Quasilocal Operators and Stability of the Essential Spectrum

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Abstract

We establish criteria for the stability of the essential spectrum for unbounded operators acting in Banach modules. The Banach module structure allows one to give a meaning to notions like vanishing at infinity or quasilocal operators which covers many situations of practical interest. Our abstract results can be applied to large classes of differential operators of any order with complex measurable coefficients, singular Dirac operators, Laplace-Beltrami operators on Riemannian manifolds with measurable metrics, operators acting on sections of vector fiber bundles over non-smooth manifolds or locally compact abelian groups.

1 Introduction

The main purpose of this paper is to establish criteria which ensure that the difference of the resolvents of two operators is compact. In order to simplify later statements, we use the following definition (our notations are quite standard; we recall however the most important ones at the end of this section).

Definition 1.1 *Let A and B be two closed operators acting in a Banach space $\mathscr{H}$. We say that B* is a compact perturbation of A *if there is $z \in \rho(A) \cap \rho(B)$ such that $(A-z)^{-1} - (B-z)^{-1}$ is a compact operator.*

Under the conditions of this definition the difference $(A-z)^{-1} - (B-z)^{-1}$ is a compact operator for all $z \in \rho(A) \cap \rho(B)$. In particular, *if B is a compact* *perturbation of A, then A and B have the same essential spectrum*, and this for any reasonable definition of the essential spectrum, see [GW]. To be precise, in this paper we define the essential spectrum of A as the set of points $\lambda \in \mathbb{C}$ such that $A - \lambda$ is not Fredholm.

We shall describe now a standard and simple, although quite powerful, method of proving that B is a compact perturbation of A. Note that we are interested in situations where A and B are differential (or pseudo-differential) operators with complex measurable coefficients which differ little on a neighborhood of infinity. An important point in such situations is that one has not much information about the domains of the operators. However, one often knows explicitly a generalized version of the "quadratic form domain" of the operator. Since we want to consider operators of any order (in particular Dirac operators) we shall work in the following framework, which goes beyond the theory of accretive forms.

Let $\mathscr{G}, \mathscr{H}, \mathscr{K}$ be reflexive Banach spaces such that $\mathscr{G} \subset \mathscr{H} \subset \mathscr{K}$ continuously and densely. We are interested in operators in $\mathscr{H}$ constructed according to the following procedure: let A_0, B_0 be continuous bijective maps $\mathscr{G} \to \mathscr{K}$ and let A, B be their restrictions to $A_0^{-1}\mathscr{H}$ and $B_0^{-1}\mathscr{H}$. These are closed densely defined operators in $\mathscr{H}$ and we take $z = 0 \in \rho(A) \cap \rho(B)$. Then in $\mathcal{B}(\mathscr{K}, \mathscr{G})$ we have

$$A_0^{-1} - B_0^{-1} = A_0^{-1}(B_0 - A_0)B_0^{-1}. \tag{1.1}$$

In particular, we get in $\mathcal{B}(\mathscr{H})$

$$A^{-1} - B^{-1} = A_0^{-1}(B_0 - A_0)B^{-1}. \tag{1.2}$$

We get the simplest compactness criterion: if $A_0 - B_0 : \mathscr{G} \to \mathscr{K}$ is compact, then B is a compact perturbation of A. But in this case we have more: the operator $A_0^{-1} - B_0^{-1} : \mathscr{K} \to \mathscr{G}$ is also compact, and this can not happen if A_0, B_0 are differential operators with distinct principal part (cf. below). This also excludes singular lower order perturbations, e.g. Coulomb potentials in the Dirac case.

The advantage of the preceding criterion is that no knowledge of the domains $\mathcal{D}(A), \mathcal{D}(B)$ is needed. To avoid the mentioned disadvantages, one may assume that one of the operators is more regular than the second one, so that the functions in its domain are, at least locally, slightly better than those from $\mathscr{G}$. Note that $\mathcal{D}(B)$ when equipped with the graph topology is such that $\mathcal{D}(B) \subset \mathscr{G}$ continuously and densely and we get a second compactness criterion by asking that $A_0 - B_0 : \mathcal{D}(B) \to \mathscr{K}$ be compact. This time again we get more than needed, because not only B is a compact perturbation of A, but also $A_0^{-1} - B_0^{-1} : \mathscr{H} \to \mathscr{G}$ is compact. However, perturbations of the principal part of a differential operator are allowed and also much more singular perturbations of the lower order terms, cf. [N1] for the Dirac case.

In this paper we are interested in situations where we have really no information concerning the domains of A and B (besides the fact that they are subspaces of $\mathscr{G}$). The case when A, B are second order elliptic operators with measurable complex coefficients acting in $\mathscr{H} = L^2(\mathbb{R}^n)$ has been studied by Ouhabaz and Stollmann in [OS] and, as far as we know, this is the only paper where the "unperturbed" operator is not smooth. Their approach consists in proving that the difference $A^{-k} - B^{-k}$ is compact for some $k \geq 2$ (which implies the compactness of $A^{-1} - B^{-1}$). In order to prove this, they take advantage of the fact that $\mathcal{D}(A^k)$ is a subset of the Sobolev space $W^{1,p}$ for some $p > 2$, which means that we have a certain gain of local regularity. Of course, L^p techniques from the theory of partial differential equations are required for their methods to work.

We shall explain now in the most elementary situation the main ideas of our approach to these questions. Let $\mathscr{H} = L^2(\mathbb{R})$ and $P = -i\frac{d}{dx}$. We consider operators of the form $A_0 = PaP + V$ and $B_0 = PbP + W$ where a, b are bounded operators on $\mathscr{H}$ such that Re a and Re b are bounded below by strictly positive numbers. V and W are assumed to be continuous operators $\mathscr{H}^1 \to \mathscr{H}^{-1}$, where $\mathscr{H}^s$ are Sobolev spaces associated to $\mathscr{H}$. Then $A_0, B_0 \in \mathcal{B}(\mathscr{H}^1, \mathscr{H}^{-1})$ and we put some conditions on V, W which ensure that A_0, B_0 are invertible (e.g. we could include the constant z in them). Thus we are in the preceding abstract framework with $\mathscr{G} = \mathscr{H}^1$ and $\mathscr{K} = \mathscr{H}^{-1} \equiv \mathscr{G}^*$. Then from (1.2) we get

$$A^{-1} - B^{-1} = A_0^{-1}P(b-a)PB^{-1} + A_0^{-1}(W - V)B^{-1}. \tag{1.3}$$

Let R be the first term on the right hand side and let us see how we could prove that it is a compact operator on $\mathscr{H}$. Note that the second term should be easier to treat since we expect V and W to be operators of order less than 2.

We have $R\mathscr{H} \subset \mathscr{H}^1$, so we can write $R = \psi(P)R_1$ for some $\psi \in B_0(\mathbb{R})$ (bounded Borel function which tends to zero at infinity) and $R_1 \in \mathcal{B}(\mathscr{H})$. This is just half of the conditions needed for compactness, in fact R will be compact if and only if one can also find $\varphi \in B_0(\mathbb{R})$ and $R_2 \in \mathcal{B}(\mathscr{H})$ such that $R = \varphi(Q)R_2$ (the notations are standard, see the paragraph after Proposition 2.23 if needed). Of course, the only factor which can help to get such a decay is $b-a$. So let us suppose that we can write $b-a = \xi(Q)U$ for some $\xi \in B_0(\mathbb{R})$ and a bounded operator U on $\mathscr{H}$. We denote $S = A_0^{-1}P$ and note that this is a bounded operator on $\mathscr{H}$, because $P : \mathscr{H} \to \mathscr{H}^{-1}$ and $A_0^{-1} : \mathscr{H}^{-1} \to \mathscr{H}^1$ are bounded. Then $R = S\xi(Q)UPB^{-1}$ and $UPB^{-1} \in \mathcal{B}(\mathscr{H})$, hence R will be compact if the operator $S \in \mathcal{B}(\mathscr{H})$ has the following property: for each $\xi \in B_0(\mathbb{R})$ there are $\varphi \in B_0(\mathbb{R})$ and $T \in \mathcal{B}(\mathscr{H})$ such that $S\xi(Q) = \varphi(Q)T$.

An operator S with the property specified above will be called *quasilocal*. For reasons that will become clear later on, we should be more precise and say "right quasilocal with respect to the module structure defined by $B_0(\mathbb{R})$". Anyway, we see that the compactness of R follows from the quasilocality of S and our main point is that it is easy to check this property under very general assumptions on A, cf. Corollary 2.14, and Proposition 2.22 for abstract criteria and Lemmas 3.7, 5.2 and 6.11 for more concrete examples. The perturbative technique used in the proof of Lemma 6.11 seems to us most interesting since it shows that for the quasilocality question it suffices in fact to consider operators with smooth coefficients.

The applications that we have in mind are of a much more general nature than the example considered above. In fact, an abstract formulation of the ideas described above, see Proposition 2.6, allows one to treat pseudo-differential operators on finite dimensional vector spaces over a local (e.g. p-adic) field (see [Sa, Ta] for the corresponding calculus), in particular differential operators of arbitrary order on $\mathbb{R}^n$, and also abstractly defined classes of operators acting on sections of vector bundles over locally compact spaces, in particular an abstract version of the Laplace operator on manifolds with locally L^∞ Riemannian metrics. Sections 4-6 are devoted to such applications. We stress once again that, in the applications to differential operators, we are interested only in situations where the coefficients are not smooth and the lower order terms are quite singular.

Plan of the paper: In Section 2 we introduce an algebraic formalism which allows us to treat in a unified and simple way operators which have an algebraically complex structure, e.g. operators acting on sections of vector fiber bundles over a locally compact space. The class of "vanishing at infinity" operators is defined through an a priori given algebra of operators on a Banach space $\mathscr{H}$, that we call multiplier algebra of $\mathscr{H}$, and this allows us to define the notion of quasilocality in a natural and general context, that of Banach modules. Several examples of multiplier algebras are given Subsections 2.4, 2.5 and 6.1. We stress that Section 2 is only an accumulation of definitions and straightforward consequences.

We mention that this algebraic framework allows one to study differential operators in L^p spaces. However, this question will not be considered in the present version of our work.

Section 3 contains several abstract compactness criteria which formalize in the context of Banach modules the ideas involved in the example discussed above.

In Section 4 we give our first concrete examples of the abstract theory. In Subsection 4.1 we discuss operators in divergence form on $\mathbb{R}^n$, hence of order $2m$ with $m \geq 1$ integer, with coefficients of a rather general form (they do not have to be functions, for example). In the next subsection we consider pseudo-differential operators on abelian groups and Dirac operators on $\mathbb{R}^n$.

Perturbations of the Laplace operator on a Riemannian manifold with locally L^∞ metric are considered in Section 5. We introduce and study an abstract model of this situation which fits very naturally in our algebraic framework. We also have results on Laplace operators acting on differential forms of any order, but we shall include them only in the final version of the paper.

In Section 6 we discuss the question of "weakly vanishing at infinity" functions, a notion which is easily expressed in terms of filters finner than the Fréchet filter. The quasilocality result presented in Theorem 6.8 is, technically speaking, the deepest assertion of this paper: the proof requires nontrivial tools from the modern theory of Banach spaces, cf. the second part of the Appendix. Theorem 6.12 is a last application of our formalism: we prove a compactness result for operators of order $2m$ in divergence form assuming that the difference between their coefficients vanishes at infinity in a weak sense. Such results were known before only in the case $m = 1$, see [OS].

In the first part of the Appendix we collect some general facts concerning operators acting in scales of spaces which are often used without comment in the rest of the paper. In the second part we prove a version of the Maurey's factorization theorem that we need in Section 6.

Notations: If $\mathscr{G}$ and $\mathscr{H}$ are Banach spaces then $\mathcal{B}(\mathscr{G}, \mathscr{H})$ is the space of bounded linear operators $\mathscr{G} \to \mathscr{H}$, the subspace of compact operators is denoted $\mathcal{K}(\mathscr{G}, \mathscr{H})$, and we set $\mathcal{B}(\mathscr{H}) = \mathcal{B}(\mathscr{H}, \mathscr{H})$ and $\mathcal{K}(\mathscr{H}) = \mathcal{K}(\mathscr{H}, \mathscr{H})$. The domain and the resolvent set of an operator S will be denoted by $\mathcal{D}(S)$ and $\rho(S)$ respectively. The norm of a Banach space $\mathscr{G}$ is denoted by $\|\cdot\|_{\mathscr{G}}$ and we omit the index if the space plays a central rôle. The adjoint space (space of antilinear continuous forms) of a Banach space $\mathscr{G}$ is denoted $\mathscr{G}^*$ and if $u \in \mathscr{G}$ and $v \in \mathscr{G}^*$ then we set $v(u) = \langle u, v\rangle$. The embedding $\mathscr{G} \subset \mathscr{G}^{**}$ is realized by defining $\langle v, u\rangle = \overline{\langle u, v\rangle}$.

If $\mathscr{G}, \mathscr{H}, \mathscr{K}$ are Banach spaces such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely and $\mathscr{H} \subset \mathscr{K}$ continuously then we always identify $\mathcal{B}(\mathscr{H})$ with a subset of $\mathcal{B}(\mathscr{G}, \mathscr{K})$ with the help of the natural continuous embedding $\mathcal{B}(\mathscr{H}) \hookrightarrow \mathcal{B}(\mathscr{G}, \mathscr{K})$

A *Friedrichs couple* $(\mathscr{G}, \mathscr{H})$ is a pair of Hilbert spaces $\mathscr{G}, \mathscr{H}$ together with a continuous dense embedding $\mathscr{G} \subset \mathscr{H}$. The *Gelfand triplet* associated to it is obtained by identifying $\mathscr{H} = \mathscr{H}^*$ with the help of the Riesz isomorphism and then taking the adjoint of the inclusion map $\mathscr{G} \to \mathscr{H}$. Thus we get $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ with continuous and dense embeddings. Now if $u \in \mathscr{G}$ and $v \in \mathscr{H} \subset \mathscr{G}^*$ then $\langle u, v\rangle$ is the scalar product in $\mathscr{H}$ of u and v and also the action of the functional v on u. As noted above, we have $\mathcal{B}(\mathscr{H}) \subset \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$.

If X is a locally compact topological space then $B(X)$ is the C^*-algebra of bounded Borel complex functions on X, with norm $\sup_{x\in X} |\varphi(x)|$, and $B_0(X)$ is the subalgebra consisting of functions which tend to zero at infinity. Then $C(X)$, $C_{\rm b}(X)$, $C_0(X)$ and $C_{\rm c}(X)$ are the spaces of complex functions on X which are continuous, continuous and bounded, continuous and convergent to zero at infinity, and continuous with compact support respectively. The characteristic function of a subset $S \subset X$ is denoted χ_S .

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2 Banach modules and quasilocal operators

2.1 Banach modules

We use the terminology of [FD] but with some abbreviations, e.g. a *morphism* is a linear multiplicative map between two algebras, and a $*$-*morphism* is a morphism between two $*$-algebras which commutes with the involutions. We recall that an *approximate unit* in a Banach algebra $\mathcal{M}$ is a net $\{J_\alpha\}$ in $\mathcal{M}$ such that $\|J_\alpha\| \leq C$ for some constant C and all α and $\lim_\alpha \|J_\alpha M - M\| = \lim_\alpha \|MJ_\alpha - M\| = 0$ for all $M \in \mathcal{M}$. It is well known that any C^*-algebra has an approximate unit. If $\mathscr{H}$ is a Banach space, we shall say that a Banach subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathscr{H})$ is *non-degenerate* if the linear subspace of $\mathscr{H}$ generated by the elements Mu, with $M \in \mathcal{M}$ and $u \in \mathscr{H}$, is dense in $\mathscr{H}$.

Definition 2.1 *A* Banach module *is a couple* $(\mathscr{H}, \mathcal{M})$ *consisting of a Banach space* $\mathscr{H}$ *and a non-degenerate Banach subalgebra* $\mathcal{M}$ *of* $\mathcal{B}(\mathscr{H})$ *which has an approximate unit. If* $\mathscr{H}$ *is a Hilbert space and* $\mathcal{M}$ *is a* C^**-algebra of operators on* $\mathscr{H}$*, we say that* $\mathscr{H}$ *is a* Hilbert module.

We shall adopt the usual *abus de language* and say that $\mathscr{H}$ is a Banach module. The distinguished subalgebra $\mathcal{M}$ will be called *multiplier algebra of* $\mathscr{H}$ and, when required by the clarity of the presentation, we shall denote it $\mathcal{M}(\mathscr{H})$. We are especially interested in the case when $\mathcal{M}$ does not have a unit: the operators from $\mathcal{M}$ are the prototype of “vanishing at infinity operators”, or the identity cannot vanish at infinity. Note that it is implicit in Definition 2.1 that if $\mathscr{H}$ is a Hilbert module then its adjoint space $\mathscr{H}^*$ is identified with $\mathscr{H}$ with the help of the Riesz isomorphism.

If $\{J_\alpha\}$ is an approximate unit of $\mathcal{M}$, then the density in $\mathscr{H}$ of the linear subspace generated by the elements Mu is equivalent to

$$\lim_\alpha \|J_\alpha u - u\| = 0 \quad \text{for all} \quad u \in \mathscr{H}. \tag{2.4}$$

But much more is true:

$$u \in \mathscr{H} \Rightarrow u = Mv \text{ for some } M \in \mathcal{M} \text{ and } v \in \mathscr{H}. \tag{2.5}$$

This follows from the Cohen-Hewitt theorem, see Theorem A.3. By using (2.4) we could avoid any reference to this result in our later arguments; this would make them more elementary but less simple.

If $\mathscr{H}$ is a Banach module and the Banach space $\mathscr{H}$ is reflexive we say that $\mathscr{H}$ is a *reflexive Banach module*. In this case the adjoint Banach space $\mathscr{H}^*$ *is equipped with a canonical Banach module structure*, its multiplier algebra being $\mathcal{M}(\mathscr{H}^*) := \{A^* \mid A \in \mathcal{M}(\mathscr{H})\}$. This is a closed subalgebra of $\mathcal{B}(\mathscr{H}^*)$ which clearly has an approximate unit and the linear subspace generated by the elements of the form A^*v, with $A \in \mathcal{M}(\mathscr{H})$ and $v \in \mathscr{H}^*$, is weak*-dense, hence dense, in $\mathscr{H}^*$. Indeed, if $u \in \mathscr{H}$ and $\langle u, A^*v\rangle = 0$ for all such A, v then $Au = 0$ for all $A \in \mathcal{M}(\mathscr{H})$ hence $u = 0$ because of (2.4).

Definition 2.2 *A couple $(\mathscr{G}, \mathscr{H})$ consisting of two Hilbert modules such that $\mathscr{G} \subset \mathscr{H}$ continuously and densely will be called a* Friedrichs module. *If $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H})$, we say that $(\mathscr{G}, \mathscr{H})$ is a* compact Friedrichs module.

There is no a priori relation between the multiplier algebras of $\mathscr{H}$ and $\mathscr{G}$ and the choice $\mathcal{M}(\mathscr{G}) = \mathcal{B}(\mathscr{G})$ is allowed. We observe that in general it is not possible to take $\mathcal{M}(\mathscr{G})$ equal to the set of operators $M \in \mathcal{M}(\mathscr{H})$ which leave $\mathscr{G}$ invariant: it may happen that this algebra has not an approximate unit.

In the situation of this definition we always identify $\mathscr{H}$ with its adjoint space, which gives us a Gelfand triplet $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$. Note that, $\mathscr{G}$ being reflexive, $\mathscr{G}^*$ is also a Hilbert module.

If $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs module then each operator M from $\mathcal{M}(\mathscr{H})$ extends to a compact operator $M : \mathscr{H} \to \mathscr{G}^*$ (this is the adjoint of the compact operator $M^* : \mathscr{G} \to \mathscr{H}$). Thus we shall have $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H}) \cap \mathcal{K}(\mathscr{H}, \mathscr{G}^*)$.

2.2 Operators vanishing at infinity

Let $\mathscr{H}$ and $\mathscr{K}$ be Banach spaces. If $\mathscr{K}$ is a Banach module then we shall denote by $\mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$ the norm closed linear subspace generated by the operators MT, with $T \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $M \in \mathcal{M}(\mathscr{K})$. We say that an operator in $\mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$ *left vanishes at infinity* (with respect to $\mathcal{M}(\mathscr{K})$, if this is not obvious from the context). If J_α is an approximate unit for $\mathcal{M}(\mathscr{K})$, then for an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ we have:

$$\begin{aligned} S \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K}) \quad &\Leftrightarrow \quad \lim_\alpha \|J_\alpha S - S\| = 0 \\ &\Leftrightarrow \quad S = MT \text{ for some } M \in \mathcal{M}(\mathscr{K}) \text{ and } T \in \mathcal{B}(\mathscr{H}, \mathscr{K}). \end{aligned} \tag{2.6}$$

The second equivalence follows from the Cohen-Hewitt theorem.

If $\mathscr{H}$ is a Banach module then one can similarly define $\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ as the norm closed linear subspace generated by the operators TM with $T \in \mathcal{B}(\mathscr{H},\mathscr{K})$ and $M \in \mathcal{M}(\mathscr{H})$. We say that the elements of $\mathcal{B}_0^r(\mathscr{H},\mathscr{K})$ *right vanish at infinity*. As above, if J_α is an approximate unit for $\mathcal{M}(\mathscr{H})$ we have

$$\begin{aligned} S \in \mathcal{B}_0^r(\mathscr{H},\mathscr{K}) \quad &\Leftrightarrow \quad \lim_\alpha \|SJ_\alpha - S\| = 0 \\ &\Leftrightarrow \quad S = TM \text{ for some } M \in \mathcal{M}(\mathscr{H}) \text{ and } T \in \mathcal{B}(\mathscr{H},\mathscr{K}). \end{aligned} \tag{2.7}$$

If both $\mathscr{H}$ and $\mathscr{K}$ are Banach modules we set

$$\mathcal{B}_0(\mathscr{H},\mathscr{K}) = \mathcal{B}_0^l(\mathscr{H},\mathscr{K}) \cap \mathcal{B}_0^r(\mathscr{H},\mathscr{K}). \tag{2.8}$$

The elements of $\mathcal{B}_0(\mathscr{H},\mathscr{K})$ are called *vanishing at infinity*.

