MODEL CATEGORIES

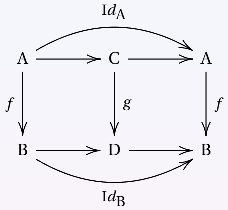
Sophie MARQUES

18 octobre 2011

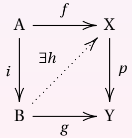
Preliminaries definitions

Let $\mathscr{C}$ be a category.

- **f is a retract of g** if and only if there is a commutative diagram of the form



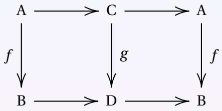
- A **functorial factorization** is an ordered pair (α, β) of functors $Map\mathscr{C} \to Map\mathscr{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in Map\mathscr{C}$.
- Let $i : A \to B$ and $p : X \to Y$ of $\mathscr{C}$. Then i has the **left lifting property** with respect to p and p has the **right lifting property** with respect to i if, for every commutative diagram



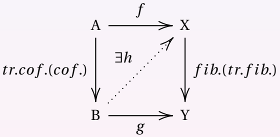
there is a lift $h : B \to X$ such that $hi = f$ and $ph = g$.

Model structure

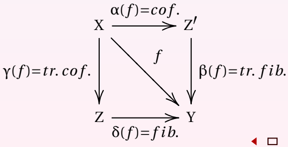
- A **model structure** on a category $\mathscr{C}$ is three subcategories of called **weak equivalences**, **cofibrations** and **fibrations**, (Define a **trivial cofibration** (resp. **trivial fibration**) to be both a cofibration (resp. fibration) and a weak equivalence) and two functorial factorizations (α,β) and (γ,δ) satisfying the following properties :
  - (**2-out-of-3**) If two of f, g and gf are weak equivalences, then so is the third.
  - (**Retracts**) If f is a retract of g and g is a weak equivalence, cofibration or fibration, then so is f.



  - (**Lifting**) Trivial cofibrations (cofibrations) have LLP with respect to fibrations (trivial fibrations).

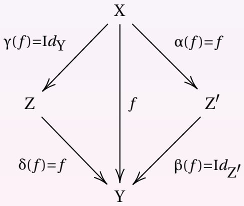


  - (**Factorization**)



Model category and first trivial example

- A **model category** is a category $\mathscr{C}$ with small limits and colimits together with a model structure.
- **Example :** Let $\mathscr{C}$ a category with small colimits and limits. One can put a model structure on $\mathscr{C}$ defining :
  - a weak equivalence if and only if it is an isomorphism.
  - every map to be both a cofibration and fibration.
  - the two factorizations (α, β) and (γ, δ) for a map f on $\mathscr{C}$ are the following :



Structures on model categories

Let $\mathscr{C}$ and $\mathscr{D}$ model categories.

- (**Product model category**) $\mathscr{C} \times \mathscr{D}$ becomes a model category in the obvious way.
- (**Dual model category**) The opposite category $\mathscr{C}^{op}$ is also a model category, where cofibration (resp. fibration resp. weak equivalence) of $\mathscr{C}^{op}$ are fibration (resp. cofibration resp. weak equivalence) of $\mathscr{C}$ and where the functorial factorisation are also inverted. We denote this category $\mathrm{D}\mathscr{C}$ and $\mathrm{D}^2\mathscr{C} = \mathscr{C}$.
- (**Category $\mathscr{C}_*$ under the terminal object** $*$) object is a map $* \xrightarrow{v} \mathrm{X}$ often write (X, v) and morphism from (X, v) to (Y, w) is a morphism $\mathrm{X} \to \mathrm{Y}$ that takes v to w. Denote the forgetful functor $\mathrm{U} : \mathscr{C}_* \to \mathscr{C}$. Define a map f in $\mathscr{C}_*$ to be a cofibration (fibration, weak equivalence) if and only if $\mathrm{U}f$ is a cofibration (fibration, weak equivalence) in $\mathscr{C}$. Then $\mathscr{C}_*$ is a model category.

Basic properties

Let $\mathscr{C}$ be a model category.

- (**Characterizations of fibrations and cofibrations by lifting properties**) A map is a cofibration (trivial cofibration) if and only if it has LLP with respect to all trivial fibrations (fibrations). Dually, a map is a fibration (trivial fibration) if and only if it has RLP with respect to all trivial cofibrations (cofibrations).
- (**Pushouts/pulllbacks**) Cofibrations (trivial cofibrations) are closed under pushouts. Dually, fibrations (trivial fibrations) are closed under pullbacks.

Definitions

Let $\mathscr{C}$ be a model category.

