Asymptotic completeness below the two-boson threshold in the massive translation invariant Nelson model

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Motivation

Asymptotic completeness in QFT attracted much attention during the last two decades:

1. Relativistic QFT: [Lechner 08, Tanimoto-W.D. 11, Gérard-W.D. 12]
2. Non-relativistic QFT:
   (a) Confined case: [Spohn 97, Dereziński-Gérard 99, Fröhlich-Griesemer--Schlein 01, De Roeck-Kupiainen 11, Faupin-Sigal 12, De Roeck - Kupiainen-Griesemer 13]
   (b) Translationally invariant case: [Fröhlich-Griesemer-Schlein 04/05]

In this talk we aim at asymptotic completeness results for massive translationally invariant models for arbitrary coupling constants.

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Outline

1. The class of models
2. The energy-momentum spectrum
3. Main result
4. Outline of the proof
5. Conclusion and outlook

Nelson model

Definition

1. $\mathcal{H} = \mathcal{K} \otimes \Gamma(\mathfrak{h})$, where $\mathcal{K} = L^2(\mathbb{R}^\nu, dy)$, $\mathfrak{h} = L^2(\mathbb{R}^\nu, dk)$.
2. $H = \Omega(-i\nabla_y) \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega) + (a(G_y) + a^*(G_y))$, where
   - (a) $\Omega(p) = \frac{p^2}{2M}$ or $\Omega(p) = \sqrt{p^2 + M^2}$.
   - (b) $\omega(k) = \sqrt{k^2 + m^2}$, $m > 0$.
   - (c) $G_y(k) = \mathrm{e}^{-iky} G(k)$, $G \in S(\mathbb{R}^\nu)$, rotationally invariant.

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Polaron model

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   (a) $\Omega(p) = \frac{p^2}{2M}$.
   (b) $\omega(k) = \mathrm{const} > 0$.
   (c) $G_y(k) = \mathrm{e}^{-iky} \frac{G(k)}{|k|}$, $G \in S(\mathbb{R}^3)$, rotationally invariant.

Remark: Polaron model describes interaction of an electron with optical phonons in a dielectric cristal.

The fiber Hamiltonians

1. H commutes with the momentum operators

$$P = -i\nabla_y \otimes 1 + 1 \otimes \mathrm{d}\Gamma(k).$$

   The joint spectral measure will be denoted by $E(\,\cdot\,)$.

2. Therefore, H has a fiber decomposition

$$H = I^* \Big(\int^{\oplus} d\xi \, H(\xi) \Big) I,$$

   where I is a unitary transformation.

3. The fiber Hamiltonians have the form

$$H(\xi) = \Omega(\xi - \mathrm{d}\Gamma(k)) + \mathrm{d}\Gamma(\omega) + a^*(G) + a(G)$$

   and act on $\mathcal{H}_\xi = \Gamma(\mathfrak{h})$.

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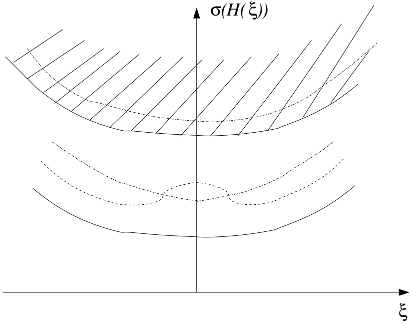
The energy-momentum spectrum

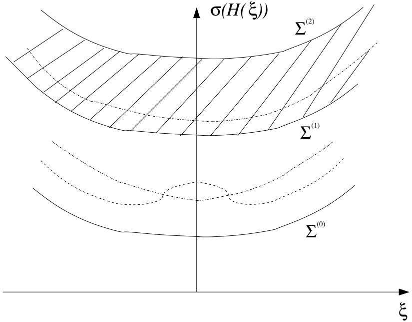


The joint spectrum Σ of (P, H) can be decomposed as follows

$$\Sigma = \Sigma_{\mathrm{pp}} \cup \Sigma_{\mathrm{ac}} \cup \Sigma_{\mathrm{sc}},$$

where $\Sigma_i = \{ (\xi, \lambda) \in \mathbb{R}^{\nu+1} \,|\, \lambda \in \sigma_i(H(\xi)) \}, \quad i \in \{ \mathrm{pp}, \mathrm{ac}, \mathrm{sc} \}.$

