Scattering theory of the Hodge-Laplacian

Batu Güneysu

Institut für Mathematik
Humboldt-Universität zu Berlin

Spectral theory and its applications

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This talk is about the paper:

- Let $M$ be an arbitrary smooth $m$-manifold
  - One of the most fundamental problems in geometry is the determination of the spectrum of the Laplace operator corresponding to a Riemannian metric $g$ on $M$, in particular the one of the Hodge-Laplace operator $\Delta^j_g$ which acts on differential $j$-forms
  - If $M$ is compact, then the spectrum $\sigma(\Delta^j_g)$ of $\Delta^j_g$ consists of eigenvalues with a finite multiplicity and thus the situation is (analytically) very simple
  - If $M$ is noncompact, then $\sigma(\Delta^j_g)$ will typically contain some continuous part which is impossible to control in general
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If $M$ is noncompact, then $\sigma(\Delta_g^{(j)})$ will typically contain some continuous part which is impossible to control in general.
However, there is a “perturbative” way to control the absolutely continuous part $\sigma_{ac}(\Delta^{(j)}_g)$ of $\sigma(\Delta^{(j)}_g)$ in the noncompact case:

Assume that there is a quasi-isometric metric $\tilde{g}$ on $M$ such that we have some good information about the absolutely continuous part $(\Delta^{(j)}_{\tilde{g}})_{ac}$ of $\Delta^{(j)}_{\tilde{g}}$. Then once we can show that the wave operators $W_{\pm}(\Delta^{(j)}_g, \Delta^{(j)}_{\tilde{g}}, I)$ exist and are complete, they induce unitary equivalences

$$(\Delta^{(j)}_{\tilde{g}})_{ac} \sim (\Delta^{(j)}_g)_{ac}, \text{ so that } \sigma_{ac}(\Delta^{(j)}_{\tilde{g}}) = \sigma_{ac}(\Delta^{(j)}_g).$$

Here $I = I_{g,\tilde{g}} : \Omega(M, g) \to \Omega(M, \tilde{g})$ is the canonical identification $\alpha \mapsto \alpha$. 
However, there is a “perturbative” way to control the absolutely continuous part \( \sigma_{ac}(\Delta^{(j)}_g) \) of \( \sigma(\Delta^{(j)}_g) \) in the noncompact case:

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Here \( I = I_{g,\tilde{g}} : \Omega(M, g) \to \Omega(M, \tilde{g}) \) is the canonical identification \( \alpha \mapsto \alpha \).
The question we address here is:

**In what sense do \( \tilde{g} \) and \( g \) have to be close to each other to ensure that \( W_{\pm}(\Delta g, \Delta \tilde{g}, I) \) exist and are complete?**

From calculating \( \Delta \tilde{g} - \Delta g \) in the (particulary important) case where one metric arises a conformal perturbation of the other, we expect the correct assumption to be a *first order* control in the deviation of \( g \) and \( \tilde{g} \).
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For functions $= 0$-forms: Classically, people have considered special topologies $M = (0, \infty) \times S^{m-1}$ with warped metrics. Then the problem is typically unitarily equivalent to a scattering problem for Sturm-Liouville operators, which is a rather old (but not necessarily easy) story.

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Theorem (Müller/Salomonsen 2007, JFA 253)

Let $g, \tilde{g}$ be complete metrics on $M$ with $|\sec g|, |\sec \tilde{g}| \leq L$, such that the covariant $C^2$-deviation $x \mapsto \frac{1}{2}|g - \tilde{g}|_g(x)$ of $g$ from $\tilde{g}$ is bounded from above by some $\beta : M \to (0, \infty)$ of moderate decay, in a way such that for appropriate constants $a, b, c, C$ one has

$$\beta^a \in L^1(M, g), \quad |\beta^b(x) \tilde{\text{inj}}_g(x)^c| \leq C \quad \text{for all } x,$$

where $\tilde{\text{inj}}_g(x) := \min\left(\frac{\pi}{12\sqrt{L}}, \text{inj}_g(x)\right)$. Then $W_\pm(\Delta^{(0)}_g, \Delta^{(0)}_{\tilde{g}}, I^{(0)})$ exist and are complete.

