Atoms in the limiting spectrum of sparse graphs

Justin Salez (lpma)
A graph $G = (V, E)$ can be represented by its adjacency matrix $A$:

$$A_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|V|}$ capture essential information. It is convenient to encode them into a probability measure on $\mathbb{R}$:

$$\mu_G = \frac{1}{|V|} \sum_{k=1}^{|V|} \delta_{\lambda_k}.$$ 

**Question:** How does $\mu_G$ typically look when $G$ is large?
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EMPIRICAL SPECTRAL DISTRIBUTION OF A GRAPH

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SPECTRUM OF A RANDOM GRAPH ON 10000 NODES
THE SEMI-CIRCLE LAW
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- Erdős-Rényi model: $n$ nodes, edges present with proba $p_n$
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\mu_{G_n} \left( \sqrt{np_n(1 - p_n)} d\lambda \right) \xrightarrow{\mathcal{P}(\mathbb{R}) \atop n \to \infty} \frac{\sqrt{4 - \lambda^2}}{2\pi} 1(|\lambda| \leq 2) d\lambda.
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- What about **sparse graphs**: \(|E| \asymp |V|\)?
graph with average degree 3 on 1000 nodes
GRAPH WITH AVERAGE DEGREE 3 ON 1000 NODES
GRAPH WITH AVERAGE DEGREE 3 ON 10000 NODES
RANDOM 3-REGULAR GRAPH ON 10000 NODES
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UNIFORM RANDOM TREE ON 250 NODES
For many sequences \( \{G_n\}_{n \geq 1} \) of sparse graphs, the spectrum \( \{\mu_{G_n}\}_{n \geq 1} \) approaches a model-dependent limit \( \mu \):

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\mu_{G_n} \xrightarrow{n \to \infty} \mu.
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- Random \( d \)-regular graph on \( n \) nodes (Kesten-McKay, 1981)
- Erdős-Rényi \( p \approx c/n \) (Khorunzhy-Shcherbina-Vengerovsky '04)
- Uniform random tree on \( n \) vertices (Bhamidi-Evans-Sen '09)

This phenomenon is just one of the many consequences of the fact that the underlying local geometry converges!
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SPECTRA OF SPARSE GRAPHS

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LOCAL WEAK CONVERGENCE (Benjamini-Schramm)
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$\mathcal{L}$ describes the local geometry of $G_n$ around a random node.
LOCAL WEAK CONVERGENCE (Benjamini-Schramm)

\[ G_n \xrightarrow{\text{loc.}}_{n \to \infty} \mathcal{L} \]

\[ \sum_{o \in V_n} 1 \{ BR(G_n, o) \} \xrightarrow{n \to \infty} L(BR(G, o)) \]

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$\mathcal{L}$: probability distribution over locally finite rooted graphs $(G, o)$. 
**LOCAL WEAK CONVERGENCE (Benjamini-Schramm)**

\[ G_n \xrightarrow{\text{loc.}} \xrightarrow{n \to \infty} L \]

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\frac{1}{|V_n|} \sum_{o \in V_n} 1_{\{B_R(G_n, o) \equiv \bullet\}} \xrightarrow{n \to \infty} L(B_R(G, o) \equiv \bullet).
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SOME SPARSE GRAPHS AND THEIR LOCAL LIMITS
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- \( G_n = \) box of size \( n \times \ldots \times n \) in the lattice \( \mathbb{Z}^d \)

- \( G_n = \) random \( d \)-regular graph on \( n \) nodes

- \( L = \) dirac at the \( d \)-regular infinite rooted tree

- \( G_n = \) Erdős-Rényi graph with \( p_n = c \) on \( n \) nodes

- \( L = \) law of a Galton-Watson tree with degree Poisson(\( c \))

- \( G_n = \) random graph with degree distribution \( \nu \) on \( n \) nodes

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- \( L = \) Infinite Skeleton Tree (Grimmett, 1980)

- \( G_n = \) preferential attachment graph on \( n \) nodes

- \( L = \) Polya-point graph (Berger-Borgs-Chayes-Sabery, 2009)
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Some Sparse Graphs and Their Local Limits

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SPECTRAL CONVERGENCE REVISITED

Can we give a sense to $\mu_G = \frac{1}{|V|} \sum \delta_{\lambda_i}$ when $G$ is replaced by $L$?