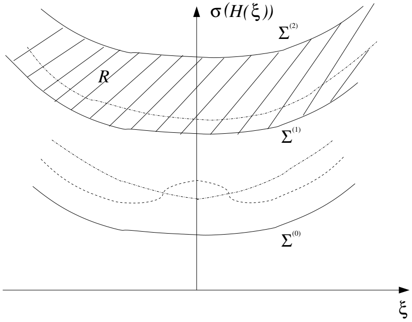
Thresholds



$$\Sigma^{(1)}(\xi) \quad := \quad \inf_{k} \{ \Sigma^{(0)}(\xi - k) + \omega(k) \}$$

$$\Sigma^{(2)}(\xi) \quad := \quad \inf_{k_1, k_2} \{ \Sigma^{(0)}(\xi - k_1 - k_2) + \omega(k_1) + \omega(k_2) \}.$$

The structure of continuous spectrum



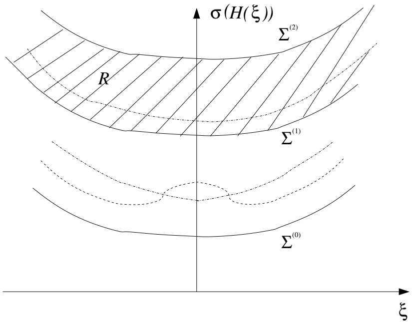
Theorem (J.S. Møller, M.G. Rasmussen, 2011)

The set

$$R := \{ (\xi, \lambda) \in \mathbb{R}^{\nu+1} \,|\, \lambda < \Sigma^{(2)}(\xi) \}$$

has trivial intersection with $\Sigma_{\rm sc}$*. Hence it is contained in* $\Sigma_{\rm ac} \cup \Sigma_{\rm pp}$.

Main Result: General formulation



Main result: Asymptotic completeness holds in R.

Preliminaries on extended objects

Definition

Extended objects are defined as follows

$$\begin{aligned}\mathcal{H}^{\mathrm{ex}} &= \mathcal{H} \otimes \Gamma(\mathfrak{h}),\\ H^{\mathrm{ex}} &= H \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega),\\ P^{\mathrm{ex}} &= P \otimes 1 + 1 \otimes \mathrm{d}\Gamma(k).\end{aligned}$$

The joint spectral measure of $(P^{\mathrm{ex}}, H^{\mathrm{ex}})$ will be denoted by $E^{\mathrm{ex}}(\,\cdot\,)$.

Preliminaries on extended objects

Definition

1. Let $U : \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$ be the canonical identification:

$$U|0\rangle = |0\rangle \otimes |0\rangle,$$
$$Ua^*(h_1, h_2) = (a^*(h_1) \otimes 1 + 1 \otimes a^*(h_2)).$$

2. Let q_0, q_∞ be bounded operators on $\mathcal{K} \otimes \mathfrak{h}$. We define operators $\hat{q} : \mathcal{K} \otimes \mathfrak{h} \to \mathcal{K} \otimes (\mathfrak{h} \oplus \mathfrak{h})$ by:

$$\hat{q}(\Psi \otimes h) := (q_0(\Psi \otimes h), q_\infty(\Psi \otimes h)).$$

3. We define $\check{\Gamma}(\hat{q}) : \mathcal{H} \to \mathcal{H}^{\mathrm{ex}}$ by $\check{\Gamma}(\hat{q}) := U\Gamma(\hat{q})$.