If $(\mathscr{G},\mathscr{H})$ is a Friedrichs module then the space $\mathcal{B}_0^l(\mathscr{G},\mathscr{G}^*)$ for example is well defined, but it could be too large for some purposes (it is equal to $\mathcal{B}(\mathscr{G},\mathscr{G}^*)$ if the multiplier algebra of $\mathscr{G}$ is $\mathcal{B}(\mathscr{G})$). For this reason we introduce the next spaces. Recall that we have a natural continuous embedding $\mathcal{B}(\mathscr{H}) \subset \mathcal{B}(\mathscr{G},\mathscr{G}^*)$. Let

$$\mathcal{B}_{00}^l(\mathscr{G},\mathscr{G}^*) = \text{ norm closure of } \mathcal{B}_0^l(\mathscr{H}) \text{ in } \mathcal{B}(\mathscr{G},\mathscr{G}^*). \tag{2.9}$$

The spaces $\mathcal{B}_{00}^r(\mathscr{G},\mathscr{G}^*)$ and $\mathcal{B}_{00}(\mathscr{G},\mathscr{G}^*)$ are similarly defined. We have

$$\mathcal{K}(\mathscr{G},\mathscr{G}^*) \subset \mathcal{B}_{00}(\mathscr{G},\mathscr{G}^*) \tag{2.10}$$

because $\mathcal{K}(\mathscr{H})$ is a dense subset of $\mathcal{K}(\mathscr{G},\mathscr{G}^*)$ and $\mathcal{K}(\mathscr{H}) \subset \mathcal{B}_0(\mathscr{H})$, see below.

Some simple properties of these spaces are described below.

Proposition 2.3 *If $\mathscr{K}$ is a reflexive Banach module and $S \in \mathcal{B}_0^l(\mathscr{H},\mathscr{K})$ then S^* belongs to $\mathcal{B}_0^r(\mathscr{K}^*,\mathscr{H}^*)$.*

Proof: We have $S = MT$ with $M \in \mathcal{M}(\mathscr{K})$ and $T \in \mathcal{B}(\mathscr{H},\mathscr{K})$ by (2.6), which implies $S^* = T^*M^*$ and we have $M^* \in \mathcal{M}(\mathscr{K}^*)$ by definition. ■

Corollary 2.4 *If $\mathscr{H}$ is a Hilbert module then $\mathcal{B}_0(\mathscr{H})$ is a C^*-algebra and S belongs to $\mathcal{B}_0(\mathscr{H})$ if and only if $S = MTN$ with $M, N \in \mathcal{M}(\mathscr{H})$ and $T \in \mathcal{B}(\mathscr{H})$.*

Proof: $\mathcal{B}_0(\mathscr{H})$ is a C^*-algebra, so $S = S_1S_2$ for some operators $S_1, S_2 \in \mathcal{B}_0(\mathscr{H})$. Thus $S_1 = MT_1$ and $S_2 = T_2N$ for some $M, N \in \mathcal{M}(\mathscr{H})$ and $T_1, T_2 \in \mathcal{B}(\mathscr{H})$, hence $S = MT_1T_2N$. ■

Proposition 2.5 *If $\mathscr{K}$ is a Banach module then $\mathcal{K}(\mathscr{H},\mathscr{K}) \subset \mathcal{B}_0^l(\mathscr{H},\mathscr{K})$. If $\mathscr{H}$ is a reflexive Banach module, then $\mathcal{K}(\mathscr{H},\mathscr{K}) \subset \mathcal{B}_0^r(\mathscr{H},\mathscr{K})$.*

Proof: If $\{J_\alpha\}$ is an approximate unit for $\mathcal{M}(\mathscr{K})$ then s-$\lim_\alpha J_\alpha u = u$ uniformly in u if u belongs to a compact subset of $\mathscr{K}$. Hence if $S \in \mathcal{K}(\mathscr{H}, \mathscr{K})$ then $\lim_\alpha \|J_\alpha S - S\| = 0$ and thus $S \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$ by (2.6). To prove the second part of the proposition, observe that if $S \in \mathcal{K}(\mathscr{H}, \mathscr{K})$ then $S^* \in \mathcal{K}(\mathscr{K}^*, \mathscr{H}^*)$, hence $S^* \in \mathcal{B}_0^l(\mathscr{K}^*, \mathscr{H}^*)$ by what we just proved, so $S^{**} \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K}^{**})$ by Proposition 2.3. So $\lim_\alpha \|S^{**} J_\alpha - S^{**}\| = 0$ if $\{J_\alpha\}$ is an approximate unit for $\mathcal{M}(\mathscr{H})$. But clearly$S = S^{**}$, hence $S \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$. ∎

Proposition 2.6 *Let $\mathscr{H}$ be a Banach module and $\mathscr{G}$ a Banach space continuously embedded in $\mathscr{H}$ and such that $\mathcal{M}(\mathscr{H}) \subset \mathcal{K}(\mathscr{G}, \mathscr{H})$. If $R \in \mathcal{B}_0^l(\mathscr{H})$ and $R\mathscr{H} \subset \mathscr{G}$, then $R \in \mathcal{K}(\mathscr{H})$.*

Proof: According to (2.6) we have $R = \lim_\alpha J_\alpha R$, the limit being taken in norm. But $R \in \mathcal{B}(\mathscr{H}, \mathscr{G})$ by the closed graph theorem and $J_\alpha \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ by hypothesis, so that $J_\alpha R \in \mathcal{K}(\mathscr{H})$. ∎

Corollary 2.7 *If $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs module and $R \in \mathcal{B}_0^l(\mathscr{H})$ is such that $R\mathscr{H} \subset \mathscr{G}$, then $R \in \mathcal{K}(\mathscr{H})$.*

2.3 Quasilocal operators

Definition 2.8 *Let $\mathscr{H}$, $\mathscr{K}$ be Banach modules and let $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$. We say that S is* left quasilocal *if for each $M \in \mathcal{M}(\mathscr{K})$ we have $MS \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$. We say that S is* right quasilocal *if for each $M \in \mathcal{M}(\mathscr{H})$ we have $SM \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$. If S is left and right quasilocal, we say that S is* quasilocal.

We denote $\mathcal{B}_q^l(\mathscr{H}, \mathscr{K})$, $\mathcal{B}_q^r(\mathscr{H}, \mathscr{K})$ and $\mathcal{B}_q(\mathscr{H}, \mathscr{K})$ these classes of operators. These are clearly closed subspaces of $\mathcal{B}(\mathscr{H}, \mathscr{G})$. The next result is obvious; a similar assertion holds for right quasilocality.

Proposition 2.9 *Let $\{J_\alpha\}$ be an approximate unit for $\mathcal{M}(\mathscr{K})$ and let S be an operator in $\mathcal{B}(\mathscr{H}, \mathscr{K})$. Then S is left quasilocal if and only if one of the following conditions is satisfied:*
(1) *$J_\alpha S \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$ for all α.*
(2) *for each $M \in \mathcal{M}(\mathscr{K})$ there are $T \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $N \in \mathcal{M}(\mathscr{H})$ such that $MS = TN$.*

The next proposition, which says that the set of quasilocal operators is stable under the usual algebraic operations, is an immediate consequence of Proposition 2.9. There is, of course, a similar statement with “left” and “right” interchanged. We denote by $\mathscr{G}, \mathscr{H}$ and $\mathscr{K}$ Banach modules.

Proposition 2.10 (1) $S \in \mathcal{B}^l_q(\mathscr{H},\mathscr{K})$ *and* $T \in \mathcal{B}^l_q(\mathscr{G},\mathscr{H}) \Rightarrow ST \in \mathcal{B}^l_q(\mathscr{G},\mathscr{K})$.
(2) *If* $\mathscr{H},\mathscr{K}$ *are reflexive and* $S \in \mathcal{B}^l_q(\mathscr{H},\mathscr{K})$, *then* $S^* \in \mathcal{B}^r_q(\mathscr{K}^*,\mathscr{H}^*)$.
(3) *If* $\mathscr{H}$ *is a Hilbert module then* $\mathcal{B}_q(\mathscr{H})$ *is a unital* C^**-subalgebra of* $\mathcal{B}(\mathscr{H})$.

Obviously $\mathcal{B}^l_0(\mathscr{H},\mathscr{K}) \subset \mathcal{B}^r_q(\mathscr{H},\mathscr{K})$ and $\mathcal{B}^r_0(\mathscr{H},\mathscr{K}) \subset \mathcal{B}^l_q(\mathscr{H},\mathscr{K})$. But our main results depend on finding other, more interesting examples of quasilocal operators.

Remark: A more natural and suggestive name for "quasilocal operators" would be *decay preserving operators*. We did not use it because the french version of this terminology is rather heavy to use.

2.4 X-modules over locally compact spaces

In the next two subsections we give examples of Banach modules important for this paper. We always denote by X a locally compact non-compact topological space; later we equip it with some more structure.

Definition 2.11 *A* Banach X-module *is a Banach space* $\mathscr{H}$ *equipped with a continuous morphism* $Q : C_0(X) \to \mathcal{B}(\mathscr{H})$ *such that the linear subspace generated by the vectors of the form* $Q(\varphi)u$, *with* $\varphi \in C_0(X)$ *and* $u \in \mathscr{H}$, *is dense in* $\mathscr{H}$. *If* $\mathscr{H}$ *is a Hilbert space and* Q *is a* $*$*-morphism, we say that* $\mathscr{H}$ *is a* Hilbert X-module.

A Friedrichs module $(\mathscr{G},\mathscr{H})$ such that $\mathscr{H}$ is a Hilbert X-module will be called *Friedrichs* X*-module*. Note that here there are no assumptions concerning the module structure of $\mathscr{G}$.

We shall use the notation $\varphi(Q) \equiv Q(\varphi)$. The Banach module structure on $\mathscr{H}$ is defined by the closure $\mathcal{M}$ in $\mathcal{B}(\mathscr{H})$ of the set of operators of the form $\varphi(Q)$ with $\varphi \in C_0(X)$. In the case of a Hilbert X-module the closure is not needed and we get a Hilbert module structure (recall that a $*$-morphism between two C^*-algebras is continuous and its range is a C^*-algebra).

We note that the morphism Q *has an extension, also denoted* Q, *to a unital continuous morphism of* $C_{\rm b}(X)$ *into* $\mathcal{B}(X)$ which is uniquely determined by the following strong continuity property: if $\{\varphi_n\}$ is a bounded sequence in $C_{\rm b}(X)$ such that $\varphi_n \to \varphi$ locally uniformly, then $\varphi_n(Q) \to \varphi(Q)$ strongly on $\mathscr{H}$. Indeed, using once again the Cohen-Hewitt theorem we see that for each $u \in \mathscr{H}$ there are $\psi \in C_0(X)$ and $v \in \mathscr{H}$ such that $u = \psi(Q)v$ hence we can define $\varphi(Q)u = (\varphi\psi)(Q)v$ for each $\varphi \in C_{\rm b}(X)$; then if e_α is an approximate unit for $C_0(X)$ with $\|e_\alpha\| \leq 1$ we get $\varphi(Q)u = \lim(\varphi e_\alpha)(Q)u$ hence $\|\varphi(Q)\| \leq \|Q\| \sup|\varphi|$.

In the case of a Hilbert X-module we can say more.

Lemma 2.12 *If $\mathscr{H}$ is a Hilbert X-module, then the $*$-morphism Q canonically extends to a $*$-morphism $\varphi \mapsto \varphi(Q)$ of $B(X)$ into $\mathcal{B}(\mathscr{H})$ having the property : if $\{\varphi_n\}$ is a bounded sequence in $B(X)$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in X$, then $\varphi_n(Q)$ converges strongly to $\varphi(Q)$ on $\mathscr{H}$.*

Proof: Q extends, by standard integration theory, to a $*$-morphism of $B(X)$ into $\mathcal{B}(\mathscr{H})$ which is uniquely determined by the following property: if $U \subset X$ is open then $\chi_U(Q) = \sup_\varphi \varphi(Q)$, where φ runs over the set of continuous functions with compact support such that $0 \leq \varphi \leq \chi_U$. We note that if X is second countable then this property is equivalent to the convergence condition from the statement of the lemma. ∎

Remark: A separable Hilbert X-module is essentially a direct integral of Hilbert spaces over X, see [Di, Ch. II], but we shall not need this fact. On the other hand, Banach X-modules appear naturally in differential geometry as spaces of sections of vector fiber bundles over a manifold X, and this is the point of interest for us.

The *support* supp $u \subset X$ of an element u of a Banach X-module $\mathscr{H}$ is defined as the smallest closed set such that its complement U has the property $\varphi(Q)u = 0$ if $\varphi \in C_{\mathrm{c}}(U)$. Clearly, the set $\mathscr{H}_{\mathrm{c}}$ of elements $u \in \mathscr{H}$ such that supp u is compact is a dense subspace of $\mathscr{H}$.

If $\mathscr{H}$ and $\mathscr{K}$ are Banach X-modules, then a map $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is called *local* if supp $Su \subset$ supp u for each $u \in \mathscr{H}$; clearly locality implies right quasilocality. Now we look for more interesting criteria of quasilocality.

Let $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $\varphi, \psi \in C(X)$, not necessarily bounded. We say that $\varphi(Q)S\psi(Q)$ is a bounded operator if there is a constant C such that

$$\|\xi(Q)\varphi(Q)S\psi(Q)\eta(Q)\| \leq C \sup|\xi| \sup|\eta|$$

for all $\xi, \eta \in C_{\mathrm{c}}(X)$. The lower bound of the admissible constants C in this estimate is denoted $\|\varphi(Q)S\psi(Q)\|$. If $\mathscr{K}$ is a reflexive Banach X-module, then the product $\varphi(Q)S\psi(Q)$ is well defined as sesquilinear form on the dense subspace $\mathscr{K}_{\mathrm{c}}^* \times \mathscr{H}_{\mathrm{c}}$ of $\mathscr{K}^* \times \mathscr{H}$ and the preceding boundedness notion is equivalent to the continuity of this form for the topology induced by $\mathscr{K}^* \times \mathscr{H}$. We similarly define the boundedness of the commutator $[S, \varphi(Q)]$.

Proposition 2.13 *Assume that $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and let $\Theta : X \to [1, \infty[$ be a continuous function such that $\lim_{x\to\infty} \Theta(x) = \infty$. If $\Theta^{-1}(Q)S\Theta(Q)$ is a bounded operator then S is left quasilocal. If $\Theta(Q)S\Theta^{-1}(Q)$ is a bounded operator then S is right quasilocal.*

Proof: Let $K \subset X$ be compact, let $U \subset X$ be a neighbourhood of infinity in X, and let $\varphi, \psi \in C_{\mathrm{b}}(X)$ such that supp $\varphi \subset K$, supp $\psi \subset U$ and $|\varphi| \leq 1, |\psi| \leq 1$.

Then $\Theta\varphi$ is a bounded function and $\psi\Theta^{-1}$ is bounded and can be made as small as we wish by choosing U conveniently. Thus given $\varepsilon > 0$ we have

$$\|\varphi(Q)S\psi(Q)\| \leq \|\varphi\Theta\| \cdot \|\Theta^{-1}(Q)S\Theta(Q)\| \cdot \|\Theta^{-1}\psi\| \leq \varepsilon$$

if U is a sufficiently small neighbourhood of infinity. ■

The boundedness of $\Theta^{-1}(Q)S\Theta(Q)$ is usually checked by estimating the commutator $[S, \Theta(Q)]$; we give an example for the case of metric spaces. Note that on metric spaces one has a natural class of regular functions, namely the Lipschitz functions, for example the functions which give the distance to subsets: $\rho_K(x) = \inf_{y\in K} \rho(x, y)$ for $K \subset X$.

We say that a locally compact metric space (X, ρ) is *proper* if the metric ρ has the property $\lim_{y\to\infty} \rho(x, y) = \infty$ for some (hence for all) points $x \in X$. Equivalently, if X is not compact but the closed balls are compact.

Corollary 2.14 *Let (X, ρ) be a proper locally compact metric space. If S belongs to $\mathcal{B}(\mathscr{H}, \mathscr{K})$ and if $[S, \theta(Q)]$ is bounded for each positive Lipschitz function θ, then S is quasilocal .*

Proof: Indeed, by taking $\theta = 1 + \rho_K$ and by using the notations of the proof of Proposition 2.13, we easily get the following estimate: there is $C < \infty$ depending only on K such that

$$\|\varphi(Q)S\psi(Q)\| \leq C(1 + \rho(K, U))^{-1},$$

where $\rho(K, U)$ is the distance from K to U. Since S^* has the same properties as S, this proves the quasilocality of S. Note that the boundedness of $[S, \rho_x(Q)]$ for some $x \in X$ suffices in this argument. ■

2.5 X-modules over locally compact groups

If X is a locally compact abelian group one can associate to it more interesting classes of Banach modules. We always assume X non-compact and we denote additively the group operation. For example, X could be $\mathbb{R}^n, \mathbb{Z}^n$, or a finite dimensional vector space over a local field, e.g. over the field of p-adic numbers. Let X^* be the abelian locally compact group dual to X. One can construct interesting Banach subalgebras of $C_0(X)$ by using the Fourier transformation and submultiplicative functions on X^*, but the approach we adopt is more intrinsic.

Definition 2.15 *If X is a locally compact abelian group, then a* Banach X-module *is a Banach space $\mathscr{H}$ equipped with a strongly continuous representation $\{V_k\}$ of the dual group X^* on $\mathscr{H}$.*

Note that we shall use the same notation V_k for the representations of X^* in different spaces $\mathscr{H}$ whenever this does not lead to ambiguities.

Such a Banach X-module has a canonical structure of Banach module that we now define. We choose Haar measures dx and dk on X and X^* normalized by the following condition: if the Fourier transform of a function φ on X is given by $(\mathcal{F}\varphi)(k) \equiv \widehat{\varphi}(k) = \int_X \overline{k(x)}\varphi(x)dx$ then $\varphi(x) = \int_{X^*} k(x)\widehat{\varphi}(k)dk$. Recall that $X^{**} = X$. Let $C^{(a)}(X) := \mathcal{F}L^1_{\mathrm{c}}(X^*)$ be the set of Fourier transforms of integrable functions with compact support on X^*. It is easy to see that $C^{(a)}(X)$ is a $*$-algebra for the usual algebraic operations; more precisely, it is a dense subalgebra of $C_0(X)$ stable under conjugation. For $\varphi \in C^{(a)}(X)$ we set

$$\varphi(Q) = \int_{X^*} V_k \widehat{\varphi}(k)dk. \tag{2.11}$$

This definition is determined by the formal requirement $k(Q) = V_k$. Then

$$\mathcal{M} := \text{ norm closure of } \{\varphi(Q) \mid \varphi \in C^{(a)}(X)\} \text{ in } \mathcal{B}(\mathscr{H}) \tag{2.12}$$

is a Banach subalgebra of $\mathcal{B}(\mathscr{H})$.

Lemma 2.16 *The algebra $\mathcal{M}$ has an approximate unit consisting of elements of the form $e_\alpha(Q)$ with $e_\alpha \in C^{(a)}(X)$.*

Proof: Indeed, let us fix a compact neighborhood K of the identity in X^*. The set of compact neighborhoods of the identity α such that $\alpha \subset K$ is ordered by $\alpha_1 \geq \alpha_2 \Leftrightarrow \alpha_1 \subset \alpha_2$. For each such α define e_α by $\widehat{e}_\alpha = \chi_\alpha/|\alpha|$, where $|\alpha|$ is the Haar measure of α. Then $\|e_\alpha(Q)\| \leq \sup_{k\in K} \|V_k\| < \infty$, from which it is easy to infer that $\lim_\alpha \|e_\alpha(Q)\varphi(Q) - \varphi(Q)\| = 0$ for all $\varphi \in C^{(a)}(X)$. ■

It is easily seen now that the couple $(\mathscr{H}, \mathcal{M})$ satisfies the conditions of Definition 2.1, which gives us the canonical Banach module structure on $\mathscr{H}$.

Remark 2.17 Assume that $\mathcal{A}$ is a Banach algebra with approximate unit and that a morphism $\Phi : \mathcal{A} \to \mathcal{M}(\mathscr{H})$ with dense image is given. Then the Cohen-Hewitt theorem shows that each $u \in \mathscr{H}$ can be written as $u = Av$ where $A \in \Phi(\mathcal{A})$ and $v \in \mathscr{H}$. We give now examples of such algebras in the preceding context. If ω is a sub-multiplicative function on X^*, i.e. a Borel map $X^* \to [1, \infty[$ satisfying $\omega(k'k'') \leq \omega(k')\omega(k'')$ (this implies local boundedness), let $C^{(\omega)}(X)$ be the set of functions φ whose Fourier transform $\widehat{\varphi}$ satisfies

$$\|\varphi\|_{C^{(\omega)}} := \int_{X^*} |\widehat{\varphi}(k)|\omega(k)dk < \infty. \tag{2.13}$$

Then $C^{(\omega)}(X)$ is a subalgebra of $C_0(X)$ and is a Banach algebra for the norm (2.13). Moreover, $C^{(a)}(X) \subset C^{(\omega)}(X)$ densely and the net $\{e_\alpha\}$ defined in the proof of Lemma 2.16 is an approximate unit of $C^{(\omega)}(X)$. If $\|V_k\|_{\mathcal{B}(\mathscr{H})} \leq c\omega(k)$ for some number $c > 0$ then $\varphi(Q)$ is well defined for each $\varphi \in C^{(\omega)}(X)$ by the relation (2.11) and $\Phi(\varphi) = \varphi(Q)$ is a continuous morphism $C^{(\omega)}(X) \to \mathcal{M}(\mathscr{H})$ with dense range. We could take $\omega(k) = \sup(1, \|V_k\|_{\mathcal{B}(\mathscr{H})})$ but if a second Banach X-module $\mathscr{K}$ is given then it is more convenient to take $\omega(k) = \sup\{1, \|V_k\|_{\mathcal{B}(\mathscr{H})}, \|V_k\|_{\mathcal{B}(\mathscr{K})}\}$.

The adjoint of a reflexive Banach X-module has a natural structure of Banach X-module. Indeed, it is known and easy to prove that a weakly continuous representation is strongly continuous. Thus we can equip the adjoint space $\mathscr{H}^*$ with the Banach X-module structure defined by the representation $k \mapsto (V_{\bar{k}})^*$, where $\bar{k}$ is the complex conjugate of k (i.e. its inverse in X^*).

The group X is, in particular, a locally compact topological space, hence the notion of Banach X-module in the sense of Definition 2.11 makes sense. But this is in fact a particular case of that of Banach X-module in the sense of Definition 2.15. Indeed, according to the comments after Definition 2.11, we get a strongly continuous representation of X^* on $\mathscr{H}$ by setting $V_k = k(Q)$. In the case of Hilbert X-modules we have a more precise fact.