- The **homotopy category** is the localization of $\mathscr{C}$ with respect to the class of weak equivalences denote $\gamma : \mathscr{C} \to \mathrm{H}o\,\mathscr{C}$. That is for all f in $\mathscr{C}$ weak equivalence $\gamma(f)$ is an isomorphism and if $F : \mathscr{C} \to \mathscr{D}$ is a functor that sends weak equivalences to isomorphisms, then there is an unique functor $\mathrm{H}o\,F : \mathrm{H}o\,\mathscr{C} \to \mathscr{D}$ such that $(\mathrm{H}o\,F) \circ \gamma = F$. (property universal => unique up to isomorphism).
- **(Lemma)** The universal property of localization induces an isomorphism of categories between the category of functor $\mathrm{H}o\,\mathscr{C} \to \mathscr{D}$ and natural transformations and the category of functors $\mathscr{C} \to \mathscr{D}$ which take weak equivalences to isomorphisms and natural transformations.
- An object of $\mathscr{C}$ is called **cofibrant** (**fibrant**) if the map from the initial (to the terminal) object is a cofibration (resp. fibration). By applying the functor β and α to the map from the initial object to X, we get a functor $X \mapsto QX$ such that QX is cofibrant, and a natural transfomation $q_X : QX \to X$ wich is a trivial fibration. We say that Q is the **cofibrant replacement functor of** $\mathscr{C}$. Similary, there is a **fibrant replacement functor** RX together with a natural trivial cofibration $X \to RX$.

Definitions

Let $\mathscr{C}$ be a model category and $f, g : B \to X$ two maps in $\mathscr{C}$.

- A **cylinder object for B** is a factorisation of the fold map $B \amalg B \to B$ into a cofibration $i_0 + i_1 : B \amalg B \to B'$ followed by a weak equivalence $s : B' \to B$. Dually, a **path object for X** is a factorization of the diagonal map $X \to X \times X$ into a weak equivalence $r : X \to X'$ followed by a fibration $(p_0,\ p_1) : X' \to X \times X$.
- A **left homotopy from f to g** is a map $H : B' \to X$ for some cylinder B' for B such that $Hi_0 = f$ and $Hi_1 = g$ written $f \sim^l g$. We say that f and g **left homotopic**. Dually, a **right homotopy from f to g** is a map $K : B \to X'$ for some cylinder X' for X such that $p_0 H = f$ and $p_1 H = g$ written $f \sim^r g$. We say that f and g **right homotopic**.
- We say that f and g are **homotopic**, if there are both left and right homotopic.
- f is a **homotopy equivalence** if there is a map $h : X \to B$ such that $hf \sim 1_B$ and $fh \sim 1_X$.

Fundamental theorem

Theorem

Denote $\mathscr{C}_{cf}$ the subcategory of $\mathscr{C}$ whose object are both cofibrant and fibrant object of $\mathscr{C}$.

- *The homotopy relation on the morphism of $\mathscr{C}_{cf}$ is an equivalence relation compatible with composition. So, the category $\mathscr{C}_{cf}/\sim$ exists and the functor $\delta : \mathscr{C}_{cf} \to \mathscr{C}_{cf}/\sim$ invert the homotopy equivalences which are exactly the weak equivalences in $\mathscr{C}_{cf}$.*
- *There is a unique isomorphism $j : \mathscr{C}_{cf}/\sim \to \mathrm{Ho}\,\mathscr{C}_{cf}$ such that $j\delta = \gamma$ (it is the identity on the objects).*
- *The inclusion $\mathscr{C}_{cf} \to \mathscr{C}$ induces an equivalence of categories $\mathscr{C}_{cf}/\sim \simeq \mathrm{Ho}\,\mathscr{C}_{cf} \to \mathrm{Ho}\,\mathscr{C}$. In addition, there is isomorphism $\mathrm{Ho}\,\mathscr{C}(\gamma X, \gamma Y) \simeq \mathscr{C}(QX, RY)/\sim$.*
- *If $f : A \to B$ is a map in $\mathscr{C}$ such that γf is an isomorphism in $\mathrm{Ho}\,\mathscr{C}$ if and only if f is a weak equivalence.*

Definitions

Let $\mathscr{C}$ and $\mathscr{D}$ to be model categories.

- A functor $F : \mathscr{C} \to \mathscr{D}$ is a **left Quillen functor** if F is a left adjoint and preserves cofibrations and trivial cofibrations.
- A functor $U : \mathscr{D} \to \mathscr{C}$ is a **right Quillen functor** if U is a right adjoint and preserves fibrations and trivial fibrations.
- Let (F, U, ϕ) is an adjunction from $\mathscr{C}$ to $\mathscr{D}$. (F, U, ϕ) is a **Quillen adjunction** if F is a left Quillen functor.
- If $F : \mathscr{C} \to \mathscr{D}$ is a left Quillen functor, define the **total left derived functor** $LF : Ho\,\mathscr{C} \to Ho\,\mathscr{D}$ of F to be the composite $Ho\,F \circ Ho\,Q : Ho\,\mathscr{C} \to Ho\,\mathscr{C}_c \to Ho\,\mathscr{D}$. Given a natural transformation $\tau : F \to F'$ of left Quillen functors define the **total derived natural transformation** $L\tau$ to be such that $(L\tau)_X = \tau_{QX}$.
- If $U : \mathscr{D} \to \mathscr{C}$ is a right Quillen functor, define the **total right derived functor** $RF : Ho\,\mathscr{D} \to Ho\,\mathscr{C}$ of U to be the composite $Ho\,U \circ Ho\,R : Ho\,\mathscr{D} \to Ho\,\mathscr{D}_f \to Ho\,\mathscr{C}$. Given a natural transformation $\tau : U \to U'$ of right Quillen functors define the **total derived natural transformation** $R\tau$ to be such that $(R_\tau)_X = \tau_{RX}$.
- A Quillen adjunction is called a **Quillen equivalence** if and only if, for any cofibrant X in $\mathscr{C}$ and fibrant Y in $\mathscr{D}$, a map $f : FX \to Y$ is a weak equivalence in $\mathscr{D}$ if and only if $\phi(f) : X \to UY$ is a weak equivalence in $\mathscr{C}$.