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Main result

Theorem

1. *There exists the wave operator* $\Omega_R^+ : E^{\mathrm{ex}}(R)\mathcal{H}_+ \to E(R)\mathcal{H}$ *given by*

$$\Omega_R^+ := \mathrm{s}\text{-}\lim_{t\to\infty} \mathrm{e}^{itH}\check{\Gamma}(1,1)^*\mathrm{e}^{-itH^{\mathrm{ex}}},$$

where $\mathcal{H}_+ := E(\Sigma_{\mathrm{pp}})\mathcal{H}\otimes\Gamma(\mathfrak{h})$.

2. Ω_R^+ *is a unitary, i.e.,*

$$\Omega_R^{+*}\Omega_R^+ = E^{\mathrm{ex}}(R)|_{\mathcal{H}_+},$$
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Mourre theory

Theorem (J.S. Møller, M.G. Rasmussen, 2011)

Let $(\xi_0, \lambda_0) \in R$, (λ_0 outside of some discrete set). Let a_{ξ_0} have the form

$$a_{\xi_0} = \frac{1}{2}\{v_{\xi_0} \cdot i\nabla_k + i\nabla_k \cdot v_{\xi_0}\},$$

where $v_{\xi_0} \in C_0^\infty(\mathbb{R}^\nu \backslash \{0\}; \mathbb{R}^\nu)$ is a suitable vector field.
Then there exist a neighbourhood N_0 of ξ_0, a neighbourhood $\mathcal{J}_0$ of λ_0, and a constant $c_{\rm m} > 0$ s.t. for any $\xi \in N_0$:

$$\mathbf{1}_{\mathcal{J}_0}(H(\xi))i[H(\xi), \mathrm{d}\Gamma(a_{\xi_0})]\mathbf{1}_{\mathcal{J}_0}(H(\xi)) \geq c_{\rm m}\mathbf{1}_{\mathcal{J}_0}(H(\xi))$$

$$\mathbf{1}_{\mathcal{J}_0}(H^{(1)}(\xi))i[H^{(1)}(\xi), 1 \otimes a_{\xi_0}]\mathbf{1}_{\mathcal{J}_0}(H^{(1)}(\xi)) \geq c_{\rm m}\mathbf{1}_{\mathcal{J}_0}(H^{(1)}(\xi))$$

where $H^{(1)} := H \otimes 1 + 1 \otimes \omega$.

Remark: For future reference we fix open set $\mathcal{O}$ s.t. $\overline{\mathcal{O}} \subset N_0 \times \mathcal{J}_0$.

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Main steps of the proof of AC

1. Given the small region of spectrum $\mathcal{O} \subset R$, in which the Mourre estimate holds with the conjugate operator $\mathrm{d}\Gamma(a_{\xi_0})$, we set

$$\tilde{a}_{\xi_0} := \frac{1}{2}\{v_{\xi_0} \cdot (i\nabla_k - y) + (i\nabla_k - y) \cdot v_{\xi_0}\}.$$

2. We choose q_0, q_∞ as smooth approximate characteristic functions of $(-\infty, c_0]$ and $[c_0, \infty)$, where $0 < c_0 \ll c_{\mathrm{m}}$, s.t. $q_0 + q_\infty = 1$. We set $q_0^t := q_0(\tilde{a}_{\xi_0}/t)$, $q_\infty^t := q_\infty(\tilde{a}_{\xi_0}/t)$.
3. Using minimal velocity estimates, which follow from Mourre theory, we show that

$$W_{\mathcal{O}}^{+*} := \mathrm{s}\text{-}\lim_{t\to\infty} \mathrm{e}^{itH^{\mathrm{ex}}} \check{\Gamma}(\hat{q}^t)\mathrm{e}^{-itH} E(\mathcal{O}) \text{ exists.}$$

4. Then $\Omega_R^+ W_{\mathcal{O}}^{+*} := \mathrm{s}\text{-}\lim_{t\to\infty} \mathrm{e}^{itH}\check{\Gamma}(1,1)^*\check{\Gamma}(q^t)\mathrm{e}^{-itH}E(\mathcal{O}) = E(\mathcal{O})$.

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Structure of the proof of time-convergence

The proof relies on Mourre theory and the method of propagation estimates (PE)

Heisenberg derivatives

$\downarrow$

Minimal velocity PE

$\downarrow$

Existence of asymptotic observables (e.g. $W_{\mathcal{O}}^{+*}$).

