POLYCYCLIC GROUPS : LOCAL AND GLOBAL ORBITS OF ALGEBRAIC GROUPS

MARQUES SOPHIE

In the following, we consider a $\mathbb{Q}$-group $\mathcal{G}$ such that the polynomial equations defining $\mathcal{G}$ as a subset of $GL_n(\mathbb{C})$ all have integer coefficient and a $\mathbb{Q}$-rational representation $\rho : \mathcal{G} \to GL_m(\mathbb{C})$ such that the rational expressions which define this map are polynomials with integer coefficients.
For a positive integer r, $\mathcal{G}(\mathbb{Z}/r\mathbb{Z})$ will denote the set of all matrices in $GL_n(\mathbb{Z}/r\mathbb{Z})$ whose entries satisfy this equations over $\mathbb{Z}/r\mathbb{Z}$ and $\bar{\rho} : \mathcal{G}(\mathbb{Z}/r\mathbb{Z}) \to M_m(\mathbb{Z}/r\mathbb{Z})$ the reduction of ρ mod r.

1. Local orbit of $\mathcal{G}$ in $\mathbb{Q}^m$

1.1. **First main theorem.** The $\mathbb{Q}$-rational representation ρ define an action of $\mathcal{G}$ on the vector space $\mathbb{C}^m$, in particular, an action of $\mathcal{G}(\mathbb{Z})$ on $\mathbb{Q}^m$.

Definition (Proposition) 1.1. *Let a and b in $\mathbb{Q}^m$. We say that a, b are in the same* ***local orbit of $\mathcal{G}$*** *if one of this two equivalent conditions are satisfied :*

1. *for all r positive integer, there exists a matrix $g_r \in \mathcal{G}(\mathbb{Z}/r\mathbb{Z})$ such that $\bar{a}\bar{\rho}(g_r) = \bar{b}$ where $\bar{a} = a \bmod r$.*
2. *for every prime p, there exists $g_p \in \mathcal{G}(\mathbb{Z}_p)$ such that $a\rho(g_p) = b$.*

The goal is first to prove the following main theorem :

Theorem 1.2. *Every local orbit of $\mathcal{G}$ in $\mathbb{Q}^m$ is the union of finitely many orbits of $\mathcal{G}(\mathbb{Z})$.*

1.2. **Sketch of proof of the first main theorem.** Let C be the local orbit of $\mathcal{G}(\mathbb{Q})$ in $\mathbb{Q}^m$, $c \in C$ and denote $\mathcal{H}$ the stabilizer of c in $\mathcal{G}$.
First, we can establish the following proposition (next section).

Proposition 1.3. *Every local $\mathcal{G}$-orbit in $\mathbb{Q}^m$ is contained in the union of finitely many orbits of $\mathcal{G}(\mathbb{Q})$.*

Denote $\mathcal{G}^\infty := \prod_p \mathcal{G}(\mathbb{Z}_p)$. In view of this proposition, to prove the main theorem, it remains to show the following lemma :

Lemma 1.4. *If for each b and $c \in C$, there exists $g \in \mathcal{G}(\mathbb{Q})$ and $\hat{g} \in \mathcal{G}^\infty$ such that $b\rho(g) = c = b\rho(\hat{g})$ ($\square$) then C meets only finitely many orbits of $\mathcal{G}(\mathbb{Z})$.*

Proof. Denote $\mathcal{G}(\mathbb{A}_f) := \{\hat{g} = (g_p) \in \prod_p \mathcal{G}(\mathbb{Q}_p) | g_p \in \mathcal{G}(\mathbb{Z}_p), \textit{for almost all prime } p\}$ the adele group of $\mathcal{G}$ and $\mathcal{D} := \mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A}_f)/\mathcal{H}^\infty$ the double coset.
We can consider the intersection $\mathcal{G}^\infty \cap \mathcal{G}(\mathbb{Q})$ in $\mathcal{G}(\mathbb{A}_f)$ and check that $\mathcal{G}^\infty \cap \mathcal{G}(\mathbb{Q}) = \mathcal{G}(\mathbb{Z})$.
We will define a map : $\psi : C \to \mathcal{D}$. For each $b \in C$, let $g_b \in \mathcal{G}(\mathbb{Q})$ and $\hat{g}_b \in \mathcal{G}^\infty$ as in ($\square$) and put $\hat{h_b} := {g_b}^{-1}\hat{g}_b \in \mathcal{G}(\mathbb{A}_f)$, one can prove that $\hat{h_b} \in \mathcal{H}(\mathbb{A}_f)$ and that the map

$$\begin{array}{cccc} \psi : & C & \to & \mathcal{D} \\ & b & \mapsto & \mathcal{H}(\mathbb{Q}).\hat{h_b}.\mathcal{H}^\infty \end{array}$$

is well defined ($\mathcal{H}(\mathbb{Q}).\hat{h_b}.\mathcal{H}^\infty$ doesn't depend in the choices of g_b and $\hat{g}_b$). $\square$

