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Ramification modérée pour des actions de schémas en groupes affines et pour des champs quotients

Tameness for actions of affine group schemes and quotient stacks

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"What does that mean–'tame'?"
"It is an act too often neglected," said the fox. It means to establish ties."
"'To establish ties'?"
"Just that," said the fox. "To me, you are still nothing more than a little boy who is just like a hundred thousand other little boys. And I have no need of you. And you, on your part, have no need of me. To you, I am nothing more than a fox like a hundred thousand other foxes. But if you tame me, then we shall need each other. To me, you will be unique in all the world. To you, I shall be unique in all the world . . ." [...]
The fox gazed at the little prince, for a long time.
"Please–tame me!" he said.
"I want to, very much," the little prince replied. "But I have not much time. I have friends to discover, and a great many things to understand."
"One only understands the things that one tames," said the fox. "Men have no more time to understand anything. They buy things all ready made at the shops. But there is no shop anywhere where one can buy friendship, and so men have no friends any more. If you want a friend, tame me . . ."
"What must I do, to tame you?" asked the little prince.
"You must be very patient," replied the fox. "First you will sit down at a little distance from me–like that–in the grass. I shall look at you out of the corner of my eye, and you will say nothing. Words are the source of misunderstandings. But you will sit a little closer to me, every day . . ." [...]
So the little prince tamed the fox. And when the hour of his departure drew near–
"Ah," said the fox, "I shall cry."
"It is your own fault," said the little prince. "I never wished you any sort of harm; but you wanted me to tame you . . ." [...]
"Men have forgotten this truth," said the fox. "But you must not forget it. You become responsible, forever, for what you have tamed. "

The Little Prince (Antoine de Saint-Exupery, 1943).

Champ de blé avec gerbes de Van Gogh

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Abstract

The purpose of this thesis is to understand how to generalize the ramification theory for actions by affine group schemes with a particular interest for the notion of tameness. As general context for this summary, we consider an affine basis $S := Spec(R)$ where R is a commutative, unitary ring, an affine, finitely presented, Noetherian scheme $X := Spec(B)$ over S, a flat, finitely presented, affine group scheme $G := Spec(A)$ over S and an action of G on X that we denote by (X, G). Finally, we denote $[X/G]$ the quotient stack associated to this action and we set $Y := Spec(B^A)$ where B^A is the ring of invariants for the action (X, G). Moreover, we suppose that the inertia stack is finite.
As reference point, we take the classical theory of ramification for rings endowed with an action of a finite, abstract group. In order to understand how to generalize this theory for actions of group schemes, we consider the actions of constant group schemes knowing that the data of such actions is equivalent to the data of rings endowed with an action of a finite abstract group, this being the classical case. We obtain thus in this new context notions generalizing the ring of invariants as a quotient, the inertia group and all their properties. The unramified case is generalized naturally by the free actions. For the tame case, which interests us particularly here, two generalizations are proposed in the literature: the one of tame actions of affine group schemes introduced by Chinburg, Erez, Pappas et Taylor in the article [CEPT96] and the one of tame stacks introduced by Abramovich, Olsson and Vistoli in [AOV08]. It was then natural to compare these two notions and to understand how to generalize the classical properties of tame objects for the actions of affine group schemes.
First of all, we traduced algebraically the tameness property on a quotient stack as the exactness of the functor of invariants. This permits to obtain easily thanks to [CEPT96] that tame actions define always tame quotient stacks. For the converse, we only manage to prove it when we suppose G to be finite, locally free over S and X flat over Y. We are able to see that the notion of tameness for a ring endowed with an action of a finite, abstract group Γ is equivalent to the fact that all the inertia group schemes at the topological points are linearly reductive if we consider the action of the constant group scheme corresponding to Γ over X. It was thus natural to wonder if this property was also true in general. In fact, the article [AOV08] characterizes the fact that the quotient stack $[X/G]$ is tame by the fact that the inertia group schemes at the geometric points are linearly reductive.
Again, if we consider the case of rings endowed with an action of a finite, abstract group, it is well known that these actions can be totally reconstructed from an action involving an inertia group. When we consider actions by constant group schemes, this is translated as a slice theorem, that is, a local description of the initial action by an action involving an inertia group. For example, we establish that the fact that an action is free is a "local property" for the fppf topology and this can be translated also as a "local" slice theorem. Thanks to [AOV08], we already know that a tame quotient stack $[X/G]$ is locally isomorphic for the fppf topology to a quotient stack $[X/H]$, where H is an extension of the inertia group in a point of Y. When G is finite over S, it was possible to show that H is also a subgroup of G. In this thesis, it was not possible to obtain a slice theorem in this generality. However, when G is commutative, finite over S, it is possible to prove the existence of a torsor, if we suppose $[X/G]$ to be tame. This permits to prove a slice theorem when G is commutative, finite over S and $[X/G]$ is tame.

Keywords : Tame actions of group schemes, tame stacks, moduli spaces, Hopf algebras and comodules, ramification theory, deformation theory.

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Résumé en Français:

Ramification modérée pour des actions de schémas en groupes affines et pour des champs quotients

L'objet de cette thèse est de comprendre comment se généralise la théorie de la ramification pour des actions par des schémas en groupes affines avec un intérêt particulier pour la notion de modération. Comme contexte général pour ce résumé, considérons une base affine $S := Spec(R)$ où R est un anneau unitaire, commutatif, $X := Spec(B)$ un schéma affine sur S, $G := Spec(A)$ un schéma en groupes affine, plat et de présentation finie sur S et une action de G sur X que nous noterons (X, G). Enfin, nous notons $[X/G]$ le champ quotient associé à cette action et $Y := Spec(B^A)$ où B^A est l'anneau des invariants pour l'action (X, G). Supposons de plus que le champ d'inertie soit fini.