Preliminaries on extended objects

1. We recall that $H^{\mathrm{ex}} = H \otimes 1 + 1 \otimes \mathrm{d}\Gamma(\omega)$ has a fiber decomposition

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Here we set

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Key lemma

Lemma 1

Let $\chi \in C_0^\infty(\mathbb{R})$ be supported below $\Sigma^{(2)}(\xi)$. Then

$$\chi(H^{\mathrm{ex}}(\xi)) = \chi(H(\xi))P_0 \oplus \chi(H^{(1)}(\xi))P_1,$$

where P_n are projections on $\mathcal{H}_\xi^{(n)}$.

Proof. One notes that spectrum of $H^{(2)}(\xi)$ is contained above $\Sigma^{(2)}(\xi)$. This follows from the formulas:

$$\begin{aligned} H^{(2)}(\xi) &= \int^{\oplus} dk_1 dk_2 \big(H(\xi - k_1 - k_2) + \omega(k_1) + \omega(k_2)\big), \\ \Sigma^{(2)}(\xi) &= \inf_{k_1, k_2} (\Sigma^{(0)}(\xi - k_1 - k_2) + \omega(k_1) + \omega(k_2)). \quad \square \end{aligned}$$

Remark

The geometric inverse of the wave operator can be decomposed as:

$$W_{\mathcal{O}}^{+*} := \mathrm{s}\text{-}\lim_{t\to\infty} \mathrm{e}^{itH^{\mathrm{ex}}} \check{\Gamma}(\hat{q}(\tilde{a}_{\xi_0}/t)\mathrm{e}^{-itH} E(\mathcal{O})$$

$$= \mathrm{s}\text{-}\lim_{t\to\infty} (I^{\mathrm{ex}})^* \left(\int^{\oplus} d\xi \, \mathrm{e}^{itH^{\mathrm{ex}}(\xi)} \check{\Gamma}(\hat{q}(a_{\xi_0}/t))\mathrm{e}^{-itH(\xi)} \right) I \, E(\mathcal{O}).$$

Thus it is enough to prove convergence of asymptotic observables at fixed momentum ξ.

Heisenberg derivatives

Definition

1. Let $\mathbb{R} \ni t \mapsto \Phi_t \in B(\Gamma(\mathfrak{h}))$ be uniformly bounded. We set
$$\mathbf{D}\Phi_t = \partial_t \Phi_t + i[H(\xi), \Phi_t].$$
2. Let $\mathbb{R} \ni t \mapsto \Phi_t^{(1)} \in B(\Gamma(\mathfrak{h}) \otimes \mathfrak{h})$ be uniformly bounded. Then
$$\mathbf{D}^{(1)}\Phi_t^{(1)} = \partial_t \Phi_t^{(1)} + i[H^{(1)}(\xi), \Phi_t^{(1)}].$$

Heisenberg derivatives

Proposition 1

Fix $\xi \in \mathbb{R}^\nu$ *and suppose that:*

1. $\chi \in C_0^\infty(\mathbb{R})_\mathbb{R}$ *supported in* $(-\infty, \Sigma_0^{(2)}(\xi))$.
2. $q \in C^\infty(\mathbb{R})_\mathbb{R}$ *s.t.* $q' \in C_0^\infty(\mathbb{R})$ *and* $q = 0$ *near zero.*
3. $j_0, j_\infty \in C^\infty(\mathbb{R})_\mathbb{R}$ *s.t.* $j_0', j_\infty' \in C_0^\infty(\mathbb{R})$, $0 \leq j_0, j_\infty \leq 1$, $j_0 = 1$ *near zero and* $j_0^2 + j_\infty^2 = 1$.
4. $\mathrm{supp}\, j_0 \cap \mathrm{supp}\, q = \emptyset$.

