Time-dependent approach to irreversibility and scattering in open quantum systems

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based on work with A. Kupiainen.

Plan of talk

Goal

Quantum systems of type: Small system + bath (or field). Can we derive microscopically thermalization, scattering, etc. ?

1. Cluster expansion for polymer model
2. Weakly perturbed Markov chain as a non-commutative polymer model.
3. From noncommutative to ordinary polymer model.
4. The quantum setup: why is it a weakly perturbed Markov chain?

Polymer models

Polymer Model on $I_N = \{1, 2, \dots, N\}$

$$Z_N = \sum_{\mathcal{A} \in 2^{I_N}} \chi(\mathcal{A}\,\text{admissible}) \prod_{A \in \mathcal{A}} \varrho(A)$$

- *Polymer weights* $\varrho(A) \in \mathbb{C}$.
- Adjacency relation on 2^{I_N}: $\quad A \sim A' \quad \Leftrightarrow \quad A \cap A' \neq \emptyset$
- $\mathcal{A}$ admissible means: $\forall A \neq A' \in \mathcal{A} : A \nsim A'$

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Examples: Product and Weakly coupled systems

'Product' $\varrho(A) = \chi(A = \{\tau\}) z(\tau) \quad \Rightarrow \quad Z_N = \prod_\tau (1 + z(\tau))$.

'Weak coupling' $\varrho(A) = \mathcal{O}(\epsilon)$ for $|A| = 2 \quad \Rightarrow \quad Z_N = ??$

Good type of expansion of 'weak' around 'product' turns out:

$$F_N := \log Z_N = \sum_\tau \log(1 + z(\tau)) + N\mathcal{O}(\epsilon), \qquad N \to \infty$$

Relation to correlation function

In case $Z_N = 1$, can interpret (if not, just normalize)

$$\mathbb{P}(\mathcal{A}) = \chi(\mathcal{A}\,\text{admissible}) \prod_{A \in \mathcal{A}} \varrho(A), \qquad Z_N = 1$$

(prob. of config. of 'real' polymers, interacting via exclusion)

Correlation function between points τ, τ'

Let $\mathrm{Supp}\mathcal{A} = \cup_{A \in \mathcal{A}} A$.

$$\nu(\tau, \tau') := \mathbb{P}(\tau, \tau' \in \mathrm{Supp}\mathcal{A}) - \mathbb{P}(\tau \in \mathrm{Supp}\mathcal{A})\mathbb{P}(\tau' \in \mathrm{Supp}\mathcal{A})$$

Does it decay: $\nu(\tau, \tau') \to 0$ as $|\tau - \tau'| \to \infty$?

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For weak coupling: Yes! $(|\nu(\tau,\tau')| \leq (C\epsilon)^{|\tau-\tau'|})$

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For weak coupling: Yes! $(|\nu(\tau,\tau')| \leq (C\epsilon)^{|\tau-\tau'|})$

This is known in STAT-MECH as 'high-temperature behaviour'.

Relation to correlation function

Correlation

$$\nu(\tau,\tau') := \mathbb{P}(\tau,\tau' \in \mathrm{Supp}\mathcal{A}) - \mathbb{P}(\tau \in \mathrm{Supp}\mathcal{A})\mathbb{P}(\tau' \in \mathrm{Supp}\mathcal{A})$$

satisfies

$$\nu(\tau,\tau') := \partial_{\kappa_{\tau'}}\partial_{\kappa_\tau} \log Z(\kappa)\Big|_{\kappa=0}$$

with $\kappa = (\kappa_\tau)_{\tau \in I_N}$

$$\begin{aligned} Z(\kappa) &= \mathbb{E}(\mathrm{e}^{\sum_\tau \kappa_\tau \chi(\tau \in \mathrm{Supp}\mathcal{A})}) \\ &= \sum_{\mathcal{A}} \prod_{A \in \mathcal{A}} \varrho_{\kappa_A}(A), \qquad \varrho_{\kappa_A}(A) = \varrho(A)\mathrm{e}^{\sum_{\tau \in A} \kappa_\tau} \end{aligned}$$

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Cluster Expansion

A way to write a $F_N = \log Z_N$ as a sum of local terms

$\Rightarrow$ No (or only very small) terms that depend on both $\kappa_\tau, \kappa_{\tau'}$.

Cluster Expansion: Naive approach

Try to expand logarithm! Take 'weakly coupled' model:

$$\varrho(A) = \begin{cases} \epsilon & A = \{\tau, \tau + 1\} \\ 0 & \text{otherwise} \end{cases}$$

Truncate at order ϵ^2

$$Z = 1 + \sum_{A} \varrho(A) + \sum_{A,A'} \chi(A \nsim A')\varrho(A)\varrho(A') + \mathcal{O}(\epsilon^4)$$

Use $\log(1 + x) = 1 + x - x^2 + \mathcal{O}(x^3)$:

$$\log Z = 1 + \sum_{A} \varrho(A) - \sum_{A,A'} \varrho(A)\varrho(A') + \sum_{A,A'} \chi(A \nsim A')\varrho(A)\varrho(A') + \mathcal{O}(\epsilon^4)$$

$$= 1 + \sum_{A} \varrho(A) - \sum_{A,A'} \chi(A \sim A')\varrho(A)\varrho(A')) + \mathcal{O}(\epsilon^4)$$

