# The random phase property and the Lyapunov spectrum

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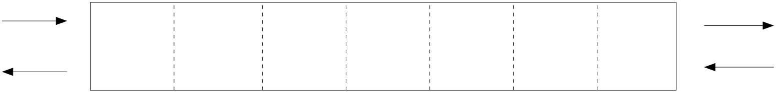
# Set-up:

- Topic: one particle quantum mechnics in quasi-1D random media
- sample with independent building blocks, each with $L$ channels



- the $n$th block has a transfer matrix $\mathcal{T}_n$ (equiv. scattering matrix)
- $\mathcal{T}_n$ is in the generalized Lorentz group $\mathrm{U}(L,L) \subset \mathrm{Mat}(2L,\mathbb{C})$

$$\mathcal{T}^*\mathcal{G}\mathcal{T} = \mathcal{G} \qquad \mathcal{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Polar decomposition in $\mathrm{U}(L,L)$ with diagonal $\Lambda \geq 0$:

$$\mathcal{T} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} \sqrt{1+\Lambda} & \sqrt{\Lambda} \\ \sqrt{\Lambda} & \sqrt{1+\Lambda} \end{pmatrix} \begin{pmatrix} u' & 0 \\ 0 & v' \end{pmatrix}$$

# Reminder on maximal entropy Ansatz (MEA):

**MEA:** in the polar decomposition of $\mathcal{T}_N \cdots \mathcal{T}_1$ the unitaries $u, v, u', v' \in \mathrm{U}(L)$ are independent and Haar distributed

- $N$ size of mesoscopic volume
- MEA leads the DMPK flow equations for $\Lambda$

**Discussion:**

- Markov process $(\mathcal{T}_N \cdots \mathcal{T}_1)_{N \geq 1}$ on $\mathrm{U}(L, L)$
- polar decomposition of $\mathcal{T}'\mathcal{T}$ from those of $\mathcal{T}'$ and $\mathcal{T}$ difficult
- state space non-compact
- approach is universal, no parameter dependence (as energy, etc.)
- no numerical test known in concrete models

# Alternative approach

**Aims:**
Model-dependent random dynamics on a compact space
Again link to RMT
Verifiable numerically by TMM procedure
Close to theory of products of random matrices

Natural action of $\mathrm{U}(L,L)$ on isotropic flag manifolds $\mathbb{F}$ (compact)
Flag manifold has set $\mathbb{I}$ of isotropic frames as cover:

$$\begin{aligned}\mathbb{I} &= \{\Phi \in \mathrm{Mat}(2L \times L, \mathbb{C}) \,|\, \Phi^*\Phi = \mathbf{1}\ ,\ \Phi^*\mathcal{G}\Phi = 0\,\} \\ &= \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} U \\ V \end{pmatrix} \middle|\ U, V \in \mathrm{U}(L) \right\}\end{aligned}$$

Identifying frames with same flag shows $\mathbb{F} = \mathbb{I}/\mathbb{T}^L$.

# Action

Action of $\mathrm{U}(L,L)$ on $\mathbb{I}$:

$$\mathcal{T}\cdot\Phi = \mathcal{T}\,\Phi\,S(\mathcal{T},\Phi)^{-1}$$

with $S(\mathcal{T},\Phi)$ upper triangular $L\times L$ with positive diagonal

Cocycle:

$$S(\mathcal{T}'\mathcal{T},\Phi) = S(\mathcal{T}',\mathcal{T}\cdot\Phi)\,S(\mathcal{T},\Phi)$$

Does not factor to cocycle on flag $\mathbb{F}$, but diagonal does!