Although a breakthrough at that time, this result is certainly not optimal: The required control is of second order in the deviation of $g$ and $\tilde{g}$. 
Indeed, using the harmonic radius function (later...) $x \mapsto r_g(x)$ with a certain Sobolev control, one can do much better:
Theorem (Hempel/Weder/Post 2013, JFA 266)

Let $g, \tilde{g}$ be complete quasi-isometric metrics on $M$ with

$$\int_M d(g, \tilde{g})(x) h^{-(m+2)}(x) \mu_g(dx) < \infty,$$

(1)

where $d(g, \tilde{g}) : M \to (0, \infty)$ is a certain function (later...) which measures a zeroth order deviation of the metrics, and where $h : M \to (0, 1]$ is an arbitrary lower bound on

$$M \ni x \mapsto \max \left( \min(r_g(x), 1), \min(r_{\tilde{g}}(x), 1) \right) \in (0, 1].$$

Then $W_\pm(\Delta_g^{(0)}, \Delta_{\tilde{g}}^{(0)}, I^{(0)})$ exist and are complete.

This zeroth order result should be the state of the art for functions.
We were interested in extending the latter result to differential forms.

Here, for some entirely algebraic reasons, we have restricted ourselves to conformal perturbations.
Given a Riemannian metric $g$ we have

- $\nabla_g$: the Levi-Civita connection
- $Q_g: \wedge^2 TM \to \wedge^2 TM$ the s.a. curvature endomorphism
- for a smooth 1-form $\alpha$, $\text{int}_g(\alpha) = \text{ext}^\dagger_g : \wedge^T M \to \wedge^T M$ is interior multiplication with $\alpha$
- the codifferential $\delta_g := d^\dagger_g : \Omega_{\mathcal{C}\infty}(M) \to \Omega_{\mathcal{C}\infty}(M)$
- the Dirac type operator $D_g := d + \delta_g : \Omega_{\mathcal{C}\infty}(M) \to \Omega_{\mathcal{C}\infty}(M)$
- the Hodge-Laplacian $\Delta_g := D_g^2 : \Omega_{\mathcal{C}\infty}(M) \to \Omega_{\mathcal{C}\infty}(M)$
- the Friedrichs realization $H_g$ of $\Delta_g$ in $\Omega_{L^2}(M, g)$
- the resolvents $R_{g, \lambda} := (H_g + \lambda)^{-1}$, $\lambda > 0$.
- everything filters through the form degree; notation: $\Omega_{L^2}(M) = \bigoplus_{j=0}^m \Omega_{L^2}^{(j)}(M, g)$, $H_g = \bigoplus_{j=0}^m H_g^{(j)}$ etc.
Our main technical tool will be harmonic coordinates with Sobolev control:

**Definition (Cheeger/Anderson)**

Let \( p \in (m, \infty) \), \( q \in (1, \infty) \), \( x \in M \). Then the \( W_g^{1,p} \)-harmonic radius at \( x \) with Euclidean distortion \( q \), \( r_g(x, p, q) \in (0, \infty] \), is defined to be supremum of all \( r > 0 \) such that there is a \( \Delta_g^{(0)} \)-harmonic chart \( \Phi : B_g(x, r) \to U \subset \mathbb{R}^m \) which, with respect to the \( \Phi \)-coordinates, satisfies the estimates

\[
q^{-1}(\delta_{ij}) \leq (g_{ij}) \leq q(\delta_{ij}) \quad \text{as symmetric bilinear forms,} \quad (2a)
\]

\[
r^{1-\frac{m}{p}} \left( \int_{U} |\partial_k g_{ij}(y)|^p \, dy \right)^{1/p} \leq q - 1 \quad \text{for all } i, j, k \in \{1, \ldots, m\}. \quad (2b)
\]
It is not obvious at all that \( r_g(x, p, q) > 0 \). Anyway, one has the following elementary fact:

**Proposition (B/G/M)**

*For any fixed \( p, q \), the function \( x \mapsto \min(1, r_g(x, p, q)) \) is 1-Lipschitz w.r.t. \( g \).*

The fact that indeed \( r_g(x, p, q) > 0 \) as claimed in the definition follows from applying the following results near \( x \):
Proposition (Cheeger/Anderson 90%; B/G/M)

