞ a $|H_\lambda|^{1/2}$-bounded operator such that $[H_\lambda, iA_\infty] = M_\infty + S_\infty$ holds in the sense of forms on $\mathcal{D}(N^{1/2})$.*

ii) *The conditions* **(M1)**–**(M4)** *are satisfied.*

iii) *There is $\lambda_0 > 0$ such that the following extended Mourre estimate*

$$M_\infty + S_\infty \geq a(\lambda)E_{\mathcal{I}}(H_\lambda) - b(\lambda)E_{\mathcal{I}^c}(H_\lambda)\langle H_\lambda\rangle. \tag{5.1}$$

holds true in the sense of forms on $\mathcal{D}(N^{1/2})$, for all $|\lambda| \leq \lambda_0$. Here, $a(\lambda)$ is positive and can be written as $\big(1 + O(\lambda)\big)$. Besides, $b(\lambda)$ is also positive.

iv) *If* **(I1b)** *holds true, then H_λ has no eigenvalue in $\mathcal{I}$, for all $|\lambda| \leq \lambda_0$.*

v) *If* **(I2)** *holds true (and not necessarily* **(I1b)***), then H_λ has no eigenvalue in the interior of $\mathcal{I}$, for all $|\lambda| \leq \lambda_0$. Moreover, one obtains the estimations of the resolvent given in Theorem 1.2.*

Proof. By Lemma 4.3, we have the first point. The point ii) is shown in Section 4.2. The point iii) follows from Proposition 5.2. Indeed, since S_∞ is form bounded with respect to H_λ, we have that for all $\eta > 0$

$$\begin{aligned}&E_{\mathcal{I}}(H_\lambda)S_\infty E_{\mathcal{I}^c}(H_\lambda) + E_{\mathcal{I}^c}(H_\lambda)S_\infty E_{\mathcal{I}}(H_\lambda) \geq \\ &\qquad - \eta E_{\mathcal{I}}(H_\lambda)S_\infty\langle H_\lambda\rangle^{-1}S_\infty E_{\mathcal{I}}(H_\lambda) - \eta^{-1}E_{\mathcal{I}^c}(H_\lambda)\langle H_\lambda\rangle.\end{aligned} \tag{5.2}$$

The point iv) follows from the Virial Theorem, Proposition 4.11. Finally, Theorem C.8 gives point v), the space $\mathscr{G}$ appearing therein is identified in Lemma 4.3. $\square$

5.2. **The inequality.** Here we establish the extended Mourre estimate away from the threshold. We use only **(I0)** and **(I1a)** and do not assume any Fermi golden rule assumption.

Proposition 5.2. *Let $\mathcal{I}_0$ be a compact interval such that $\sigma(K)\cap\mathcal{I}_0=\emptyset$. Let $\mathcal{I}$ be an open interval included in $\mathcal{I}_0$. Let $M_\infty := N + 1\otimes P_\Omega \geq 1$ and let $S_\infty := -1\otimes P_\Omega - \lambda\phi(ia_\infty\alpha)$. For λ small enough, we get*

$$M_\infty + E_{\mathcal{I}}(H_\lambda)S_\infty E_{\mathcal{I}}(H_\lambda) \geq \big(1+O(\lambda)\big)E_{\mathcal{I}}(H_\lambda), \tag{5.3}$$

holds true in the sense of forms on $\mathcal{D}(N^{1/2})$.

Proof. Let $\mathcal{J}$ be a compact set containing $\mathcal{I}$ and contained in the interior of $\mathcal{I}_0$. Note that (1.4) gives $E_{\mathcal{J}}(H_0)1_{\mathscr{K}}\otimes P_\Omega=0$. By Proposition 2.2, we derive:

$$E_{\mathcal{J}}(H_0)S_\infty E_{\mathcal{J}}(H_0) = \lambda E_{\mathcal{J}}(H_0)\phi(ia_\infty\alpha)E_{\mathcal{J}}(H_0) = O(\lambda)E_{\mathcal{J}}(H_0). \tag{5.4}$$

As $M_\infty\geq 1$, it remains to prove that $E_{\mathcal{I}}(H_\lambda)S_\infty E_{\mathcal{I}}(H_\lambda)=O(\lambda)E_{\mathcal{I}}(H_\lambda)$. We insert $E_{\mathcal{J}}(H_0)+E_{\mathcal{J}^c}(H_0)$ on the right and on the left of S_∞. By (5.4), all the four terms are actually $O(\lambda)E_{\mathcal{I}}(H_\lambda)$. Indeed, Proposition 2.2 gives for instance that

$$E_{\mathcal{I}}(H_\lambda)E^c_{\mathcal{J}}(H_0)S_\infty E_{\mathcal{J}}(H_0)E_{\mathcal{I}}(H_\lambda) = O(\lambda)E_{\mathcal{I}}(H_\lambda).$$

For the right hand side, take $h\in\mathcal{C}_c^\infty(\mathcal{J})$ so that $h|_{\mathcal{I}}=1$. We have

$$E_{\mathcal{I}}(H_\lambda)E_{\mathcal{J}^c}(H_0) = E_{\mathcal{I}}(H_\lambda)\big(h(H_\lambda)-h(H_0)\big)E_{\mathcal{J}^c}(H_0) = O(\lambda),$$

by Lemma 5.4. □

Remark 5.3. This proof would not work over one of thresholds $\{k_i\}_{i=0,\dots,n}$. Here, we use in a drastic way that $E_{\mathcal{J}}(H_0)1\otimes P_\Omega=0$. However, when $\sigma(K)\cap\mathcal{I}=\{k_i\}$, this expression is never 0 and is of norm 1. A brutal estimation would give

$$M + E_{\mathcal{I}}(H_\lambda)S_\infty E_{\mathcal{I}}(H_\lambda) \geq O(\lambda)E_{\mathcal{I}}(H_\lambda). \tag{5.5}$$

We have no control on the sign. This is no surprise as we know that one may uncouple the two parts of the system and an eigenvalue can remain, see Section 6. To control the sign, one needs to gain some positivity just above $P_{k_i}\otimes P_\Omega$. This would be the rôle of the Fermi golden rule and of the operator B_ε.