Cluster Expansion: Naive approach

$$\log Z = 1 + \sum_{A} \varrho(A) - \underbrace{\sum_{A,A'} \chi(A \sim A') \varrho(A) \varrho(A')}_{=0 \text{ if } diam(A \cup A') > 3} + \mathcal{O}(\epsilon^4)$$

This means that $\log Z$ is sum of local terms, depending on at most 3 neighboring points. $\Rightarrow$ Sucess! (Locality $\Rightarrow$ Correlation decay)

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Of course, this is nonsense because $\mathcal{O}(\epsilon^4)$ is rather $\mathcal{O}(N^2\epsilon^4)$.
Yet, the expansion turns out to be correct for small ϵ!

Cluster Expansion: A theorem

Assumption on ϱ: 'close to independence'

$$\sup_{\tau} \sum_{A \ni \tau} \mathrm{e}^{a|A|} |\varrho(A)| \leq a, \qquad \text{(Kotecky.Preiss criterion)}$$

Result

Provided KP holds,

$$\log Z = \sum_{A} \varrho^{T}(A), \qquad \text{with } \sum_{A \ni \tau} |\varrho^{T}(A)| \leq a$$

with $\varrho^{T}(A)$ function of $\varrho(A'), A' \subset A$

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- $\sum_{A \ni \tau} |\varrho^{T}(A)| \leq a$ gives some locality (summability) in F_N
- Most powerful if a can be chosen independent of N.
- Note that exp. decay in $|A|$ for $\varrho(A)$ is required.
- Polymers $A = \{\tau\}$ can always be scaled out, not covered here.

Correlation decay from cluster expansion

Assume that *KP*-criterion holds in a stronger way

Assumption on ϱ: encode decay ($d(A) = diam(A)$)

$$\sup_{\tau} \sum_{A \ni \tau} \mathrm{e}^{a|A|} |\varrho_\alpha(A)| \leq a, \qquad \varrho_\alpha(A) = \varrho(A) d(A)^\alpha$$

Results with and without α

$$\log Z = \sum_A \varrho^T(A), \qquad \text{with } \sum_{A \ni \tau} d(A)^\alpha |\varrho^T(A)| \leq a$$

$$\begin{aligned}
\partial_{\tau_\kappa} \partial_{\tau'_\kappa} \log Z &= \sum_{A:\{\tau,\tau'\} \subset A} \partial_{\tau_\kappa} \partial_{\tau'_\kappa} \varrho^T(A) \\
&= |\tau - \tau'|^{-\alpha} \sum_{A:\{\tau,\tau'\} \subset A} |\partial_{\tau_\kappa} \partial_{\tau'_\kappa} \varrho^T_\alpha(A)| \\
&\sim |\tau - \tau'|^{-\alpha} a. \qquad (\text{if } KP \text{ uniform in } \kappa \in D \subset \mathbb{C}.)
\end{aligned}$$

Let Ω be a finite set ($|\Omega| = d$) and U a transition kernel on $\ell^1(\Omega)$, i.e.

- U is an $d \times d$- matrix with entries $U(s, s') \geq 0$
- Conservation of probability. $U^* 1 = 1$.

Interpretation: If $\rho \in \ell^1(\Omega)$ is a pdf. on Ω then $U^k \rho$ is the pdf. after k time-steps.

U has spectral gap $\Rightarrow$ chain exponentially ergodic

If $\sigma(U)$ consists of simple eigenvalue 1 and all other eigenvalues μ have $|\mu| < 1 - g$, then

$$U^k - |\rho_*\rangle\langle 1| = \mathcal{O}((1-g)^k), \qquad k \to \infty$$

for some pdf. ρ_* ($=$ unique invariant pdf.)

Weakly random Markov chains

Let transition kernels $U = U_\tau$ depend on timestep τ through some randomness ω, such that

- U_τ for different τ are 'weakly dependent' (formalize later)
- Joint law U_τ is time-translation invariant.
- $T = \mathbb{E}_\omega(U_\tau)$ (also transition kernel) has a gap.

Interpretation: Still $U_N \dots U_1$ is transition kernel.

Chain still (exponentially) ergodic?

For example; Consider the 'average total transition map'

$$\mathcal{Z}_N := \mathbb{E}_\omega(U_N \dots U_1) \quad \stackrel{?}{\to} \quad |\tilde{\rho}_*\rangle\langle 1|$$

- Don't expect exp. decay unless correlations U_τ decay exp.
- $\exists$ prob. solutions, but want brutal method (later: prob $\to$ $\mathbb{C}$-numbers)

Cluster expansion would help if applicable

Assume that we obtain ($v, w \in \ell^1(\Omega)$)

$$\langle v|\mathcal{Z}_N|w\rangle = \sum_{\mathcal{A}} \prod_{A\in\mathcal{A}} \varrho(A) = \mathrm{e}^{\sum_A \varrho^T(A)}$$

where

- $\rho(A)$ (and hence $\varrho(A)$) depend on w only if $1 \in A$ and on v only if $N \in A$.
- some decay: $\sum_{A\ni\tau} |\varrho^T(A)| d(A)^\alpha \leq a$.

