Generalized Nash Inequalities

D. Bakry

Joint works with F. Bolley, I. Gentil, P. Maheux

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Describe some robust and general method to control spectra for symmetric diffusion operators

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In the model case $\Gamma(f,f) = g^{ij}\partial_i f \partial_j f$, where g^{ij} is the Riemann metric.

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for a *density* p_t, and

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- Examples

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Sobolev $\Longrightarrow$ Logarithmic Sobolev $\Longrightarrow$ Poincaré

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Converse : use : $H(t)$ is convex.
Indeed stronger : $\ln H$ is convex

$$H'' = 4\int (LP_t f)^2 d\mu, \; H' = -2\int P_t f L(P_t f) d\mu$$

$$\Longrightarrow (\mathrm{Cauchy-Shwartz}) \Longrightarrow HH'' \geqslant H'^2.$$

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- Sobolev $\iff$ $\|P_t f\|_\infty \leqslant K(t)\|f\|_1$, $K(t) = Ct^{-n/2}$, $0 < t \leqslant 1$. (Ultracontractivity)
- Logarithmic Sobolev "almost enough" to get discrete spectrum (but not ultracontractivity)

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$\Psi = Cx^{n/(n+2)} (x \to \infty)$.

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take asymptotics in $t = 0$ and optimize in α.

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$$\int f^2 d\mu \leqslant r\mathcal{E}(f) + \Big(\int \Pi(f) d\mu\Big)^2 \tag{1}$$

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Then, $\sigma_{ess} \in [\frac{1}{r}, \infty)$.

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If $\sigma_{ess} = \emptyset$, then there exists $w \in \mathcal{L}^2(\mu)$ and $r \mapsto \beta(r)$ on $(0, \infty)$ such that

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Typically $w = \sum_i a_i f_i$, where f_i eigenvectors and (a_i) decreasing such that $\sum_i |a_i| < \infty$; then $\beta(r) = n(r)/a^2_{n(r)}$, such that $(f_1, \cdots, f_{n(r)})$ span the spectral space $E_{1/r}$.

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Holds as soon as the A_k are nicely separated and the embedding from $\mathcal{H}^1(A_k)$ into $\mathcal{L}^2(A_k)$ is compact.

Weighted Nash inequalities

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: Question : what is the relation with Ultracontractivity?

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Bound on the spectrum and the density

$$\sum_n e^{-\lambda_n t} \leqslant K^2(t)e^{ct}, \; p_t(x,y) \leqslant K(t)e^{ct}w(x)w(y)$$

$$K^{-1}(s) = \int_{\Psi^{-1}(s)}^\infty \frac{\Psi'(x)}{x}dx.$$

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Hint Use convexity of $t \mapsto \ln \|P_t f\|_2$.

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$W = c|x|^\alpha$, $1 < \alpha < 2$. Not ultracontractive but Weighted Nash with $\Psi(x) = C(1+x)^\lambda$ $(0 < \lambda < 1)$ and $w = e^{W/2}|x|^{-\gamma}$, $\gamma > 0$.

Thank you for your attention