Here we have used the elementary:

Lemma 5.4. *Let $h\in\mathcal{C}_c^\infty(\mathbb{R})$ and $s\leq 1/2$. Let V be symmetric operator being H_0-form bounded operator, with constant lower than 1. Then, there is C such that*

$$\big\|\langle H_0\rangle^s\big(h(H_0)-h(H_0+\lambda V)\big)\big\| \leq C|\lambda|.$$

6. A Mourre estimate at the thresholds

In this section we would like to study the absence of eigenvalue above one of the thresholds. From a physical point of view, as soon as the interaction is on, one expects the embedded eigenvalues to disappear into the complex plane and to turn into resonances. This is however not mathematically true as one may uncouple the Bosonic Field and the atom. Take for instance ω bounded, $\alpha\in\mathcal{B}(\mathscr{K},\mathscr{K}\otimes\mathfrak{h})$, given by $\alpha(x):=1\otimes b$, for all $x\in\mathscr{K}$ and where $\omega b\in\mathfrak{h}$. After a dressing transformation, see for instance [D][Theorem 3.5], the operator H_λ is unitarily equivalent to the free operator $K\otimes 1_{\Gamma(\mathfrak{h})}+1_{\mathscr{K}}\otimes \mathrm{d}\Gamma(\tilde\omega_\lambda)$, for some $\tilde\omega_\lambda\in\mathcal{B}(\mathfrak{h})$. Therefore, H_λ has the same eigenvalues as H_0 for all λ. Note that this is no restriction to suppose that ω is bounded thanks to the exponential law, see for instance [BSZ][Section 3.2]. We couple the two systems through a Fermi golden rule assumption.

6.1. **The Fermi golden rule hypothesis.** We choose one eigenvalue k_{i_0} of H_{el} for $i_0 > 0$. Let $P := P_{k_{i_0}} \otimes P_\Omega$ and let $\overline{P} := 1 - P$. Note that P is of finite rank. We give an implicit hypothesis on α and explain how to check it in Appendix A.

Definition 6.1. We say that the *Fermi golden rule* holds true at energy k for a couple (H_0, α) if there exist positive ε_0, c_1 and c_2 such that

$$c_1 P \geq P\phi(\alpha)\overline{P}\,\mathrm{Im}(H_0 - k + i\varepsilon)^{-1}\overline{P}\phi(\alpha)P \geq c_2 P, \tag{6.1}$$

holds true in the sense of forms, for all $\varepsilon_0 > \varepsilon > 0$.

Due to the Fock space structure, one may omit $\overline{P}$ in (6.1) but we keep it to emphasize the link between hypotheses of this type in other fields (like for Schrödinger operators). Since P is of finite rank, this property follows from (1.2).

The upper and the lower bounds of (6.1) would be crucial in our analysis. We shall keep track of the lower bound in the sequel so as to emphasis the gain of positivity it occurs. We set few notations.

$$R_\varepsilon := \big((H_0 - k_{i_0})^2 + \varepsilon^2\big)^{-1/2},\ \overline{R_\varepsilon} := \overline{P} R_\varepsilon \text{ and } F_\varepsilon := \overline{R_\varepsilon}^2. \tag{6.2}$$

Note that $\varepsilon R_\varepsilon^2 = \mathrm{Im}(H_0 - k_{i_0} + i\varepsilon)^{-1}$ and that R_ε commutes with P. We get:

$$(c_1/\varepsilon)P \geq P\phi(\alpha)F_\varepsilon\phi(\alpha)P \geq (c_2/\varepsilon)P, \tag{6.3}$$

for $\varepsilon_0 > \varepsilon > 0$. It follows:

$$\|R_\varepsilon\| = 1/\varepsilon \text{ and } \|P\phi(\alpha)\overline{R}_\varepsilon\| \leq c_1^{1/2}\varepsilon^{-1/2}. \tag{6.4}$$

As pointed out in Remark 5.3, we seek some more positivity for the commutator above the energy $P = P_{k_i} \otimes P_\Omega$. We proceed like in [BFSS] and set

$$B_\varepsilon := \mathrm{Im}(\overline{R_\varepsilon}^2\phi(\alpha)P).$$

It is a finite rank operator, see Lemma 4.5 for more properties. Observe now that we gain some positivity as soon as $\lambda \neq 0$:

$$P[H_\lambda, i\lambda B_\varepsilon]P = \lambda^2 P\phi(\alpha)F_\varepsilon\phi(\alpha)P \geq (c_2\lambda^2/\varepsilon)P. \tag{6.5}$$

It is therefore natural to modify our conjugate operator. We set

$$\hat{A}_n := A_n + \lambda\theta B_\varepsilon,\ \text{ for } n \in \mathbb{N}^* \cup \{\infty\}. \tag{6.6}$$

It depends on the two parameters $\lambda \in \mathbb{R}$, $\varepsilon > 0$ and on an extra technical $\theta > 0$. For the sake of clarity, we do not write these extra dependences. Heuristically, the operator A_∞ would give the positivity around the threshold and the B_ε would complete it just above. We mention that $\hat{A}_\infty$ is maximal symmetric and generates a semigroup of isometries, see Lemma 3.2.

6.2. **Main result.** We prove the extended Mourre estimate over the threshold k_{i_0}. This is the heart of the paper. The proof relies on the Feshbach method. We exploit the freedom we have so far on ε and θ: set $\varepsilon := \varepsilon(\lambda)$ and $\theta =: \theta(\lambda)$ and suppose that $\lambda = o(\varepsilon)$, $\varepsilon = o(\theta)$ and $\theta = o(1)$ as λ tends to 0. We summarize this into:

$$|\lambda| \ \ll\ \varepsilon \ \ll\ \theta \ \ll\ 1,\ \text{as } \lambda \text{ tends to } 0. \tag{6.7}$$

In [BFSS], this condition is more involved and the size of the interval comes into the play. We stress that the conjugate operator $\hat{A}_\infty$ depends on these three parameters.

Theorem 6.2. *Let $\mathcal{I}_0$ be a compact interval containing k_{i_0} and no other k_i. Assume the Fermi golden rule hypothesis* (6.1) *and* (6.7) *hold true. Suppose also that* **(I0)** *and* **(I1a)** *are satisfied. Then, for all open interval $\mathcal{I} \subset \mathcal{I}_0$:*

i) *There are $M_\infty \geq 1$ and $\hat{S}_\infty$ a $|H_\lambda|^{1/2}$-bounded operator such that $[H_\lambda, i\hat{A}_\infty] = M_\infty + \hat{S}_\infty$ holds in the sense of forms on $\mathcal{D}(N^{1/2})$.*

ii) *There is $\lambda_0 > 0$ such that the following extended Mourre estimate*

$$M_\infty + \hat{S}_\infty \geq a(\lambda)E_{\mathcal{I}}(H_\lambda) - b(\lambda)E_{\mathcal{I}^c}(H_\lambda)\langle H_\lambda\rangle \tag{6.8}$$

holds true in the sense of forms on $\mathcal{D}(N^{1/2})$, for all $\lambda \in (0, \lambda_0)$. Here, one has $a(\lambda) = \lambda^2\theta c_2/5\varepsilon$ and $b(\lambda) > 0$.