Then philosphy on correlation decay applies $\Rightarrow$ Interpret v as observable, then dependence on w decays:

$$\| |\tilde{\rho}_*\rangle\langle 1| - \mathcal{Z}_N \| \leq CN^{-\alpha}, \qquad N \to \infty$$

for some pdf. $\tilde{\rho}_*$.

Expansion of $\mathcal{Z}_N$

Set

$$U_\tau = T + B_\tau, \qquad T := \mathbb{E}(U_\tau)$$

B_τ is small $\Rightarrow$ expand in powers of B.

$$\begin{aligned}\mathcal{Z}_N &= \mathbb{E}(U_N \dots U_1) \\ &= \sum_{A \subset I_N} \mathbb{E}(\underbrace{\dots B_{\tau_2} \dots B_{\tau_1} \dots}_{T \text{ whenever} \dots}), \qquad A = \{\tau_1, \tau_2, \dots, \tau_m\})\end{aligned}$$

Need more formalism to write this:

Formalism: Tensor Lattice (but all is finite!)

Let

$$\mathscr{R} := \mathcal{B}(\ell^1(\Omega)), \qquad \text{(space of } d \times d\text{-matrices)}$$

and

$$\mathscr{R}_{I_N} = \otimes^N \mathscr{R} \sim \mathscr{R}_N \otimes \ldots \otimes \mathscr{R}_2 \otimes \mathscr{R}_1$$

with subalgebras $\mathscr{R}_A$ for $A \subset I_N$. Contraction $\mathcal{T}$

$$\mathcal{T} : \mathscr{R}_{\{k,k+1,\ldots,k+l\}} \to \mathscr{R} : \qquad \mathcal{T}(O_{k+l} \otimes \ldots \otimes O_k) = O_{k+l} \ldots O_k$$

and extend by linearity. Redefine

$$B_\tau \overset{new}{=} 1 \ldots 1 \otimes B_\tau \otimes 1 \ldots 1, \qquad T_\tau \overset{new}{=} 1 \ldots 1 \otimes T_\tau \otimes 1 \ldots 1$$

Then $B_\tau, T_\tau \in \mathscr{R}_{I_N}$:

$$\underbrace{B_3 T B_1}_{previously} \Rightarrow \underbrace{\mathcal{T}(T_2 B_3 B_1)}_{now} \quad (= \mathcal{T}(B_1 B_3 T_2))$$

Use of formalism

$$
\begin{aligned}
\mathcal{Z}_N &= \mathbb{E}(U_N \dots U_1) \\
&= \sum_{A \subset I_N} \mathbb{E}\mathcal{T}\left[(\prod_{\tau \in A^c} T_\tau)(\prod_{\tau \in A} B_\tau) \right] \\
&= \sum_{A \subset I_N} \mathcal{T}\mathbb{E}\left[(\prod_{\tau \in A^c} T_\tau)(\prod_{\tau \in A} B_\tau) \right] \\
&= \sum_{A \subset I_N} \mathcal{T}\left[(\prod_{\tau \in A^c} T_\tau)\mathbb{E}(\prod_{\tau \in A} B_\tau) \right] \\
&=: \sum_{A \subset I_N} \mathcal{T}\left[(\prod_{\tau \in A^c} T_\tau) G_A \right]
\end{aligned}
$$

Ways to write operations on big matrices: $\mathcal{T}$ selects some entries, $\mathbb{E}$ averages all entries. G_A is a matrix of moments.

Formalism: Lattice systems

$$\mathcal{Z}_N =: \sum_{A \subset I_N} \mathcal{T} \left[\Big(\prod_{\tau \in A^c} T_\tau \Big) G_A \right]$$

Not very meaningful. G_A need not be small when $diam(A)$ is large.
We need correlations G_A^c instead of moments G_A!

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Relation between moments and correlations

$$G_{A'} = \sum_{\text{partitions } \mathcal{A} \text{ of } A'} \prod_{A \in \mathcal{A}} G_A^c$$

$\Rightarrow$ defines correlations G_A^c by recursion:

$$G_\tau = G_{\tau'}^c, \qquad G_{\{\tau',\tau\}} = G_{\tau'}^c G_\tau^c + G_{\{\tau',\tau\}}^c$$

hence $G_{\{\tau',\tau\}}^c = G_{\{\tau',\tau\}} - G_{\tau'} G_\tau$

Expand $\mathcal{Z}_N$ in G_A^c

$$\mathcal{Z}_N =: \sum_{A' \subset I_N} \mathcal{T}\left[\Big(\prod_{\tau \in (A')^c} T_\tau \Big) G_{A'} \right]$$

$$=: \sum_{A' \subset I_N} \sum_{\text{partitions } \mathcal{A} \text{ of } A'} \mathcal{T}\left[\Big(\prod_{\tau \in (A')^c} T_\tau \Big) \prod_{A \in \mathcal{A}} G_A^c \right]$$