Some useful results

Let $\mathscr{C}$ and $\mathscr{D}$ to be model categories.
Suppose, moreover $(F, U, \phi), (F, U', \phi'), (F', U, \phi'') : \mathscr{C} \to \mathscr{D}$ are Quillen adjunction.

- $L(F, U, \phi) := (LF, RU, R\phi)$ is an adjunction wich is called **derived adjunction**.
- (F, U, ϕ) is a Quillen equivalence if and only if $L(F, U, \phi)$ is an adjoint equivalence of categories.
- (F, U, ϕ) is a Quillen equivalence if and only if (F, U', ϕ') is so. Dually, (F, U, ϕ) is a Quillen equivalence if and only if (F', U, ϕ'')
- Suppose $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ are left Quillen functors. Then if two out of three of F, G and GF are Quillen equivalences, so is the third.

Example : $\mathscr{C}_*$

Let $\mathscr{C}$ be a model category

- The disjoint base point functor $\mathscr{C} \to \mathscr{C}_*$ wich send an object X to $X \amalg *$ is part of a Quillen adjunction, where the right adjoint is the forgetful functor.
- A Quillen adjunction $(F, U, \phi) : \mathscr{C} \to \mathscr{D}$ induces a Quillen adjunction $(F_*, U_*, \phi_*) : \mathscr{C}_* \to \mathscr{D}_*$. Furthermore, $F_*(X_+)$ is naturally isomorphic to $(FX)_+$.
- Suppose $F : \mathscr{C} \to \mathscr{D}$ is a Quillen equivalence, and suppose in addition that the terminal object $*$ of $\mathscr{C}$ is cofibrant and that F preserves the terminal object. Then $F_* : \mathscr{C}_* \to \mathscr{D}_*$ is a Quillen equivalence.

Definitions

- A **2-category** K has a superclass of objects K_0, of morphisms K_1 and a composition morphism and of 2-morphism K_2 and 2 compositions morphisms, together with domains, codomains and identities. The objects and the morphisms form a category $\mathscr{K}$ called the underlying category of K. For fixed A and B two object of K_1. The morphisms $A \to B$ and the 2-morphisms between then form a category $K_v(A,B)$ under the vertical composition. Furthermore, under horizontal composition, the functor and the 2-morphisms $A \to B$ form also a category $K_h(A,B)$. Finally, this two different composition of 2-morphisms are compatibles and for a and b two object of K_0 (resp. K_1) the collection of morphism for a to b form a class.
- An invertible 2-morphism is a **2-isomorphism**.
- A **2-functor between two 2-categories** is a correspondence that preserves identities, domains, codomains, and all compositions.
- Let K and L to be 2-categories. A **pseudo-2-functor** $F : K \to L$ is three maps $K_0 \to L_0$, $K_1 \to L_1$ and $K_2 \to L_2$ all denoted by F together with 2-isomophisms $\alpha : F(1_A) \to 1_{FA}$ for all objects A of K and 2-isomorphisms $m_{g,f} : Fg \circ Ff \to F(g \circ f)$ for all pairs (g, f) of morphisms of K such that the associative and unit coherence diagram commute. F preserves domains and codomains and it is functorial with respect to vertical composition. Moreover, m is natural with respect to horizontal composition.

Examples of 2-categories and 2-functors

- **2-categories**
  - Categories, functors, and natural transformations denote it $\mathcal{C}at$.
  - **Categories, adjunctions, and natural transformations, denote it** $\mathcal{C}at_{ad}$
  - **Model categories, Quillen adjunction and natural transformations, denote it** $\mathcal{M}od$
- **2-functors**
  - Forgetful 2-functors from $\mathcal{M}od$ to $\mathcal{C}at_{ad}$ and from $\mathcal{C}at_{ad}$ to $\mathcal{C}at$
- **Pseudo-2-functor**
  - **(Theorem) The homotopy category, derived adjunction, and derived natural transformation define a pseudo-2-functor $\mathbf{Ho} : \mathbf{Mod} \to \mathbf{Cat_{ad}}$.**