Lemma 2.18 *If $\mathscr{H}$ is a Hilbert space then giving a Hilbert X-module structure on $\mathscr{H}$ is equivalent with giving a Banach X-module structure on $\mathscr{H}$ such that the representation $\{V_k\}_{k\in X^*}$ is unitary. The relation between the two structures is determined by the condition $V_k = k(Q)$.*

Proof: If $\mathscr{H}$ is a Hilbert X-module then we can define $V_k = k(Q) \in \mathcal{B}(\mathscr{H})$ and check that $\{V_k\}_{k\in X^*}$ is a strongly continuous unitary representation of X^* on $\mathscr{H}$ with the help of Lemma 2.12. Reciprocally, it is well known that such a representation allows one to equip $\mathscr{H}$ with a Hilbert X-module structure. The main point is that the estimate $\|\varphi(Q)\| \leq \sup|\varphi|$ holds, see [Lo]. ■

If X is a locally compact abelian group, then Banach X-modules which are not Hilbert X-modules often appear in the following context.

Definition 2.19 *If X is a locally compact abelian group then a* stable Friedrichs X-module *is a Friedrichs X-module $(\mathscr{G}, \mathscr{H})$ satisfying $V_k\mathscr{G} \subset \mathscr{G}$ for all $k \in X^*$ and such that if $u \in \mathscr{G}$ and if $K \subset X^*$ is compact then $\sup_{k\in K} \|V_k u\|_{\mathscr{G}} < \infty$.*

Here $V_k = k(Q)$. It is clear that $V_k\mathscr{G} \subset \mathscr{G}$ implies $V_k \in \mathcal{B}(\mathscr{G})$ and that the local boundedness condition implies that the map $k \mapsto V_k \in \mathcal{B}(\mathscr{G})$ is a weakly, hence strongly, continuous representation of X^* on $\mathscr{G}$ (not unitary in general). The local boundedness condition is automatically satisfied if X^* is second countable.

Thus, *if $(\mathscr{G}, \mathscr{H})$ is a stable Friedrichs X-module, then $\mathscr{G}$ is equipped with a canonical Banach X-module structure*. Then, by taking adjoints, we get a natural Banach X-module structure on $\mathscr{G}^*$ too. Our definitions are such that after the identifications $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ the restriction to $\mathscr{H}$ of the operator V_k acting in $\mathscr{G}^*$ is just the initial V_k. Indeed, we have $V_k^* = V_k^{-1} = V_{\bar{k}}$ in $\mathscr{H}$. Thus there is no ambiguity in using the same notation V_k for the representation of X^* in the three spaces $\mathscr{G}, \mathscr{H}$ and $\mathscr{G}^*$.

Remark 2.20 We stress that if $(\mathscr{G}, \mathscr{H})$ is a stable Friedrichs X-module then $\mathscr{G}$ is always equipped with the Banach module structure associated to its X-module structure defined above (we recall that in the case of a general Friedrichs X-module there was no restriction on the module structure of $\mathscr{G}$). As a consequence, *we have $\mathcal{B}_0^l(\mathscr{K}, \mathscr{G}) \subset \mathcal{B}_0^l(\mathscr{K}, \mathscr{H})$ for an arbitrary Banach space $\mathscr{K}$, hence also $\mathcal{B}_q^r(\mathscr{K}, \mathscr{G}) \subset \mathcal{B}_q^r(\mathscr{K}, \mathscr{H})$ if $\mathscr{K}$ is a Banach module*. Indeed, if $S \in \mathcal{B}_0^l(\mathscr{K}, \mathscr{G})$ then $S = \varphi(Q)T$ for some $\varphi \in C^{(\omega)}(X)$ with $\omega(k) = \sup(1, \|V_k\|_{\mathcal{B}(\mathscr{G})})$, see Remark 2.17, and some $T \in \mathcal{B}(\mathscr{K}, \mathscr{G})$. But clearly such a $\varphi(Q)$ belongs to the multiplier algebra of $\mathscr{H}$ and $T \in \mathcal{B}(\mathscr{K}, \mathscr{H})$.

We show now that, in the case of Banach X-modules over locally compact groups, quasilocality is related to regularity in the sense of the next definition.

Definition 2.21 *Let $\mathscr{H}$ and $\mathscr{K}$ be Banach X-modules. We say that a continuous operator $S : \mathscr{H} \to \mathscr{K}$ is of class $C^{\mathrm{u}}(Q)$, and we write $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$, if the map $k \mapsto V_k^{-1} S V_k \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is norm continuous.*

Note that norm continuity at the origin implies norm continuity everywhere. The class of regular operators is stable under algebraic operations:

Proposition 2.22 *Let $\mathscr{G}, \mathscr{H}, \mathscr{K}$ be Banach X-modules.*
(i) If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ and $T \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{H})$ then $ST \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{K})$.
(ii) If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ is bijective, then $S^{-1} \in C^{\mathrm{u}}(Q; \mathscr{K}, \mathscr{H})$.
(iii) If $S \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ and $\mathscr{H}, \mathscr{G}$ are reflexive, then $S^* \in C^{\mathrm{u}}(Q; \mathscr{K}^*, \mathscr{H}^*)$.

Proof: We prove only (ii). If we set $S_k = V_k^{-1} S V_k$ then $V_k^{-1} S^{-1} V_k = S_k^{-1}$, hence

$$\|V_k^{-1} S^{-1} V_k - S^{-1}\| = \|S_k^{-1} - S^{-1}\| = \|S_k^{-1}(S - S_k)S^{-1}\| \le C\|S - S_k\|$$

which tends to zero as $k \to 0$. ■

Proposition 2.23 *If $T \in C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{K})$ then T is quasilocal.*

Proof: We show that $\varphi(Q)T \in \mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$ if $\varphi \in C^{(a)}(X)$. A similar argument gives $T\varphi(Q) \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$. Set $T_k = V_k T V_k^{-1}$, then

$$\varphi(Q)T = \int_{X^*} \widehat{\varphi}(k) V_k T dk = \int_{X^*} T_k \widehat{\varphi}(k) V_k dk.$$

Since $k \mapsto T_k$ is norm continuous on the compact support of $\widehat{\varphi}$, for each $\varepsilon > 0$ we can construct, with the help of a partition of unity, functions $\theta_i \in C_c(X^*)$ and operators $S_i \in \mathcal{B}(\mathscr{H}, \mathscr{K})$, such that $\|T_k - \sum_{i=1}^n \theta_i(k) S_i\| < \varepsilon$ if $\widehat{\varphi}(k) \neq 0$. Thus

$$\|\varphi(Q)T - \sum_{i=1}^n \int_{X^*} \theta_i(k) S_i \widehat{\varphi}(k) V_k dk\| \leq \varepsilon \sum_{i=1}^n \int_{X^*} |\widehat{\varphi}(k)| \|V_k\|_{\mathcal{B}(\mathscr{H})} dk.$$

Now, since $\mathcal{B}_0(\mathscr{H}, \mathscr{K})$ is a norm closed subspaces, it suffices to show that the operator $\int_{X^*} \theta_i(k) S_i \widehat{\varphi}(k) V_k dk$ belongs to $\mathcal{B}_0^r(\mathscr{H}, \mathscr{K})$ for each i. But if ψ_i is the inverse Fourier transform of $\theta_i \widehat{\varphi}$ then this is $S_i \psi_i(Q)$ and $\psi_i \in C^{(a)}(X)$. ■

We recall that if X is an abelian locally compact group then there is enough structure in order to develop a rich pseudo-differential calculus in $L^2(X)$, but we give only elementary examples. If φ and ψ are bounded Borel functions on X and X^* respectively then, following standard quantum mechanical conventions, we denote by $\varphi(Q)$ the operator of multiplication by φ in $L^2(X)$ and we set $\psi(P) = \mathcal{F}^{-1} M_\psi \mathcal{F}$, where M_ψ is the operator of multiplication by ψ in $L^2(X^*)$. Then one gets more general pseudo-differential operators of order zero by considering C^*-algebras generated by products $\varphi(Q)\psi(P)$. We recall that the C^*-algebra generated by such products with φ and ψ bounded Borel and convergent to zero at infinity is the algebra of compact operators on $L^2(X)$.

Let $C_b^u(X)$ and $C_b^u(X^*)$ be the algebras of bounded uniformly continuous functions on X and X^* respectively. Below the space $L^2(X)$ is equipped with its natural Hilbert X-module structure.

Proposition 2.24 *The C^*-algebra generated by the operators $\varphi(Q)$ and $\psi(P)$ with $\varphi \in C_b^u(X)$ and $\psi \in C_b^u(X^*)$ consists of quasilocal operators on $L^2(X)$.*

Proof: Since the set of quasilocal operators in $\mathcal{B}(L^2(X))$ is a C^*-algebra, it suffices to show that each $\varphi(Q)$ and $\psi(P)$ is quasilocal. For $\varphi(Q)$ the assertion is trivial while for $\psi(P)$ we apply Proposition 2.23. ■

3 Abstract compactness results

In this section $(\mathscr{G}, \mathscr{H})$ will always be a *compact Friedrichs module*, see Definition 2.2. As usual, we associate to it a Gelfand triplet $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ and we set

$\|\cdot\| = \|\cdot\|_{\mathscr{H}}$. We are interested in criteria which ensure that an operator B is a compact perturbation of an operator A, both operators being unbounded operators in $\mathscr{H}$ obtained as restrictions of some bounded operators $\mathscr{G} \to \mathscr{G}^*$. More precisely, the following is a general assumption (suggested by the statement of Theorem 2.1 in [OS]) which will always be fulfilled:

$$\textbf{(}AB\textbf{)} \begin{cases} A, B \text{ are closed densely defined operators in } \mathscr{H} \text{ with } \rho(A) \cap \rho(B) \neq \emptyset \\ \text{and having the following properties: } \mathcal{D}(A) \subset \mathscr{G} \text{ densely, } \mathcal{D}(A^*) \subset \mathscr{G}, \\ \mathcal{D}(B) \subset \mathscr{G} \text{ and } A, B \text{ extend to continuous operators } \widetilde{A}, \widetilde{B} \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*). \end{cases}$$

The rôle of the assumption (AB) is to allow us to give a rigorous meaning to the formal relation, where $z \in \rho(A) \cap \rho(B)$,

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1}(B - A)(B - z)^{-1}. \tag{3.14}$$

Recall that $z \in \rho(A)$ if and only if $\bar{z} \in \rho(A^*)$ and then $(A^* - \bar{z})^{-1} = (A - z)^{-1*}$. Thus we have $(A - z)^{-1*}\mathscr{H} \subset \mathscr{G}$ by the assumption (AB) and this allows one to deduce that $(A - z)^{-1}$ extends to a unique continuous operator $\mathscr{G}^* \to \mathscr{H}$, that we shall denote for the moment by R_z. From $R_z(A - z)u = u$ for $u \in \mathcal{D}(A)$ we get, by density of $\mathcal{D}(A)$ in $\mathscr{G}$ and continuity, $R_z(\widetilde{A} - z)u = u$ for $u \in \mathscr{G}$, in particular

$$(B - z)^{-1} = R_z(\widetilde{A} - z)(B - z)^{-1}.$$

On the other hand, the identity

$$(A - z)^{-1} = (A - z)^{-1}(B - z)(B - z)^{-1} = R_z(\widetilde{B} - z)(B - z)^{-1}$$

is trivial. Subtracting the last two relations we get

$$(A - z)^{-1} - (B - z)^{-1} = R_z(\widetilde{B} - \widetilde{A})(B - z)^{-1}$$

Since R_z is uniquely determined as extension of $(A - z)^{-1}$ to a continuous map $\mathscr{G}^* \to \mathscr{H}$, we shall keep the notation $(A - z)^{-1}$ for it. With this convention, the rigorous version of (3.14) that we shall use is:

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1}(\widetilde{B} - \widetilde{A})(B - z)^{-1}. \tag{3.15}$$

Theorem 3.1 *Let A, B satisfy assumption (AB) and let us assume that there are a Banach module $\mathscr{K}$ and operators $S \in \mathcal{B}(\mathscr{K}, \mathscr{G}^*)$ and $T \in \mathcal{B}_0^l(\mathscr{G}, \mathscr{K})$ such that $\widetilde{B} - \widetilde{A} = ST$ and $(A - z)^{-1}S \in \mathcal{B}_q^r(\mathscr{K}, \mathscr{H})$ for some $z \in \rho(A) \cap \rho(B)$. Then the operator B is a compact perturbation of the operator A and $\sigma_{\rm ess}(B) = \sigma_{\rm ess}(A)$.*

Proof: It suffices to show that $R \equiv (A-z)^{-1} - (B-z)^{-1} \in \mathcal{B}_0^l(\mathscr{H})$, because the domains of A and B are included in $\mathscr{G}$, hence $R\mathscr{H} \subset \mathscr{G}$, which finishes the proof because of Corollary 2.7. Now due to (3.15) and to the factorization assumption, we can write R as a product $R = [(A-z)^{-1}S][T(B-z)^{-1}]$ where the first factor is in $\mathcal{B}_q^r(\mathscr{K}, \mathscr{H})$ and the second in $\mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$, so the product is in $\mathcal{B}_0^l(\mathscr{H})$. ■

Remarks 3.2 (1) We could have stated the assumptions of Theorem 3.1 in an apparently more general form, namely $B - A = \sum_{k=1}^n S_k T_k$ with operators $S_k \in \mathcal{B}(\mathscr{K}_k, \mathscr{G}^*)$ and $T_k \in \mathcal{B}(\mathscr{G}, \mathscr{K}_k)$. But we are reduced to the stated version of the assumption by considering the Hilbert module $\mathscr{K} = \oplus \mathscr{K}_k$ and $S = \oplus S_k, T = \oplus T_k$.
(2) If $V \in \mathcal{K}(\mathscr{G}, \mathscr{G}^*)$ and if $\mathscr{K}$ is an infinite dimensional module, then there are operators $S \in \mathcal{B}(\mathscr{K}, \mathscr{G}^*)$ and $T \in \mathcal{K}(\mathscr{G}, \mathscr{K})$ such that $V = ST$ (the proof is an easy exercise). This and the preceding remark show that compact contributions to $\widetilde{B} - \widetilde{A}$ are trivially covered by the factorization assumption.

Example 3.3 One can construct interesting classes of operators with the properties required in (AB) as follows. Let $\mathscr{G}_a$, $\mathscr{G}_b$ be Hilbert spaces such that $\mathscr{G} \subset \mathscr{G}_a \subset \mathscr{H}$ and $\mathscr{G} \subset \mathscr{G}_b \subset \mathscr{H}$ continuously and densely. Thus we have two scales

$$\mathscr{G} \subset \mathscr{G}_a \subset \mathscr{H} \subset \mathscr{G}_a^* \subset \mathscr{G}^*,$$
$$\mathscr{G} \subset \mathscr{G}_b \subset \mathscr{H} \subset \mathscr{G}_b^* \subset \mathscr{G}^*.$$

Then let $A_0 \in \mathcal{B}(\mathscr{G}_a, \mathscr{G}_a^*)$ and $B_0 \in \mathcal{B}(\mathscr{G}_b, \mathscr{G}_b^*)$ such that $A_0 - z : \mathscr{G}_a \to \mathscr{G}_a^*$ and $B_0 - z : \mathscr{G}_b \to \mathscr{G}_b^*$ are bijective for some number z. According to Lemma A.1 we can associate to A_0, B_0 closed densely defined operators $A = \widehat{A_0}, B = \widehat{B_0}$ in $\mathscr{H}$, such that the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are dense subspaces of $\mathscr{G}_a$ and the domains $\mathcal{D}(B)$ and $\mathcal{D}(B^*)$ are dense subspaces of $\mathscr{G}_b$. If we also have $\mathcal{D}(A) \subset \mathscr{G}$ densely, $\mathcal{D}(A^*) \subset \mathscr{G}$ and $\mathcal{D}(B) \subset \mathscr{G}$, then all the conditions of the assumption (AB) are fulfilled with $\widetilde{A} = A_0|\mathscr{G}$ and $\widetilde{B} = B_0|\mathscr{G}$.

The case when one of the operators, for example A, is self-adjoint is worth to be considered separately. As explained in the Appendix, the conditions on A in assumption (AB) are satisfied if $\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ densely. Moreover, if A is semibounded, then this condition is also necessary. In particular, we have:

Corollary 3.4 *Let A, B be self-adjoint operators on $\mathscr{H}$ such that*

$$\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2}) \text{ and } \mathcal{D}(B) \subset \mathscr{G} \subset \mathcal{D}(|B|^{1/2}) \text{ densely.}$$

Let $\widetilde{A}, \widetilde{B}$ be the unique extensions of A, B to operators in $\mathscr{B}(\mathscr{G}, \mathscr{G}^*)$. Assume that there is a Hilbert module $\mathscr{K}$ and that $\widetilde{B} - \widetilde{A} = S^*T$ for some $S \in \mathcal{B}(\mathscr{G}, \mathscr{K})$ and $T \in \mathcal{B}_0^l(\mathscr{G}, \mathscr{K})$ such that $S(A-z)^{-1} \in \mathcal{B}_q^l(\mathscr{H}, \mathscr{K})$ for some $z \in \rho(A) \cap \rho(B)$. Then B is a compact perturbation of A and $\sigma_{\rm ess}(B) = \sigma_{\rm ess}(A)$.

The results which follow are either corollaries of Theorem 3.1 or are versions of the theorem based on essentially the same proof. We shall use the results and the terminology of the Appendix. We begin with the simplest corollary which nevertheless covers interesting examples. Note that X is always assumed non-compact.

Theorem 3.5 *Assume that $(\mathscr{G}, \mathscr{H})$ is a compact stable Friedrichs X-module over a locally compact abelian group X and that condition (AB) is satisfied. Assume, furthermore, that $\widetilde{A} - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some $z \in \rho(A) \cap \rho(B)$ and that $\widetilde{A} \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{G}^*)$. If $\widetilde{B} - \widetilde{A} \in \mathcal{B}_0^l(\mathscr{G}, \mathscr{G}^*)$, then the operator B is a compact perturbation of the operator A.*

Proof: We apply Theorem 3.1 with $\mathscr{K} = \mathscr{G}^*$, S the identity operator and $T = \widetilde{B} - \widetilde{A}$. Then $(\widetilde{A} - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{G})$ by (ii) of Proposition 2.22, hence $(\widetilde{A} - z)^{-1} \in \mathcal{B}_q(\mathscr{G}^*, \mathscr{G})$ by Proposition 2.23. But this is stronger than $(\widetilde{A} - z)^{-1} \in \mathcal{B}_q^r(\mathscr{G}^*, \mathscr{H})$, as follows from the Remark 2.20. ■

The next results are convenient for applications to differential operators in divergence form. In these statements we implicitly use Lemma A.1: we note that the operators $D^* a D$ and $D^* b D$ considered below belong to $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ and we denote by Δ_a and Δ_b the operators on $\mathscr{H}$ associated to them. The notation $\mathcal{B}_{00}^l(\mathscr{E}, \mathscr{E}^*)$ is introduced in (2.9).

Theorem 3.6 *Let $(\mathscr{G}, \mathscr{H})$ be a compact Friedrichs module, let $(\mathscr{E}, \mathscr{K})$ be an arbitrary Friedrichs module, and assume that we are given operators $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$ and $a, b \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ and a complex number z such that:*
(1) *The operators $D^* a D - z$ and $D^* b D - z$ are bijective maps $\mathscr{G} \to \mathscr{G}^*$,*
(2) $a - b \in \mathcal{B}_{00}^l(\mathscr{E}, \mathscr{E}^*)$,
(3) $D(\Delta_a^* - \bar{z})^{-1} \in \mathcal{B}_q^l(\mathscr{H}, \mathscr{K})$.
Then Δ_b is a compact perturbation of Δ_a.

Proof: We give a proof independent of Theorem 3.1, although we could apply this theorem. From Lemma A.1 it follows that the operators $\Delta_a - z$ and $\Delta_b - z$ extend to bijections $\mathscr{G} \to \mathscr{G}^*$ and the identity

$$R := (\Delta_a - z)^{-1} - (\Delta_b - z)^{-1} = (\Delta_a - z)^{-1} D^*(b - a) D (\Delta_b - z)^{-1}$$

holds in $\mathcal{B}(\mathscr{G}^*, \mathscr{G})$, hence in $\mathcal{B}(\mathscr{H})$. Since the domains of Δ_a and Δ_b are included in $\mathscr{G}$, we have $R\mathscr{H} \subset \mathscr{G}$. Thus, according to Corollary 2.7, it suffices to show that $R \in \mathcal{B}_0^l(\mathscr{H})$. Since the space $\mathcal{B}_0^l(\mathscr{H})$ is norm closed and since by hypothesis we can approach $b - a$ in norm in $\mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ by operators in $\mathcal{B}_0^l(\mathscr{K})$, it suffices to show that

$$(D(\Delta_a^* - \bar{z})^{-1})^* c D (\Delta_b - z)^{-1} \in \mathcal{B}_0^l(\mathscr{H})$$

if $c \in \mathcal{B}_0^l(\mathscr{K})$. But this is clear because $cD(\Delta_b - z)^{-1} \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{K})$ and $(D(\Delta_a^* - \bar{z})^{-1})^* \in \mathcal{B}_q^r(\mathscr{K}, \mathscr{H})$ by Proposition 2.10. ■

By (2.10) we have $\mathcal{K}(\mathscr{E}, \mathscr{E}^*) \subset \mathcal{B}_{00}^l(\mathscr{E}, \mathscr{E}^*)$, but the case $a - b \in \mathcal{K}(\mathscr{E}, \mathscr{E}^*)$ is trivial from the point of view of this paper. Although the space $\mathcal{B}_{00}^l(\mathscr{E}, \mathscr{E}^*)$ is much larger than $\mathcal{K}(\mathscr{E}, \mathscr{E}^*)$, we can allow still more general perturbations and obtain more explicit results if we impose more structure on the modules, cf. Remark 4.2. We now describe such an improvement for the case of X-modules, where X is a locally compact abelian group. We shall need the following fact.