If $G = (V, E)$ is a graph finite, we have for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} \mu_G(d\lambda) = \frac{1}{|V|} \sum_{o \in V} (A_G - z)^{-1}_{oo}.$$
SPECTRAL CONVERGENCE REVISITED

Can we give a sense to $\mu_G = \frac{1}{|V|} \sum_i \delta_{\lambda_i}$ when $G$ is replaced by $\mathcal{L}$?

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If $\mathcal{L}$ is the law of a random rooted graph $(G, o)$, define $\mu_{\mathcal{L}}$ by

$$\int_{\mathbb{R}} \frac{1}{\lambda - z} \mu_{\mathcal{L}}(d\lambda) = \mathbb{E} \left[ \langle e_o | (A_G - z)^{-1} e_o \rangle \right].$$
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$$

**Fact:**

$G_n \xrightarrow{loc.} \mathcal{L} \quad \Rightarrow \quad \mu_{G_n} \xrightarrow{\mathcal{P}(\mathbb{R})} \mu_{\mathcal{L}}$
RECURSION IN THE CASE OF TREES

\[ T = T_1 T_2 T_d \]

\[ T = 1 2 d \]

\[ (A_T - z) - 1 \]

\[ \mu_L \]
Recursion in the case of trees

\[ (A_T - z)_{oo}^{-1} = \frac{-1}{z + \sum_i (A_{T_i} - z)_{ii}^{-1}} \]
RECURSION IN THE CASE OF TREES

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- Explicit resolution for infinite regular trees
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- Explicit resolution for infinite regular trees
- Recursive distributional equation for Galton-Watson trees
Explicit resolution for infinite regular trees
Recursive distributional equation for Galton-Watson trees
In principle, this equation contains everything about $\mu_L$
ILLUSTRATION: THE NULLITY OF SPARSE GRAPHS
Conjecture (Bauer-Golinelli '01). For $G_n : \text{Erdős-Rényi } (n, \frac{c}{n})$,

$$\mu_{G_n}(\{0\}) \xrightarrow{n \to \infty} \lambda^* + e^{-c\lambda^*} + c\lambda^* e^{-c\lambda^*} - 1,$$

where $\lambda^* \in [0, 1]$ is the smallest root of $\lambda = e^{-ce^{-c\lambda}}$. 
ILLUSTRATION: THE NULLITY OF SPARSE GRAPHS

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**Theorem** (Bordenave-Lelarge-S. '11)
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$$\mu_{G_n}(\{0\}) \xrightarrow{n \to \infty} \lambda^* + e^{-c\lambda^*} + c\lambda^* e^{-c\lambda^*} - 1,$$

where $\lambda^* \in [0, 1]$ is the smallest root of $\lambda = e^{-ce^{-c\lambda}}$.

**Theorem** (Bordenave-Lelarge-S. '11)

$G_n \xrightarrow{\text{loc.}} \mathcal{L} \implies \mu_{G_n}(\{0\}) \to \mu_{\mathcal{L}}(\{0\})$.
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1. $G_n \xrightarrow{loc.} \mathcal{L} \implies \mu_{G_n}(\{0\}) \to \mu_{\mathcal{L}}(\{0\})$.
2. When $\mathcal{L}$ is a GW-tree with degree distribution $\nu$,

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\mu_{\mathcal{L}}(\{0\}) = \min_{\lambda = \lambda^{**}} \left\{ f'(1)\lambda \lambda^* + f(1 - \lambda) + f(1 - \lambda^*) - 1 \right\},
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with $f(z) = \sum_k \nu(k)z^k$ and $\lambda^* = \frac{f'(1-\lambda)}{f'(1)}$. 
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- $G_n \xrightarrow{loc.} \mathcal{L} \implies \mu_{G_n}(\{0\}) \to \mu_{\mathcal{L}}(\{0\})$.
- When $\mathcal{L}$ is a GW-tree with degree distribution $\nu = \text{Poisson}(c)$,

$$\mu_{\mathcal{L}}(\{0\}) = \min_{\lambda = \lambda^{**}} \left\{ f'(1)\lambda \lambda^* + f(1 - \lambda) + f(1 - \lambda^*) - 1 \right\},$$

with $f(z) = \sum_k \nu(k)z^k = e^{c-cz}$ and $\lambda^* = \frac{f'(1-\lambda)}{f'(1)} = e^{-c\lambda}$. 
Let's keep things simple: $L = GW$-tree with degree Poisson($c$).