iii) *If* **(I1b)** *holds true, then H_λ has no eigenvalue in $\mathcal{I}$.*

iv) *If* **(I2)** *holds true (and not necessarily* **(I1b)***), then H_λ has no eigenvalue in the interior of $\mathcal{I}$, for all $|\lambda| \leq \lambda_0$. Moreover, one obtains the estimation of the resolvents given in Theorem 1.2.*

Remark 6.3. By taking θ and ε as power of λ, one may take $a(\lambda) = \lambda^{1+\eta}/5$, for some $\eta > 0$. We do not reach the power 1 as expected in Remark 5.3. This is due to the non-linearity in λ of the conjugate operator. Note also, this is very small and then one does not expect a fast propagation of the state, i.e. the eigenvalue turns into a resonance.

The proof of this theorem needs few steps and is given in Section 6.4. We first go into the Feshbach method and deal with unperturbed spectral measure in Proposition 6.5. Next, in Proposition 6.8, we change the spectral measure.

6.3. **The infrared decomposition.** As suggested by (6.5), one expects to have to slip the space with the projector P to take advantage of this positivity. To do so, we use the Feshbach method. As our result is local in energy, we fix a compact interval $\mathcal{J}$ which contains the selected eigenvalue k_{i_0} and no others. We consider the Hilbert space $\mathscr{H}_{\mathcal{J}} := E_{\mathcal{J}}(H_0)\mathscr{H}$. Let $\mathscr{H}_{\mathcal{J}}^{\mathrm{v}} := P\mathscr{H}_{\mathcal{J}}$ and $\mathscr{H}_{\mathcal{J}}^{\overline{\mathrm{v}}}$ its orthogonal in $\mathscr{H}_{\mathcal{J}}$. The v subscript stands for vacuum. Given H bounded in $\mathscr{H}_{\mathcal{J}} = \mathscr{H}_{\mathcal{J}}^{\overline{\mathrm{v}}} \oplus \mathscr{H}_{\mathcal{J}}^{\mathrm{v}}$, we write it following this decomposition in a matricial way:

$$H = \begin{pmatrix} H^{\overline{\mathrm{v}}\overline{\mathrm{v}}} & H^{\overline{\mathrm{v}}\mathrm{v}} \\ H^{\mathrm{v}\overline{\mathrm{v}}} & H^{\mathrm{v}\mathrm{v}} \end{pmatrix}. \tag{6.9}$$

We recall the Feshbach method, see [BFS] and see also [DJ][Section 3.2] for more results of this kind.

Proposition 6.4. *Assume that $z \notin \sigma(H^{\overline{\mathrm{v}}\overline{\mathrm{v}}})$. We define*

$$G_{\mathrm{v}}(z) \quad := \quad z1^{\mathrm{v}\mathrm{v}} - H^{\mathrm{v}\mathrm{v}} - H^{\mathrm{v}\overline{\mathrm{v}}}\big(z1^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - H^{\overline{\mathrm{v}}\overline{\mathrm{v}}}\big)^{-1}H^{\overline{\mathrm{v}}\mathrm{v}}.$$

Then, $z \in \sigma(H)$ if and only if $0 \in \sigma\big(G_{\mathrm{v}}(z)\big)$.

The reader should keep in mind that $\mathcal{J}$ would be chosen slightly bigger than the interval $\mathcal{I}$. This lost comes from the change of spectral measure from H_0 to H_λ. The aim of the section is to show the following proposition about $\hat{S}_\infty$, see (4.7).

Proposition 6.5. *Let $\mathcal{J}$ be a compact interval containing k and no other k_i. Suppose the Fermi golden rule* (6.1) *and* (6.7)*, then one has*

$$E_{\mathcal{J}}(H_0)\hat{S}_\infty E_{\mathcal{J}}(H_0) \geq (c_2\lambda^2\theta\varepsilon^{-1}/3 - 1)E_{\mathcal{J}}(H_0) \tag{6.10}$$

holds true in the sense of forms, for λ small enough.

We go through a series of lemmata and give the proof at the end of the section. The -1 of the r.h.s. seems at first sight disturbing as we seek for some positivity. It would be balanced when we will add the operator $M_\infty \geq 1$, see Section 6.4. In the first place, we estimate the parts of $\hat{S}_\infty$.

Lemma 6.6. *With respect to the decomposition* (6.9), *as* λ *goes to* 0, *we have*

$$E_{\mathcal{J}}(H_0)\big(\lambda\phi(a_\infty\alpha) - P\big)E_{\mathcal{J}}(H_0) = \begin{pmatrix} O(\lambda) & O(\lambda) \\ O(\lambda) & -1 \end{pmatrix}.$$

Proof. The part in P follows directly from (1.4). The one in α results from Proposition 2.2 and the fact that $P\phi(a_\infty\alpha)P = 0$. $\square$

Lemma 6.7. *Suppose that the Fermi golden rule* (6.1) *holds true. Then, the form* $[H_\lambda, B_\varepsilon]$ *defined on* $\mathcal{D}(H_\lambda)\times\mathcal{D}(H_\lambda)$ *extends to a finite rank operator on* $\mathscr{H}$, *still denoted by* $[H_\lambda, B_\varepsilon]$. *As* λ *tends to* 0, *we have*

$$\big\| [H_\lambda, \lambda\theta B_\varepsilon] \big\|_{\mathcal{B}(\mathscr{H})} \leq O(\lambda\theta\varepsilon^{-1/2}) + O(\lambda^2\theta\varepsilon^{-3/2}). \tag{6.11}$$

Besides, with respect to the decomposition (6.9), *we have:*

$$E_{\mathcal{J}}(H_0)[H_0, \lambda\theta B_\varepsilon]E_{\mathcal{J}}(H_0) = \begin{pmatrix} 0 & O(\lambda\theta\varepsilon^{-1/2}) \\ O(\lambda\theta\varepsilon^{-1/2}) & 0 \end{pmatrix}$$

and

$$E_{\mathcal{J}}(H_0)[\lambda\phi(\alpha), \lambda\theta B_\varepsilon]E_{\mathcal{J}}(H_0) = \begin{pmatrix} O(\lambda^2\theta\varepsilon^{-3/2}) & O(\lambda^2\theta\varepsilon^{-3/2}) \\ O(\lambda^2\theta\varepsilon^{-3/2}) & \lambda^2\theta F_\varepsilon \end{pmatrix}.$$