**Markov process** of $\mathbb{I}$: $\Phi_n = \mathcal{T}_n\cdot\Phi_{n-1}$

**Use:** Calculation of Lyapunov spectrum (as in TMM)

$$\begin{aligned}\gamma_p &= \lim_{N\to\infty}\frac{1}{N}\log\ \|\Lambda^p\,\mathcal{T}_N\cdots\mathcal{T}_1\| \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log\langle e_p|S(\mathcal{T}_n,\Phi_{n-1})|e_p\rangle\end{aligned}$$

# Random phase property (RPP)

**Rough RPP:** $\Phi_N$ Haar distributed on $\mathbb{I} \cong \mathrm{U}(L) \times \mathrm{U}(L)$

- MEA implies the rough RPP
- But rough RPP is WRONG in concrete situations (details later)
- Need to go to normal system of coordinates and open channels

Interest in weak coupling regime of randomness:

$$H = H_0 + \lambda\, H_1 \qquad H_1 \text{ random}$$

$$\mathcal{T}_n = \mathcal{T} + \mathcal{O}(\lambda) \qquad \text{with } \mathcal{T} \text{ non-random}$$

Normal system of coordinates:

$$\mathcal{M}^{-1}\mathcal{T}_n\mathcal{M} = \mathcal{R}\, e^{\lambda \mathcal{P}_n + \mathcal{O}(\lambda^2)}$$

with $\mathcal{R}$ direct sum of $2 \times 2$ blocks (as symplectic diagonalization)

# Random phase property (RPP)

Elliptic/open channels and hyperbolic/closed/evanescent channels

$$\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta} \end{pmatrix} \qquad \begin{pmatrix} \cosh(\eta) & \sinh(\eta) \\ \sinh(\eta) & \cosh(\eta) \end{pmatrix}$$

$\mathcal{R}$ checker board sum of such blocks (Jordan blocks excluded)

$\pi_e$ and $\pi_h$ projections in $\mathbb{C}^L$ on elliptic/hyperbolic channels

**RPP:** Unique (!) invariant measure of Markov process on $\mathbb{I}$

$$\Phi_n = \mathcal{R}\, e^{\lambda \mathcal{P}_n} \cdot \Phi_{n-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

satisfies with errors of order $\mathcal{O}(\lambda)$:

(R1) $\pi_e U \pi_h = \pi_h U \pi_e = 0$

(R2) $\pi_h U \pi_h$ fixed permutation

(R3) $\pi_e U \pi_e$ Haar distributed on $\mathrm{U}(L_e)$ where $L_e = \dim(\pi_e)$

(R4) $U$ and $V$ independent and identically distributed (no TRI)

$U = \overline{V}$ or $U = I^* \overline{V} I$ (TRI with even or odd spin)

# General implications of RPP

**Program:**
- General implications of RPP
- Numerics and application for Anderson model
- How to prove the RPP

**Theorem**

*Suppose RPP holds for* $\mathcal{T}_n = \mathcal{R}\, e^{\lambda \mathcal{P}_n} \in \mathrm{U}(L,L)$*. For* $p > L_e$

$$\gamma_p \;=\; \frac{\lambda^2}{4L_e^2}\,\mathbf{E}\,\mathrm{Tr}(\Pi_e(\mathcal{P}^* + \mathcal{P})\Pi_e\mathcal{P}\Pi_e)\,\left(L - p + \frac{1}{\beta}\right) \;+\; \mathcal{O}(\lambda^3)$$

*where* $\Pi_e = \mathrm{diag}(\pi_e, \pi_e)$ *and* $\beta = 1, 2, 4$*.*

- Equidistance of Lyapunov spectrum
- Dependence of inverse localization length $\gamma_L$ on universality class

# Anderson model on a strip

Hilbert space $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^L \ni (\psi_n)_{n\in\mathbb{Z}}\,,\ \psi_n \in \mathbb{C}^L$

$$(H\psi)_n \;=\; \psi_{n+1} + \psi_{n-1} + (e^{i\varphi} S + e^{-i\varphi} S^* + \lambda V_n)\psi_n$$

$S$ cyclic shift on $\mathbb{C}^L$, $\varphi$ magnetic flux
$V_n = \mathrm{diag}(v_{n,1}, \ldots, v_{n,L})$ with i.i.d. centered entries