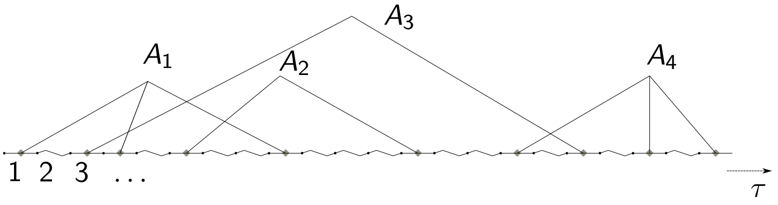
$$=: \sum_{\mathcal{A} \text{ admissible}} \mathcal{T}\left[\Big(\prod_{\tau \notin \mathrm{Supp}(\mathcal{A})} T_\tau \Big) \prod_{A \in \mathcal{A}} G_A^c \right]$$

Has the same form as polymer model, with weights $\tilde{\varrho}(A) = G_A^c$, but matrix-valued. A natural extra-severe 'KP'-condition is

$$\sum_{A \ni \tau} \epsilon^{-(|A|-1)} \| G_A^c \| d(A)^\alpha \leq 1, \qquad \text{for some } \epsilon \ll 1.$$

$\Rightarrow$ What to do: $\nexists$ noncommutative cluster expansion

Graphical rep: Noncommutative polymers



: $\tau \in \mathrm{Supp}\mathcal{A}$, : $\tau \notin \mathrm{Supp}\mathcal{A}$.

Operator weight (Contribution to $\mathcal{Z}_N$): each gives T, each connected component of gives G^c_A.

Failure: Weight is not a product of numbers depending only on A

$$\langle v|\mathcal{Z}_N|w\rangle \neq \sum_{\mathcal{A}} \prod_{A\in\mathcal{A}} f(A)$$

Way out: Use gap of T

Gap of T

Ergodicity assumption: $T = RT + R_\perp T$ with $R = RT = |\rho_*\rangle\langle 1|$ one-dimensional projection and $\|(R_\perp T)^k\| \leq C(1-g)^k$.

Insert in expansion:

$$\mathcal{Z}_N = \sum_{\mathcal{A},\mathcal{J}} \mathcal{T}\left[\prod_{A\in\mathcal{A}} G_A^c \prod_{\tau\in\mathrm{Supp}\mathcal{J}} (R_\perp T)_\tau \prod_{\tau\in I_N\setminus(\mathrm{Supp}\mathcal{J}\cup\mathrm{Supp}\mathcal{A})} (R)_\tau\right].$$

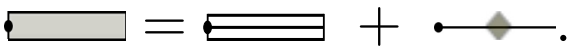
- $\mathcal{A}$ is admissible collection, $\mathcal{J}$ is collection of intervals
- $\mathrm{Supp}\mathcal{A} \cap \mathrm{Supp}\mathcal{J} = \emptyset$.
- R decouples: $RO_m R \ldots RO_2 RO_1 R = R\prod_j \mathrm{Tr}(RO_j) \Rightarrow$ Product of operators turns into product of numbers
- Price to pay: new polymers: intervals in $\mathcal{J}$, but their weight decays exp. : $\|(R_\perp T)^k\| \leq C(1-g)^k$

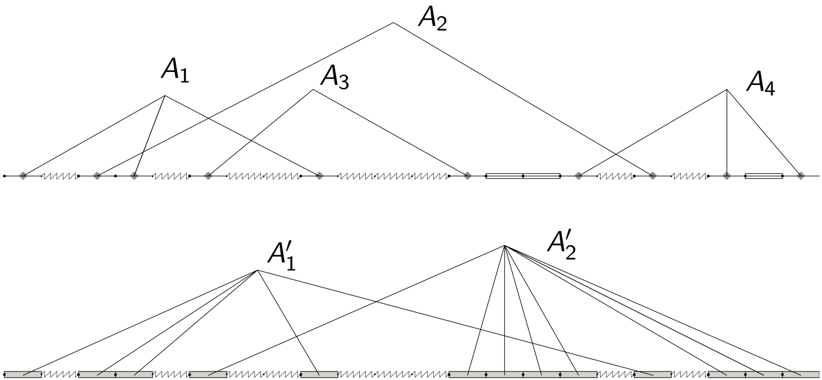
Way out: Use gap of T

First split

$$T = TR + TR_\perp =$$

then resum new and old polymers





New polymers

Expansion for

$$\frac{\langle v|\mathcal{Z}_N|w\rangle}{\langle v|\rho_*\rangle\langle 1|w\rangle} = \sum_{\mathcal{A}'}\prod_{A\in\mathcal{A}'}\varrho(A)$$

with, for bulk A ($1\notin A, N\notin A$)

$$\varrho(A)R = \sum_{\substack{\mathcal{A},\mathcal{J}\\(\mathcal{A},\mathcal{J})connected}} \mathcal{T}\left[\prod_{A\in\mathcal{A}} G_A^c \prod_{\tau\in\mathcal{J}}(R_\perp T)_\tau \prod_{\tau\in\partial A} R_\tau\right]$$

- $\partial A = \{\tau\in A^c : \mathrm{dist}(\tau, A) = 1\}$
- Note: $\mathrm{Supp}\mathcal{A}\cup\mathrm{Supp}\mathcal{J}$ is completely sandwiched between R
- $(\mathcal{A},\mathcal{J})$ connected means: set of sets $A\in\mathcal{A}$ and intervals $J\in\mathcal{J}$ are connected by the adjacency relation

$$S\sim S' \Leftrightarrow \mathrm{dist}(S,S') = 1$$

.