Assume that $\text{Ric}_g(x) \geq -\frac{1}{\beta^2}$ and $\text{inj}_g(x) \geq \tilde{h}(x)$, where $\beta > 0$ is a constant and $\tilde{h} : M \rightarrow (0, \infty)$ is a continuous. Then:

a) If $\tilde{h}$ is $g$-Lipschitz, then for any $p, q$ there is $C = C(m, p, q) > 0$ such that for all $x \in M$ one has

$$\min(r_g(x, p, q), 1) \geq C \min\left(1, \frac{\tilde{h}(x)}{1 + \|d\tilde{h}\|_{\infty, g}}, \beta\right).$$

b) If there is a $x_0 \in M$, and $c_1 > 0$, $c_2 \geq 0$ such that $\tilde{h} \geq c_1 e^{-c_2 d_g(\cdot, x_0)}$, then for any $p, q$ there is $C = C(m, p, q) > 0$ such that for all $x \in M$ one has

$$\min(r_g(x, p, q), 1) \geq C \min\left(1, \frac{c_1 e^{-c_2 d_g(x, x_0)}}{e^{c_2}}, \beta\right).$$
The importance of Sobolev harmonic coordinates: By embedding theorems, we get a Hölder control on $g_{ij}$. To make an effective use of this observation in the form-case, we add:

**Definition**

For any $K > 0$ and any function $h: M \rightarrow (0, 1]$, let

$$\mathcal{M}_{K, h}(M) = \left\{ \tilde{g} \mid \tilde{g} \text{ is a complete metric on } M \text{ with } Q_{\tilde{g}} \geq -K \right\}.$$

and $\min(1, r_*(\cdot, p, q)) \geq h$ for some $p \in (m, \infty), q \in (1, \sqrt{2})$. 

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Carleman type resolvent estimate:

**Theorem (B/G/M)**

Assume that $g \in \mathcal{M}_{K,h}(M)$ for some pair $(K, h)$. Then for all $n \in \mathbb{N}$ with $n \geq m/4 + 2$ there is a $C = C(m, n) > 0$, such that for all $\lambda > K \max_{j=0,\ldots,m} j(m-j) + 1$, the operator $R_{g,\lambda}^n$ is an integral operator, with a Borel integral kernel

$$M \times M \ni (x, y) \mapsto R_{g,\lambda}^n(x, y) \in \text{Hom} \left( \wedge T^*_y M, \wedge T^*_x M \right)$$

which satisfies the estimate

$$\int_M \left| R_{g,\lambda}^n(x, y) \right|^2 \mathcal{J}_2 \mu_g(\text{dy}) \leq C h(x)^{-m} \text{ for all } x \in M.$$
The proof is rather complicated. The key observations are:

1. \( V_g^{(j)} := \Delta_g^{(j)} - \nabla_g^\dagger \nabla_g \cdot j \) is zeroth order and s.a. by Weitzenböck’s formula.

2. The Gallot-Meyer estimate states that under \( Q_g \geq -K \) one has \( V_g^{(j)} \geq -K \cdot j(m - j) \).

3. Now one can use probabilistic domination results for covariant Schrödinger semigroups \( e^{-t(\nabla^\dagger \nabla + V)} \) (e.g. my paper in JFA 262) to control \( R_{g,\lambda}^{(j),n} \) by \( R_{g,1}^{(0),n} \). The latter can be controlled by \( \min(1, r_g(\cdot, p, q)) \).
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Given a smooth function \( \psi : M \to \mathbb{R} \) let \( g_\psi \) denote the conformally equivalent metric \( g_\psi := e^{2\psi}g \). Then \( g \) and \( g_\psi \) are quasi-isometric if and only if \( \psi \) is bounded and then we have the canonical identification operator \( I = I_{g,g_\psi} : \Omega_{L^2}(M, g) \to \Omega_{L^2}(M, g_\psi) \).

Given a Borel function \( h : M \to (0, \infty) \) and a smooth function \( \psi : M \to \mathbb{R} \) define

\[
d(g, \psi)(x) := \max\{\sinh(2|\psi(x)|), |d\psi(x)|_g\}, \quad x \in M,
\]

\[
d_h(g, \psi) := \int_M d(g, \psi)(x) h(x)^{-(m+2)} \mu_g(dx) \quad \in [0, \infty].
\]

We call \( \psi \) a \( h \)-scattering perturbation of \( g \), if one has \( d_h(g, \psi) < \infty \).
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We call \( \psi \) a \textit{h-scattering perturbation of} \( g \), if one has \( d_h(g, \psi) < \infty \).
Now we can formulate our main result for forms:

**Theorem (B/G/M)**

Let \( \psi : M \to \mathbb{R} \) be smooth with \( \psi, |d\psi|_g \) bounded, and assume that \( g, g_{\psi} \in \mathcal{M}_{K,h}(M) \) for some pair \((K, h)\), in a way such that \( \psi \) is a \( h \)-scattering perturbation of \( g \). Then the wave operators

\[
W_\pm(H_{g_{\psi}}, H_g, I) = s\lim_{t \to \pm \infty} e^{itH_{g_{\psi}}} I e^{-itH_g} P_{\text{ac}}(H_g)
\]

exist and are complete, and everything filters (a posteriori... \( \Rightarrow \) total forms and Dirac type operators!) through the form degree.
Corollary

Assume that $g$ is complete with $Q_g \geq -K$ for some $K > 0$ and that $\tilde{g}$ is a metric on $M$ which is conformally equivalent to $g$ and which coincides with $g$ at infinity. Then the assumptions of our main result are satisfied.