Then, in view of this observations, one can prove the following lemma :

Lemma 1.5. *The fibers of ψ are exactly the orbits of $\mathcal{G}(\mathbb{Z})$ in C.*

Finally, the main theorem is a direct consequence of this theorem of Borel

Theorem 1.6 (Conséquence du théorème 7.1, Borel and Serre (1964))**.** *For every* $\mathbb{Q}$*-group* $\mathcal{H}$*, the set* $\mathcal{H}(\mathbb{Q})\backslash\mathcal{H}(\mathbb{A}_f)/\mathcal{H}^\infty$ *is finite.*

1.3. **Sketch of proof of the proposition 1.3.** Let C be the local orbit of $\mathcal{G}(\mathbb{Q})$ in $\mathbb{Q}^m$, $c \in C$ and denote $\mathcal{H}$ the stabilizer of c in $\mathcal{G}$.

(1) For k a field, $\bar{k}$ a algebraic closure of k, $\Gamma_k := Gal(\bar{k}|k)$, denote $Z^1(k, \mathcal{H}) := Z^1(\Gamma_k, \mathcal{H}(\bar{\mathbb{Q}}))$ the set of the 1-cocycles, denote $\sim$ the usual equivalence relation on $Z^1(k, \mathcal{H})$, $H^1(k, \mathcal{H}) := Z^1(k, \mathcal{H})/\sim$ the first cohomology group. Choose $\overline{\mathbb{Q}_p}$ containing $\overline{\mathbb{Q}}$, pour tout premier p, so that $\mathcal{H}(\overline{\mathbb{Q}}) \leq \mathcal{H}(\overline{\mathbb{Q}_p})$ and consider the restriction map $r_p : \Gamma_{\mathbb{Q}_p} \to \Gamma_{\mathbb{Q}}$. We can check that the map

$$w : H^1(\mathbb{Q}, \mathcal{H}) \to \prod_p H^1(\mathbb{Q}_p, \mathcal{H})$$

which associate to $[\sigma]$ the class of one element σ of $Z^1(\mathbb{Q}, \mathcal{H})$, the product over the prime p of the class $[\sigma_p]$ of $\sigma_p := \sigma \circ r_p$ element of $Z^1(\mathbb{Q}_p, \mathcal{H})$ is well defined. And the important result for our purpose is the following theorem due to Borel and Serre :

Theorem 1.7 (Théorème 7.1, Borel and Serre (1964))**.** *The map* w *has finite fibers.*

(2) We will define a map $\theta : C \to H^1(\mathbb{Q}, \mathcal{H})$. For this, first, one can establish the following proposition which is a consequence of the Hilbert's Nullstellansatz :

Proposition 1.8. *If* a*,* $b \in \mathbb{Q}^m$ *lie in the same orbit* $\mathcal{G}(\mathbb{Q}_p)$*, for some prime* p*, then they lie in the same orbit of* $\mathcal{G}(\overline{\mathbb{Q}})$*.*

As a result, for each $a \in C$, we have an element $g_a \in \mathcal{G}(\mathbb{Q})$ with $a\rho(g_a) = c$. Then, one can prove that the map $\Delta_a : \Gamma_{\mathbb{Q}} \to \mathcal{H}(\overline{\mathbb{Q}})$ which sends a over ${g_a}^{-\gamma} g_a$ well defined belongs to $Z^1(\mathbb{Q}, \mathcal{H})$ and doesn't depend on the choice of $g_a \in \mathcal{G}(\mathbb{Q})$. So, this define a map $\theta : C \to H^1(\mathbb{Q}, \mathcal{H})$ maping a over $[\Delta_a]$ the class of Δ_a in $H^1(\mathbb{Q}, \mathcal{H})$. Finally, one can prove without too much difficulties the following lemma :

Lemma 1.9. *The fibers of* θ *are exactly the orbits of* $\mathcal{G}(\mathbb{Q})$ *in* C*.*

(3) By (1) and (2), the proposition follows. In fact, one can prove that $w(\theta(C)) \subset \prod_p H^1(\mathbb{Q}_p, \mathcal{H})$ is reduced to one element the fibre of this element $\theta(C)$ is finite by (1) and the theorem is proved by (2).