Then

$$\mathbf{D}(\chi \mathrm{d}\Gamma(q^t)\chi) = \check{\Gamma}^{(1)}(j^t)^* \chi^{(1)} \mathbf{D}^{(1)}(1 \otimes q^t)\chi^{(1)} \check{\Gamma}^{(1)}(j^t) + O(t^{-2}),$$

where we set $\chi := \chi(H(\xi))$, $\chi^{(\ell)} := \chi(H^{(\ell)}(\xi))$ *and* $q^t := q(a_{\xi_0}/t)$.

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$$\mathbf{D}(\chi \mathrm{d}\Gamma(q^t)\chi) = \check{\Gamma}^{(1)}(j^t)^* \chi^{(1)} \mathbf{D}^{(1)}(1 \otimes q^t)\chi^{(1)} \check{\Gamma}^{(1)}(j^t) + O(t^{-2}),$$

where we set $\chi := \chi(H(\xi))$, $\chi^{(\ell)} := \chi(H^{(\ell)}(\xi))$ *and* $q^t := q(a_{\xi_0}/t)$.

Heisenberg derivatives

Proposition 1a

Fix $\xi \in \mathbb{R}^\nu$ *and suppose that:*

1. $\chi \in C_0^\infty(\mathbb{R})_\mathbb{R}$ *supported in* $(-\infty, \Sigma_0^{(2)}(\xi))$.
2. $q \in C^\infty(\mathbb{R})_\mathbb{R}$ *s.t.* $q' \in C_0^\infty(\mathbb{R})$ *and* $q = 0$ *near zero.*
3. $j_0, j_\infty \in C^\infty(\mathbb{R})_\mathbb{R}$ *s.t.* $j_0', j_\infty' \in C_0^\infty(\mathbb{R})$, $0 \le j_0, j_\infty \le 1$, $j_0 = 1$ *near zero and* $j_0^2 + j_\infty^2 = 1$.

Then

$$\begin{aligned}\mathbf{D}(\chi \mathrm{d}\Gamma(q^t)\chi) = {} & \Gamma(j_0^t)\mathbf{D}(\chi \mathrm{d}\Gamma(q^t)\chi)\Gamma(j_0^t) \\ & + \check{\Gamma}^{(1)}(j^t)^* \chi^{(1)} \mathbf{D}^{(1)}(1 \otimes q^t)\chi^{(1)} \check{\Gamma}^{(1)}(j^t) + O(t^{-2}),\end{aligned}$$

where we set $\chi := \chi(H(\xi))$, $\chi^{(\ell)} := \chi(H^{(\ell)}(\xi))$ *and* $q^t := q(a_{\xi_0}/t)$.

Heisenberg derivatives

Proposition 2

Fix $\xi \in \mathbb{R}^\nu$ and suppose that:

1. $\chi \in C_0^\infty(\mathbb{R})_\mathbb{R}$ *supported in* $(-\infty, \Sigma_0^{(2)}(\xi))$.
2. $q \in C^\infty(\mathbb{R})_\mathbb{R}$ *s.t.* $q' \in C_0^\infty(\mathbb{R})$, $0 \leq q \leq 1$ *and* $q = 1$ *in a neighbourhood* Δ *of zero.*
3. $j_0, j_\infty \in C^\infty(\mathbb{R})_\mathbb{R}$ *s.t.* $j_0', j_\infty' \in C_0^\infty(\mathbb{R})$, $0 \leq j_0, j_\infty \leq 1$, $j_0 = 1$ *in* Δ *and* $j_0^2 + j_\infty^2 = 1$.

Then

$$\mathbf{D}(\chi\Gamma(q^t)\chi) = \check{\Gamma}^{(1)}(j^t)^*\chi^{(1)}(\Gamma(q^t)\otimes 1)\mathbf{D}^{(1)}(1\otimes q^t)\chi^{(1)}\check{\Gamma}^{(1)}(j^t) + O(t^{-2}),$$

where we set $\chi := \chi(H(\xi))$, $\chi^{(\ell)} := \chi(H^{(\ell)}(\xi))$ *and* $q^t := q(a_{\xi_0}/t)$.