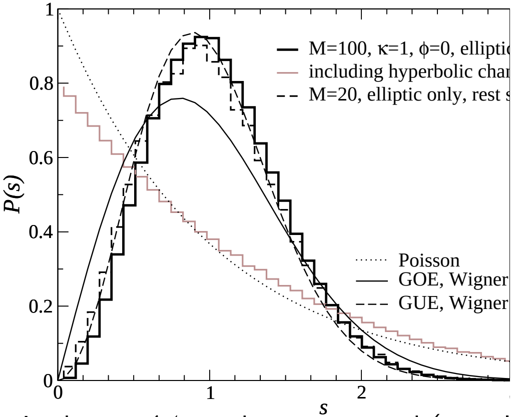
Schrödinger equation $H\psi = E\psi$ reformulated

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} \;=\; \begin{pmatrix} E\mathbf{1} - e^{i\varphi} S - e^{-i\varphi} S^* - \lambda V_n & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}$$

Transfer matrices after Cayley transform $\mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$

$$\mathcal{T}^E_n \;=\; \mathcal{C} \begin{pmatrix} E\mathbf{1} - e^{i\varphi} S - e^{-i\varphi} S^* - \lambda V_n & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \mathcal{C}^* \in \mathrm{U}(L, L)$$

# Numerical test of RPP



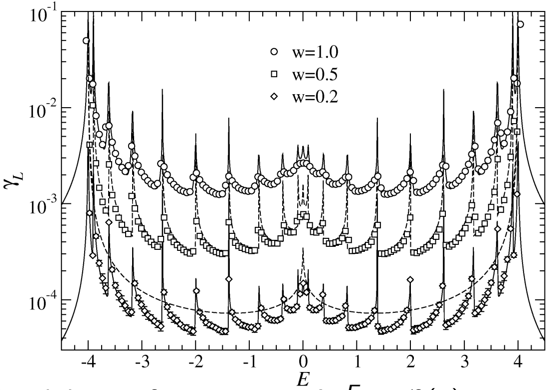

Basis change $\mathcal{M}$ can be constructed (symplectic diagonalization)

Plot of level spacing of $U_N$ for $N = 2000$ and $L = 20$ and $L = 100$

# Lyapunov exponents

Application: set $2\cos(k_l) = E - 2\cos(\frac{2\pi l}{L})$ for $l = 1, \ldots, L$

$$\gamma_p = \frac{\lambda^2}{4L} \left( \frac{1}{L_e} \sum_l \frac{1}{|\sin(k_l)|} \right)^2 \left( L - p + \frac{1}{\beta} \right) + \mathcal{O}(\lambda^3)$$




Breakdown of agreement: $L\gamma_L^E \sim \mathcal{O}(1)$

# How to prove the RPP?

**Abstract approach:** Given a random family of Lie group elements

$$\mathcal{T}_{\lambda,\sigma} = \mathcal{R} \exp(\lambda \mathcal{P}_\sigma) \in \mathcal{G}$$

where $\mathcal{R}$ generates compact group $\langle \mathcal{R} \rangle$ (no hyperbolic channels)
$\mathcal{P}_\sigma$ i.i.d. in Lie algebra with $\mathbf{E}(\mathcal{P}_\sigma) = 0$
Group acts on compact homogeneous space $\mathbb{I}$
$\mathbb{I}$ has invariant volume $\mu$
Induced Markov process on $\mathbb{I}$

$$\Phi_n = \mathcal{T}_{\lambda,n} \cdot \Phi_{n-1}$$

Interest: perturbative calculation (in $\lambda$) of averaged Birkhoff sums

$$I_{\lambda,N}(f) = \frac{1}{N} \mathbf{E} \sum_{n=1}^{N} f(\Phi_n)$$

Known: Dunford-Schwartz operator ergodic theorem

# Abstract Theorem

**Theorem**

*Suppose that*

$$\mathrm{Lie}(Re^{\lambda P}R^{-1} \,|\, R \in \langle\mathcal{R}\rangle, P \in \mathrm{supp}(\mathcal{P}_\sigma))$$