New polymers satisfy *KP*

- The A with weight G^c_A satisfy

$$\sum_{A \ni \tau} \epsilon^{-(|A|-1)} \|G^c_A\| d(A)^\alpha \leq 1, \qquad \text{for some } \epsilon \ll 1,$$

  i.e. they have $d(A)^{-\alpha}$ decay in diameter and $\epsilon^{|A|}$ decay in size.
- The intervals $J \in \mathcal{J}$ have weight $(R_\perp T)^{|J|}$: exponential decay in diameter and size, but decay rate $\mathcal{O}(1)$, not ϵ.

New KP:

$$\sum_{A \ni \tau} \mathrm{e}^{a|A|} |\varrho(A)| d(A)^\alpha \leq C\epsilon$$

Polymers ϱ inherit bad properties from their parents:

- $d(A)^{-\alpha}$ decay in diameter.
- $\mathrm{e}^{-a|A|}$ decay in size (decay rate $\mathcal{O}(1)$)
- However, still at least one ϵ because every new polymer is made out of at least one G^c_A (this is a lie)

We get the result

The new KP:

$$\sum_{A \ni \tau} \mathrm{e}^{a|A|} |\varrho(A)| d(A)^{\alpha} \leq C\epsilon$$

is indeed of the form

$$\sum_{A \ni \tau} \mathrm{e}^{a|A|} |\varrho(A)| d(A)^{\alpha} \leq a$$

so machinery applies and we get the result:

$$\| |\tilde{\rho}_*\rangle\langle 1| - \mathcal{Z}_N \| \leq CN^{-\alpha}, \qquad N \to \infty$$

for some pdf. $\tilde{\rho}_*$. Moreover,

$$\|\tilde{\rho}_* - \rho_*\| = \mathcal{O}(\epsilon)$$

'Generalized' Spin-boson model

- Hilbert space $\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{F}}$
  $\mathcal{H}_{\mathrm{S}} = \mathbb{C}^m$ (atom space) and $\mathcal{H}_{\mathrm{F}} = \Gamma(L^2(\mathbb{R}^3))$ (photon field).
- Hamiltonian: $H_{\mathrm{S}} : \mathrm{diag}(E_1.E_2, \ldots, E_m)$ and weak coupling.

$$H = H_{\mathrm{S}} \otimes 1 + 1 \otimes H_{\mathrm{F}} + \lambda H_{\mathrm{SF}}, \qquad 0 < |\lambda| < 1$$

- Free photon Hamiltonian $H_{\mathrm{F}} = \int \mathrm{d}q |q| a_q^* a_q$.
- Atom-photon coupling: $D = D^* \in \mathcal{H}_{\mathrm{S}}$

$$H_{\mathrm{SF}} = D \otimes \int \mathrm{d}q \, (\phi(q) \otimes a_q + \overline{\phi}(q) \otimes a_q^*)$$

  with form factor $\phi(q) \sim |q|^{-1+\alpha/2+\delta}$ as $q \to 0$.
- If $\alpha > 0$, H bounded below. If $\alpha > 1$, H has ground state.

Result

- Fermi Golden Rule condition (explained later)
- Correlation decay (explained later)

$$\int \mathrm{d}t |\zeta(t)| (1+|t|)^{\alpha} < \infty, \qquad \zeta(t) := \int \mathrm{d}q |\phi(q)|^2 \mathrm{e}^{\mathrm{i}|q|t}$$

Write $\langle A \rangle_t = \mathrm{Tr}\, \rho \mathrm{e}^{-\mathrm{i}tH} A \mathrm{e}^{\mathrm{i}tH}$ with ρ density matrix and A observable.

Approach to steady state (DR, Kupiainen)

Assume Fermi Golden Rule and Correlation decay with $\alpha > 0$, then

$$|\langle A \rangle_t - \langle A \rangle_\infty| \leq C(1+|t|)^{-\alpha}, \qquad \text{for } |\lambda| \ll 1$$

for sufficiently local A and ρ_0. (Example, $\rho_0 = \rho_{\mathrm{S}} \otimes |\Omega\rangle\langle\Omega|$ and $A = A_{\mathrm{S}} \otimes 1$). State $\langle A \rangle_\infty$ equals $\langle \Psi_{gs}, A\Psi_{gs} \rangle$ whenever Ψ_{gs} exists.

Some easier questions: Markovian limits

Let $\rho_{\mathrm{S},t} = \mathrm{Tr}_{\mathrm{F}}\, \mathrm{e}^{-\mathrm{i}tH} \rho_0 \mathrm{e}^{\mathrm{i}tH}$ and assume all eigenvalues of H_{S} simple.