Proof. We give some estimates independent of $\mathcal{J}$. We expand the commutators, this could be justified by considering the commutator in the form sense on $\mathcal{D}(H_\lambda)$.

$$\begin{aligned} &[\mathrm{d}\Gamma(\omega), \overline{R_\varepsilon}^2\phi(\alpha)P] = [H_0 - k_{i_0}, \overline{R_\varepsilon}^2\phi(\alpha)P] \\ &= \overline{P}(H_0 - k_{i_0})R_\varepsilon\overline{R_\varepsilon}\phi(\alpha)P + \overline{P}R_\varepsilon\overline{R_\varepsilon}\phi(\alpha)P(H_0 - k_{i_0}) = \overline{P}O(\varepsilon^{-1/2})P + 0. \end{aligned} \tag{6.12}$$

Indeed, the first term derives from (6.4) and $\|(H_0 - k_{i_0})R_\varepsilon\| = O(1)$. For the second one, note that $(H_0 - k_{i_0})P = 0$.

We turn to the second estimation and apply Proposition 2.2. We get $\phi(\alpha)R_\varepsilon = \phi(\alpha)R_1R_1^{-1}R_\varepsilon = O(\varepsilon^{-1})$. By (6.4), we have

$$\begin{aligned} [\phi(\alpha), \overline{R_\varepsilon}^2\phi(\alpha)P] &= PF_\varepsilon P + \overline{P}\phi(\alpha)R_\varepsilon\overline{R_\varepsilon}\phi(\alpha)P + \overline{P}R_\varepsilon\overline{R_\varepsilon}\phi(\alpha)P\phi(\alpha)(P + \overline{P}) \\ &= PF_\varepsilon P + \overline{P}O(\varepsilon^{-3/2})P + \overline{P}O(\varepsilon^{-3/2})\overline{P}. \end{aligned} \tag{6.13}$$

Gathering lines (6.12) and (6.13), we get (6.11). We finish by adding $E_{\mathcal{J}}(H_0)$. $\square$

We go into the Feshbach method and conclude.

Proof of Proposition 6.5. We set $C_\lambda := E_{\mathcal{J}}(H_0)\hat{S}_\infty E_{\mathcal{J}}(H_0)$. First observe that for all $\mu \leq -3/4$, we get $C_\lambda^{\mathrm{vv}} - \mu$ is invertible in $\mathcal{B}(\mathscr{H}^{\mathrm{vv}})$ and $\|(C_\lambda^{\mathrm{vv}} - \mu)^{-1}\|_{\mathcal{B}(\mathscr{H}^{\mathrm{vv}})} \leq 2$. Indeed, from Lemma 6.6 and 6.7, we have that C_λ^{vv} is bounded from below by $O(\lambda^2\theta\varepsilon^{-3/2}) + O(\lambda)$. This is bigger than $-1/2$ by (6.7), for λ small enough.

We now estimate from below the internal energy of C_λ, uniformly in $\mu \leq 3/4$. By Lemmata 6.6 and 6.7, the first part and the Fermi golden Rule (6.3), we infer

$$\begin{aligned} & C_\lambda^{\mathrm{vv}} - C_\lambda^{\mathrm{v}\overline{\mathrm{v}}}(C_\lambda^{\overline{\mathrm{v}}\overline{\mathrm{v}}} - \mu)^{-1} C_\lambda^{\overline{\mathrm{v}}\mathrm{v}} + 1 \geq \\ & \geq c_2\lambda^2\theta\varepsilon^{-1} + \big(O(\lambda\theta\varepsilon^{-1/2}) + O(\lambda^2\theta\varepsilon^{-3/2}) + O(\lambda)\big)^2 \\ & = c_2\lambda^2\theta\varepsilon^{-1}\big(O(\theta) + O(\lambda\theta\varepsilon^{-1}) + O(\varepsilon^{1/2}) + O(\lambda^2\theta\varepsilon^{-2}) + O(\lambda\varepsilon^{-1/2}) + O(\theta^{-1}\varepsilon)\big) \\ & \geq c_2\lambda^2\varepsilon^{-1}/2, \text{ for } \lambda \text{ small enough.} \end{aligned}$$

We have used (6.7) for the last line.

We are now able to conclude. Since $\mathcal{J}$ contains k_{i_0} and no other k_i. We have $E_{\mathcal{J}}(H_0)P_\Omega = P$ by (1.4). Let $\mu < c_2\lambda^2\theta\varepsilon^{-1}/2 - 1$. Note that $\mu \leq -3/4$ for λ small enough by (6.7). Thanks to the previous lower bound, we can apply Proposition 6.4 with respect to the decomposition (6.9) for C_λ and with $z = \mu$ to get the result. $\square$

6.4. **The extended Mourre estimate.** At the end of the section, we establish the extended Mourre estimate. We start by enhancing Proposition 6.5.

Proposition 6.8. *Let $\mathcal{I}$ be a compact interval containing k_{i_0} and not other k_i. Assume the Fermi golden rule* (6.1) *and* (6.7)*. Then,*

$$E_{\mathcal{I}}(H_\lambda)\hat{S}_\infty E_{\mathcal{I}}(H_\lambda) \geq c_2(\lambda^2\theta\varepsilon^{-1}/4 - 1)E_{\mathcal{I}}(H_\lambda)$$

holds true in the sense of forms for λ small enough.

Proof. Let $\mathcal{J}$ be a compact interval as in Proposition 6.5 such that $\mathcal{I}$ is included in its interior and contains k_{i_0}. By (6.7), it is enough to prove

$$\begin{aligned} & E_{\mathcal{I}}(H_\lambda)\big(\lambda\phi(a_\infty\alpha) + [H_\lambda, i\lambda\theta B_\varepsilon] - P_\Omega\big)E_{\mathcal{I}}(H_\lambda) \quad \geq \\ & \qquad \big(c_2\lambda^2\theta\varepsilon^{-1}/3 + O(\lambda^2) + O(\lambda^2\theta\varepsilon^{-1/2}) + O(\lambda^3\theta\varepsilon^{-3/2}) - 1\big)E_{\mathcal{I}}(H_\lambda). \end{aligned} \tag{6.14}$$