$\mu_L = \mu_{pp} + \mu_{sc} + \mu_{ac}$

Open problem: determine the support of each type of spectrum.

Theorem (Bordenave-Sen-Virag'13): $\mu_{pp}(R) < 1$ as soon as $c > 1$.

We will focus on the pure-point part, i.e. the atoms of $\mu_L$. This question was first raised by Ben Arous (2010).

Remark: every finite tree has positive probability under $L$.

$\forall$ all tree eigenvalues are atoms of $\mu_L$ (e.g. $0, 1, \sqrt{3}, 2 \cos \frac{2\pi}{5},...$)
SPECTRA OF GRAPH LIMITS: LITTLE IS KNOWN

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SPECTRUM OF INTEGER MATRICES

A = \{symmetric integer matrices with spectral norm \leq \Delta\}.

Theorem (Lück'02, Veselić'05, Abért-Thom-Virág'11). Fix \(\lambda \in \mathbb{R}\).

\[
\sup_{A \in A} \left| \mu_A(\lambda - \epsilon, \lambda + \epsilon) - \mu_A(\{\lambda\}) \right| \xrightarrow{\epsilon \to 0} 0.
\]

Corollary. If \(G_n \xrightarrow{\text{loc}} L\), then not only \(\mu_{G_n} \xrightarrow{\text{loc}} \mu_L\) but also

\[
\forall \lambda \in \mathbb{R}, \mu_{G_n}(\{\lambda\}) \xrightarrow{n \to \infty} \mu_L(\{\lambda\}).
\]

In particular, \(\mu_L(\{\lambda\}) = 0\) unless \(\lambda\) is a totally real algebraic integer (= root of some real-rooted monic integer polynomial).
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We are left with the following (crude) inner and outer-bounds:

\{ \text{tree eigenvalues} \} \subseteq \text{Atoms} (\mu_L) \subseteq \{ \text{totally real alg. integers} \}

Theorem (S. 2013): the inner and outer-bounds coincide!

Remark: the weaker assertion that every totally real algebraic integer is an eigenvalue of some symmetric integer matrix is known as Hofmann's conjecture (1975). It was proved by Estes (1992).

Corollary: many graph limits have the set of totally real algebraic integers as atomic support. This includes all Galton-Watson trees with \( \text{supp}(\nu) = N \), as well as the Infinite Skeleton Tree.
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PROOF IDEA: RECURSIVE FORMULATION

To a rooted tree $T$ with root $o$, associate a rational function $f_T(x) := 1 - \Phi_T(x) x \Phi_T/o(x)$ with

$$\Phi_T(x) = \det(x - A_T).$$

$\Delta \lambda \neq 0$ is a tree eigenvalue $\iff 1$ can be generated from 0 by repeated applications of $(x_1, \ldots, x_d) \mapsto \lambda_2 \sum_{i=1}^{d} 1 - x_i(x)$ for $d \in \mathbb{N}$. 
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$T = \begin{array}{c}
\text{T1} \\
\text{T2} \\
\text{d}
\end{array}$

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EXAMPLE: THE GOLDEN RATIO

\[ \lambda = 1 + \sqrt{\frac{5}{2}} \]

Remark: \( \lambda \) is a totally real algebraic integer since \( \lambda^2 = \lambda + 1 \).

Question: is \( \lambda \) a tree eigenvalue?

Iterating three times \( x \mapsto 1 + \lambda x - x \) successively gives:

\[
0 \rightarrow 1 \lambda^2 \rightarrow 1 \lambda^2 \times 1 \lambda - 1 \lambda^2 = 1 \\
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Conclusion: \( \lambda \) is an eigenvalue of \( T = \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \)
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GENERAL CASE

- Fix a totally real algebraic integer $\lambda \neq 0$.
- Consider the smallest set $F \subseteq \mathbb{R}$ satisfying:
  1. $0 \in F$
  2. $x \in F \setminus \{1\} \Rightarrow x\lambda^2 (1-x) \in F$
  3. $x, y \in F \Rightarrow x + y \in F$

Theorem (S. 2013):
$F$ is the field generated by $\lambda^2$.

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