General Idea (Van Hove '50, Davies '74)

$$\lim_{t=\lambda^{-2}\tau, \lambda \to 0} \rho_{\mathrm{S},t} = \mathrm{e}^{\tau \mathcal{L}} \rho_{\mathrm{S},0}$$

with $\mathcal{L}$ a Lindbladian (Quantum Markov generator). Limit corresponds to $t_{\mathrm{Fcorrelations}} \ll t_{dissipation} \sim \lambda^{-2}$ (phonons are like white noise).

Write $\rho_{\mathrm{S},0} = \mathrm{diag}(\mu_1(0), \ldots, \mu_m(0)) + \rho_{\text{off-diag},0}$, then

$$\mathrm{e}^{\tau \mathcal{L}} \rho_{\mathrm{S},0} = \mathrm{diag}(\mu_1(\tau), \ldots, \mu_m(\tau)) + \rho_{\text{off-diag},\tau},$$

where $\overline{\mu}(\tau)$ is a jump process on $\sigma(H_{\mathrm{S}})$. Jumps $e \to e'$, $\sim$ emission of photon with $|q| = e - e'$.
Furthermore, Decoherence: $\|\rho_{\text{off-diag},\tau}\| \sim \mathrm{e}^{-c\tau}$

Character of jump process

- The jump rate $j(e \to e')$ is calculated from second-order perturbation theory:

$$j(e \to e') = 2\pi |\langle e|D|e'\rangle|^2 \int \mathrm{d}q \delta(e - e' - |q|)|\phi(q)|^2$$

- Iff directed graph with edges $(e \sim e') \Leftrightarrow j(e \to e') \neq 0$ is connected, then the jump process converges to the state $e_{gs} = \min\{e \in \mathrm{sp} H_{\mathrm{S}}\}$ exponentially fast (Perron-Frobenius theorem): $\Rightarrow$ atom cascades down to ground state. This (together with uniqueness of e_{gs}) is our **Fermi Golden Rule condition**. In fact, only real necessity: $\mathcal{L}$ has a gap!

Non-Markovian corrections

Since the photon field is not white noise, the true evolution is not Markovian:

Correlation function $\zeta(t) := \langle \Omega, \Phi(t)\Phi(0)\Omega \rangle = \int \mathrm{d}q |\phi(q)|^2 \mathrm{e}^{\mathrm{i}|q|t}$

where $\Phi(t) := \int \mathrm{d}q \phi(q) \mathrm{e}^{\mathrm{i}t|q|} a_q + h.c.$.

- $\zeta(t)$ cannot decay exponentially. The decay is $\zeta(t) = O(t^{-(1+\alpha)})$.
- In general, one should not expect time-correlation of observables to decay faster than $\zeta(t)$. However, in the Markovian approximation, atom observables decorrelate exponentially. (Long-standing confusion in physics literature: Adler-Wainwright, Slow decorrelation in gases causes anomalous diffusion in $d = 1, 2$)

Reduced evolution $\mathcal{Z}_N$

Yesterday: Study $\mathcal{Z}_N = \mathbb{E}(U_N \dots U_1)$.
Now: essentially idem (reduced evolution)

$$\mathcal{Z}_N : \mathcal{B}_1(\mathcal{H}_\mathrm{S}) \to \mathcal{B}_1(\mathcal{H}_\mathrm{S}), \quad \mathcal{Z}_N \rho_\mathrm{S} := \mathrm{Tr}_\mathrm{F}\, \mathrm{e}^{-\mathrm{i}(N/\lambda^2)L}(\rho_\mathrm{S} \otimes |\Omega\rangle\langle\Omega|)$$

where $L = [H, \cdot]$ and $L_0 = [H_\mathrm{S} + H_\mathrm{F}, \cdot]$. Set

$$U_\tau = \mathrm{e}^{\mathrm{i}(\tau/\lambda^2)L_0} \mathrm{e}^{-\mathrm{i}(1/\lambda^2)L} \mathrm{e}^{-\mathrm{i}((\tau-1)/\lambda^2)L_0}$$

then (with $\rho_0 = \rho_\mathrm{S} \otimes |\Omega\rangle\langle\Omega|$)

$$\begin{aligned} \mathcal{Z}_N \rho_\mathrm{S} = \mathrm{Tr}_\mathrm{F}\, \mathrm{e}^{\mathrm{i}(N/\lambda^2)L_0} \mathrm{e}^{-\mathrm{i}(N/\lambda^2)L} \rho_0 &= \mathrm{Tr}_\mathrm{F}(U_N \dots U_2 U_1 \rho_0) \\ &=: \mathbb{E}(U_N \dots U_2 U_1)\rho_\mathrm{S} \end{aligned}$$

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Hence; averaging over randomness = tensoring with vacuum state and taking partial trace

Formalism for expansion

Set $\mathscr{R} = \mathscr{R}_{\mathrm{S}} = \mathcal{B}(\mathcal{B}_1(\mathscr{H}_{\mathrm{S}}))$ and $\mathscr{R}_{\mathrm{F}} = \mathcal{B}(\mathcal{B}_1(\mathscr{H}_{\mathrm{F}}))$, and the tensor lattice

$$\mathscr{R}_{I_N} = \mathscr{R}_N \otimes \ldots \otimes \mathscr{R}_1,$$

and subalgebras $\mathscr{R}_A \subset \mathscr{R}_{I_N}$ for $A \subset I_N$.