We start with the left hand side of (6.14) and introduce $E_{\mathcal{J}}(H_0) + E_{\mathcal{J}^c}(H_0)$ on the right and on the left of $([H_\lambda, i\lambda\theta B_\varepsilon] + \lambda\phi(a_\infty\alpha) - P_\Omega)$. Note that both of spectral measures are bounded in $\mathcal{D}(H_0)$, endowed with the graph norm. We need to control the mixed term. Using Lemma 5.4 and (6.11), we get

$$\begin{aligned} & E_{\mathcal{I}}(H_\lambda)E_{\mathcal{J}^c}(H_0)[H_\lambda, i\lambda\theta B_\varepsilon]E_{\mathcal{J}}(H_0)E_{\mathcal{I}}(H_\lambda) = \\ & \qquad\qquad \big(O(\lambda^2\theta\varepsilon^{-1/2}) + O(\lambda^3\theta\varepsilon^{-3/2})\big)E_{\mathcal{I}}(H_\lambda), \end{aligned}$$

and a better term for $E_{\mathcal{I}}(H_\lambda)E_{\mathcal{J}^c}(H_0)[H_\lambda, i\lambda\theta B_\varepsilon]E_{\mathcal{J}^c}(H_0)E_{\mathcal{I}}(H_\lambda)$. Since the term $\phi(a_\infty\alpha)\langle H_0\rangle^{-1/2}$ is bounded in $\mathscr{H}$ by Proposition 2.2, Lemma 5.4 gives

$$E_{\mathcal{I}}(H_\lambda)E_{\mathcal{J}^c}(H_0)\lambda\phi(a_\infty\alpha)E_{\mathcal{J}}(H_0)E_{\mathcal{I}}(H_\lambda) = O(\lambda^2)E_{\mathcal{I}}(H_\lambda),$$

and a better term for the full-mixed term. As H_0 commute with P_Ω, we infer $E_{\mathcal{I}}(H_\lambda)E_{\mathcal{J}^c}(H_0)P_\Omega E_{\mathcal{J}}(H_0)E_{\mathcal{I}}(H_\lambda) = 0$. Now using Proposition (6.5) we obtain

$$\begin{aligned} & E_{\mathcal{I}}(H_\lambda)\big([H_\lambda, i\lambda\theta B_\varepsilon] + \lambda\phi(a_\infty\alpha) - P_\Omega\big)E_{\mathcal{I}}(H_\lambda) \quad \geq \\ & \qquad (c_2\lambda^2\theta\varepsilon^{-1}/3 - 1)E_{\mathcal{I}}(H_\lambda)E_{\mathcal{J}}(H_0)E_{\mathcal{I}}(H_\lambda) \\ & \qquad + \big(O(\lambda^2) + O(\lambda^2\theta\varepsilon^{-1/2}) + O(\lambda^3\theta\varepsilon^{-3/2})\big)E_{\mathcal{I}}(H_\lambda). \end{aligned}$$

Finally, the estimation (6.14) follows by noticing that $E_{\mathcal{I}}(H_\lambda)E_{\mathcal{J}}(H_0)E_{\mathcal{I}}(H_\lambda)$ is equal to $\big(1 + O(\lambda^2)\big)E_{\mathcal{I}}(H_\lambda)$, again by Lemma 5.4. $\square$

We are now able to prove the announced result.

Proof of Theorem 6.2. The operator M_∞ and $\hat{S}_\infty$ are given in (4.4) and (4.7). Points i) and ii) are given in Section 4.3. By Proposition 6.8 and since $M_\infty \geq 1$,

$$M_\infty + E_{\mathcal{I}}(H_\lambda)\hat{S}_\infty E_{\mathcal{I}}(H_\lambda) \geq c_2\lambda^2\theta\varepsilon^{-1}/4\, E_{\mathcal{I}}(H_\lambda)$$

holds true in the form sense on $\mathcal{D}(N^{1/2})$. Then, (5.2) gives iii). The point iv) follows from the Virial Theorem, Proposition 4.11. Finally Theorem C.8 gives point v). Indeed, the space $\mathscr{G}$ appearing therein is identified in Lemma 4.3. In remains to notice that the spaces (C.4) given for $\hat{A}_\infty$ and A_∞ are the same. This follows from the fact that these operators have the same domain in $\mathscr{G}^*$, by Lemma 4.5 and that the spaces $\mathscr{G}_s^*$ are given by complex interpolation. □

Appendix A. Level shift operator

In this paper, we never make the hypothesis that we analyse an eigenvalue which could be different than the ground state energy of H_0. The point is that it is well known that it is supposed to remain, even if the perturbation is switched on, see for instance [AH, BFS, G0]. This leads to a contradiction to the hypothesis made on the Fermi golden rule. Therefore, in this section, we explain how one may check the Fermi golden rule assumption (6.1), why it is not fulfilled at ground state energy. This would also explain the compatibility with **(I0)**–**(I2)**. The computations we lead are standard, we keep it simple. See also [BFS, DJ2, JP].

Let e_i be an orthonormal basis of eigenvectors of K relative to the eigenvalue k_i. To simplify the computation, say that k_{i_0} is simple. Since k_{i_0} is simple and since $\phi(\alpha)(e_{i_0}\otimes\Omega) = \alpha(e_{i_0}) \in \mathscr{K}\otimes\mathfrak{h}$, (6.1) is equivalent to:

$$c_1 \geq \left\langle \alpha(e_{i_0}), \mathrm{Im}(H_0 - k_{i_0} + i\varepsilon)^{-1}\alpha(e_{i_0})\right\rangle \geq c_2 > 0, \text{ for } 0 < \varepsilon \leq \varepsilon_0.$$

We have $\alpha(e_{i_0}) = \sum_{i=1,\dots,n} e_i \otimes f_{i,i_0} \in \mathscr{K}\otimes\mathfrak{h}$, where $f_{i,i_0} = \left\langle e_i \otimes 1_{\mathfrak{h}}, \alpha(e_{i_0})\right\rangle$. As $\mathfrak{h} = L^2(\mathbb{R}^d, dk)$, we write f_{i,i_0} as a function of k. We go into polar coordinates, see (1.3) and infer

$$c_1 \geq \sum_{i=1,\dots,n} \int_0^\infty \int_{S^{d-1}} \varepsilon \frac{|f_{i,i_0}|^2(r\theta)r^{d-1}}{(r+\lambda_i-\lambda_{i_0})^2+\varepsilon^2}\, d\sigma\, dr \geq c_2 > 0$$

Suppose now that $(r,\theta) \mapsto |f_{i,i_0}|^2(r\theta)r^{d-1}$ is continuous and in L^1. Then by dominated convergence, we let ε go to zero and get:

$$c_1 \geq \sum_{i=1,\dots,i_0} c_i(\lambda_{i_0}-\lambda_i)^{d-1} \int_{S^{d-1}} \varepsilon |f_{i,i_0}|^2\big(\theta(\lambda_{i_0}-\lambda_i)\big)\, d\sigma \geq c_2 > 0 \tag{A.1}$$

Here note that, up to the constant c_i, $r \mapsto \varepsilon/\big((r+\lambda_i-\lambda_{i_0})^2+\varepsilon^2\big)$ is a Dirac sequence if and only if $\lambda_i \leq \lambda_{i_0}$.