*acts transitively on* $\mathbb{I}$. *Then there is a* $\mu$*-a.s. positive,* $L^1$*-normalized function* $\rho \in C^\infty(\mathbb{I})$*, such that for any* $f \in C^\infty(\mathbb{I})$ *consisting of low frequencies w.r.t.* $\mathcal{R}$

$$I_{\lambda,N}(f) = \int d\mu\, \rho\, f \;+\; \mathcal{O}(\lambda, \frac{1}{N\lambda^2})\,.$$

$\langle\mathcal{R}\rangle$ compact abelian group $\Rightarrow$ isom. $\varphi : \langle\mathcal{R}\rangle \to \mathbb{Z}_n \times (\mathbb{R}/2\pi\mathbb{Z})^k$

$f$ consists of low frequencies w.r.t. $\mathcal{R}$ if the Fourier series of the function $R \mapsto f(R\cdot\Phi)$, $R \in \langle\mathcal{R}\rangle$ is finite, uniformly in $\Phi$

**Remarks:**

- Main hypothesis replaces Furstenberg's irreducibility condition
- In Anderson model, $\mathcal{P}$ has only $L$ random entries, $\dim(\mathbb{I}) = 2L^2$
  Nevertheless, hypothesis is satisfied
- Proof provides technique to check RPP, namely $\rho = 1$
- At least the perturbative invariant measure $\rho\mu$ is unique and a.c. w.r.t. to the Riemannian volume measure

**Corollary**

*For any family in $\lambda$ of invariant measures $\nu_\lambda$,*

$$\mathrm{w}^*\text{-}\lim_{\lambda \to 0} \nu_\lambda \;=\; \mu\,\rho$$

Basic idea of proof in case $\mathcal{R} = \mathbf{1}$

For Lie algebra element $P$ define the vector field

$$\partial_P f(\Phi) \;=\; \frac{d}{dt}\Big|_{t=0} f(e^{tP}\cdot\Phi)$$

Consider $\mathcal{L} = \mathbf{E}_\sigma(\partial^2_{\mathcal{P}_\sigma})$

$\mathcal{L}$ also second derivative of Markov operator

$$\mathbf{E}_\sigma F(\mathcal{T}_{\lambda,\sigma}\cdot\Phi) \;=\; F(\Phi) + \frac{\lambda^2}{2}\mathcal{L}(F)(\phi) + \mathcal{O}(\lambda^3) \qquad F\in C^\infty(\mathbb{I})$$

Taking Birkhoff sum of both sides gives:

$$I_{\lambda,N}(\mathcal{L}F) = \mathcal{O}(\lambda,(\lambda^2 N)^{-1})$$

Adjoint $\mathcal{L}^*$ in $L^2(\mathbb{I},\mu)$, operator of Fokker-Planck type

Main claim: there is smooth a.s. positive $\rho$ with

$$\ker\mathcal{L}^* = \mathbb{C}\rho \qquad C^\infty(\mathbb{I}) = \mathbb{C}1_{\mathbb{I}} \oplus \mathcal{L}(C^\infty(\mathbb{I}))$$

Then result follows from $f = \int d\mu(\rho f) + \mathcal{L}F$ for $f\in C^\infty(\mathbb{I})$

Proof of main claim: $\mathcal{L}$ can be brought in Hörmander form
Main hypothesis implies that $\mathcal{L}$ and $\mathcal{L}^*$ are Hörmander operators
Use subelliptic estimates (compact resolvent)
Bony's maximum principle (unique groundstate)
hypoellipticity (smoothness), dissipativity ($\Re e\langle f|\mathcal{L}|f\rangle \leq c\|f\|^2$)

**Corollary**

*If $\mathcal{L}^*(\mathbf{1}_{\mathbb{I}}) = 0$, then perturbative invariant measure is Haar measure (RPP holds).*

Iteration in case $\mathcal{R} = \mathbf{1}$ gives:

$$I_{\lambda,N}(f) = \sum_{m=0}^{M-1} \lambda^m \int d\mu\, \rho_m f \,+\, \mathcal{O}(\lambda^M, \frac{1}{N\lambda^2})\,.$$