Lemma 3.7 *Let X be an abelian locally compact group and let $(\mathscr{G}, \mathscr{H})$ and $(\mathscr{E}, \mathscr{K})$ be stable Friedrichs X-modules. Let $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$ and $a \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ be operators of class $C^{\mathrm{u}}(Q)$ such that $D^*aD - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some complex number z and denote Δ_a the operator on $\mathscr{H}$ associated to D^*aD. Then the operator $D(\Delta_a - z)^{-1} \in \mathcal{B}(\mathscr{H}, \mathscr{E})$ is quasilocal.*

Proof: The lemma is an easy consequence of Propositions 2.22 and 2.23. Indeed, due to Proposition 2.23, it suffices to show that the operator $D(\Delta_a - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{H}, \mathscr{E})$. We shall prove more, namely that $D(D^*aD - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{E})$. Since D is of class $C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{E})$, and due to (i) of Proposition 2.22, it suffices to show that $(D^*aD - z)^{-1}$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}^*, \mathscr{G})$. But $D^*aD - z$ is of class $C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{G}^*)$ by (i) and (iii) of Proposition 2.22 and is a bijective map $\mathscr{G} \to \mathscr{G}^*$, so the result follows from (ii) of Proposition 2.22. ■

Theorem 3.8 *Let X be an abelian locally compact group and let $(\mathscr{G}, \mathscr{H})$ be a compact stable Friedrichs X-module and $(\mathscr{E}, \mathscr{K})$ a stable Friedrichs X-modules. Assume that $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$ and $a, b \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ are operators of class $C^{\mathrm{u}}(Q)$ such that the operators $D^*aD - z$ and $D^*bD - z$ are bijective maps $\mathscr{G} \to \mathscr{G}^*$ for some complex number z. If $a - b \in \mathcal{B}_0^l(\mathscr{E}, \mathscr{E}^*)$ then Δ_b is a compact perturbation of Δ_a.*

Proof: The proof is a repetition of that of Theorem 3.6. The only difference is that we write directly

$$R = (D(\Delta_a^* - \bar{z})^{-1})^*(b - a)D(\Delta_b - z)^{-1}$$

and observe that $(b - a)D(\Delta_b - z)^{-1} \in \mathcal{B}_0^l(\mathscr{H}, \mathscr{E}^*)$ and that $(D(\Delta_a^* - \bar{z})^{-1})^*$ as an operator $\mathscr{E}^* \to \mathscr{H}$ is quasilocal by (2) of Proposition 2.10 and by the fact that the operator $D(\Delta_a^* - \bar{z})^{-1} : \mathscr{H} \to \mathscr{E}$ is quasilocal, cf Lemma 3.7. ■

4 Pseudo-differential operators

4.1 Operators in divergence form on Euclidean spaces

Our first example is in the context of Theorem 3.8. Here $X = \mathbb{R}^n$ equipped with the Lebesgue measure and $\mathscr{H} = L^2(X)$ with the obvious Hilbert X-module structure. If $s \in \mathbb{R}$ we denote by $\mathscr{H}^s$ the usual Sobolev space.

For each $s > 0$ the couple $(\mathscr{H}^s, \mathscr{H})$ is a clearly a compact Friedrichs module. Indeed, for each $\varphi \in C_0(X)$ the operator $\varphi(Q) : \mathscr{H}^s \to \mathscr{H}$ is compact. But we have more: $(\mathscr{H}^s, \mathscr{H})$ is also a stable Friedrichs X-module with respect to the additive group structure on X. In fact, if we identify as usual X^* with X with the help of the exponential function, the representation of X in $\mathscr{H}$ which defines the Hilbert X-module structure of $\mathscr{H}$ is $(V_k u)(x) = \exp(i\langle x, k\rangle)u(x)$, where $\langle x, k\rangle$ is the scalar product. Then we easily get $V_k\mathscr{H}^s \subset \mathscr{H}^s$ and $\|V_k\| \leq C(1 + |k|)^s$.

Let us describe the objects which appear in Theorem 3.8 in the present context. We fix an integer $m \geq 1$ and take $\mathscr{G} = \mathscr{H}^m$. Let $\mathscr{K} = \bigoplus_{|\alpha|\leq m} \mathscr{H}_\alpha$, where $\mathscr{H}_\alpha \equiv \mathscr{H}$, with the natural direct sum Hilbert X-module structure. Here α are multi-indices $\alpha \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then we define

$$\mathscr{E} = \bigoplus_{|\alpha|\leq m} \mathscr{H}^{m-|\alpha|} = \{(u_\alpha)_{|\alpha|\leq m} \in \mathscr{K} \mid u_\alpha \in \mathscr{H}^{m-|\alpha|}\}$$

equipped with the Hilbert direct sum structure. It is obvious that $(\mathscr{E}, \mathscr{K})$ is a stable Friedrichs X-module (but not compact).

We set $P_k = -i\partial_k$, where ∂_k is the derivative with respect to the k-th variable, and $P^\alpha = P_1^{\alpha_1} \dots P_n^{\alpha_n}$ if $\alpha \in \mathbb{N}^n$. Then for $u \in \mathscr{G}$ let $Du = (P^\alpha u)_{|\alpha|\leq m} \in \mathscr{K}$. Since

$$\|Du\|^2 = \sum_{|\alpha|\leq m} \|P^\alpha u\|^2 = \|u\|^2_{\mathscr{H}^m}$$

we see that $D : \mathscr{G} \to \mathscr{K}$ is a linear isometry. Moreover, we have defined $\mathscr{E}$ such as to have $D\mathscr{G} \subset \mathscr{E}$, hence $D \in \mathcal{B}(\mathscr{G}, \mathscr{E})$. We shall prove now that $D \in C^{\mathrm{u}}(Q; \mathscr{G}, \mathscr{E})$ (in fact, much more). We have, with natural notations,

$$V_k^{-1} D V_k = (V_k^{-1} P^\alpha V_k)_{|\alpha|\leq m} = ((P + k)^\alpha)_{|\alpha|\leq m}$$

and this a polynomial in k with coefficients in $\mathcal{B}(\mathscr{G}, \mathscr{E})$, hence the assertion.

We shall identify $\mathscr{H}^* = \mathscr{H}$ and $\mathscr{K}^* = \mathscr{K}$, which implies $\mathscr{G}^* = \mathscr{H}^{-m}$ and

$$\mathscr{E}^* = \oplus_{|\alpha|\leq m} \mathscr{H}^{|\alpha|-m}.$$

The operator $D^* \in \mathcal{B}(\mathscr{E}^*, \mathscr{G}^*)$ acts as follows:

$$D^*(u_\alpha)_{|\alpha|\leq m} = \sum_{|\alpha|\leq m} P^\alpha u_\alpha \in \mathscr{H}^{-m},$$

because $u_\alpha \in \mathscr{H}^{|\alpha|-m}$.

By taking into account the given expressions for $\mathscr{E}$ and $\mathscr{E}^*$ we see that we can identify an operator $a \in \mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ with a matrix of operators $a = (a_{\alpha\beta})_{|\alpha|,|\beta|\le m}$, where $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$ and

$$a(u_\beta)_{|\beta|\le m} = \big(\sum_{|\beta|\le m} a_{\alpha\beta} u_\beta \big)_{|\alpha|\le m}.$$

Then we clearly have

$$D^* a D = \sum_{|\alpha|,|\beta|\le m} P^\alpha a_{\alpha\beta} P^\beta. \tag{4.16}$$

which is a general version of a differential operator in divergence form. We must, however, emphasize, that our $a_{\alpha\beta}$ are not necessarily functions, they could be pseudo-differential or more general operators.

In view of the statement of the next theorem, we note that, since the Sobolev spaces are Banach X-modules, the class of regularity $C^{\mathrm{u}}(Q; \mathscr{H}^s, \mathscr{H}^t)$ is well defined for all real s, t. A bounded operator $S : \mathscr{H}^s \to \mathscr{H}^t$ belongs to this class if and only if the map $k \mapsto V_{-k} S V_k \in \mathcal{B}(\mathscr{H}^s, \mathscr{H}^t)$ is norm continuous. In particular, this condition is trivially satisfied if S is the operator of multiplication by a function, because then V_k commutes with S. Since the coefficients $a_{\alpha\beta}$ of the differential expression (4.16) are usually assumed to be functions, this is barely a restriction in the setting of the next theorem. The condition $S \in \mathcal{B}_0^l(\mathscr{H}^s, \mathscr{H}^t)$ is also well defined and it is easily seen that it is equivalent to

$$\lim_{r\to\infty} \|\theta(Q/r) S\|_{\mathscr{H}^s \to \mathscr{H}^t} \to 0 \tag{4.17}$$

where θ is a C^∞ function on X equal to zero on a neighborhood of the origin and equal to one on a neighborhood of infinity. Now we can state the following immediate consequence of Theorem 3.8.

Proposition 4.1 *Let $a_{\alpha\beta}$ and $b_{\alpha\beta}$ be operators of class $C^{\mathrm{u}}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$ and such that the operators $D^* a D - z$ and $D^* b D - z$ are bijective maps $\mathscr{H}^m \to \mathscr{H}^{-m}$ for some complex z. Let Δ_a and Δ_b be the operators in $\mathscr{H}$ associated to $D^* a D$ and $D^* b D$ respectively. Assume that*

$$\lim_{r\to\infty} \|\theta(Q/r)(a_{\alpha\beta} - b_{\alpha\beta})\|_{\mathscr{H}^{m-|\alpha|} \to \mathscr{H}^{|\alpha|-m}} = 0 \tag{4.18}$$

for each α, β, where θ is a function as above. Then Δ_b is a compact perturbation of Δ_a and the operators Δ_a and Δ_b have the same essential spectrum.

Example: In the simplest case the coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the principal parts (i.e. $|\alpha| = |\beta| = m$) are functions. Then the conditions become: $a_{\alpha\beta}$ and $b_{\alpha\beta}$ belong to $L^\infty(X)$ and $|a_{\alpha\beta}(x) - b_{\alpha\beta}(x)| \to 0$ as $|x| \to \infty$. Of course, the assumptions on the lowest order coefficients are much more general.

Example: We show here that "highly oscillating potentials" do not modify the essential spectrum. If $m = 1$ then the terms of order one of D^*aD are of the form $S = \sum_{k=1}^{n}(P_k v'_k + v''_k P_k)$, where $v'_k \in \mathcal{B}(\mathscr{H}^1, \mathscr{H})$ and $v''_k \in \mathcal{B}(\mathscr{H}, \mathscr{H}^{-1})$. Choose $v_k \in \mathcal{B}(\mathscr{H}^1, \mathscr{H})$ symmetric in $\mathscr{H}$ and let $v'_k = iv_k, v''_k = -iv_k$. Then $S = [iP, v] \equiv \mathrm{div} v$, with natural notations, can also be thought as a term of order zero. Now assume that v_k are bounded Borel functions and consider a similar term $T = [iP, w]$ for D^*bD. Then the condition $|v_k(x) - w_k(x)| \to 0$ as $|x| \to \infty$ suffices to ensure the stability of the essential spectrum. However, the difference $S - T$ could be a function which does not tend to zero at infinity in a simple sense, being only "highly oscillating". An explicit example in the case $n = 1$ is the following: a perturbation of the form $\exp(x)(1 + |x|)^{-1}\cos(\exp(x))$ is allowed because it is the derivative of $(1 + |x|)^{-1}\sin(\exp(x))$ plus a function which tends to zero at infinity.

In order to apply Proposition 4.1 we need that $D^*aD - z : \mathscr{H}^m \to \mathscr{H}^{-m}$ be bijective for some $z \in \mathbb{C}$, and similarly for b. A standard way of checking this is to require the following *coercivity condition*:

$$\textbf{(}C\textbf{)} \left\{ \begin{array}{l} \text{there are } \mu, \nu > 0 \text{ such that for all } u \in \mathscr{H}^m : \\ \sum_{|\alpha|,|\beta| \leq m} \mathrm{Re}\, \langle P^\alpha u, a_{\alpha\beta} P^\beta u \rangle \geq \mu \|u\|^2_{\mathscr{H}^m} - \nu \|u\|^2_{\mathscr{H}} \end{array} \right.$$

Example: One often imposes a stronger ellipticity condition that we describe below. Observe that the coefficients of the highest order part of D^*aD defined by $A_0 = \sum_{|\alpha|=|\beta|=m} P^\alpha a_{\alpha\beta} P^\beta$ are operators $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H})$. Then ellipticity means:

$$\textbf{(}Ell\textbf{)} \left\{ \begin{array}{l} \text{there is } \mu > 0 \text{ such that if } u_\alpha \in \mathscr{H} \text{ for } |\alpha| = m \text{ then} \\ \sum_{|\alpha|=|\beta|=m} \mathrm{Re}\, \langle u_\alpha, a_{\alpha\beta} u_\beta \rangle \geq \mu \sum_{|\alpha|=m} \|u_\alpha\|^2_{\mathscr{H}}. \end{array} \right.$$

But we emphasize that, our conditions on the lower order terms being very general (e.g. the $a_{\alpha\beta}$ could be differential operators, so the terms of formally lower order could be of order $2m$ in fact), we have to supplement the ellipticity condition **(**Ell**)** with a condition saying that the rest of the terms $A_1 = \sum_{|\alpha|+|\beta|<2m} P^\alpha a_{\alpha\beta} P^\beta$ is small with respect to A_0. For example, we may require the existence of some $\delta < \mu$ and $\gamma > 0$ such that

$$|\sum_{|\alpha|+|\beta|<2m} \mathrm{Re}\, \langle P^\alpha u, a_{\alpha\beta} P^\beta u \rangle| \leq \delta \|u\|^2_{\mathscr{H}^m} + \gamma \|u\|^2_{\mathscr{H}}. \tag{4.19}$$

This is satisfied if $A_1\mathscr{H}^m \subset \mathscr{H}^{-m+\theta}$ for some $\theta > 0$, because for each $\varepsilon > 0$ there is $c(\varepsilon) < \infty$ such that $\|u\|_{\mathscr{H}^{m-\theta}} \le \varepsilon\|u\|_{\mathscr{H}^m} + c(\varepsilon)\|u\|_{\mathscr{H}}$.

Remark 4.2 To understand the relation between $\mathcal{B}^l_{00}(\mathscr{E},\mathscr{E}^*)$ and $\mathcal{B}^l_0(\mathscr{E},\mathscr{E}^*)$ it suffices to consider that between $\mathcal{B}^l_{00}(\mathscr{H}^s,\mathscr{H}^{-t})$ and $\mathcal{B}^l_0(\mathscr{H}^s,\mathscr{H}^{-t})$ for $s,t \ge 0$, where $\mathcal{B}^l_{00}(\mathscr{H}^s,\mathscr{H}^{-t})$ is the closure of $\mathcal{B}^l_0(\mathscr{H})$ in $\mathcal{B}(\mathscr{H}^s,\mathscr{H}^{-t})$. If $s = t = 0$ then these spaces are the same, hence we get the same conditions on the coefficients $a_{\alpha\beta} - b_{\alpha\beta}$ of the principal part ($|\alpha| = |\beta| = m$) of the operator $a - b$ if we use Theorem 3.6 or 3.8. But if $s + t > 0$ then $\mathcal{B}^l_{00}(\mathscr{H}^s,\mathscr{H}^{-t})$ does not contain operators of order $s + t$, while $\mathcal{B}^l_0(\mathscr{H}^s,\mathscr{H}^{-t})$ contains such operators.

4.2 A class of pseudo-differential operators on abelian groups

In this subsection X will be a locally compact non-compact non-discrete abelian group. We also fix a finite dimensional complex Hilbert space E and take $\mathscr{H} = L^2(X;E)$ equipped with its natural Hilbert X-module structure. Note that, according to our conventions, the unitary representation of X^* is given by the multiplication operators $V_k = k(Q)$.

Let $w : X^* \to [1,\infty[$ be a continuous function satisfying $w(k) \to \infty$ as $k \to \infty$ and such that $w(k'k) \le \omega(k')w(k)$ holds for some function ω and all k',k. We shall assume that ω is the smallest function satisfying the preceding estimate. It is clear then that ω is sub-multiplicative in the sense defined in Remark 2.17 (see [Ho, Section 10.1] for this construction).

Then $w(P)$ is a self-adjoint operator on $\mathscr{H}$ with $w(P) \ge 1$. We denote $\mathscr{H}^w = \mathcal{D}(w(P))$ and equip it with the Banach X-module structure given by the norm $\|u\|_w = \|w(P)u\|$ and the representation $V_k|\mathscr{H}^w$. Obviously, this space is a generalization of the usual notion of Sobolev spaces.

Lemma 4.3 *$(\mathscr{H}^w,\mathscr{H})$ is a compact stable Friedrichs X-module.*

Proof: If $\varphi \in C_0(X)$ then $\varphi(Q)w(P)^{-1}$ is a compact operator because w^{-1} belongs to $C_0(X)$, hence $\varphi(Q) \in \mathcal{K}(\mathscr{H}^w,\mathscr{H})$. Then observe that $V_k^{-1}w(P)V_k = w(kP)$ and $w(kP) \le \omega(k)w(P)$. Thus V_k leaves stable $\mathscr{H}^w$ and we have the estimate $\|V_k\|_{\mathcal{B}(\mathscr{H}^w)} \le \omega(k)$. ■

We shall consider now an operator A on $\mathscr{H}$ such that there are w as above and an operator $\widetilde{A} \in \mathcal{B}(\mathscr{H}^w,\mathscr{H}^{w*})$ such that $\widetilde{A} - z : \mathscr{H}^w \to \mathscr{H}^{w*}$ is bijective for some complex z and such that A is the operator induced by $\widetilde{A}$ in $\mathscr{H}$ (see the Appendix). For example, the constant coefficients case with $E = \mathbb{C}$ corresponds to the choice $A = h(P)$ with $h : X^* \to \mathbb{C}$ a Borel function such that $c'w^2 \le 1 + |h| \le c''w^2$ and such that the range of h is not dense in $\mathbb{C}$.

Theorem 3.5 is quite well adapted to show the stability of the essential spectrum of such operators under perturbations which are small at infinity. We stress that the differential operators covered by these results can be of any order and that in the usual case when the coefficients are complex measurable functions a condition of the type $\widetilde{A} \in C^{\mathrm{u}}(Q; \mathscr{H}^w, \mathscr{H}^{w*})$ is very general, if not automatically satisfied (see the remark at the end of this subsection). Hence the only condition really relevant in this context is $\widetilde{B} - \widetilde{A} \in \mathcal{B}_0^l(\mathscr{H}^w, \mathscr{H}^{w*})$ and the main point is that it allows perturbations of the higher order coefficients even in the non-smooth case.

It is clear that these results can be used to establish the stability of the essential spectrum of pseudo-differential operators on finite dimensional vector spaces over local fields, cf. [Sa, Ta], under perturbations of the same order.

We shall give only one explicit example of some physical interest, that of Dirac operators. Let $X = \mathbb{R}^n$ and let $\alpha_0 \equiv \beta, \alpha_1, \ldots, \alpha_n$ be symmetric operators on E such that $\alpha_j\alpha_k + \alpha_k\alpha_j = \delta_{jk}$. Then the free Dirac operator is $D = \sum_{k=1}^n \alpha_k P_k + m\beta$ for some real number m. The natural compact stable Friedrichs X-module now is $(\mathscr{H}^{1/2}, \mathscr{H})$ where $\mathscr{H}^s$ are usual Sobolev spaces of E-valued functions.

Proposition 4.4 *Let V, W be measurable functions on X with values symmetric operators on E and such that the operators of multiplication by V and W define continuous maps $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ and $V - W \in \mathcal{B}_0(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$. Assume that $D + V + i$ and $D + W + i$ are bijective maps $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$. Then $D + V$ and $D + W$ induce self-adjoint operators A and B in $\mathscr{H}$, B is a compact perturbation of A, and $\sigma_{\mathrm{ess}}(B) = \sigma_{\mathrm{ess}}(A)$.*

This follows immediately from Theorem 3.5. We stress that the main new feature of this result is that the "unperturbed" operator A is locally as singular as the "perturbed" one B. The assumptions imposed on V, W are quite general, compare with [Ar, AY, Kl, N1, N2].

Remark: We shall discuss here the relation between the abstract class of operators A considered in this subsection and the notion of hypoellipticity due to Hörmander. For this we shall consider the case of differential operators on $\mathbb{R}^n$ (which is identified with its dual group in the standard way). Assume first that h is a polynomial on $\mathbb{R}^n$ and that $A = h(P)$. Then the function defined by $w(k)^4 = \sum_\alpha |h^{(\alpha)}(k)|^2$ satisfies $w(k' + k) \leq (1 + c|k'|)^{m/2} w(k)$, where c is a number and m is the order of h, see [Ho, Example 10.1.3]. Now the "form domain" of the operator $h(P)$ in $L^2(\mathbb{R}^n)$ is the space $\mathscr{G} = \mathcal{D}(|h(P)|^{1/2})$ and this domain is stable under $V_k = \exp i\langle k, Q\rangle$ if and only the function w satisfies $w^2 \leq c(1 + |h|)$, see Lemma 7.6.7 in [ABG]. On the other hand, Definition 11.1.2 and Theorem 11.1.3 from [Ho] show that A is hypoelliptic if and only if $h^{(\alpha)}(k)/h(k) \to 0$ when $k \to \infty$, for all $\alpha \neq 0$. So in this case we have $c'w^2 \leq 1 + |h| \leq c''w^2$ and the operator $A = h(P)$ satisfies the conditions of this subsection if $h(\mathbb{R}^n)$ is not dense in $\mathbb{C}$. If $n = 2$ then $h(k) = k_1^4 + k_2^2$ is a simple example of polynomial which has all these properties but is not elliptic. See [GM, Subsections 2.7-2.10] for the case of matrix valued functions h.

5 Abstract Riemannian manifolds

Let $\mathscr{H}, \mathscr{K}$ be two Hilbert spaces identified with their adjoints and d a closed densely defined operator mapping $\mathscr{H}$ into $\mathscr{K}$. Let $\mathscr{G} = \mathcal{D}(\mathrm{d})$ equipped with the graph norm, so $\mathscr{G} \subset \mathscr{H}$ continuously and densely and $\mathrm{d} \in \mathcal{B}(\mathscr{G}, \mathscr{K})$.

Then the quadratic form $\|\mathrm{d}u\|^2_{\mathscr{K}}$ on $\mathscr{H}$ with domain $\mathscr{G}$ is positive densely defined and closed. Let Δ be the positive self-adjoint operator on $\mathscr{H}$ associated to it. In fact $\Delta = \mathrm{d}^*\mathrm{d}$, where the adjoint d^* of d is a closed densely defined operator mapping $\mathscr{K}$ into $\mathscr{H}$.