To satisfy the Fermi golden rule, it is enough to have a non-zero term in (A.1). When $d \geq 2$, we stress that the sum is taken till $i_0 - 1$ and therefore is empty at ground state energy. When the 1-particle space is over $\mathbb{R}$, it cannot be satisfied at this level of energy as well. Indeed, one would obtain a contradiction with the hypothesis **(I0)** and the continuity of $(r,\theta) \mapsto |f_{i,i_0}|^2(r\theta)$.

Appendix B. Properties of C_0-semigroups

In this section, we gather various facts about C_0-semigroups we use along this article. Let $\mathscr{H}$ be a Hilbert space. Recall that w$-$lim denotes the weak limit.

Definition B.1. We say $\mathbb{R}^+ \ni t \mapsto W_t$, with $W_t \in \mathcal{B}(\mathscr{H})$ is a C_0-*semigroup* if

(1) $W_0 = \mathrm{Id}$ and $W_{s+t} = W_s W_t$, for all $s, t \geq 0$,
(2) $\mathrm{w-}\lim_{t\to 0^+} W_t = \mathrm{Id}$.

Automatically, this implies that $\mathbb{R}^+ \supset t \mapsto W_t$ is strongly continuous, see [HP][Theorem 10.6.5]. We keep the convention of [GGM2] and define the generator of $\{W_t\}_{t\geq 0}$ as being the closed densely defined operator A defined on

$$\mathcal{D}(A) := \{u \in \mathscr{H} \mid \lim_{t\to 0^+} (it)^{-1}(W_t - \mathrm{Id})u \text{ exists}\}.$$

We set Au this limit. Formally, one reads $W_t = e^{itA}$. The map $\mathbb{R}^+ \supset t \mapsto W_t^*$ being weakly continuous, $\{W_t^*\}_{t\geq 0}$ is also a C_0-semigroup. Its generator is $-A^*$. We recall the *Nelson Lemma*, see for instance [BR][Corollary 3.1.7].

Lemma B.2 (Nelson Lemma)**.** *Let $\mathcal{D}$ be a dense subset of $\mathscr{H}$ and let $\{W_t\}_{t\geq 0}$ be a C_0-semigroup. If $W_t\mathcal{D} \subset \mathcal{D}$ then $\mathcal{D}$ is a core for the generator of $\{W_t\}_{t\geq 0}$.*

Let $\mathscr{G}$ and $\mathscr{H}$ be two Hilbert spaces such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely. Using the Riesz isomorphism, we identify $\mathscr{H}$ with $\mathscr{H}^*$, where the latter is the set of anti-linear forms acting on $\mathscr{H}$. We infer the following scale of spaces $\mathscr{G} \subset \mathscr{H} \simeq \mathscr{H}^* \subset \mathscr{G}^*$ with continuous and dense embeddings. In order to define the restriction of W_t on $\mathscr{G}$, we set:

Definition B.3. Given a C_0-semigroup $\{W_t\}_{t\geq 0}$ on $\mathscr{H}$. We say that $\mathscr{G}$ is *b-stable* (boundedly stable) under the action of $\{W_t\}_{t\geq 0}$ if

i) $W_t\mathscr{G} \subset \mathscr{G}$, for all $t \in \mathbb{R}^+$,
ii) $\sup_{t\in[0,1]} \|W_t u\|$ is bounded for all $u \in \mathscr{G}$.

Remark B.4. Note that unlike for C_0-groups, the second condition is required to ensure the continuity in 0. These two conditions are equivalent to the fact that $\{W_t|_{\mathscr{G}}\}_{t\geq 0}$ is a C_0-semigroup on $\mathscr{G}$.

Assuming that $\mathscr{G}$ is b-stable under the action of $\{W_t\}_{t\geq 0}$, we denote by $A_{\mathscr{G}}$ its generator. Thus, $A_{\mathscr{G}}$ is the restriction of A and its domain is given by

$$\mathcal{D}(A_{\mathscr{G}}) = \{u \in \mathscr{G} \cap \mathcal{D}(A) \mid Au \in \mathscr{G}\}.$$

If $\mathscr{G}^*$ is also b-stable under $\{W_t^*\}_{t\geq 0}$, we denote by $A_{\mathscr{G}^*}$ the generator of $\{W_t\}_{t\geq 0}$ extended to $\mathscr{G}^*$. As above A is a restriction of $A_{\mathscr{G}^*}$ and thanks to the Nelson lemma, we have that A is the closure of $A_{\mathscr{G}}$ in $\mathscr{H}$ and that $A_{\mathscr{G}^*}$ is the closure of A in $\mathscr{G}^*$. We would drop the subscript $\mathscr{G}$ when no confusion could arise.

We recall the following result of perturbation, see [K][Theorem IX.2.1].

Proposition B.5. *Let B be a bounded operator in a Hilbert space $\mathscr{H}$. Then A is the generator a C_0-semigroup if and only if $A + B$ is also one.*

Appendix C. The Mourre method

C.1. **The C^1 class.** Given a self-adjoint operator A, the so-called $C^1(A)$ class of regularity is a key notion within the Mourre's theory, see [ABG] and [GG]. This guarantees some properties of domains and that the commutator of an operator H with A would be H-bounded. In this paper, we have to deal with maximal symmetric conjugate operators and thus have to extend the standard class exposed in details in [ABG][Section 6.2]. As some refinements appear, we present an overview of the properties and refer to [GGM][Section 2] for proofs.

Within this section, we consider a closed densely defined operator A acting in a Hilbert space $\mathscr{H}$. Note this implies that $\mathcal{D}(A^*)$ is dense in $\mathscr{H}$. We first defined the class of bounded operators belonging to $\mathcal{C}^1(A)$. Let $S \in \mathcal{B}(\mathscr{H})$. We denote by $[S, A]$ the sesquilinear form defined on $\mathcal{D}(A^*) \times \mathcal{D}(A)$ by

$$\langle u, [S, A]v\rangle := \langle A^* u, Sv\rangle - \langle S^* u, Av\rangle, \text{ for } u \in \mathcal{D}(A^*), v \in \mathcal{D}(A).$$

Definition C.1. An operator $S \in \mathcal{B}(\mathscr{H})$ belongs to $\mathcal{C}^1(A)$ if the sesquilinear form $[S, A]$ is continuous for the topology of $\mathscr{H} \times \mathscr{H}$. We denote by $[S, A]^\circ$ the unique bounded operator in $\mathscr{H}$ extending this form.