Indeed, since $\psi$ is compactly supported by assumption, we can take $h(x) := \min(1, r_g(x, p, q), r_{g\psi}(x, p, q))$ for all $p > m$, $1 < q < \sqrt{2}$, which is a positive continuous function, to make $\psi$ a $h$-scattering perturbation of $g$. 
Corollary

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Assume that \( \psi : M \to \mathbb{R} \) is smooth and bounded, \( g \) is complete such that \( |\sec_g|, |\sec_{g,\psi}| \leq L \) for some \( L > 0 \), that there is some \( \beta : [0, \infty) \to (0, \infty) \) exponentially bounded from below, and a point \( x_0 \in M \) such that with \( \beta(x) := \beta(1 + d_g(x, x_0)) \) one has:

(i) There are constants \( b \in (0, 1) \) with \( \beta^b \in L^1(M, g) \), and \( C_1 > 0 \) such that for all \( x \in M \),

\[
\widetilde{\text{inj}}_g(x) := \min\left(\frac{\pi}{12\sqrt{L}}, \text{inj}_g(x)\right) \geq C_1 \cdot \beta(x)^{\frac{1-b}{m+2}}. \tag{3}
\]

(ii) For some constant \( C > 0 \) one has

\[
\frac{1}{2} |g - g_\psi| := |g - g_\psi|_g + |\nabla g - \nabla g_\psi|_g \leq C \cdot \beta \tag{4}
\]

Then the assumptions of our main result are satisfied.
The latter result can be considered as a generalization in the conformal case of the initial Müller/Salamonsen result to forms. Note however that, being a first order result, it is better even on functions.
Some steps in the proof of our main result...
An essential tool is to use a decomposition formula (the algebra of which forced us to restrict ourselves to the conformal case) efficiently with harmonic coordinates:
**Proposition**

Assume that $\psi$ and $|d\psi|_g$ are bounded, let $\lambda > 0$, $n \geq 1$ and let $g$ (and thus $g_\psi$) be complete. Then the bounded operator

$$R_{g_\psi, \lambda}^n (H_{g_\psi} I - IH_g) R_{g, \lambda}^n : \Omega_{L^2}(M, g) \to \Omega_{L^2}(M, g_\psi)$$

can be decomposed as

$$R_{g_\psi, \lambda}^n (H_{g_\psi} I - IH_g) R_{g, \lambda}^n =$$

$$R_{g_\psi, \lambda}^n \left( D_{g_\psi} \cdot 2 \sinh(2\psi) I D_g + D_{g_\psi} I (1 - e^{-2\psi}) d - d \circ (1 - e^{2\psi}) I D_g ight.$$

$$+ D_{g_\psi} \int_{g_\psi} (d\psi) \tau I - \tau \int_g (d\psi) D_g \right) R_{g, \lambda}^n. \quad (5)$$

Here $\tau := \bigoplus_{j=0}^m (m - 2j) 1 \wedge_j T^* M : \wedge T^* M \to \wedge T^* M$. 

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Now we combine the latter decomposition formula with our Carleman type resolvent estimate and the commutator relations $[A, R_{g,\lambda}^n] = 0$, where $A \in \{D_g, d, \delta_g\}$, to get that $R_{g,\psi,\lambda}(H_g \psi I - IH_g) R_{g,\lambda}^n$ is trace class for large $n \sim$ the assumptions of Belopol’skii-Birman’s theorem are satisfied.

The decomposition formula heavily requires that the underlying Hamiltonians are of the form $L^*L$. That is why we work with total forms and the Dirac type operator $D_g$ and $H_g = D_g^2 = D_g^*D_g$ instead of on a fixed form degree. On functions, all of this is very simple as $\Delta^{(0)} = d_{tg}^*d$ where the differential $d$ does not depend on the metric (and this leads to a zeroth order condition in this case).
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conformal perturbations. But you have to solve the following
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on \( M \). You can always write the one as a multiplicative
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Further studies for functions: Local Dirichlet forms (not clear
at all)? Weighted infinite graphs (this should at least admit a
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Thank you for listening!