Now let $\lambda \in \mathcal{B}(\mathscr{H})$ and $\Lambda \in \mathcal{B}(\mathscr{K})$ be self-adjoint and such that $\lambda \geq c$ and $\Lambda \geq c$ for some real $c > 0$. Then we can define new Hilbert spaces $\widetilde{\mathscr{H}}$ and $\widetilde{\mathscr{K}}$ as follows:

$$
(*) \quad \begin{cases} \widetilde{\mathscr{H}} = \mathscr{H} \text{ as vector space and } \langle u \mid v \rangle_{\widetilde{\mathscr{H}}} = \langle u \mid \lambda v \rangle_{\mathscr{H}}, \\ \widetilde{\mathscr{K}} = \mathscr{K} \text{ as vector space and } \langle u \mid v \rangle_{\widetilde{\mathscr{K}}} = \langle u \mid \Lambda v \rangle_{\mathscr{K}}. \end{cases}
$$

Since $\mathscr{H} = \widetilde{\mathscr{H}}$ and $\mathscr{K} = \widetilde{\mathscr{K}}$ as topological vector spaces; the operator $\mathrm{d} : \mathscr{G} \subset \widetilde{\mathscr{H}} \to \widetilde{\mathscr{K}}$ is still a closed densely defined operator, hence the quadratic form $\|\mathrm{d}u\|^2_{\widetilde{\mathscr{K}}}$ on $\widetilde{\mathscr{H}}$ with domain $\mathscr{G}$ is positive, densely defined and closed. We shall denote by $\widetilde{\Delta}$ the positive self-adjoint operator on $\widetilde{\mathscr{H}}$ associated to it.

We can express $\widetilde{\Delta}$ in more explicit terms as follows. Denote $\widetilde{\mathrm{d}}$ the operator d when viewed as operator acting from $\widetilde{\mathscr{H}}$ to $\widetilde{\mathscr{K}}$. Then $\widetilde{\Delta} = \widetilde{\mathrm{d}}^*\widetilde{\mathrm{d}}$, where $\widetilde{\mathrm{d}}^* : \mathcal{D}(\widetilde{\mathrm{d}}^*) \subset \widetilde{\mathscr{K}} \to \widetilde{\mathscr{H}}$ is the adjoint of $\widetilde{\mathrm{d}} = \mathrm{d}$ with respect to the new Hilbert space structures (the spaces $\widetilde{\mathscr{H}}, \widetilde{\mathscr{K}}$ being also identified with their adjoints). It is easy to check that $\widetilde{\mathrm{d}}^* = \lambda^{-1}\mathrm{d}^*\Lambda$. Thus $\widetilde{\Delta} = \lambda^{-1}\mathrm{d}^*\Lambda\mathrm{d}$.

Now let (X, ρ) be a proper locally compact metric space (see Subsection 2.4) and let us assume that $\mathscr{H}$ and $\mathscr{K}$ are Hilbert X-modules.

Definition 5.1 *A closed densely defined map* $\mathrm{d} : \mathscr{H} \to \mathscr{K}$ *is a* first order operator *if there is* $C \in \mathbb{R}$ *such that for each bounded Lipschitz function* $\varphi : X \to \mathbb{R}$ *the form* $[\mathrm{d}, \varphi(Q)]$ *is a bounded operator and* $\|[\mathrm{d}, \varphi(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leq C \operatorname{Lip} \varphi$.

Here

$$\text{Lip}\,\varphi = \inf_{x \neq y} |\varphi(x) - \varphi(y)| \rho(x,y)^{-1}.$$

In more explicit terms, we require

$$|\langle \mathrm{d}^* u, \varphi(Q)v\rangle_{\mathscr{H}} - \langle u, \varphi(Q)\mathrm{d}v\rangle_{\mathscr{K}}| \leq C\,\text{Lip}\,\varphi\,\|u\|_{\mathscr{K}}\|v\|_{\mathscr{H}}$$

for all $u \in \mathcal{D}(\mathrm{d}^*)$ and $v \in \mathcal{D}(d)$. Thus $\langle \mathrm{d}^* u, \varphi(Q)v\rangle - \langle u, \varphi(Q)\mathrm{d}v\rangle$ is a sesquilinear form on the dense subspace $\mathcal{D}(\mathrm{d}^*) \times \mathcal{D}(\mathrm{d})$ of $\mathscr{K} \times \mathscr{H}$ which is continuous for the topology induced by $\mathscr{H} \times \mathscr{K}$. Hence there is a unique continuous operator $[\mathrm{d}, \varphi(Q)] : \mathscr{H} \to \mathscr{K}$ such that

$$\langle \mathrm{d}^* u, \varphi(Q)v\rangle_{\mathscr{H}} - \langle u, \varphi(Q)\mathrm{d}v\rangle_{\mathscr{K}} = \langle u, [\mathrm{d}, \varphi(Q)]v\rangle_{\mathscr{K}}$$

for all $u \in \mathcal{D}(\mathrm{d}^*)$, $v \in \mathcal{D}(\mathrm{d})$ and $\|[\mathrm{d}, \varphi(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leq C\,\text{Lip}\,\varphi$.

Lemma 5.2 *The operator* $\mathrm{d}(\Delta+1)^{-1}$ *is quasilocal.*

Proof: We shall prove that $S := \mathrm{d}(\Delta+1)^{-1}$ is a quasilocal operator with the help of Corollary 2.14, more precisely we show that $[S, \varphi(Q)]$ is a bounded operator if φ is a positive Lipschitz function. Let $\varepsilon > 0$ and $\varphi_\epsilon = \varphi(1+\varepsilon\varphi)^{-1}$. Then φ_ε is a bounded function with $|\varphi_\varepsilon| \leq \varepsilon^{-1}$ and

$$|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| = \frac{|\varphi(x) - \varphi(y)|}{(1+\varepsilon\varphi(x))(1+\varepsilon\varphi(y))} \leq |\varphi(x) - \varphi(y)|$$

hence $\text{Lip}\,\varphi_\varepsilon \leq \text{Lip}\,\varphi$. Let $v \in \mathcal{D}(\mathrm{d})$ we have for all $u \in \mathcal{D}(\mathrm{d}^*)$:

$$\begin{aligned} |\langle \mathrm{d}^* u, \varphi_\epsilon(Q)v\rangle_{\mathscr{H}}| &= |\langle u, \varphi_\epsilon(Q)\mathrm{d}v\rangle_{\mathscr{K}} + \langle u, [\mathrm{d}, \varphi_\varepsilon(Q)]v\rangle_{\mathscr{K}}| \\ &\leq \|u\|_{\mathscr{K}}(\varepsilon^{-1}\|\mathrm{d}v\|_{\mathscr{K}} + C\,\text{Lip}\,\varphi_\varepsilon\,\|u\|_{\mathscr{H}}). \end{aligned}$$

Hence $\varphi_\varepsilon(Q)v \in \mathcal{D}(\mathrm{d}^{**}) = \mathcal{D}(\mathrm{d})$ because d is closed. Thus $\varphi_\varepsilon(Q)\mathcal{D}(\mathrm{d}) \subset \mathcal{D}(\mathrm{d})$ and by the closed graph theorem we get $\varphi_\varepsilon(Q) \in \mathcal{B}(\mathscr{G})$, where $\mathscr{G}$ is the domain of d equipped with the graph topology. This also implies that $\varphi_\varepsilon(Q)$ extends to an operator in $\mathcal{B}(\mathscr{G}^*)$ (note that $\varphi_\varepsilon(Q)$ is symmetric in $\mathscr{H}$).

Now, if we think of d as a continuous operator $\mathscr{G} \to \mathscr{K}$, then it has an adjoint $\mathrm{d}^* : \mathscr{K} \to \mathscr{G}^*$ which is the unique continuous extension of the operator $\mathrm{d}^* : \mathcal{D}(\mathrm{d}^*) \subset \mathscr{K} \to \mathscr{H} \subset \mathscr{G}^*$. Thus the canonical extension of Δ to an element of $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ is the product of $\mathrm{d} : \mathscr{G} \to \mathscr{K}$ with $\mathrm{d}^* : \mathscr{K} \to \mathscr{G}^*$ (note $\mathcal{D}(\mathrm{d})$ is the form domain of Δ). Then it is trivial to justify that we have in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$:

$$[\Delta, \varphi_\varepsilon(Q)] = [\mathrm{d}^*, \varphi_\varepsilon(Q)]\mathrm{d} + \mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)].$$

Here $[\mathrm{d}^*, \varphi_\varepsilon(Q)] = [\varphi_\varepsilon(Q), \mathrm{d}]^* \in \mathcal{B}(\mathscr{K}, \mathscr{H})$. Since $\Delta + 1 : \mathscr{G} \to \mathscr{G}^*$ is a linear homeomorphism, we then have in $\mathcal{B}(\mathscr{G}^*, \mathscr{G})$:

$$\begin{aligned}
[\varphi_\varepsilon(Q), (\Delta+1)^{-1}] &= (\Delta+1)^{-1}[\Delta, \varphi_\varepsilon(Q)](\Delta+1)^{-1} \\
&= (\Delta+1)^{-1}[\varphi_\varepsilon(Q), \mathrm{d}]^* \mathrm{d}(\Delta+1)^{-1} \\
&+ (\Delta+1)^{-1}\mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)](\Delta+1)^{-1}.
\end{aligned}$$

Finally, taking once again into account the fact that $\varphi_\varepsilon(Q)$ leaves $\mathscr{G}$ invariant, we have:

$$\begin{aligned}
[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}] = \; & [\varphi_\varepsilon(Q), \mathrm{d}](\Delta+1)^{-1} \\
& + \mathrm{d}(\Delta+1)^{-1}[\varphi_\varepsilon(Q), \mathrm{d}]^* \mathrm{d}(\Delta+1)^{-1} \\
& + \mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*[\mathrm{d}, \varphi_\varepsilon(Q)](\Delta+1)^{-1}.
\end{aligned}$$

Hence:

$$\begin{aligned}
\|[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \leq \; & \|[\varphi_\varepsilon(Q), \mathrm{d}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H})} \\
& + \|\mathrm{d}(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|[\varphi_\varepsilon(Q), \mathrm{d}]^*\|_{\mathcal{B}(\mathscr{K},\mathscr{H})}\|\mathrm{d}(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \\
& + \|\mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{K})}\|[\mathrm{d}, \varphi_\varepsilon(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{H})}.
\end{aligned}$$

The most singular factor here is

$$\|\mathrm{d}(\Delta+1)^{-1}\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{K})} \leq \|\mathrm{d}\|_{\mathcal{B}(\mathscr{G},\mathscr{K})}\|(\Delta+1)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})}\|\mathrm{d}^*\|_{\mathcal{B}(\mathscr{K},\mathscr{G}^*)}$$

and this is finite. Thus we get for a finite constant C_1:

$$\begin{aligned}
\|[\varphi_\varepsilon(Q), \mathrm{d}(\Delta+1)^{-1}]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} &\leq C_1\|[\mathrm{d}, \varphi_\varepsilon(Q)]\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} \\
&\leq C_1\, C\, \mathrm{Lip}\, \varphi_\varepsilon \leq C_1\, C\, \mathrm{Lip}\, \varphi
\end{aligned}$$

Now let $u \in \mathscr{K}_\mathrm{c}$ and $v \in \mathscr{H}_\mathrm{c}$. We get:

$$\begin{aligned}
|\langle \varphi(Q)u, \mathrm{d}(\Delta+1)^{-1}v\rangle - \langle u, \mathrm{d}(\Delta+1)^{-1}\varphi(Q)v\rangle| &= \\
= \lim_{\varepsilon\to 0} |\langle \varphi_\varepsilon(Q)u, \mathrm{d}(\Delta+1)^{-1}v\rangle - \langle u, \mathrm{d}(\Delta+1)^{-1}\varphi_\varepsilon(Q)v\rangle| & \\
\leq C_1\, C\, \mathrm{Lip}\, \varphi &
\end{aligned}$$

Thus $[\varphi(Q), \mathrm{d}(\Delta+1)^{-1}]$ is a bounded operator. ■

Theorem 5.3 *Let (X, ρ) be a proper locally compact metric space. Assume that $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs X-module and that $\mathscr{K}$ is a Hilbert X-module. Let $\mathrm{d}, \lambda, \Lambda$ be operators satisfying the following conditions:*

(i) d *is a closed first order operator from* $\mathscr{H}$ *to* $\mathscr{K}$ *with* $\mathcal{D}(\mathrm{d}) = \mathscr{G}$*;*
(ii) λ *is a bounded self-adjoint operator on* $\mathscr{H}$ *with* $\inf \lambda > 0$ *and such that* $\lambda - 1 \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ *(e.g.* $\lambda - 1 \in \mathcal{B}_0(\mathscr{H})$*);*
(iii) Λ *is a bounded self-adjoint operator on* $\mathscr{K}$ *with* $\inf \Lambda > 0$ *and such that* $\Lambda - 1 \in \mathcal{B}_0(\mathscr{K})$.
Then the self-adjoint operators Δ *and* $\widetilde{\Delta}$ *have the same essential spectrum.*

Proof: In this proof, we shall consider $\widetilde{\Delta}$ as an operator acting on $\mathscr{H}$. Since $\widetilde{\mathscr{H}} = \mathscr{H}$ as topological vector spaces and the notion of spectrum is purely topological, $\widetilde{\Delta}$ is a closed densely defined operator on $\mathscr{H}$ and it has the same spectrum as the self-adjoint $\widetilde{\Delta}$ on $\widetilde{\mathscr{H}}$. Moreover, if we define the essential spectrum $\sigma_{\text{ess}}(A)$ as the set of $z \in \mathbb{C}$ such that either $\ker(A - z)$ is infinite dimensional or the range of $A - z$ is not closed, we see that the essential spectrum is a topological notion, so $\sigma_{\text{ess}}(\widetilde{\Delta})$ is the same, whether we think of $\widetilde{\Delta}$ as operator on $\mathscr{H}$ or on $\widetilde{\mathscr{H}}$. Finally, with this definition of σ_{ess} we have $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$ if $(A - z)^{-1} - (B - z)^{-1}$ is compact operator for some $z \in \rho(A) \cap \rho(B)$.

Thus it suffices to prove that $(\Delta + 1)^{-1} - (\widetilde{\Delta} + 1)^{-1} \in \mathcal{K}(\mathscr{H})$. Now we observe that

$$\widetilde{\Delta} + 1 = \lambda^{-1} d^* \Lambda d + 1 = \lambda^{-1}(\mathrm{d}^* \Delta \mathrm{d} + \lambda)$$

and $\Delta_\Lambda = \mathrm{d}^* \Lambda \mathrm{d}$ is the positive self-adjoint operator on $\mathscr{H}$ associated to the closed quadratic form $\|\mathrm{d}u\|^2_{\widetilde{\mathscr{K}}}$ on $\mathscr{H}$ with domain $\mathscr{G}$. Thus $(\widetilde{\Delta} + 1)^{-1} = (\Delta_\Lambda + \lambda)^{-1} \lambda$ and

$$(\widetilde{\Delta} + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1} = (\Delta_\Lambda + \lambda)^{-1}(\lambda - 1) = [(\lambda - 1)(\Delta_\Lambda + \lambda)^{-1}]^*$$

The range of $(\Delta_\Lambda + \lambda)^{-1}$ is included in the form domain of $\Delta_\Lambda + \lambda$, which is $\mathscr{G}$. The map $(\Delta_\Lambda + \lambda)^{-1} : \mathscr{H} \to \mathscr{G}$ is continuous, by closed graph theorem, and $\lambda - 1 : \mathscr{G} \to \mathscr{H}$ is compact. Hence $(\widetilde{\Delta} + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1}$ is compact. Similarly:

$$(\Delta + 1)^{-1} - (\Delta_\Lambda + \lambda)^{-1} = (\mathrm{d}^* \mathrm{d} + 1)^{-1} - (\mathrm{d}^* \Delta \mathrm{d} + 1)^{-1} \in \mathcal{K}(\mathscr{H})$$

For this we use Theorem 3.6 with: $\mathscr{E} = \mathscr{K}$, $D = \mathrm{d}$, $a = 1$, $b = \Lambda$ and $z = -1$. Since $\mathrm{d}^* \mathrm{d}$ and $\mathrm{d}^* \Lambda d$ are positive self-adjoint operators on $\mathscr{H}$ with the same form domain $\mathscr{G}$, the first condition of Theorem 3.6 is satisfied. Then the second condition holds because $\Lambda - 1 \in \mathcal{B}_0(\mathscr{K})$. Thus it remains to observe that the operator $\mathrm{d}(\Delta + 1)^{-1}$ is quasilocal by Lemma 5.2. ∎

Remark: The map $\varphi \mapsto \varphi(Q)$ provides $\widetilde{\mathscr{H}}$ with a Banach X-module structure. $\widetilde{\mathscr{H}}$ is a Hilbert X-module for this structure if and only if λ is $C_0(X)$-linear. Indeed, the adjoint of $\varphi(Q)$ in $\widetilde{\mathscr{H}}$ is $\lambda^{-1}\overline{\varphi}(Q)\lambda$ and $\lambda^{-1}\overline{\varphi}(Q)\lambda = \overline{\varphi}(Q)$ is equivalent to $[\lambda, \overline{\varphi}(Q)] = 0$.

We shall consider now an application of Theorem 5.3 to concrete Riemannian manifolds. It will be clear from what follows that we could treat Lipschitz manifolds with measurable metrics (see [DP, Hi, Te, We] for example), but the case of C^1 manifolds with locally bounded metrics suffices as an example.

Let X be a *non-compact* differentiable manifold of class C^1 and T^*X be its cotangent manifold, a topological vector fiber bundle over X whose fiber over x will be denoted T_x^*X. If $u : X \to \mathbb{R}$ is differentiable then $\mathrm{d}u(x) \in T_x^*X$ is its differential at the point x and its differential $\mathrm{d}u$ is a section of T^*X. Thus for the moment d is a linear map defined on the space of real $C^1(X)$ functions to the space of sections of T^*X.

We now assume that X is equipped with a measurable locally bounded Riemannian structure. To be precise, each T_x^*X is equipped with a scalar product $\langle\cdot|\cdot\rangle_x$ and the associated norm $\|\cdot\|_x$ satisfying the following condition:

$$(R) \begin{cases} \text{if } v \text{ is a continuous section of } T^*X \text{ over a compact set } K \text{ such that} \\ v(x) \neq 0 \text{ for } x \in K, \text{ then } x \mapsto \|v(x)\|_x \text{ is a bounded Borel map on} \\ K \text{ and } \|v(x)\|_x \geq c \text{ for some number } c > 0 \text{ and all } x \in K. \end{cases}$$

This structure allows one to construct a metric compatible with the topology on X (if the scalar products do not depend continuously on x, this is not a completely trivial matter, see the references above). Since X was assumed to be non-compact, the metric space X is proper in the sense defined in Subsection 2.4 if and only if it is a complete metric space.

It will also be convenient to complexify these structures (i.e. replace T_x^*X by $T_x^*X \otimes \mathbb{C}$ and extend the scalar product to the complexification as usual) but to keep the same notations (we could, of corse, work with real Hilbert spaces, but this would not be coherent with the conventions of the rest of the paper).

Now let μ be a positive measure on X such that:

$$(M) \begin{cases} \mu \text{ is absolutely continuous and its density is locally bounded} \\ \text{and locally bounded from below by strictly positive constants.} \end{cases}$$

We shall take $\mathscr{H} = L^2(X, \mu)$ and $\mathscr{K}$ equal to the completion of the space of continuous sections of T^*X with compact support under the natural norm

$$\|v\|^2_{\mathscr{K}} = \int_X \|v(x)\|^2_x d\mu(x).$$

In fact, $\mathscr{K}$ is the space of (suitably defined) square integrable sections of T^*X.

The operator of exterior differentiation d induces a linear map $C^1_{\mathrm{c}}(X) \to \mathscr{K}$ which is easily seen to be closable as operator from $\mathscr{H}$ to $\mathscr{K}$ (this is a purely local problem and the hypotheses we put on the metric and the measure allow us to reduce ourselves to the Euclidean case). We shall keep the notation d for its closure and we note that its domain $\mathscr{G}$ is the natural first order Sobolev space $\mathscr{H}^1$ defined in this context as the closure of $C^1_{\mathrm{c}}(X)$ under the norm

$$\|u\|^2_{\mathscr{H}^1} = \int_X \Big(|u(x)|^2 + \|\mathrm{d}u(x)\|^2_x\Big) d\mu(x).$$

Note that the self-adjoint operator $\Delta = \mathrm{d}^*\mathrm{d}$ is a slightly generalized form of the Laplace operator associated to the Riemannian structure of X because μ is not necessarily the Riemannian volume element (but we could choose it so).

We shall now consider perturbations of this structure. We assume that the perturbation preserves the local structure, although Theorem 5.3 allows us to go much further.

Proposition 5.4 *Let X be a non-compact manifold of class C^1 equipped with a Riemannian structure and a measure satisfying the conditions (R) and (M) and such that X is complete for the associated metric. Let λ be a bounded Borel function on X such that $\lambda(x) \geq c$ for some number $c > 0$ and $\lim_{x\to\infty} \lambda(x) = 1$. Assume that a new Riemann structure verifying (R) is given on X such that the associated norms $\|\cdot\|'_x$ verify $\alpha(x)\|\cdot\|_x \leq \|\cdot\|'_x \leq \beta(x)\|\cdot\|_x$ for some functions α, β such that $\lim_{x\to\infty} \alpha(x) = \lim_{x\to\infty} \beta(x) = 1$. Let Δ be as above and Δ' be the analog operator associated to the second Riemann structure and to the measure $\mu' = \lambda\mu$. Then $\sigma_{\mathrm{ess}}(\Delta) = \sigma_{\mathrm{ess}}(\Delta')$.*

Proof: We check that the assumptions of Theorem 5.3 are satisfied. We noted above that X is a proper metric space for the metric associated to the initial Riemann structure. The spaces $\mathscr{H}, \mathscr{K}$ have obvious X-module structures and for each $\varphi \in C_{\mathrm{c}}(X)$ the operator $\varphi(Q) : \mathscr{H}^1 \to \mathscr{H}$ is compact. Indeed, by using partitions of unity, we may assume that the support of φ is contained in the domain of a local chart and then we are reduced to a known fact in the Euclidean case. Thus $(\mathscr{G}, \mathscr{H})$ is a compact Friedrichs X-module. To see that d is a first order operator we observe that if φ is Lipschitz then $[\mathrm{d}, \varphi]$ is the operator of multiplication by the differential $\mathrm{d}\varphi$ of φ and the estimate ess-sup $\|\mathrm{d}\varphi(x)\|_x \leq$ Lip φ is easy to obtain. The conditions on λ in Theorem 5.3 are trivially verified. So it remains to consider the operator Λ. For each $x \in X$ there is a unique operator $\Lambda_0(x)$ on $T^*_x X$ such that $\langle u|v\rangle'_x = \langle u|\Lambda_0(x)v\rangle_x$ for all $u, v \in T^*_x X$ and we have $\alpha(x)^2 \leq \Lambda_0(x) \leq \beta(x)^2$ by hypothesis. Here the inequalities must be interpreted with respect to the initial scalar product on $T^*_x X$. Thus the operator Λ on $\mathscr{K}$ is just the operator of multiplication by the function $x \mapsto \lambda(x)\Lambda(x)$ and the condition (iii) of Theorem 5.3 is clearly satisfied. ■

We note that if μ is the measure canonically associated to the initial Riemann structure then we can choose λ such that μ' be the measure associated to the second Riemann structure. In particular, if we have two locally L^∞ Riemannian metrics on a non-compact C^1 manifold, if the structures are asymptotically equivalent in the sense made precise in Proposition 5.4, and if the manifold is complete for one of the metrics (hence for the other too), then the Laplacians associated to the two metrics have the same essential spectrum. We stress that this is known, and easy to prove if one uses some local regularity estimates for elliptic equations, if one of the metrics is locally Lipschitz or Hölder continuous (in the second case, the required regularity estimate is not so easy, however). On the other hand, it is clear that our arguments, although quite elementary, cover situations when X is not of class C^1 and the metrics are only L^p. In fact, the arguments work without any modification if X is a Lipschitz manifold and a countable atlas has been chosen, because then the tangent space are well defined almost everywhere and the absolute continuity notions that we have used make sense.