We now extend the definition to unbounded operator by asking the resolvent $R(z) := (S - z)^{-1}$ to be $\mathcal{C}^1(A)$. We precise the statement. We first recall that given S a closed densely defined operator on $\mathscr{H}$, the *A-regular resolvent set* of S is the set $\rho(S, A) \subset \mathbb{C} \setminus \sigma(S)$ such that $R(z)$ is of class $\mathcal{C}^1(A)$.

Definition C.2. Let S be a closed and densely defined operator on $\mathscr{H}$. We say that S *is of class* $\mathcal{C}^1(A)$ if there are a constant C and a sequence of complex numbers $z_\nu \in \rho(S, A)$ such that $|z_\nu| \to \infty$ and $\|R(z_\nu)\| \le C|z_\nu|^{-1}$. If S is of class $\mathcal{C}^1(A)$ and $\rho(S, A) = \mathbb{C} \setminus \sigma(S)$ then we say that S is of *full class* $\mathcal{C}^1(A)$.

In many cases these two definitions coincide. Indeed, given $S \in \mathcal{C}^1(A)$, one shows that if A is regular or if S is self-adjoint with a spectral gap then S is in the full class $\mathcal{C}^1(A)$. We recall that a closed densely defined operator B is *regular* if there is a constant C and $\alpha_n \in \mathbb{C} \setminus \sigma(B)$ such that $\|(B - \alpha_n)\| \le C|\alpha_n|^{-1}$ and such that $|\alpha_n| \to \infty$. The generators of C_0-semigroups are regular for instance.

Definition C.3. Let A and S be two closed and densely defined operators in $\mathscr{H}$. We define $[A, S]$ as the sesquilinear form acting on $\big(\mathcal{D}(A^*) \cap \mathcal{D}(S^*)\big) \times \big(\mathcal{D}(A) \cap \mathcal{D}(S)\big)$ and given by $\langle u, [S, A]v\rangle := \langle A^* u, Sv\rangle - \langle S^* u, Av\rangle$.

Proposition C.4. *Let $S \in \mathcal{C}^1(A)$. Then $\mathcal{D}(A^*) \cap \mathcal{D}(S^*)$ and $\mathcal{D}(A) \cap \mathcal{D}(S)$ are cores for S and S^* respectively and the form $[A, S]$ has a unique extension to a continuous sesquilinear form denoted by $[A, S]^\circ$ on $\mathcal{D}(S^*) \cap \mathcal{D}(S)$. Moreover,*

$$[A, R(z)]^\circ = -R(z)[A, S]^\circ R(s), \text{ for all } z \in \rho(S, A),$$

where on the right hand side, $[A, S]^\circ$ is considered as an element of $\mathcal{B}\big(\mathcal{D}(S), \mathcal{D}(S^*)\big)$.

We stress the fact that $[A, S]$ extends to an element of $\mathcal{B}\big(\mathcal{D}(S), \mathcal{D}(S^*)\big)$ is not enough to ensure $S \in \mathcal{C}^1(A)$, see [GG]. Some conditions of compatibilities are to be added, see [GGM][Proposition 2.21]. This could also be bypassed by knowing some invariance under a C_0-semigroup generated by A.

Definition C.5. Let $\{W_{1,t}\}_{t\in\mathbb{R}^+}, \{W_{2,t}\}_{t\in\mathbb{R}^+}$ be two C_0-semigroups on the Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ with generator A_1 and A_2. We say that $B \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ is of *class* $\mathcal{C}^1(A_1, A_2)$ if:

$$\|W_{2,t} S - S W_{1,t}\|_{\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)} \le ct, \, 0 \le t \le 1.$$

If $\mathscr{G} \subset \mathscr{H}$ are two Hilbert spaces continuously and densely embedded and if a C_0-semigroup $\{W_t\}_{t\in\mathbb{R}^+}$, with generator A on $\mathscr{H}$, b-stabilizes $\mathscr{G}$ and $\mathscr{G}^*$, we denote the class $\mathcal{C}^1(A_{\mathscr{G}}, A_{\mathscr{G}^*})$ by $\mathcal{C}^1(A; \mathscr{G}, \mathscr{G}^*)$. We have the following result.

Proposition C.6. $S \in \mathcal{C}^1(A_1, A_2)$ *if and only if the sesquilinear form ${}_2[S, A]_1$ on $\mathcal{D}(A_2^*) \times \mathcal{D}(A_1)$ defined by $\langle u_2, {}_2[S, A]_1 u_1\rangle := \langle S^* u_2, A_1 u_1\rangle - \langle A_2^* u_2, S u_1\rangle$ is bounded* *for the topology of* $\mathscr{H}_2 \times \mathscr{H}_1$. *Let* ${}_2[S, A]_1^\circ$ *be the closure of this form in* $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$. *We have:*

$$
{}_2[S, A]_1^\circ = \text{s-}\lim_{t \to 0^+} (SW_{1,t} - W_{2,t}S).
$$

Note that for $S \in \mathcal{B}(\mathscr{H})$, with $\mathscr{H}_i = \mathscr{H}$ and $W_{i,t} = W_t$, one has $S \in \mathcal{C}^1(A_1, A_2)$ if and only if $B \in \mathcal{C}^1(A)$.

C.2. **Regularity assumptions for the limiting absorption principle.** In this part, we recall a set of assumptions presented in [GGM] so as to ensure a limiting absorption principle, see Theorem C.8. Consider H a self-adjoint operator, H' symmetric closed and densely defined and A closed and densely defined. These operators are linked by $H' = [H, iA]$ in a sense defined lower. Denote also $\mathscr{D} := \mathcal{D}(H) \cap \mathcal{D}(H')$ endowed with the intersection topology, namely the topology associated to the norm $\| \cdot \| + \|H \cdot \| + \|H' \cdot \|$.

We start by some assumptions on H and on H'.

- **(M1)** *H is of full class $\mathcal{C}^1(H')$, $\mathscr{D} = \mathcal{D}(H) \cap \mathcal{D}(H'^*)$ and this is a core for H'.*
- **(M2)** *There are $\mathcal{I} \subset \mathbb{R}$ open and bounded and $a, b > 0$ such that*

$$
H' \geq \big(a 1_{\mathcal{I}}(H) - b 1_{\mathcal{I}^c}(H)\big)\langle H \rangle \tag{C.1}
$$

holds true in the sense of forms on $\mathscr{D}$.