Heisenberg derivatives

Proposition 2a

Fix $\xi \in \mathbb{R}^\nu$ and suppose that:

1. $\chi \in C_0^\infty(\mathbb{R})_\mathbb{R}$ *supported in* $(-\infty, \Sigma_0^{(2)}(\xi))$.
2. $q \in C^\infty(\mathbb{R})_\mathbb{R}$ *s.t.* $q' \in C_0^\infty(\mathbb{R})$, $0 \leq q \leq 1$ *and* $q = 1$ *in a neighbourhood* Δ *of zero.*
3. $j_0, j_\infty \in C^\infty(\mathbb{R})_\mathbb{R}$ *s.t.* $j_0', j_\infty' \in C_0^\infty(\mathbb{R})$, $0 \leq j_0, j_\infty \leq 1$, $j_0 = 1$ *in* Δ *and* $j_0^2 + j_\infty^2 = 1$.

Then

$$\mathbf{D}(\chi\Gamma(q^t)\chi) = \frac{1}{t}\check{\Gamma}^{(1)}(j^t)^*\chi^{(1)}C_t(1 \otimes q'^t)\chi^{(1)}\check{\Gamma}^{(1)}(j^t) + O(t^{-2}),$$

where $t \mapsto C_t$ *satisfies* $C_t(N+1) = O(1)$ *and* $[C_t, 1 \otimes p^t] = O(t^{-1})$ *for any* $p \in C^\infty(\mathbb{R})$ *with* $p' \in C_0^\infty(\mathbb{R})$.

Propagation estimates

Proposition 3

1. *Let* $(\xi_0, \lambda_0) \in R$, $\xi_0 \in N_0$, $\lambda_0 \in \mathcal{J}_0$, c_{m} *be as in the Mourre estimate.*
2. *Let* $\chi \in C_0^\infty(\mathbb{R})_{\mathbb{R}}$ *be supported in* $\mathcal{J}_0$ *and* $\xi \in N_0$.
3. *Let* $0 < \varepsilon < c_0 < c_{\mathrm{m}}$, $r > \varepsilon$ *and* $\mathcal{I} := [-r, c_0] \backslash [-\varepsilon, \varepsilon]$.

Then there exists $c > 0$ *such that for all* $\Psi \in \mathcal{F}$:

$$\int_1^\infty \frac{dt}{t} \langle \Psi_t, \check{\Gamma}^{(1)}(j^t)^* \chi^{(1)} (1 \otimes \mathbf{1}_{\mathcal{I}}(a_{\xi_0}/t)) \chi^{(1)} \check{\Gamma}^{(1)}(j^t) \Psi_t \rangle \leq c \|\Psi\|^2,$$

where $\Psi_t := \mathrm{e}^{-itH(\xi)}\Psi$ *and* $j = (j_0, j_\infty)$ *as defined in Proposition 1a.*

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Propagation estimates

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3. *Let* $0 < \varepsilon < c_0 < c_{\mathrm{m}}$, $r > \varepsilon$ *and* $\mathcal{I}_0 := [-r, c_0]$.

Then there exists $c > 0$ *such that for all* $\Psi^{(1)} \in \mathcal{F} \otimes \mathfrak{h}$*:*

$$\int_1^\infty \frac{dt}{t} \langle \Psi_t^{(1)}, \chi^{(1)}(1 \otimes \mathbf{1}_{\mathcal{I}_0}(a_{\xi_0}/t))\chi^{(1)}\Psi_t^{(1)} \rangle \leq c \|\Psi^{(1)}\|^2,$$

where $\Psi_t^{(1)} := \mathrm{e}^{-itH^{(1)}(\xi)}\Psi^{(1)}$ *and* $\chi^{(1)} := \chi(H^{(1)}(\xi))$.

Propagation estimates

Proposition 4

1. *Let* $(\xi_0, \lambda_0) \in R$, $\xi_0 \in N_0$, $\lambda_0 \in \mathcal{J}_0$, c_{m} *be as in the Mourre estimate.*
2. *Let* $\chi \in C_0^\infty(\mathbb{R})_{\mathbb{R}}$ *be supported in* $\mathcal{J}_0$ *and* $\xi \in N_0$.