6 Weakly vanishing perturbations

6.1 General remarks

The algebraic framework introduced in Section 2 and the abstract Theorems 3.1 and 3.6 should allow one to go beyond the primitive idea of "vanishing at infinity perturbation" that we considered so far. Indeed, we recall that, according to our general definitions, the multiplier algebra of a Banach module should be the prototype of the notion of vanishing at infinity operator. The purpose of this section is to give examples of such extensions.

Let X be a locally compact non-compact topological space and let $\mathscr{H}$ be a Hilbert X-module. Then the C^*-algebra of the operators $\varphi(Q)$ with $\varphi \in C_0(X)$ is the initial multiplier algebra of $\mathscr{H}$ but, due to Lemma 2.12, we can also consider on $\mathscr{H}$ the Hilbert module structure defined by the algebra consisting of the operators $\varphi(Q)$ with φ an arbitrary bounded Borel function on X. These operators cannot be considered as vanishing at infinity, but we could consider some other subalgebras of $B(X)$. It is easy to see that each function $\varphi \in B_0(X)$ can be written as a product $\varphi = \theta\psi$ with $\theta \in C_0(X)$ and $\psi \in B_0(X)$ (this is obvious if one accepts the Cohen-Hewitt Theorem A.3). Thus we get no improvement by going from $C_0(X)$ to $B_0(X)$. Hence we have to point out a class of functions which vanish at infinity in a weaker sense.

A natural idea is to extend the usual notion of neighborhood of infinity. It is usual to define the filter of neighborhoods of infinity as the family of subsets of X with relatively compact complement; we shall call this the *Fréchet filter*. If

$\mathcal{F}$ is a filter on X finer than the Fréchet filter then a function $\varphi : X \to \mathbb{C}$ such that $\lim_{\mathcal{F}} \varphi = 0$ can naturally be thought as convergent to zero at infinity in a generalized sense (we recall that $\lim_{\mathcal{F}} \varphi = 0$ means that for each $\varepsilon > 0$ the set of points x such that $|\varphi(x)| < \varepsilon$ belongs to $\mathcal{F}$). It is clear that

$$B_{\mathcal{F}}(X) := \{\varphi \in B(X) \mid \lim_{\mathcal{F}} \varphi = 0\} \tag{6.20}$$

is a C^*-algebra and that we can consider on $\mathscr{H}$ the Hilbert module structure defined by the multiplier algebra $\mathcal{M}_{\mathcal{F}} := \{\varphi(Q) \mid \varphi \in B_{\mathcal{F}}(X)\}$. We will be interested in the corresponding classes of vanishing at infinity or quasilocal operators. To be precise, we shall speak in this context of (left or right) *$\mathcal{F}$-vanishing at infinity* operators or of (left or right) *$\mathcal{F}$-quasilocal operators*. Below and later on we use the notation $N^{\mathrm{c}} = X \setminus N$.

Lemma 6.1 *Let $\mathscr{H}, \mathscr{K}$ be Hilbert X-modules. Then an operator $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ is right $\mathcal{F}$-quasilocal if and only if for each Borel set N with $N^{\mathrm{c}} \in \mathcal{F}$ and for each $\varepsilon > 0$ there is a Borel set $F \in \mathcal{F}$ such that $\|\chi_F(Q) S \chi_N(Q)\| \leq \varepsilon$.*

Proof: We note first that the family of operators χ_N, where N runs over the family of Borel sets with complement in $\mathcal{F}$, is an approximate unit for $B_{\mathcal{F}}(X)$. Indeed, if $\varepsilon > 0$ and $\varphi \in B_{\mathcal{F}}(X)$ then the set $N = \{x \mid |\varphi(x)| > \varepsilon\}$ has the properties required above and $\sup_x |\varphi(x)(1 - \chi_N(x))| \leq \varepsilon$. Thus, according to Proposition 2.9, S is right $\mathcal{F}$-quasilocal if and only if $S\chi_N(Q)$ is left $\mathcal{F}$-vanishing at infinity for each N. Now the result follows from (2.6). ∎

The main restriction we have to impose on $\mathcal{F}$ comes from the fact that the Friedrichs couple $(\mathscr{G}, \mathscr{H})$ which is involved in the definition of the class of operators that we study must be such that $\varphi(Q) \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ if $\varphi \in B_{\mathcal{F}}(X)$. That this is an important restriction follows from the following easily proven result:

Lemma 6.2 *Let X be an Euclidean space, $\mathscr{H} = L^2(X)$, and let $\mathscr{G} = \mathscr{H}^s$ be a Sobolev space of order $s > 0$. If $\varphi \in B(X)$ then $\varphi(Q) \in \mathcal{K}(\mathscr{G}, \mathscr{H})$ if and only if*

$$\lim_{a \to \infty} \int_{|x-a| \leq 1} |\varphi(x)| dx = 0. \tag{6.21}$$

The importance of such a condition in questions of stability of the essential spectrum has been noticed in [He, LV, OS, We]. That it is a natural condition follows also from the characterizations that we shall give below in a more general context.

Let X be a locally compact non-compact abelian group. We shall say that a function $\varphi \in B(X)$ is *weakly vanishing (at infinity)* if

$$\lim_{a \to \infty} \int_{a+K} |\varphi(x)| dx = 0 \text{ for each compact set } K. \tag{6.22}$$

We shall denote by $B_{\mathrm{w}}(X)$ the set of functions φ satisfying (6.22). This is clearly a C^*-algebra. Note that it suffices that the convergence condition in (6.22) be satisfied for only one compact set K with non-empty interior.

Let us now express the condition (6.22) in terms of convergence to zero along a certain filter. We denote $|K|$ the exterior (Haar) measure of a set $K \subset X$ and we set $K_a = a + K$ if $a \in X$. A subset N is called w-*small (at infinity)* if there is a compact neighborhood K of the origin such that $\lim_{a\to\infty} |N \cap K_a| = 0$. The complement of a w-small set will be called w-*large (at infinity)*. The family $\mathcal{F}_{\mathrm{w}}$ of all w-large sets is clearly a filter on X finer than the Fréchet filter.

Observe that a Borel set is w-small if and only if its characteristic function weakly vanishes at infinity. Denote $f * g$ the convolution of two functions on X.

Lemma 6.3 *For a function $\varphi \in B(X)$ the following conditions are equivalent:* (1) *φ is weakly vanishing;* (2) *$\theta * |\varphi| \in C_0(X)$ if $\theta \in C_{\mathrm{c}}(X)$;* (3) *$\lim_{\mathcal{F}_{\mathrm{w}}} \varphi = 0$;* (4) *$\varphi(Q)\psi(P)$ is a compact operator on $L^2(X)$ for all $\psi \in C_0(X)$.*

Proof: The equivalence of (1) and (2) is clear because $\int_{K_a} |\varphi| dx = (\chi_K * |\varphi|)(a)$. Then (3) means that for each $\varepsilon > 0$ the Borel set N where $|\varphi(x)| > \varepsilon$ is w-small. Since $\chi_N \le \varphi/\varepsilon$, the implication (2) $\Rightarrow$ (3) is clear, while the reciprocal implication follows from $\chi_K * |\varphi| \le \sup|\varphi| \chi_K * \chi_N + \varepsilon |K|$. If (4) holds, let us choose ψ such that its Fourier transform $\widehat{\psi}$ be a positive function in $C_{\mathrm{c}}(X)$ and let $f \in C_{\mathrm{c}}(X)$ be positive and not zero. Since $\psi(P)f$ is essentially the convolution of $\widehat{\psi}$ with f, there is a compact set K with non-empty interior such that $\psi(P)f \ge c\chi_K$ with a number $c > 0$. Let U_a be the unitary operator of translation by a in $L^2(X)$, then $U_a f \to 0$ weakly when $a \to \infty$, hence $\|\varphi(Q)U_a\psi(P)f\| = \|\varphi(Q)\psi(P)U_a f\| \to 0$. Since $U_a^*\varphi(Q)U_a = \varphi(Q-a)$ we get $\|\varphi(Q-a)\chi_K\| \to 0$, hence (1) holds.

Finally, let us prove that (1) $\Rightarrow$ (4). It suffices to prove that $\varphi(Q)\psi(P)$ is compact if $\widehat{\psi} \in C_{\mathrm{c}}(X)$ and for this it suffices that $\bar{\psi}(P)|\varphi|^2(Q)\psi(P)$ be compact. Since $\xi := |\varphi|^2 \in B_{\mathrm{w}}(X)$ and since $\psi(P)$ is the operator of convolution by a function in $\theta \in C_{\mathrm{c}}(X)$, we are reduced to proving that the integral operator S with kernel $S(x,y) = \int \bar{\theta}(z-x)\xi(z)\theta(z-y)dz$ is compact. If $K = \operatorname{supp}\theta$ and Λ is the compact set $K - K$, then clearly there is a number C such that

$$|S(x,y)| \le C \int_{K_x} \xi(z)dz\chi_\Lambda(x-y) \equiv \phi(x)\chi_\Lambda(x-y)$$

where $\phi \in C_0(X)$. The last term here is a kernel which defines a compact operator T. Thus $\eta(Q)S$ is a Hilbert-Schmidt operator for each $\eta \in C_{\mathrm{c}}(X)$ and from the preceding estimate we get $\|(S - \eta(Q)S)u\| \le \|(1-\eta(Q))T|u|\|$ for each

$u \in L^2(X)$. Thus $\|S - \eta(Q)S\| \leq \|(1 - \eta(Q))T\|$ and the right hand side tends to zero if $\eta \equiv \eta_\alpha$ is an approximate unit for $C_0(X)$. ■

We define now a second class of functions which vanishes at infinity in a generalized sense, and this for an arbitrary Borel space X equipped with a positive measure μ such that $\mu(X) = \infty$. Let us say that a set $F \subset X$ is of *cofinite measure* if its complement F^c is of finite (exterior) measure. The family of sets of cofinite measure is clearly a filter $\mathcal{F}_\mu$ and if X is a locally compact space and μ a Radon measure then $\mathcal{F}_\mu$ is finer than the Fréchet filter. Moreover, if X is an abelian locally compact non-compact group then $\mathcal{F}_\mu \subset \mathcal{F}_\mathrm{w}$ and the inclusion is strict. If φ is a function on X then $\lim_{\mathcal{F}_\mu} \varphi = 0$ means that for each $\varepsilon > 0$ the set where $|\varphi(x)| \geq \varepsilon$ is of finite measure. We denote $B_\mu(X)$ the C^*-subalgebra of $B(X)$ consisting of functions with this property.

Proposition 6.4 *Let (X, μ) be a positive measure space with $\mu(X) = \infty$ and let us equip $L^2(X)$ with the Hilbert module structure defined by $B_\mu(X)$. If $S \in \mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ for some $p < 2$, then S is right $\mathcal{F}_\mu$-quasilocal.*

Proof: We first show that $\mathcal{M}_\mu := \{\varphi(Q) \mid \varphi \in B_\mu(X)\}$ defines indeed a Hilbert module structure on $\mathscr{H} = L^2(X)$. Let $\mathcal{N}_\mu$ be the set of Borel subsets of finite measure of X. Then $\{\chi_N\}_{N \in \mathcal{N}_\mu}$ is an approximate unit of $B_\mu(X)$ because for each $\varphi \in B_\mu(X)$ and each $\varepsilon > 0$ we have $N = \{x \mid |\varphi(x)| \geq \varepsilon\} \in \mathcal{N}_\mu$ and $\sup |\varphi - \chi_N \varphi| \leq \varepsilon$. That the action of $\mathcal{M}_\mu$ on $\mathscr{H}$ is non-degenerate follows from the density of $L^1 \cap L^\infty$ in L^2 and the fact that each $u \in L^1 \cap L^\infty$ can be written as $u = \varphi v$ with $\varphi = \sqrt{|u|} \in L^2 \cap L^\infty \subset \mathcal{M}_\mu$ and $v = \sqrt{|u|}\,\mathrm{sign}\, u \in L^2$.

Now let $S \in \mathcal{B}(L^2(X))$ such that S induces a continuous operator in $L^p(X)$ for some number p such that $1 < p < 2$. We shall prove that for each $N \in \mathcal{N}_\mu$ the operator $T = S\chi_N(Q)$ has the property: for each $\varepsilon > 0$ there is a Borel set $F \in \mathcal{F}_\mu$ such that $\|\chi_F(Q)T\| \leq \varepsilon$. According to Lemma 6.1, this implies the right $\mathcal{F}_\mu$-quasilocality of S.

Since N is of finite measure, $\chi_N(Q)$ is a bounded operator $L^2 \to L^p$, hence $T \in \mathcal{B}(L^2, L^p)$. The rest of the proof is a straightforward application of the following factorization theorem, due to Bernard Maurey [Ma]:

Let $1 < p < 2$ and let T be an arbitrary continuous linear map from a Hilbert space $\mathscr{H}$ into L^p. Then there is $R \in \mathcal{B}(\mathscr{H}, L^2)$ and there is a function $g \in L^q$, where $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, such that $T = g(Q)R$.

In our case $\mathscr{H} = L^2$. Let $a > 0$ real and let F be the set of points x such that $|g(x)| \leq a$. Since $g \in L^q$ with $q < \infty$, we have $F \in \mathcal{F}_\mu$ and

$$\|\chi_F(Q)T\|_{\mathcal{B}(L^2)} = \|\chi_F(Q)g(Q)R\|_{\mathcal{B}(L^2)} \leq a\|R\|_{\mathcal{B}(L^2)}.$$

Thus it suffices to choose a such that $a\|R\|_{\mathcal{B}(L^2)} = \varepsilon$. ■

We introduce now classes of vanishing at infinity functions of a more topological nature. We consider only the case of an Euclidean space X, the extension to the case of locally compact groups or metric spaces being obvious. We set $B_a(r) = \{x \in X \mid |x-a| < r\}$, $B_a = B_a(1)$ and $B(r) = B_0(r)$.

Let us fix a *uniformly discrete set* $L \subset X$, i.e. a set such that $\inf |a-b| > 0$ where the infimum is taken over couples of distinct points $a, b \in L$. Let $L_\varepsilon = L + B(\varepsilon)$ be the set of points at distance $< \varepsilon$ from L. We say that a subset $N \subset X$ is *L-thin* if for each $\varepsilon > 0$ there is $r < \infty$ such that $N \setminus B(r) \subset L_\varepsilon$. In other terms, N is L-thin if there is a family $\{\delta_a\}_{a\in L}$ of positive real numbers with $\delta_a \to 0$ as $a \to \infty$ such that $N \subset \bigcup B_a(\delta_a)$. The complement of such a set will be called *L-fat*. We denote $\mathcal{F}_L$ the family of L-fat sets, we note that $\mathcal{F}_L$ is a filter on X contained in $\mathcal{F}_{\mathrm{w}}$ and finer than the Fréchet filter, and we denote $B_L(X)$ the set of bounded Borel functions such that $\lim_{\mathcal{F}_L} \varphi = 0$. So $\varphi \in B(X)$ belongs to $B_L(X)$ if and only if the set $\{|\varphi| \geq \lambda\}$ is L-thin for each $\lambda > 0$. The advantage of this filter is that we have a simple criterion of $\mathcal{F}_L$-quasilocality.

Proposition 6.5 *Let $X = \mathbb{R}^n$ and let S be a bounded operator on $L^2(X)$ such that on the region $x \neq y$ its distribution kernel is a function satisfying the estimate $|S(x,y)| \leq c|x-y|^{-m}$ for some $m > n$. Then S is $\mathcal{F}_L$-quasilocal.*

Proof: Let $\theta \in C_{\mathrm{b}}(X)$ such that $\theta(x) = 0$ on a neighborhood of the origin and $S_\theta(x,y) = \theta(x-y)S(x,y)$. If $\xi(x) = \theta(x)|x|^{-m}$ then for the operator S_θ of kernel $S_\theta(x,y)$ we have $\|S_\theta u\| \leq c\|\xi * |u|\|$ hence $\|S_\theta\| \leq c\|\xi\|_{L^1}$ By choosing a convenient sequence of functions θ we see that S is the norm limit of a sequence of operators which besides the properties from the statement of the proposition are such that $S(x,y) = 0$ if $|x-y| > R(S)$. Since the set of $\mathcal{F}_L$-quasilocal operators is closed in norm (see Subsection 2.3), we may assume in the rest of the proof that the kernel of S has this property. In fact, in order to simplify the notations and without loss of generality, we shall assume $S(x,y) = 0$ if $|x-y| > 1$.

Let N be an L-thin Borel set and let $\varepsilon > 0$. We shall construct an L-fat Borel set with $F \subset N^{\mathrm{c}}$ such that $\|\chi_N(Q)S\chi_F(Q)\| \leq \varepsilon$. Since the adjoint operator S^* has the same properties as S, this suffices to prove quasilocality.

We shall only need two simple estimates. First, if $\rho_x(G)$ is the distance from a Borel set G to a point x, then

$$\int_G \frac{dy}{|x-y|^{2m}} \leq C(m,n)\rho_x(G)^{n-2m}. \tag{6.23}$$

Then, if B_0, B are two balls with the same center and radiuses δ and $\delta + \varepsilon$, then

$$\int_{B_0} \rho_x(B^{\mathrm{c}})^{n-2m} dx \leq C(m,n)\varepsilon^{n-2m}\delta^n. \tag{6.24}$$

We shall choose $\varepsilon = \delta^{n/2m}$. Then $\chi_{B_0}(Q)S\chi_{B^c}(Q)$ is an operator with integral kernel and we can estimate its Hilbert-Schmidt norm as follows:

$$\begin{aligned}\|\chi_{B_0}(Q)S\chi_{B^c}(Q)\|_{HS}^2 &= \int_{X\times X}\chi_{B_0}(x)|S(x,y)|^2\chi_{B^c}(y)dxdy \\ &\le c\int_{B_0}dx\int_{B^c}\frac{dy}{|x-y|^{2m}} \le C\int_{B_0}\rho_x(B^c)^{n-2m}dx \\ &\le C'\varepsilon^{n-2m}\delta^n = C'\delta^\lambda \qquad (6.25)\end{aligned}$$

where $\lambda = n^2/2m > 0$.

We can assume that $N = \bigcup_a B_a(\delta_a)$, where the sequence of numbers δ_a satisfies $\delta_a \to 0$ as $a \to \infty$. Denote $N_a = B_a(\delta_a)$ and $M_a = B_a(\delta_a + \varepsilon_a)$, where we choose $\varepsilon_a = \delta_a^{n/2m}$ as above. Choose r such that the balls N_a are pairwise disjoint and $\delta_a + \varepsilon_a < 1$ if $|a| > r$ and let R such that $\chi_{N_a}(Q)S\chi_{B(R)^c}(Q) = 0$ if $|a| \le r$. Let $M = \bigcup M_a$ and $F = M^c \setminus B(R)$, so that F is a closed L-fat set. Then for any $u \in L^2(X)$ we have:

$$\|\chi_N(Q)S\chi_F(Q)u\|^2 = \sum_{|a|>r}\|\chi_{N_a}(Q)S\chi_F(Q)u\|^2.$$

Since S is of range 1 we have $\chi_{N_a}(Q)S\chi_{B_a(2)^c}(Q) = 0$ if $\delta_a < 1$. Thus

$$\|\chi_N(Q)S\chi_F(Q)u\|^2 \le \sum_{|a|>r}\|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\|^2\,\|\chi_{B_a(2)}(Q)u\|^2$$

The number of $b \in L$ such that $B_b(2)$ meets $B_a(2)$ is a bounded function of a, hence there is a constant C depending only on L such that

$$\|\chi_N(Q)S\chi_F(Q)u\| \le C\sup_{|a|>r}\|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\|\,\|u\|.$$

We have $F \subset M^c \subset M_a^c$ hence

$$\|\chi_{N_a}(Q)S\chi_{F\cap B_a(2)}(Q)\| \le \|\chi_{N_a}(Q)S\chi_{M_a^c}(Q)\|_{HS} \le C'\delta_a^{\lambda/2}$$

because of (6.25). So the norm $\|\chi_N(Q)S\chi_F(Q)\|$ can be made as small as we wish by choosing r large enough. ■

Corollary 6.6 *Let $X = \mathbb{R}^n$, μ the Lebesgue measure, and L a uniformly discrete subset of $\mathbb{R}^n$. Then a pseudo-differential operator of class S^0 on $L^2(X)$ is both $\mathcal{F}_\mu$-quasilocal and $\mathcal{F}_L$-quasilocal.*

Proof: In the first case we use Proposition 6.4 by taking into account that a pseudo-differential operator of class S^0 belongs to $\mathcal{B}(L^p(X))$ for all $1 < p < \infty$ and that the adjoint of such an operator is also pseudo-differential of class S^0. For the second case, note that the distribution kernel of such an operator verifies the estimates $|S(x,y)| \le C_k |x-y|^{-n}(1+|x-y|)^{-k}$ for any $k > 0$, see [Ho]. ∎

We shall consider now a general class of filters defined in terms of the metric and measure space structure of the euclidean X. To each function $\nu : X \to]0,\infty[$ such that $\liminf_{a\to\infty} \nu(a) = 0$ we associate a set of subsets of X as follows:

$$\mathscr{N}_\nu = \{N \subset X \mid \limsup_{a\to\infty} \nu(a)^{-1}|N \cap B_a| < \infty\}. \tag{6.26}$$

We recall that B_a is the unit ball centered at a. Clearly $\mathscr{F}_\nu = \{F \subset X \mid F^{\mathrm{c}} \in \mathscr{N}_\nu\}$ is a filter on X finner than the Fr´echet filter. Our purpose is to give a criterion of $\mathscr{F}_\nu$-quasilocality. For this we make a preliminary remark concerning the class of $C^{\mathrm{u}}(Q)$. We shall say that an operator $S \in \mathcal{B}(L^2(X))$ is of *finite range* if there is $r < \infty$ such that its distribution kernel satisfies $S(x,y) = 0$ for $|x-y| > r$.