If $\mathcal{R} \neq \mathbf{1}$, one can use instead

$$\hat{\mathcal{L}}F \;=\; \int_{\langle\mathcal{R}\rangle} dR\, \mathbf{E}_\sigma(\partial^2_{R\mathcal{P}_\sigma R^{-1}} F)$$

# Wegner $L$-orbital model

As Anderson model, but no shift and $V_n$ full random hermitian

**Proposition**

*For* $|E| < 2, E \neq 0$, *then* $\mathrm{RPP}$ *holds*

$$\gamma_p^E \;=\; \lambda^2 \,\frac{1+2(L-p)}{2(4-E^2)} \;+\; \mathcal{O}\Big(\frac{\lambda^3}{\min\{|E|,|E\pm 2|\}}\Big)$$

Remarks:

- Case $L = 1$ is Thouless formula (Pastur and Figotin, and above)
- Perturbatively equidistant Lyapunov spectrum
- Scaling by factor $L \sim L_e$ different from Anderson

# Application to anomalies (no RPP holds)

## Proposition

*$E = 0$, Kappus-Wegner anomaly for Anderson or L-orbital model*

$$\sum_{p=1}^{L} \gamma_p^E \;=\; \lambda^2 L^2 \int d\Phi\, \rho(\Phi)\, f(UV^*) + \mathcal{O}(L\lambda^3)$$

*where both $f$ and $\rho \neq 1$ are explicit*

## Proposition

*$L = 1$, i.e. Anderson model. For band edge $E = 2$:*

$$\gamma^E \;=\; \lambda^{2/3} \int d\Phi\, \rho(\Phi)\, g(UV^*) \;+\; \mathcal{O}(\lambda)$$

*for some smooth $g : S^1 \to \mathbb{R}$*

Remark: Derrida-Gardener (1987)

# Oscillation theorem

$$(H_N\psi)_n = T_{n+1}\psi_{n+1} + V_n\psi_n + T_n\psi_{n-1} \text{ with } T_{N+1} = T_0 = 0$$

## Theorem

*$H^N$ on finite volume $L \times N$, $E \in \mathbb{R}$*
*$W^E_N = U^E_N(V^E_N)^*$ unitary at $N$ with $W^E_0 = \mathbf{1}$*

- *$L$ lifted eigenphases $\theta^E_{N,\ell} \in \mathbb{R}$ of $W^E_N$ real analytic in $E$*
- *$E$ eigenvalue of $H^N$ of multiplicity $m$*
*iff $\theta^E_{N,\ell} = \pi \mod 2\pi$ for $m$ eigenphases*
- *speed matrix $S^E_N = \frac{1}{\imath}(W^E_N)^*\partial_E W^E_N$ positive definite*
- *each $\theta^E_{N,\ell}$ increasing function of $E$*
- *each $\theta^E_{N,\ell}$ makes $N$ turns for $E \in \mathbb{R}$*
- *$H^N$ is real $\Rightarrow$ $W^E_N$ symmetric and $S^E_N$ real*

**Remark:** reduction of dimension by 1 for eigenvalue calculation

**Preliminary numerical results**
for Anderson on large squares ($L = N$) and cubes ($L = N^2$)

- RPP does not hold for $W_N^E$, no flat DOS of phases
- but already separation of hyperbolic and elliptic channels
- elliptic phases of $W_N^E$ RMT-level spacing
- DOS of $S_N^E$ fat tails in localization regime (as in quasi-1D)
- $S_N^E$ approximately Pastur-Marchenko in metallic phase
- eigenbasis of $W_N^E$ and $S_N^E$ not correlated

**Szenario** for localized phase:
high speeds give Poisson statistics

**Szenario** for metallic phase:
level-spacing of $W_N^E$ and small phase speeds of same magnitude lead to GOE-level spacing of $H^N$

# Resumé

- RPP describes distribution on a compact space in normal coordinates
- Holds for Wegner $L$-orbital (proof) and Anderson (numerics)
- Formulas for Lyapunov spectrum
- Techniques for study of finite size systems