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For $V = V_{\mathrm{S}} \otimes V_{\mathrm{F}}$, $V' = V'_{\mathrm{S}} \otimes V'_{\mathrm{F}} \in \mathscr{R} \otimes \mathscr{R}_{\mathrm{F}}$, put

$$V' \odot V := V'_{\mathrm{S}} \otimes V_{\mathrm{S}} \otimes (V'_{\mathrm{F}} V_{\mathrm{F}}) \in \mathscr{R}^{\otimes 2} \otimes \mathscr{R}_{\mathrm{F}}$$

(tensor in S, product in F: think of F as space of disorder). Extend by linearity.

More generally, for $V_\tau \in \mathscr{R}_\tau \otimes \mathscr{R}_{\mathrm{F}}$, we define

$$V_{\tau_m} \odot \ldots \odot V_{\tau_2} \odot V_{\tau_1} \in \mathscr{R}_A \otimes \mathscr{R}_{\mathrm{F}}, \qquad A = \{\tau_1 \ldots, \tau_m\}$$

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Formalism for expansion

Recall

$$V_{\tau_m} \odot \ldots \odot V_{\tau_2} \odot V_{\tau_1} \in \mathscr{R}_A \otimes \mathscr{R}_{\mathrm{F}}, \qquad A = \{\tau_1 \ldots, \tau_m\}$$

Expectation: $\mathbb{E}$: averages over disorder,i.e. removes disorder variables

$$\mathbb{E} : \mathscr{R}_A \otimes \mathscr{R}_{\mathrm{F}} \to \mathscr{R}_A : \qquad \mathbb{E}(W)\rho_{\mathrm{S}} := \mathrm{Tr}_{\mathrm{F}}\, W(\rho_{\mathrm{S}} \otimes |\Omega\rangle\langle\Omega|)$$

For example, with $|A| = 1$;

$$T := \mathbb{E}(U_\tau), \qquad B_\tau := U_\tau - T$$

Think again of T_τ, B_τ, U_τ as $\in \mathscr{R}_{I_N}$ acting only on $\mathscr{R}_\tau$. Moments

$$G(A) := \mathbb{E}(B_{\tau_m} \odot \ldots \odot B_{\tau_2} \odot B_{\tau_1})$$

Formalism for expansion: Correlations

Moments

$$G_A := \mathbb{E}(B_{\tau_m} \odot \ldots \odot B_{\tau_2} \odot B_{\tau_1})$$

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Relation between moments and correlations

$$G_{A'} = \sum_{\text{partitions } \mathcal{A} \text{ of } A'} \prod_{A \in \mathcal{A}} G^c_A$$

$\Rightarrow$ defines correlations G^c_A by recursion:

$$G_\tau = G^c_{\tau'}, \qquad G_{\{\tau',\tau\}} = G^c_{\tau'} G^c_\tau + G^c_{\{\tau',\tau\}}$$

hence $G^c_{\{\tau',\tau\}} = G_{\{\tau',\tau\}} - G_{\tau'} G_\tau$

Final expansion

Contraction $\mathcal{T}$

$$\mathcal{T} : \mathscr{R}_{\{k,k+1,\ldots,k+l\}} \to \mathscr{R} : \qquad \mathcal{T}(O_{k+l} \otimes \ldots \otimes O_k) = O_{k+l} \ldots O_k$$

$$\mathcal{Z}_N = \mathbb{E}(U_N \ldots U_1) \tag{1}$$

$$= \mathbb{E}((T_N + B_N) \ldots (T_1 + B_1)) \tag{2}$$

$$\sum_{\mathcal{A}} \mathcal{T}\Big(\prod_{A \in \mathcal{A}} G_A^c \prod_{\tau \notin \mathrm{Supp}(\mathcal{A})} T_\tau\Big) \tag{3}$$

(sum over admissible $\mathcal{A}$). Note that algebra is identical to yesterday.

Facts about T and G

What was important yesterday?

1. T has gap $\ell^1(\Omega) \to \ell^1(\Omega)$
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1. T has gap $\ell^1(\Omega) \to \ell^1(\Omega)$
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These two statements will hold without change

1. T has simple eigenvalue 1 and gap as operator $\mathcal{B}_1(\mathcal{H}_{\mathrm{S}}) \to \mathcal{B}_1(\mathcal{H}_{\mathrm{S}})$.
2. Same *KP* condition

$$\sum_{A \ni \tau} \epsilon^{-(|A|-1)} \| G_A^c \| d(A)^{\alpha} \leq 1, \qquad \text{for some } \epsilon \ll 1,$$

   where

   - Decay parameter α given by $\int \mathrm{d}t \zeta(t)(1+t)^{\alpha} < \infty$.
   - Small parameter $\epsilon \sim |\lambda|^{2\alpha}$.