Proposition 6.7 *The set of linear continuous finite range operators on $L^2(X)$ is a dense $*$-subalgebra of $C^{\mathrm{u}}(Q)$.*

Proof: The fact that the set of finite range operators in $\mathcal{B}(L^2(X))$ is a $*$-algebra is easy to check. We prove now that a finite range operator $S \in \mathcal{B}(L^2(X))$ is of class $C^{\mathrm{u}}(Q)$. Let us denote $Z = \mathbb{Z}^n$ and for each $a \in Z$ let $K_a = a + K$, where $K =]-1/2, 1/2]^n$, so that K_a is a unit cube centered at a and we have $X = \bigcup_{a\in Z} K_a$ disjoint union. Let χ_a be the characteristic function of K_a and let us abbreviate $\chi_a = \chi(Q)$. If r is as above, we similarly define $L =]-r-1, r+1]^n$, $L_a = a + L$ and denote φ_a the characteristic function of L_a. Note that there is a number N such that any cube L_a intersects at most N other cubes L_b.

It suffices to prove that for each linear function $\xi : X \to \mathbb{R}$ the commutator $[\xi(Q), S]$ is bounded, because this is equivalent to the fact that the map $k \mapsto V_k^* S V_k$ is Lipschitz. We have $\sum_u \chi_a = 1$ strongly on L^2 and $[\xi(Q), S]\chi_a = \varphi_a[\xi(Q), S]\chi_a$ due to the assumption concerning the range of S. Thus there is a constant C depending only on N such that for $u \in L^2$ with compact support:

$$\begin{aligned} \|[\xi(Q), S]u\|^2 &\le C \sum \|\varphi_a[\xi(Q), S]\chi_a u\|^2 \\ &= C \sum \|\varphi_a[\xi(Q) - \xi(a), S]\chi_a \cdot \chi_a u\|^2 \\ &\le C \sum C' \|\chi_a u\|^2 = CC' \|u\|^2 \end{aligned}$$

Now we shall prove that any operator of class $C^{\mathrm{u}}(Q)$ is a norm limit of finite range operators and this in the more general setting of Hilbert X-modules. Let X

be an abelian locally compact group and let $\mathscr{H}, \mathscr{K}$ be Hilbert X-modules. We fix a Haar measure dk on X^* and if $S \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ and $\Theta \in L^1(X^*)$ we define

$$S_\Theta = \int_{X^*} V_k^* S V_k \Theta(k) dk. \tag{6.27}$$

The integral is well defined because $k \mapsto V_k^* S V_k \in \mathcal{B}(\mathscr{H})$ is a bounded strongly continuous map. In order to explain the main idea of the proof we shall make a formal computation involving the spectral measure $E(A) = \chi_A(Q)$, see Lemmas 2.12 and 2.18 (we shall use the same notation for the spectral measures in $\mathscr{H}$ and $\mathscr{K}$). We have for $k \in X^*$ and $\varphi(Q) \in B(X)$

$$\varphi(Q) V_k^* = \varphi(Q) k(Q)^* = (\varphi \overline{k})(Q) = \int \varphi(x) \overline{k}(x) E(dx).$$

Note also that for $x, y \in X$ we have $\overline{k}(x) k(y) = k(-x) k(y) = k(y-x)$. Let $\widehat{\Theta}(x) = \int \overline{k(x)} \Theta(k) dk$ be the Fourier transform of Θ. Then we have for all $\varphi, \psi \in B(X)$:

$$\begin{aligned} \varphi(Q) S_\Theta \psi(Q) &= \int_{X^*} \Theta(k) dk \int_X \int_X \varphi(x) \overline{k}(x) k(y) \psi(y) E(dx) S E(dy) \\ &= \int_X \int_X \widehat{\Theta}(x-y) \varphi(x) \psi(y) E(dx) S E(dy). \end{aligned} \tag{6.28}$$

We can rigorously justify this computation and give a meaning to the last integral by taking into account that $E(A) S E(B)$ induces a finitely additive measure on the algebra generated by rectangles $A \times B$ in $X \times X$ (note that $\widehat{\Theta} \in C_0(X)$). If S is Hilbert-Schmidt then the measure is in fact σ-additive and the result becomes obvious. We shall, however, avoid these questions and we shall directly prove only what we need. Namely, we show the following:

($*$) $\begin{cases} \text{If the support of } \widehat{\Theta} \text{ is a compact set } \Lambda \text{ and if supp } \varphi \cap (\Lambda + \text{supp } \psi) = \varnothing \\ \text{then } \varphi(Q) S_\Theta \psi(Q) = 0. \end{cases}$

Observe that if ($*$) holds for a certain set of operators S then it also holds for the strongly closed linear subspace of $\mathcal{B}(\mathscr{H}, \mathscr{K})$ generated by it. So it suffices to prove ($*$) for S an operator of rank one $Sf = v\langle u, f\rangle$ with some fixed $u \in \mathscr{H}$ and $v \in \mathscr{K}$. Now the computation giving (6.28) obviously makes sense in the weak topology and gives for $f \in \mathscr{H}$ and $g \in \mathscr{K}$:

$$\langle g, \varphi(Q) S_\Theta \psi(Q) f\rangle = \int_X \int_X \widehat{\Theta}(x-y) \varphi(x) \psi(y) \langle g, E(dx) u\rangle \langle u, E(dy) f\rangle,$$

hence $(*)$ holds for such S.

Finally, note that if $S \in C^{\mathrm{u}}(Q)$ then S is norm limit of operators of the form S_Θ. For this it suffices to take $\Theta = |K|^{-1}\chi_K$ where K runs over the set of open relatively compact neighbourhoods of the neutral element of X^*, $|K|$ being the Haar measure of K. Then, by approximating conveniently Θ in L^1 norm, one shows that S is norm limit of operators S_Θ such that $\widehat{\Theta}$ has compact support. ■

Remark: This proposition gives a new proof of Proposition 2.23 for the case of Hilbert X-modules. Indeed, it is obvious that a finite range operator is quasilocal.

Theorem 6.8 *Let $X = \mathbb{R}^n$ and let $\nu : X \to]0, \infty[$ such that $\liminf_{a\to\infty} \nu(a) = 0$ and $\sup_{|b-a|\leq r} \nu(b)/\nu(a) < \infty$ for each real r. If $S \in \mathcal{B}(L^2(X))$ is of class $C^{\mathrm{u}}(Q)$ and if $S \in \mathcal{B}(L^p(X))$ for some $p < 2$, then S is right $\mathscr{F}_\nu$-quasilocal.*

Proof: We can approximate in norm in $\mathcal{B}(L^2(X))$ the operator S by operators which are in $\mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ and have finite range. Indeed, the approximation procedure (6.27) used in the proof of Proposition 6.7 is such that it leaves $\mathcal{B}(L^2(X)) \cap \mathcal{B}(L^p(X))$ invariant (because V_k are isometries in L^p too). Since the set of right $\mathscr{F}_\nu$-quasilocal operators is norm closed in $\mathcal{B}(L^2(X))$, we may assume in the rest of the proof that S is of finite range. According to Lemma 6.1, it suffices to show that, for a given Borel set $N \in \mathscr{N}_\nu$ and for any number $\varepsilon > 0$, there is a Borel set $M \in \mathscr{N}_\nu$ such that $\|\chi_{M^c}(Q)S\chi_N(Q)\| < \varepsilon$.

In the rest of the proof we shall freely use the notations introduced in the in the second part of the Appendix (see also the proof of Proposition 6.7). In particular, q is defined by $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$. If $f \in L^2(X)$ we have

$$\|\chi_N f\|_{L^p(K_a)} \leq \|\chi_N\|_{L^q(K_a)} \|f\|_{L^2(K_a)} \leq |N \cap K_a|^{1/q} \|f\|_{L^2(K_a)}.$$

Since $N \in \mathscr{N}_\nu$ we can find a constant c such that $|N \cap K_a| \leq c\nu(a)$ (note that the definition (6.26) does not involve the restriction of ν to bounded sets). Thus, if we take $\lambda_a = \nu(a)^{-1/q}$ for $a \in Z \equiv \mathbb{Z}^n$, we get $\chi_N f \in \mathscr{L}$ with the notations of the Appendix. In other terms, we see that we have $\chi_N(Q) \in \mathcal{B}(L^2(X), \mathscr{L})$. Let $T = S\chi_N(Q)$ and let us assume that we also have $S \in \mathcal{B}(\mathscr{L})$. Then $T \in \mathcal{B}(L^2(X), \mathscr{L})$ and we can apply the Maurey type factorization theorem Theorem A.8, where $\mathscr{H} = L^2(X)$. Thus we can write $T = g(Q)R$ for some $R \in \mathcal{B}(L^2(X))$ and some function $g \in \mathscr{M}$, which means that $G := \sup_{a\in Z} \nu(a)^{-1/q} \|g\|_{L^q(K_a)}$ is a finite number. If $t > 0$ and $M = \{x \mid g(x) > t\}$ then we get for all $a \in Z$:

$$|M \cap K_a| = \|\chi_M\|^q_{L^q(K_a)} \leq \|g/t\|^q_{L^q(K_a)} \leq (G/t)^q \nu(a).$$

Note that the second condition imposed on ν in Theorem 6.8 ca be stated as follows: there is an increasing strictly positive function δ on $[0, \infty[$ such that $\nu(b) \leq$

$\delta(|b-a|)\nu(a)$ for all a, b. Indeed, we may take $\delta(r) = \sup_{|b-a|\le r} \nu(b)/\nu(a)$. Now let $a \in X$ and let $D(a)$ be the set of $b \in Z$ such that K_b intersects B_a. Clearly $D(a)$ contains at most 2^n points b all of them satisfying $|b-a| \le \sqrt{n}+1$. Hence:

$$|M \cap K_a| \le \sum_{b \in D(a)} |M \cap K_b| \le 2^n \sup_{b \in D(a)} (G/t)^q \nu(b) \le 2^n (G/t)^q \delta(\sqrt{n}+1)\nu(a),$$

which proves that M belongs to $\mathscr{N}_\nu$. On the other hand, we have:

$$\|\chi_{M^c}(Q)T\| = \|\chi_{M^c}(Q)g(Q)R\| \le \|\chi_{M^c}g\|_{L^\infty}\|R\| \le t\|R\|.$$

To finish the proof of the theorem it suffices to take $t = \varepsilon/\|R\|$.

We still have to prove that $S \in \mathcal{B}(\mathscr{L})$. Since S is of finite range, there is a number r such that $\chi_a(Q)\chi_b(Q) = 0$ if $|a-b| \ge r$. Then for any $f \in \mathscr{L}$:

$$\sum_a \lambda_a^2 \|\chi_a S f\|_{L^p}^2 = \sum_a \lambda_a^2 \|\sum_{|b-a|<r} \chi_a S \chi_b f\|_{L^p}^2 \le C \sum_{|b-a|<r} \lambda_a^2 \|\chi_a S \chi_b f\|_{L^p}^2$$

where C is a number depending only on r and n. Since S is bounded in L^p the last term is less than $CC' \sum_{|b-a|<r} \lambda_a^2 \|\chi_b f\|_{L^p}^2$ for some constant C'. Finally, from $\nu(b) \le \delta(|b-a|)\nu(a) \le \delta(r)\nu(a)$ we get

$$\sum_{|a-b|<r} \lambda_a^2 = \sum_{|a-b|<r} \nu(a)^{-2/q} \le L(r)\delta(r)^{2/q}\lambda_b^2$$

where $L(r)$ is the maximum number of points from Z inside a ball of radius r. Thus we have $\|S\|_{\mathcal{B}(\mathscr{L})}^2 \le CC'L(r)\delta(r)^{2/q}$. ■

Corollary 6.9 *Let $X = \mathbb{R}^n$ and let S be a pseudo-differential operator of class S^0. Then S is $\mathcal{F}_{\mathrm{w}}$-quasilocal, i.e. for each $\varphi \in B_{\mathrm{w}}(X)$ there are $\psi_1, \psi_2 \in B_{\mathrm{w}}(X)$ and $T_1, T_2 \in \mathcal{B}(L^2(X))$ such that $\varphi(Q)S = T_1\psi_1(Q)$ and $S\varphi(Q) = \psi_2(Q)T_2$.*

Proof: Since the adjoint of S is also a pseudo-differential operator of class S^0, it suffices to show that S is right $\mathcal{F}_{\mathrm{w}}$-quasilocal. We have $S \in \mathcal{B}(L^p(X))$ for all $1 < p < \infty$ and S is of class $C^{\mathrm{u}}(Q)$ because the commutators $[Q_j, S]$ are bounded operators for all $1 \le j \le n$. Thus we can apply Theorem 6.8 and deduce that for any function ν as in the statement of the theorem, for any $\varepsilon > 0$, and for any $N \in \mathscr{N}_\nu$ there is $M \in \mathscr{N}_\nu$ such that $\|\chi_{M^c}(Q)S\chi_N(Q)\| \le \varepsilon$. Now let N be a Borel w-small set, i.e. such that $|N \cap B_a| \to 0$ if $a \to \infty$. We shall prove that there is a function ν with the properties required in Theorem 6.8 and with $\lim_{a\to\infty} \nu(a) = 0$ such that $N \in \mathscr{N}_\nu$. This finishes the proof of the corollary because the relation $M \in \mathscr{N}_\nu$ implies now that M is w-small.

We construct ν as follows. The relation $\theta(r) = \sup_{|a|\geq r} |N \cap B_a|$ defines a positive decreasing function on $[0, \infty[$ which tends to zero at infinity and such that $|N \cap B_a| \leq \theta(|a|)$ for all $a \in X$. We set $\xi(t) = \theta(0)$ if $0 \leq t < 1$ and for $k \geq 0$ integer and $2^k \leq t < 2^{k+1}$ we define $\xi(t) = \max\{\xi(2^{k-1})/2, \theta(2^k)\}$. So ξ is a strictly positive decreasing function on $[0, \infty[$ which tends to zero at infinity and such that $\theta \leq \xi$. Moreover, if $2^k \leq s < 2^{k+1}$ and $2^{k+p} \leq t < 2^{k+p+1}$ then

$$\xi(t) = \xi(2^{k+p}) \geq \xi(2^{k+p-1})/2 \geq \ldots \geq 2^{-p}\xi(2^k) = 2^{-p}\xi(s)$$

hence $\xi(s) \geq \xi(t) \geq \frac{s}{2t}\xi(s)$ if $1 \leq s \leq t$. We take $\nu(a) = \xi(|a|)$, so ν is a bounded strictly positive function on X with $\lim_{a\to\infty} \nu(a) = 0$ and $|N \cap B_a| \leq \nu(a)$ for all a. If a, b are points with $|a|, |b| \geq 1$ and $|a - b| \leq r$ then $\nu(b)/\nu(a) \leq 1$ if $|a| \leq |b|$ and if $|a| > |b|$ then

$$\frac{\nu(b)}{\nu(a)} = \frac{\xi(|b|)}{\xi(|a|)} \leq \frac{2|a|}{|b|} \leq 2(1 + r).$$

Thus the second condition imposed on ν in Theorem 6.8 is also satisfied. ■

Remark 6.10 We stress that we shall need this corollary for a very simple class of operators, namely $S = \psi(P)$ with $\psi(k) = k^\alpha(\sum_{|\beta|\leq m} k^{2\beta})^{-1/2}$ and $|\alpha| \leq m$.

6.2 Applications

We shall give an application of the formalism presented in Subsection 6.1 in the framework of Subsection 4.1. We consider on $\mathscr{H}$ the class of "vanishing at infinity" functions corresponding to the C^*-algebra of multipliers $B_{\mathrm{w}}(X)$. The conditions of decay at infinity (4.18) imposed in Proposition 4.1 come from the consideration of $\mathscr{H}$ equipped with the Hilbert module structure defined by the algebra $B_0(X)$. Note that if we equip $\mathscr{H}$ with the Hilbert module structure defined by the algebra $B_{\mathrm{w}}(X)$ the property of compactness of the Friedrichs module $(\mathscr{G}, \mathscr{H})$ remains valid, cf. Lemma 6.2 and the space $\mathscr{K}$ inherits a natural direct sum Hilbert module structure.

Our purpose is to apply Theorem 3.6 in this setting. The only thing which remains to be checked is the left $\mathcal{F}_{\mathrm{w}}$-quasilocality of the operator $D(\Delta_a^* - \bar{z})^{-1}$. We shall establish such a result below assuming that the lower order coefficients are also bounded operators, but it is clear that this assumption can be replaced by much more general ones. Note also that in this subsection we are less precise and identify the operators D^*aD and Δ_a, although they act in different spaces.

In the next lemma we consider only the filter $\mathcal{F}_{\mathrm{w}}$. Of course, the result remains true if $\mathcal{F}_{\mathrm{w}}$ is replaced by $\mathcal{F}_\mu$ or $\mathcal{F}_L$.

Lemma 6.11 *Let $\Delta_a = \sum_{|\alpha|,|\beta|\leq m} P^\alpha a_{\alpha\beta} P^\beta$ with $a_{\alpha\beta} \in \mathcal{B}(\mathscr{H})$ $\mathcal{F}_{\mathrm{w}}$-quasilocal (e.g. $a_{\alpha\beta} \in B(X)$) and let as assume that the operator $\Delta_a : \mathscr{H}^m \to \mathscr{H}^{-m}$ is coercive, i.e. there are numbers $\mu, \nu > 0$ such that for all $u \in \mathscr{H}^m$:*

$$\mathrm{Re}\,\langle u, \Delta_a u\rangle \geq \mu \|u\|^2_{\mathscr{H}^m} - \nu \|u\|^2_{\mathscr{H}}. \tag{6.29}$$

Then $P^\alpha(\Delta_a + z)^{-1}$ is $\mathcal{F}_{\mathrm{w}}$-quasilocal if $|\alpha| \leq m$ and if Re $z > 0$ *is large.*

Proof: We shall denote by Δ the operator Δ_a corresponding to the case when a is the identity matrix, so $\Delta = D^*D = \sum_{|\alpha|\leq m} P^{2\alpha}$ (of course, this is not the Laplace operator). In fact, Δ is the canonical (Riesz) positive isomorphism of $\mathscr{G}$ onto $\mathscr{G}^*$ and (6.29) means Re $\Delta_a \geq \mu\Delta - \nu$. Note that we can include ν in the term of order zero of Δ_a, hence there is no loss of generality if we assume $\nu = 0$. Later computations look simpler if $\mu = 1$ and we can reduce ourselves to this situations by replacing a by a/μ. Thus we may assume that we have the estimate Re $\Delta_a \geq \Delta$. Now let us decompose $\Delta_a = \Delta + D^*(a-1)D \equiv \Delta + V$ and, if θ is a positive number, let us set $A_\theta = \Delta + \theta V$. Then $A_\theta \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ and we have Re $A_\theta \geq \Delta$, so that if Re $z \geq 0$ then $A_\theta + z : \mathscr{G} \to \mathscr{G}^*$ is bijective and $\|(A_\theta + z)^{-1}\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})} \leq 1$ (see the Appendix). It follows easily that the function $\theta \mapsto (A_\theta + z)^{-1} \in \mathcal{B}(\mathscr{G}^*, \mathscr{G})$ is real analytic on $]0, \infty[$ which implies that the function $\theta \mapsto P^\alpha(A_\theta + z)^{-1} \in \mathcal{B}(\mathscr{H})$ is real analytic too. The set of $\mathcal{F}_{\mathrm{w}}$-quasilocal operators is a closed subspace of the Banach space $\mathcal{B}(\mathscr{H})$ and an analytic function which on an open set takes values in a closed subspace remains in that subspace for ever. Thus it suffi ces to how that the operator $P^\alpha(A_\theta + z)^{-1}$ is $\mathcal{F}_{\mathrm{w}}$-quasilocal for small values of θ. The operator $P^\alpha(A_\theta + z)^{-1}$ is also a holomorphic function of z in the region Re $z > 0$, so by a similar argument we see that it suffi ces to consider $z \geq 0$. Below we shall take $z = 0$, the argument in general is identical.

For reasons of simplicity, we change again the notations: we set $b = \theta(a-1)$, we assume $\|b\|_{\mathcal{B}(\mathscr{K})} < 1$, and denote $V = D^*bD$ and $A = \Delta + V$. Let $S = \Delta^{-1/2}$, where Δ is considered as self-adjoint operator on $\mathscr{H}$. Note that S is an isometry of $\mathscr{G}^*$ onto $\mathscr{H}$ and of $\mathscr{H}$ onto $\mathscr{G}$. Then we have:

$$A^{-1} = S(1 + SVS)^{-1}S = \sum_{k\geq 0} (-1)^k S(SVS)^k S$$

the series being norm convergent in $\mathcal{B}(\mathscr{G}^*, \mathscr{G})$. Indeed, $\|DS\|_{\mathcal{B}(\mathscr{H},\mathscr{K})} = 1$, hence

$$\|S(SVS)^k S\|_{\mathcal{B}(\mathscr{G}^*,\mathscr{G})} = \|(SD^*bDS)^k\|_{\mathcal{B}(\mathscr{H})} \leq \|b\|^k_{\mathcal{B}(\mathscr{K})}$$

and $\|b\|_{\mathcal{B}(\mathscr{K})} < 1$. Thus $P^\alpha A^{-1}$ is a sum of terms $P^\alpha(-1)^k S(SVS)^k S$ which converges in norm, so it suffi ces that each of them be $\mathcal{F}_{\mathrm{w}}$-quasilocal. But

$$P^\alpha S(SVS)^k S = (P^\alpha S)(SD^*)b(DS)\dots(SD^*)b(DS)S$$

and each factor in the product is $\mathcal{F}_{\rm w}$-quasilocal: for b this is an hypothesis (or trivial if the $a_{\alpha\beta}$ are functions), and for $P^\alpha S$, DS and SD^* because of Corollary 6.9. ■

Below we give just an example of application of Theorem 3.6. The conditions on the lower order coefficients can be improved without difficulty.