2. Second main theorem

Let $\mathfrak{a}$ a ring of integer.

Theorem 2.1. *Every orbit of* $\mathcal{G}(\mathfrak{a})$ *meets only finitely many orbits of* $\mathcal{G}(\mathbb{Z})$ *in* $\mathbb{Z}^m$*.*

Since when we increase the ring $\mathfrak{a}$, $\mathcal{G}(\mathfrak{a})$ only gets bigger, we can suppose from now that k is normal in $\mathbb{Q}$. Denote $\Gamma := Gal(k|\mathbb{Q})$, this group acts on $\mathcal{G}(\mathfrak{a})$ and since $\mathfrak{a} \cup \mathbb{Q} = \mathbb{Z}$, the group of the invariant by this action is $\mathcal{G}(\mathbb{Z})$. For $c \in \mathbb{Z}$, let $C := c\rho(\mathcal{G}(\mathfrak{a})) \cup \mathbb{Z}^m$ and $\mathcal{H}$ be the stabilizer of c in $\mathcal{G}$.
Following the proof of the first theorem, one can define a map

$$\begin{array}{rccl} \theta : & C & \to & H^1(\Gamma, \mathcal{H}(\mathfrak{a})) \\ & a & \mapsto & [\Delta_a] \end{array}$$

such that its fibers are exactly the orbits of $\mathcal{G}(\mathbb{Z})$ in C. The theorem that we want to prove is then the consequence of the following result due to Borel and Serre.

Theorem 2.2. *Let $\mathcal{H}$ be a $\mathbb{Q}$-group, k a finite normal extension field of $\mathbb{Q}$ with ring of integers $\mathfrak{a}$ and $\Gamma = Gal(k|\mathbb{Q})$ then $H^1(\Gamma, \mathcal{H}(\mathfrak{a}))$is finite.*

Sketch of proof. In order to obtain this theorem, we need the following two lemma proved in Segal's book. The first lemma gives a description of the cohomology group as a semi-direct product :

Lemma 2.3. *Let H be a group, Γ a finite group acting on H. Then there is a $1:1$ corespondance between the set of the conjugacy classes of the complements to H in $H \rtimes \Gamma$ and the set $H^1(\Gamma, H)$.*

The second lemma gives a condition sufficient such that the set of the conjugacy class describe in the previous lemma is finite :

Lemma 2.4. *Let H be a group, Γ a finite group acting on H. If $H \rtimes \Gamma$ is isomorphic to an arithmetic group then there the set of the conjugacy classes of the complements to H in $H \rtimes \Gamma$ is finite.*

Finally, to prove the theorem it remains to prove the $\mathcal{H}(\mathfrak{a}) \rtimes \Gamma$ is isomorphic to an arithmetic group. For this, one can give a description of this group as a semi-direct product of matrix group over $\mathbb{Z}$ denoted by $\mathcal{H}^+ \rtimes \Gamma^*$ using a $\mathbb{Z}$ basis $(u_1, ..., u_d)$ of $\mathfrak{a}$ which permit to construct an embedding of $GL_n(\mathfrak{a})$ (resp. $GL_n(k)$ into $GL_n(\mathbb{Z})$ (resp. $GL_n(\mathbb{Q})$). And then this semi-direct product is a arithmetic group by the following lemma :

Lemma 2.5. *Let H be a subgroup of $GL_n(\mathbb{Z})$. H is a arithmetic group (in some $\mathbb{Q}$-group of degree n) if and only if H has finite index in its own closure in $GL_n(\mathbb{Z})$.*

Indeed, since Γ^* is finite, it suffixes to prove that $\mathcal{H}(\mathfrak{a})^*$ is closed in $GL_n(\mathbb{Z})$ which is true since it can be describe as the zero set of polynomials which take value in $GL_n(\mathbb{Q})$. □

3. References

Borel and Serre (1964), Théorème de finitude en cohomologie galoisienne. Comment. math. Helv. 39, 111-64.

Segal (1983) , D. Polycyclic Groups. Cambridge, England : Cambridge University Press, 1983.

(MARQUES) Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France et Università degli studi di Padova, Dipartimento di Matematica, Via Trieste 63, 35121, Padova, Italia dans le cadre de ALGANT.DOC.

E-mail address: Sophie.Marques@math.u-bordeaux1.fr (http://www.math.u-bordeaux1.fr/~smarques)