Then there exist $c > 0$ *and* $0 < \varepsilon_0 < c_{\mathrm{m}}/2$ *s.t. for any* $r > 0$ *and* $\Psi \in \mathcal{F}$:

$$\int_1^\infty \frac{dt}{t} \left\| \Gamma\left(\mathbf{1}_{[-r, \varepsilon_0]}(a_{\xi_0}/t)\right) \chi(H(\xi)) \Psi_t \right\|^2 \leq c \|\Psi\|^2,$$

where $\Psi_t = \mathrm{e}^{-itH(\xi)} \Psi$.

Asymptotic observables

Proposition 5

1. *Let* $(\xi_0, \lambda_0) \in R$, $\xi_0 \in N_0$, $\lambda_0 \in \mathcal{J}_0$, c_{m} *be as in the Mourre estimate.*
2. *Let* $\chi \in C_0^\infty(\mathbb{R})_{\mathbb{R}}$ *be supported in* $\mathcal{J}_0$ *and* $\xi \in N_0$.
3. *Let* $q \in C^\infty(\mathbb{R})$ *be s.t.* $q' \in C_0^\infty(\mathbb{R})$, $0 \leq q \leq 1$, $q = 1$ *in* $[-\varepsilon, \varepsilon]$ *and* $\operatorname{supp} q' \subset (-\infty, c_{\mathrm{m}}) \backslash [-\varepsilon, \varepsilon]$ *for some* $0 < \varepsilon < c_{\mathrm{m}}$.

Then the following strong limit exists

$$Q^+(H(\xi))\chi := \mathrm{s}\text{-}\lim_{t\to\infty} \mathrm{e}^{itH(\xi)}\Gamma(q^t)\mathrm{e}^{-itH(\xi)}\chi,$$

and commutes with bounded Borel functions of $H(\xi)$. *(Here we set* $\chi := \chi(H(\xi))$*). Moreover, if* $\operatorname{supp} q \subset (-\infty, \varepsilon_0)$, *where* ε_0 *appeared in Proposition 4, then* $Q^+(H(\xi))\chi = 0$.

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Asymptotic observables

Proposition 6

1. *Let* $(\xi_0, \lambda_0) \in R$, $\xi_0 \in N_0$, $\lambda_0 \in \mathcal{J}_0$, c_{m} *be as in the Mourre estimate.*
2. *Let* $\chi \in C_0^\infty(\mathbb{R})_{\mathbb{R}}$ *be supported in* $\mathcal{J}_0$ *and* $\xi \in N_0$.
3. *Let* j_0, j_∞ *be as in Proposition 1a and s.t.* $\mathrm{supp}\, j_0', j_\infty' \subset (-\infty, c_{\mathrm{m}})$.
4. *Let* $\hat{q} = (q_0, q_\infty) := (j_0^2, j_\infty^2)$ *(in particular* $q_0 + q_\infty = 1$*).*

Then the following strong limit exists:

$$W^+(\hat{q}^t)(\xi)^* \chi := \mathrm{s}\text{-}\lim_{t\to\infty} \mathrm{e}^{itH^{\mathrm{ex}}(\xi)} \check{\Gamma}(\hat{q}^t) \mathrm{e}^{-itH(\xi)} \chi,$$

where we set $\chi := \chi(H(\xi))$. *These operators intertwine (bounded Borel functions of)* $H(\xi)$ *and* $H^{\mathrm{ex}}(\xi)$.

Conclusion and outlook

- We have shown asymptotic completeness below the two-boson threshold for a class of massive non-relativistic QFT.
- The results hold for arbitrary coupling strength and in space of arbitrary dimension.
- The class of models includes the massive Nelson model and the polaron model.
- Future directions:
  1. Asymptotic completeness above the two-boson threshold. (Problem of regularity of embedded mass shells).
  2. Asymptotic completeness in local relativistic QFT. (Project with C. Gérard, arXiv:1211.3393).

W.D. and J.S. Møller arXiv:1210.6645

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