TAME ACTIONS OF AFFINE GROUP SCHEMES

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1. Well known case, classical number theory

Consider a Γ-extension of rings of integers B/C (i.e. an action of Γ on B with $C := B^{\Gamma} = \{b \in B | \gamma.b = b\}$ the ring of invariants under the action of Γ). For all $\mathfrak{p}$ a prime ideal in B, denote by $k(\mathfrak{p})$ the residue field at $\mathfrak{p}$.
As is well known (see for example " Cassels and Frolich ; Algebraic number theory) the following assertions are equivalent :

(1) B/C is tame.
(2) The trace map $Tr : B \to C$ which maps $b \in B$ to $\sum_{\gamma \in \Gamma} \gamma.b$ is surjective
(3) For all prime ideals of B , the order of the inertia group is prime to the residue field's characteristic where the inertia group is defined by

$$\Gamma_0(\mathfrak{p}) := \{\gamma \in \Gamma | \gamma.\mathfrak{p} = \mathfrak{p} \text{ and } \gamma.x = x, \ \forall x \in k(\mathfrak{p})\}$$

2. Tame actions of affine group schemes

We fix first the notation.
– Let $S := Spec(R)$ an affine base where R is a commutative ring.
– An affine scheme $X := Spec(B)$ over S.
– An affine group scheme $G := Spec(A)$ over S, recall that A is an Hopf algebra, denote $\Delta : A \to A \otimes_R A$ the comultiplication.

The data of an **action of affine group scheme**, denote by (X, G), is equivalent to the data of a structure of an A-comodule on B given by an R-linear map $\rho_B : B \to B \otimes_R A$.
A **tame action** is an action such that there exist a A-comodule map $\alpha : A \to B$ such that

$$\rho_B \circ \alpha = (\alpha \otimes Id_B)\Delta$$

which is unitary i.e. $\alpha(1_A) = 1_B$.

3. Constant group scheme, trace subjectivity, generalization

Take $G := Spec(Map(\Gamma, R))$ where Γ is an abstract finite group. One can prove that the data of an action of G on X it's the same as the data of a Γ-action on B. Furthermore, the ring of invariants $C := B^A = \{b \in B | \rho_B(b) = b \otimes 1\}$ of the action (X, G) is equal to the ring of invariants B^{Γ} of the action (B, Γ).
One can prove in this case that the tameness condition on the action is equivalent to the trace surjectivity. This extends the equivalence (1) $\Leftrightarrow$ (2) of the first part.
In the general case of the second part, we can prove the existence of a projector $p : B \to C$ which is nearly the same to the trace surjectivity property. The existence of the projector permit to show an important result for the following, if you suppose that you have a tame action (X, G), one can prove that the invariant functor $(-)^A : (B, A)\text{-}Mod \to C\text{-}Mod$ is exact. Recall that an element of $(B, A)\text{-}Mod$ is a left B-module, a right A-comodule with compatibility rules. Denote $Y := Spec(B^A)$. Moreover, the S-schemes morphism $\rho : X \to Y$ is a categorical universal quotient for the action.

4. Inertia group

We can define the inertia group of an action of affine group scheme at a T-point given by a S-morphism $\xi : Spec(T) \to X$ to be the fiber product

$$\begin{array}{ccc} I_G(\xi) & \xrightarrow{p_2} & Spec(T) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \Delta\circ\xi} \\ X \times_S G & \longrightarrow & X \times_S X \end{array}$$

For a topological point $\mathfrak{p} \in X$, the S-morphism is $\xi : Spec(k(\mathfrak{p})) \to X$, the canonical morphism, we denote in this case the inertia group by $I_G(\mathfrak{p})$. In the constant case, the inertia group at a topological point $\mathfrak{p} \in X$ is the constant group scheme associated to the inertia group $\Gamma_0(\mathfrak{p})$ of the Γ-extension B/C at the prime ideal $\mathfrak{p}$. For an extension of B/C like on the first part, the third condition of the equivalence is equivalent to requiring that the algebraic group $k(\mathfrak{p})[\Gamma_0(\mathfrak{p})]$ is semisimple that is, in algebraic geometric word, $I_G(\mathfrak{p})$ is an linearly reductive group scheme over $k(\mathfrak{p})$. We want to improve this fact.

5. Our result

To simplify, Abramovich, Olsson and Vistoli (AOV) introduced a new notion of tameness in their recent article "Tame stacks in positive characteristic". We will see the meaning of this notion, after recalling the context of our results. On the following, we assume :

(1) S is noetherian.
(2) G is flat, of finite type over S.
(3) X is flat of finite type over S.
(4) all the inertia groups are finite, for a given action (X, G)

Lemma 5.1. *Under this hypothesis, the tameness condition on the quotient stack* $[X/G]$ *(which is an Artin stack) on AOV sense is equivalent to the exactness of the invariant functor* $(-)^A$. *(On geometric term, this mean the exactness of the functor* $\rho_* : Qcoh^G X \to Qcoh Y$*, where* $Qcoh^G X$ *is the set of the* G*-equivariant quasi coherent sheaves.)*

Using A0V results, we prove that the tameness condition on the quotient stack $[X/G]$ on AOV sense is equivalent to have all inertia groups linearly reductive at geometric point which is equivalent to have this result at all T-point with T-noetherian. So, since we have the exactness condition verified for the two condition of tameness defined before, the two tameness condition implies the result that we want on the inertia groups.
Moreover, we have an equivalence between the two notion of tameness if you suppose moreover that G is finite with a flat condition on B over C.

6. Application : etale slices

We recall that an action admits an etale (fppf) slice if :

(1) there exist a categorical quotient $Y := X/G$.
(2) for all $y \in Y$, we suppose that,
    (a) there exist an etale S-morphism $Y' \to Y$ containing y in its image.
    (b) there exist G_0 a closed subgroup scheme of $G_{(Y')}$ called the slice group at y which stabilized a point $x \in X$ over y, i.e. $G_{0k(x)}$ is a subgroup of $I_G(x)$.

(c) there exist Z Y'-scheme with a G_0-action such that $Y' := Z/G$ and the action (X, G) is induced by the action (Z, G_0) that is,

$$(Z \times_{Y'} (G \times_S Y'))/G_0 \simeq X \times_Y Y'$$

In other words, the slices are a neighborhood for the action ("We can recover all the action knowing the action on the inertia groups").
For example, action by constant scheme admit fppf slices (see for example Raynaud "Anneaux locaux henséliens"). This is interesting specially in the tame case because inertia group have in this case a very simple structure. By our result, we can prove a less general result, tame action admit étale generalized linearly reductive slices. That is,

(1) there exist a categorical quotient $Y := X/G$.
(2) for all $y \in Y$, we suppose that,
  (a) there exist $Y' \to Y$ an etale S-morphism containing y in its image.
  (b) there exist G_0 a group scheme linearly reductive over Y' which stabilized a point $x \in X$ over y, i.e. $G_{0k(x)}$ is a subgroup of $I_G(x)$.
  (c) there exist Z Y'-scheme with a G_0-action such that $Y' := Z/G$ and

$$[X/G] \times_Y Y' \simeq [Z/G_0]$$

So, the quotient stack of a tame action is locally for the étale topology like the quotient stack by a linearly reductive group. The interest is that linearly reductive groups are semidirect product of tame constant scheme by a diagonalisable group scheme, locally for the fpqc topology.

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