The last one is the *strict Mourre estimate*. In order to check the first hypothesis, we rely on [GGM][Lemma 2.26], see also [S][Lemma 2.6]:

Lemma C.7. *Let H, M be self-adjoint operators such that $H \in \mathcal{C}^1(M)$ and that $\mathcal{D}(H) \cap \mathcal{D}(M)$ is a core of M. Let R be a symmetric operator such that $\mathcal{D}(R) \supset \mathcal{D}(H)$. Set H' the closure of $M + R$ defined on $\mathcal{D}(R) \cap \mathcal{D}(M)$. Then H is of full class $\mathcal{C}^1(H')$ and $\mathcal{D}(H) \cap \mathcal{D}(H')$ is a core for H' and $\mathcal{D}(H) \cap \mathcal{D}(H') = \mathcal{D}(H) \cap \mathcal{D}(H'^*) = \mathcal{D}(H) \cap \mathcal{D}(M)$.*

Assuming **(M2)**, one chooses $c > 0$ such that $H' + c\langle H \rangle \geq \langle H \rangle$ (take for instance $c = b + 1$). Since $H' + c\langle H \rangle$ is symmetric and positive, it possesses a Friedrichs extension $G \geq \langle H \rangle$. We name the form domain of G:

$$
\mathscr{G} := \mathcal{D}(G^{1/2}), \text{ endowed with the graph norm } \| \cdot \|_{\mathscr{G}}. \tag{C.2}
$$

Note that $\mathscr{G}$ is also obtained by completing the space $\mathscr{D}$ with the help of the norm $\|u\|_{\mathscr{G}} = \sqrt{\langle u, (H' + c\langle H \rangle)u \rangle}$. We identify these spaces in Lemma 4.3.

We now recall the dual norm $\| \cdot \|_{\mathscr{G}^*}$ of $\mathscr{G}$. Given $u \in \mathscr{H}$, we set

$$
\|u\|_{\mathscr{G}^*} := \sup_{v \in \mathscr{D}, \, \|v\|_{\mathscr{G}} \leq 1} |\langle u, v \rangle| = \|G^{-1/2}u\|. \tag{C.3}
$$

Using the Riesz isomorphism, we identify $\mathscr{H}$ with $\mathscr{H}^*$ the space of anti-linear forms on $\mathscr{H}$. The space $\mathscr{G}^*$ is given by the completion of $\mathscr{H}$ with respect to the norm $\| \cdot \|_{\mathscr{G}^*}$. We get the following scale space:

$$
\mathscr{D} \subset \mathscr{G} \subset \mathscr{H} \simeq \mathscr{H}^* \subset \mathscr{G}^* \subset \mathscr{D}^*,
$$

with dense and continuous embeddings.

We turn to the assumptions concerning the conjugate operator A and higher commutators. Suppose A to be the generator of $\{W_t\}_{t \in \mathbb{R}^+}$

- **(M3)** *The C_0-semigroup $\{W_t\}_{t \in \mathbb{R}^+}$ is of isometries and b-stabilizes $\mathscr{G}$ and $\mathscr{G}^*$,*
- **(M4)** *$H \in \mathcal{C}^1(A; \mathscr{G}, \mathscr{G}^*)$,*

(M5) $H \in \mathcal{C}^{1,1}(A; \mathscr{G}, \mathscr{G}^*)$.

The hypothesis **(M4)** implies that

$$\lim_{t \to 0^+} \big(\langle u, W_t H u \rangle - \langle H u, W_t u \rangle \big) = \langle u, H' u \rangle, \text{ for all } u \in \mathscr{D}.$$

The hypothesis **(M5)** means that $H \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ and that

$$\int_0^1 \big\| [W_t, [W_t, H]] \big\|_{\mathcal{B}(\mathscr{G}, \mathscr{G}^*)} \frac{dt}{t^2} < \infty.$$

This is equivalent to the fact that H belongs to $\big(\mathcal{C}^2(A; \mathscr{G}, \mathscr{G}^*), \mathcal{B}(\mathscr{G}, \mathscr{G}^*)\big)_{1/2,1}$. We refer to [ABG, T] for real interpolation.

One may also consider the stronger $H' \in \mathcal{C}^1(A; \mathscr{G}, \mathscr{G}^*)$, i.e.

(M5') $H \in \mathcal{C}^2(A; \mathscr{G}, \mathscr{G}^*)$.

We now give the result. Let $A_{\mathscr{G}^*}$ be the generator of $\{W_t\}_{t \in \mathbb{R}^+}$ generator in $\mathscr{G}^*$. For $s \in (0,1)$, we set:

$$\mathscr{G}_s^* := \mathcal{D}\big(|A_{\mathscr{G}^*}|^s\big) \text{ and } \mathscr{G}_{-s} := (\mathscr{G}_s^*)^*. \tag{C.4}$$

Here, the absolute value is taken with respect to the Hilbert structure of $\mathscr{G}^*$. Given $\mathcal{J}$ an interval, we define $J_0^{\pm} := \{\lambda \pm i\mu, \lambda \in \mathcal{J} \text{ and } \mu > 0\}$. Finally, set $R(z) := (H - z)^{-1}$. From [GGM], we obtain:

Theorem C.8. *Assume that* **(M1)**–**(M5)** *hold true. Let $\mathcal{J}$ be a compact interval included in $\mathcal{I}$. Then if $z \in \mathcal{J}_0^{\pm}$, $R(z)$ induces a bounded operator in $\mathcal{B}(\mathscr{G}_s^*, \mathscr{G}_{-s})$, for all $s \in (1/2, 1]$. Moreover the limit $R(\lambda \pm i0) = \lim_{\mu \to \pm 0} R(\lambda + i\mu)$ exists in the norm topology of $\mathcal{B}(\mathscr{G}_s^*, \mathscr{G}_{-s})$, locally uniformly in $\lambda \in \mathcal{J}$ and the maps $\lambda \mapsto R(\lambda \pm i0) \in \mathcal{B}(\mathscr{G}_s^*, \mathscr{G}_{-s})$ are Hölder continuous of order $s - 1/2$.*

This theorem can be improved by considering weights in some Besov spaces related to the conjugate operator. We refer to [GGM] for more details. Note that the theory exposed in [GGM] is formulated with the hypothesis **(M5')** but, as mentioned in [GGM] and proceeding like in [ABG] for instance, the hypothesis **(M5)** is enough to apply the theory.

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