Now: Outline how to get these properties

Gap for T

This is a consequence of

Theorem on master equation (Davies, 74)

Assume that $\zeta(t) \in L^1(\mathbb{R}_+)$, then

$$\mathrm{Tr}_{\mathrm{F}}\, U_\tau(\rho_{\mathrm{S}} \otimes |\Omega\rangle\langle\Omega|) \quad \underset{\lambda\to 0}{\longrightarrow} \quad \mathrm{e}^{\mathcal{L}}\rho_{\mathrm{S}}$$

The left-hand side is T, so we get

$$T - \mathrm{e}^{\mathcal{L}} \text{ is small for small } \lambda$$

If $\mathcal{L}$ generates ergodic semigroup, then $\mathrm{e}^{\mathcal{L}}$ has a gap. Spectral perturbation theory of isolated e.v. then gives

$$T \text{ has gap for small } \lambda$$

Expansion

Duhamel expansion $L = L_0 + \lambda L_{\mathrm{SF}}$:

$$\mathcal{Z}_N \rho_{\mathrm{S}} = \sum_{m=0}^{\infty} (-\mathrm{i}\lambda)^m \int_{0<t_1<\ldots<t_m<(N/\lambda^2)} \mathrm{d}\underline{t}$$

$$\mathrm{Tr}_{\mathrm{F}}\big(L_{\mathrm{SF}}(t_m)\ldots L_{\mathrm{SF}}(t_2) L_{\mathrm{SF}}(t_1) \rho_{\mathrm{S}} \otimes |\Omega\rangle\langle\Omega|\big)$$

with $L_{\mathrm{SF}}(t_i) = \mathrm{e}^{-\mathrm{i}t_i L_0} L_{\mathrm{SF}} \mathrm{e}^{\mathrm{i}t_i L_0}$. Use Wick theorem and group terms into

$$\mathcal{Z}_N = \int \mathrm{d}\underline{u}\mathrm{d}\underline{v}\, \mathcal{K}(\underline{u}, \underline{v})$$

- $\underline{u} = (u_1, \ldots, u_m), \underline{v} = (v_1, \ldots, v_m)$ such that $u_i \leq u_{i+1}$ and $u_i \leq v_i$
- Bound $\|\mathcal{K}(\underline{u}, \underline{v})\| \leq \prod_j C\lambda^2 |\zeta(v_j - u_j)|$

Link between $\mathcal{K}$ and G_A^c

$$\mathcal{Z}_N = \int \mathrm{d}\underline{u}\mathrm{d}\underline{v}\, \mathcal{K}(\underline{u}, \underline{v})$$

$$(\underline{u}, \underline{v}) \Rightarrow \text{Partition } \mathcal{A}' \text{ of } I_N \;\Rightarrow \mathcal{A}' = (\mathcal{A}, \underbrace{\{\tau\}, \{\tau'\}, \ldots \{\tau''''\}}_{\text{singletons}})$$

so

$$(\underline{u}, \underline{v}) \quad \Rightarrow \quad \text{Admissible } \mathcal{A}(\underline{u}, \underline{v})$$

Then

$$\int_{\mathcal{A}(\underline{u},\underline{v})=\mathcal{A}} \mathrm{d}\underline{u}\mathrm{d}\underline{v}\, \mathcal{K}(\underline{u}, \underline{v}) = \mathcal{T}(\prod_{A\in\mathcal{A}} G_A^c \prod_{\tau\notin \mathrm{Supp}\mathcal{A}} T_\tau)$$

Bound on G_A^c

$$\|G_A^c\| \leq \int_{\mathcal{A}(\underline{u},\underline{v})=\{A\}} \mathrm{d}\underline{u}\mathrm{d}\underline{v}\, \|\mathcal{K}(\underline{u},\underline{v})\|$$

Estimate this integral

- There has to be set of pairs $(u_i, v_i), i \in J_1 \subset \{1, \ldots, m\}$ that makes the necessary links. $\Rightarrow$ smallness comes from those
- Ohter pairs $(u_i, v_i), i \in J_2$ can repeat these links or stay within one block. $\Rightarrow$ no smallness from them, but have to control sum over them and bound it by $C^{|A|}$ (which then turns out be harmless since we get in a natural way $\epsilon^{|A|-1}$)

Usually, in Feynman diagrams, one cannot control sum over all (absolute values of) diagrams (i.e. sets of pairs). The fact that we can do it in this model is what makes the analysis so easy!

Results in the time-dependent approach

The following results follow with only trivial changes in the setup: ,for example, conjugating the Hamiltonian

$$H \to \mathrm{e}^{\kappa N} H \mathrm{e}^{-\kappa N}, \quad \kappa \in \mathbb{R}, \qquad \text{(obtain number bound)}$$

and under the same assumptions:

- Mixing at positive T and in non-equilibrium setup.
- Statistics (CLT,LDP) of energy transport.

(Hence, situations with non-self adjoints Liouvillians included: seems to have robustness of spectral PT for isolated e.v.)

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With more effort, control photon numbers in localized spatial regions: Expansion gets messy but idea is identical. (Asymptotic completeness then follows from this)

Problems in the time-dependent approach

If one considers a model with ionization threshold, technique breaks down completely: diagrams in infinite space instead of finite spin space cannot be controlled.

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The technique is blind for global spectral information: e.g. cannot take advantage of the presense of photon mass to prove anything at all.