Theorem 6.12 *Let Δ_a be as in Lemma 6.11 and let $b = (b_{\alpha\beta})_{|\alpha|,|\beta|\le m}$ with $b_{\alpha\beta}$ bounded operators $\mathscr{H}^{m-|\beta|} \to \mathscr{H}^{|\alpha|-m}$ such that Δ_b is coercive. For $|\alpha|+|\beta| = m$ assume that $b_{\alpha\beta}-a_{\alpha\beta}$ is left $\mathcal{F}_{\rm w}$-vanishing at infinity (which holds if $b_{\alpha\beta}-a_{\alpha\beta} \in B_{\rm w}(X)$). If $|\alpha|+|\beta| < m$ we assume $b_{\alpha\beta} - a_{\alpha\beta} \in \mathcal{K}(\mathscr{H}^{m-|\beta|}, \mathscr{H}^{|\alpha|-m})$. Then the operator Δ is a compact perturbation of Δ_a, in particular Δ_a and Δ_b have the same essential spectrum.*

Proof: We check the conditions of Theorem 3.6. Because of the coercivity assumptions, condition (1) is fulfilled, and (2) is satisfied by Lemma 6.11. The part of condition (3) involving the coefficients such that $|\alpha|+|\beta| = m$) is satisfied by definition, for the lower order coefficients it suffices to use (2.10). ■

Remark 6.13 If $b_{\alpha\beta} \in B(X)$ and $b_{\alpha\beta} - a_{\alpha\beta} \in B_{\rm w}(X)$ for all α, β, then the compactness conditions on the lower order coefficients are satisfied. Indeed, if $\varphi \in B_{\rm w}(X)$ then $\varphi(Q) : \mathscr{H}^s \to \mathscr{H}^{-t}$ is compact if $s, t \ge 0$ and one of them is not zero, see Lemma 6.2.

A Appendix

1. This Appendix consists of two parts: in the first one we discuss some elementary abstract facts which are used without comment in the main text and in the second one we present a Maurey type factorization theorem adapted to our needs.

Let $(\mathscr{G}, \mathscr{H})$ be a Friedrichs couple and $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ the Gelfand triplet associated to it. To an operator $S \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$ (which is the same as a continuous sesquilinear form on $\mathscr{G}$) we associate an operator $\widehat{S}$ acting in $\mathscr{H}$ according to the rules: $\mathcal{D}(\widehat{S}) = S^{-1}(\mathscr{H})$, $\widehat{S} = S|\mathcal{D}(\widehat{S})$. Due to the identification $\mathscr{G}^{**} = \mathscr{G}$, the operator S^* is an element of $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$, so $\widehat{S^*}$ makes sense. On the other hand, if $\widehat{S}$ is densely defined in $\mathscr{H}$ then the adjoint $\widehat{S}^*$ of $\widehat{S}$ with respect to $\mathscr{H}$ is also well defined and we clearly have $\widehat{S^*} \subset \widehat{S}^*$.

Lemma A.1 *If $S - z : \mathscr{G} \to \mathscr{G}^*$ is bijective for some $z \in \mathbb{C}$, then $\widehat{S}$ is a closed densely defined operator, we have $\widehat{S}^* = \widehat{S^*}$ and $z \in \rho(\widehat{S})$. Moreover, the domains $\mathcal{D}(\widehat{S})$ and $\mathcal{D}(\widehat{S}^*)$ are dense subspaces of $\mathscr{G}$.*

Proof: Clearly we can assume $z = 0$. From the bijectivity of $S : \mathscr{G} \to \mathscr{G}^*$ and the inverse mapping theorem it follows that S and S^* are homeomorphisms of $\mathscr{G}$ onto $\mathscr{G}^*$. Since $\mathscr{H}$ is dense in $\mathscr{G}^*$, we see that $\mathcal{D}(\widehat{S})$ and $\mathcal{D}(\widehat{S^*})$ are dense in $\mathscr{G}$, hence in $\mathscr{H}$. Since $\widehat{S^*} \subset \widehat{S}^*$, the operator $\widehat{S}^*$ is also densely defined in $\mathscr{H}$. Thus $\widehat{S}$ is densely defined and closable. We now show that it is closed. Consider a sequence of elements $u_n \in \mathcal{D}(\widehat{S})$ such that $u_n \to u$ and $\widehat{S}u_n \to v$ in $\mathscr{H}$. Then $Su_n \to v$ in $\mathscr{G}^*$ hence, S^{-1} being continuous, $u_n \to S^{-1}v$ in $\mathscr{G}$, so in $\mathscr{H}$. Hence $u = S^{-1}v \in \mathcal{D}(\widehat{S})$ and $\widehat{S}u = v$.

We have proved that $\widehat{S}$ is densely defined and closed and clearly $0 \in \rho(\widehat{S})$. Then we also have $0 \in \rho(\widehat{S}^*)$, so $\widehat{S}^* : \mathcal{D}(\widehat{S}^*) \to \mathscr{H}$ is bijective. Since $\widehat{S^*} : \mathcal{D}(\widehat{S^*}) \to \mathscr{H}$ is also bijective and $\widehat{S}^*$ is an extension of $\widehat{S^*}$, we get $\widehat{S^*} = \widehat{S}^*$. ■

A standard example of operator satisfying the condition required above is a *coercive* operator, i.e. such that Re $\langle u, Su\rangle \geq \mu\|u\|^2_{\mathscr{G}} - \nu\|u\|^2_{\mathscr{H}}$ for some strictly positive constants μ, ν and all $u \in \mathscr{G}$. Indeed, replacing S by $S + \nu$, we may assume Re $\langle u, Su\rangle \geq \mu\|u\|^2_{\mathscr{G}}$. Since S^* verifies the same estimate, this clearly gives $\|Su\|_{\mathscr{G}^*} \geq \mu\|u\|_{\mathscr{G}}$ and $\|S^*u\|_{\mathscr{G}^*} \geq \mu\|u\|_{\mathscr{G}}$ for all $u \in \mathscr{G}$. Thus S and S^* are injective operators with closed range, which implies that they are bijective.

If A is a self-adjoint operator on $\mathscr{H}$ then there is a natural Gelfand triplet associated to it, namely $\mathcal{D}(|A|^{1/2}) \subset \mathscr{H} \subset \mathcal{D}(|A|^{1/2})^*$. Then A extends to a continuous operator $A_0 : \mathcal{D}(|A|^{1/2}) \to \mathcal{D}(|A|^{1/2})^*$ which fulfills the conditions of Lemma A.1 and one has $\widehat{A}_0 = A$. In our applications it is interesting to know whether there are other Gelfand triplets $\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*$ with $\mathcal{D}(A) \subset \mathscr{G}$ and such that A extends to a continuous operator $\mathscr{G} \to \mathscr{G}^*$. For not semibounded operators, e.g. for Dirac operators, many other possibilities exist such that $\mathscr{G}$ is not comparable to $\mathcal{D}(|A|^{1/2})$. But if A is semibounded, then the class of spaces $\mathscr{G}$ is rather restricted, as the next lemma shows.

Lemma A.2 *Assume that A is a bounded from below self-adjoint operator on $\mathscr{H}$ and such that $\mathcal{D}(A) \subset \mathscr{G}$ densely. Then A extends to a continuous operator $\widetilde{A} : \mathscr{G} \to \mathscr{G}^*$ if and only if $\mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ and in this case $\widetilde{A} = A_0|_{\mathscr{G}}$.*

Proof: We prove only the nontrivial implication of the lemma. So let us assume that A extends to some $\widetilde{A} \in \mathcal{B}(\mathscr{G}, \mathscr{G}^*)$. Replacing A by $A + \lambda$ with λ a large enough number, we can assume that $A \geq 1$. For $u \in \mathcal{D}(A)$ we have

$$\|A^{1/2}u\|_{\mathscr{H}} = \sqrt{\langle u, Au\rangle} = \sqrt{\langle u, \widetilde{A}u\rangle} \leq C\|u\|_{\mathscr{G}},$$

where $C^2 = \|\widetilde{A}\|_{\mathscr{G}\to\mathscr{G}^*}$. Since $\mathcal{D}(A)$ is dense in $\mathscr{G}$, it follows that the inclusion map $\mathcal{D}(A) \to \mathcal{D}(A^{1/2})$ extends to a continuous linear map $J : \mathscr{G} \to \mathcal{D}(A^{1/2})$.

If $u \in \mathscr{G}$ then there is a sequence $\{u_n\}$ in $\mathcal{D}(A)$ such that $u_n \to u$ in $\mathscr{G}$. Then $J(u_n) \to J(u)$ in $\mathcal{D}(A^{1/2})$. Since $\mathscr{G}$ and $\mathcal{D}(A^{1/2})$ are continuously embedded in $\mathscr{H}$ we shall have $u_n \to u$ in $\mathscr{H}$ and $u_n = J(u_n) \to J(u)$ in $\mathscr{H}$, hence $J(u) = u$ for all $u \in \mathscr{G}$. In other terms, $\mathscr{G} \subset \mathcal{D}(A^{1/2})$. ■

We note that, under the conditions of the lemma, the inclusions $\mathcal{D}(A) \subset \mathscr{G}$ and $\mathscr{G} \subset \mathcal{D}(|A|^{1/2})$ are continuous (by the closed graph theorem), so we have a scale

$$\mathcal{D}(A) \subset \mathscr{G} \subset \mathcal{D}(|A|^{1/2}) \subset \mathscr{H} \subset \mathcal{D}(|A|^{1/2})^* \subset \mathscr{G}^* \subset \mathcal{D}(A)^*$$

with continuous and dense embeddings (because $\mathcal{D}(A)$ is dense in $\mathcal{D}(|A|^{1/2})$).

In view of its importance in this paper, we state below the Cohen-Hewitt factorization theorem [FD, Ch. V–9.2].

Theorem A.3 *Let $\mathscr{C}$ be a Banach algebra with an approximate unit, let $\mathscr{H}$ be a Banach space, and let $Q : \mathscr{C} \to \mathcal{B}(\mathscr{H})$ be a continuous morphism. Denote $\mathscr{H}_0$ the closed linear subspace of $\mathscr{H}$ generated by the elements of the form $Q(\varphi)v$ with $\varphi \in \mathscr{C}$ and $v \in \mathscr{H}$. Then for each $u \in \mathscr{H}_0$ there are $\varphi \in \mathscr{C}$ and $v \in \mathscr{H}$ such that $u = Q(\varphi)v$.*

2. In this second part of the appendix we shall prove a version of the factorization theorem due to Bernard Maurey (see the proof of Proposition 6.4 here and Theorem 8 in [Ma]). Our proof follows closely that of Maurey; we shall, however, give all the details, since the Banach space techniques involved in it are not very usual in the context of spectral theory. We fi rst recall the Ky Fan's Lemma, see [DJT, 9.10].

Proposition A.4 *Let $\mathcal{K}$ be a compact convex subset of a Hausdorff topological vector space and let $\mathscr{F}$ be a convex set of functions $F : \mathcal{K} \to]-\infty, +\infty]$ such that each $F \in \mathscr{F}$ is convex and lower semicontinuous. If for each $F \in \mathscr{F}$ there is $g \in \mathcal{K}$ such that $F(g) \leq 0$, then there is $g \in \mathcal{K}$ such that $F(g) \leq 0$ for all $F \in \mathscr{F}$.*

We need a second general fact that we state below. Let (X, μ) be a σ-fi nite positive measure space and let $L^0(X)$ be the space of μ-equivalence classes of complex valued measurable functions on X with the topology of convergence in measure. Let $\mathscr{L}$ be a Banach space with $\mathscr{L} \subset L^0(X)$ linearly and continuously and such that if $f \in L^0(X)$, $g \in \mathscr{L}$ and $|f| \leq |g|$ (μ-a.e.) then $f \in \mathscr{L}$ and $\|f\|_{\mathscr{L}} \leq \|g\|_{\mathscr{L}}$.

Proposition A.5 *There is a number C, independent of $\mathscr{L}$, such that for any Hilbert space $\mathscr{H}$ and any $T \in \mathcal{B}(\mathscr{H}, \mathscr{L})$ the following inequality holds*

$$\|(\textstyle\sum_j |Tu_j|^2)^{1/2}\|_{\mathscr{L}} \leq C\|T\|_{\mathcal{B}(\mathscr{H},\mathscr{L})}(\textstyle\sum_j \|u_j\|^2)^{1/2} \tag{1.30}$$

for all finite families $\{u_j\}$ of vectors in $\mathscr{H}$.

This a rather standard consequence of Khinchin's inequality [DJT, 1.10]. The result is stated in [Pi] with an explicit value for C.

From now on we work in a setting adapted to our needs in Section 6, although it is clear that we could treat by the same methods a general abstract situation. Let $X = \mathbb{R}^n$ equipped with the Lebesgue measure, denote $Z = \mathbb{Z}^n$, and for each $a \in Z$ let $K_a = a + K$, where $K =]-1/2, 1/2]^n$, so that K_a is a unit cube centered at a and we have $X = \bigcup_{a\in Z} K_a$ disjoint union. Let χ_a be the characteristic function of K_a and if $f : X \to \mathbb{C}$ let $f_a = f|K_a$. We fix a number $1 < p < 2$ and a family $\{\lambda_a\}_{a\in Z}$ of strictly positive numbers $\lambda_a > 0$ and we define $\mathscr{L} \equiv \ell^2_\lambda(L^p)$ as the Banach space of all (equivalence classes) of complex functions f on X such that

$$\|f\|_{\mathscr{L}} := \Big(\sum_{a\in Z} \|\lambda_a \chi_a f\|^2_{L^p}\Big)^{1/2} < \infty. \tag{1.31}$$

Here $L^p = L^p(X)$ but note that, by identifying $\chi_a f \equiv f_a$, we can also interpret $\mathscr{L}$ as a conveniently normed direct sum of the spaces $L^p(K_a)$, see [DJT, page XIV]. If $\lambda_a = 1$ for all a we set $\ell^2_\lambda(L^p) = \ell^2(L^p)$. Observe that $\ell^2(L^2) = L^2(X)$.

Let q be given by $\frac{1}{p} = \frac{1}{2} + \frac{1}{q}$, so that $1 < p < 2 < q < \infty$. We also need the space $\mathscr{M} \equiv \ell^\infty_\lambda(L^q)$ defined by the condition

$$\|g\|_{\mathscr{M}} := \sup_{a\in Z} \|\lambda_a \chi_a g\|_{L^q} < \infty. \tag{1.32}$$

The definitions are chosen such that $\|gu\|_{\mathscr{L}} \le \|g\|_{\mathscr{M}} \|u\|_{L^2}$ where $L^2 = L^2(X)$. As explained at [DJT, page XV], the space $\mathscr{M}$ is naturally identified with the dual space of the Banach space $\mathscr{M}_* \equiv \ell^1_{\lambda^{-1}}(L^{q'})$, where $\frac{1}{q} + \frac{1}{q'} = 1$, defined by the norm

$$\|h\|_{\mathscr{M}_*} := \sum_{a\in Z} \|\lambda_a^{-1} \chi_a h\|_{L^{q'}}.$$

Below, when we speak about w^*-topology on $\mathscr{M}$ we mean the $\sigma(\mathscr{M}, \mathscr{M}_*)$-topology. Clearly

$$\mathscr{M}_1^+ = \{g \in \mathscr{M} \mid g \ge 0, \|g\|_{\mathscr{M}} \le 1\}$$

is a convex compact subset of $\mathscr{M}$ for the w^*-topology.

Lemma A.6 *For each $f \in \mathscr{L}$ there is $g \in \mathscr{M}_1^+$ such that $\|f\|_{\mathscr{L}} = \|g^{-1} f\|_{L^2}$.*

Proof: We can assume $f \ge 0$. Since $1 = \frac{p}{2} + \frac{p}{q}$, we have:

$$\|f_a\|_{L^p} = \|f_a\|^{p/2}_{L^p} \|f_a\|^{p/q}_{L^p} = \|f_a^{p/2}\|_{L^2} \|f_a^{p/q}\|_{L^q} = \|f_a^{-p/q} f\|_{L^2} \|f_a^{p/q}\|_{L^q}$$

with the usual convention $0/0 = 0$. Now we define g_a on K_a as follows. If $f_a = 0$ then we take any $g_a \geq 0$ satisfying $\lambda_a \|g_a\|_{L^q} = 1$. If $f_a \neq 0$ let

$$g_a = \lambda_a^{-1}\big(f_a/\|f_a\|_{L^p}\big)^{p/q} = \lambda_a^{-1}\|f_a^{p/q}\|_{L^q}^{-1} f_a^{p/q}.$$

Thus we have $\lambda_a\|g_a\|_{L^q} = 1$ for all a, in particular $\|g\|_{\mathscr{M}} = 1$. By the preceding computations we also have $\|f_a\|_{L^p} = \|g_a^{-1}f_a\|_{L^2}\|g_a\|_{L^q}$ and so

$$\|f\|_{\mathscr{L}}^2 = \sum \lambda_a^2 \|f_a\|_{L^p}^2 = \sum \lambda_a^2 \|g_a\|_{L^q}^2 \|g_a^{-1} f_a\|_{L^2}^2 = \sum \|g_a^{-1} f_a\|_{L^2}^2$$

which is just $\|g^{-1}f\|_{L^2}^2$. ■

The main technical result follows.

Proposition A.7 *Let $(f^u)_{u\in U}$ be a family of functions in $\mathscr{L}$ such that, for each $\alpha = (\alpha_u)_{u\in U}$ with $\alpha_u \in \mathbb{R}$, $\alpha_u \geq 0$ and $\alpha_u \neq 0$ for at most a finite number of u, the function $f^\alpha := (\sum_u |\alpha_u f^u|^2)^{1/2}$ satisfies $\|f^\alpha\|_{\mathscr{L}} \leq \|\alpha\|_{\ell^2(U)}$. Then there is $g \in \mathscr{M}_1^+$ such that $\|g^{-1}f^u\|_{L^2} \leq 1$ for all $u \in U$.*

Proof: For each α as in the statement of the proposition we define a function $F_\alpha : \mathscr{M}_1^+ \to]-\infty, +\infty]$ as follows:

$$F_\alpha(g) = \|g^{-1}f^\alpha\|_{L^2}^2 - \|\alpha\|_{\ell^2(U)}^2 = \sum_u \alpha_u^2\big(\|g^{-1}f^u\|_{L^2}^2 - 1\big).$$

Our purpose is to apply Proposition A.4 with $\mathscr{K} = \mathscr{M}_1^+$ equipped with the w^*-topology and $\mathscr{F}$ equal to the set of all functions F_α defined above. We saw before that $\mathscr{K}$ is a convex compact set. From the second representation of F_α given above it follows that $\mathscr{F}$ is a convex set. Each F_α is a convex function because $\|g^{-1}f^\alpha\|_{L^2}^2 = \int g^{-2}(f^\alpha)^2 \mathrm{d}x$ and the map $t \mapsto t^{-2}$ is convex on $[0, \infty[$. We shall prove in a moment that F_α is lower semicontinuous. From Lemma A.6 it follws that there is $g_\alpha \in \mathscr{K}$ such that $\|f^\alpha\|_{\mathscr{L}} = \|g_\alpha^{-1}f^\alpha\|_{L^2}$. Then by our assumptions we have

$$F_\alpha(g_\alpha) = \|f^\alpha\|_{\mathscr{L}}^2 - \|\alpha\|_{\ell^2(U)}^2 \leq 0.$$

From Ky Fan's Lemma it follows that one can choose $g \in \mathscr{K}$ such that $F_\alpha(g) \leq 0$ for all α, which finishes the proof of the proposition.

It remains to show the lower semicontinuity of F_α. For this it suffices to prove that $g \mapsto \|g^{-1}f\|_{L^2}^2 \in [0, \infty]$ is lower semicontinuous on $\mathscr{K}$ if $f \in \mathscr{L}, f \geq 0$. But

$$\|g^{-1}f\|_{L^2}^2 = \sum_a \int_{K_a} g_a^{-2} f_a^2 \mathrm{d}x$$

and the set of lower semicontinuous functions $\mathscr{K} \to [0, \infty]$ is stable under sums and upper bounds of arbitrary families. Hence it suffices to prove that each map $g \mapsto \int_{K_a} g_a^{-2} f_a^2 \mathrm{d}x$ is lower semicontinuous. This map can be written as a composition $\phi \circ J_a$ where $J_a : \mathscr{M} \to L^q(K_a)$ is the restriction map $J_a g = g_a$ and $\phi : L^q(K_a) \to [0, \infty]$ is defined by $\phi(\theta) = \int_{K_a} \theta^{-2} f_a^2 \mathrm{d}x$. The map J_a is continuous if we equip $L^q(K_a)$ with the weak topology and $\mathscr{M}$ with the w^*-topology because it is the adjoint of the norm continuous map $L^{q'}(K_a) \to \mathscr{M}_*$ which sends u into the function equal to u on K_a and 0 elsewhere. Thus it suffices to show that ϕ is lower semicontinuous on the positive part of $L^q(K_a)$ equipped with the weak topology and for this we can use exactly the same argument as Maurey. We must prove that the set $\{\theta \in L^q(K_a) \mid \theta \geq 0, \phi(\theta) \leq r\}$ is weakly closed for each real r. Since ϕ is convex, this set is convex, so it suffices to show that it is norm closed. But this is clear by the Fatou Lemma. ■

Theorem A.8 *Let $\mathscr{H}$ be a Hilbert space and $T : \mathscr{H} \to \mathscr{L}$ a linear continuous map. Then there exist a linear continuous map $R : \mathscr{H} \to L^2(X)$ and a positive function $g \in \mathscr{M}$ such that $T = g(Q)R$.*

Proof: Let U be the unit ball of $\mathscr{H}$ and for each $u \in U$ let $f^u = Tu$. From Proposition A.5 we get

$$\|f^\alpha\|_{\mathscr{L}} = \|(\textstyle\sum_u |T(\alpha_u u)|^2)^{1/2}\|_{\mathscr{L}} \leq A(\textstyle\sum_u \|\alpha_u u\|^2)^{1/2} \leq A(\textstyle\sum_u |\alpha_u|^2)^{1/2}$$

where $A = C\|T\|_{\mathcal{B}(\mathscr{H},\mathscr{L})}$. Since there is no loss of generality in assuming $A \leq 1$, we see that the assumptions of Proposition A.7 are satisfied. So there is $g \in \mathscr{M}_1^+$ such that $\|g^{-1}Tu\|_{L^2(X)} \leq 1$ for all $u \in U$. Thus it suffices to define R by the rule $Ru = g^{-1}Tu$ for all $u \in \mathscr{H}$. ■

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