Comme point de référence, nous prenons la théorie classique de la ramification pour des anneaux munis d'une action par un groupe fini abstrait. Afin de comprendre comment généraliser cette théorie pour des actions par des schémas en groupes, nous considérons les actions par des schémas en groupes constants en se rappelant que la donnée de telles actions est équivalente à celle d'un anneau muni d'une action par un groupe fini abstrait nous ramenant au cas classique. Nous obtenons ainsi dans ce nouveau contexte des notions généralisant l'anneau des invariants en tant que quotient, les groupes d'inertie et toutes leurs propriétés. Le cas non ramifié se généralise naturellement avec les actions libres. En ce qui concerne le cas modéré, qui nous intéresse particulièrement pour cette thèse, deux généralisations sont proposées dans la littérature. Celle d'actions modérées par des schémas en groupes affines introduite par Chinburg, Erez, Pappas et Taylor dans l'article [CEPT96] et celle de champ modéré introduite par Abramovich, Olsson et Vistoli dans [AOV08]. Il a été alors naturel d'essayer de comparer ces deux notions et de comprendre comment se généralisent les propriétés classiques d'objets modérés à des actions par des schémas en groupes affines.

Tout d'abord, nous avons traduit algébriquement la propriété de modération sur un champ quotient comme l'exactitude du foncteur des invariants. Ce qui nous a permis d'obtenir aisément à l'aide de [CEPT96] qu'une action modérée définit toujours un champ quotient modéré. Quant à la réciproque, nous avons réussi à l'obtenir seulement lorsque nous supposons de plus que G est fini et localement libre sur S et que X est plat sur Y. Nous pouvons voir que la notion de modération pour l'anneau B muni d'une action par un groupe fini abstrait Γ est équivalente au fait que tous les groupes d'inertie aux points topologiques sont linéairement réductifs si l'on considère l'action par le schéma en groupes constant correspondant à Γ sur X. Il a été donc naturel de se demander si cette propriété est encore vraie en général. Effectivement, l'article [AOV08] caractérise le fait que le champ quotient $[X/G]$ est modéré par le fait que les groupes d'inertie aux points géométriques sont linéairement réductifs.

À nouveau, si l'on considère le cas des anneaux munis d'une action par un groupe fini abstrait, il est bien connu que l'action peut être totalement reconstruite à partir de l'action d'un groupe inertie. Lorsque l'on considère le cas des actions par les schémas en groupes constants, cela se traduit comme un théorème de slices, c'est-à-dire une description locale de l'action initiale par une action par un groupe d'inertie. Par exemple, lorsque G est fini, localement libre sur S, nous établissons que le fait qu'une action soit libre est une propriété locale pour la topologie fppf, ce qui peut se traduire comme un théorème de slices. Grâce à [AOV08], nous savons déjà qu'un champ quotient modéré $[X/G]$ est localement isomorphe pour la topologie fppf à un champ quotient $[X/H]$ où H est une extension du groupe d'inertie en un point de Y. Lorsque G est fini sur S, il nous a été possible de montrer que H est aussi un sous-groupe de G. Dans la présente thèse, il n'a pas été possible d'obtenir un théorème de slices dans cette généralité. Cependant, lorsque G est commutatif, fini sur S, il est possible de montrer l'existence d'un torseur, si l'on suppose que le champ quotient soit modéré. Ceci nous a permis de prouver un théorème de slices lorsque G est commutatif, fini sur S et $[X/G]$ est modéré.

Mots clés : Actions modérées de schémas en groupes, Champs modérés, Espaces de modules, Algèbres de Hopf et comodules, Théorie de la ramification, Théorie de la déformation.

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Riassunto in Italiano:

Ramificazione moderata per azioni di schemi in gruppi affini e per *stacks* quoziente.

Lo scopo di questa tesi è capire come si generalizza la teoria della ramificazione per azioni di schemi in gruppi affini con un interesse particolare per la nozione di moderazione. Come contesto generale per questo riassunto, consideriamo una base affine $S := Spec(R)$ dove R è un anello unitario e commutativo, $X := Spec(B)$ uno schema affine, noetheriano e di presentazione finita su S, $G := Spec(A)$ uno schema in gruppi affine, piatto e di presentazione finita su S e un'azione di G su X che denoteremo (X, G). Infine, denotiamo con $[X/G]$ lo stack quoziente associato a questa azione e $Y := Spec(B^A)$ dove B^A è l'anello degli invarianti per l'azione (X, G). Supponiamo inoltre che il campo d'inerzia sia finito.
Come punto di riferimento prendiamo la teoria classica della ramificazione per anelli muniti d'un'azione d'un gruppo finito astratto. Al fine di comprendere come generalizzare questa teoria per azioni di schemi in gruppi, consideriamo le azioni di schemi in gruppi costanti ricordando che il dato di tali azioni è equivalente al dato d'un anello dotato d'un'azione d'un gruppo finito astratto, riconducendosi al caso classico. Otteniamo così in questo nuovo contesto delle nozioni che generalizzano l'anello degli invarianti in quanto quoziente, i gruppi d'inerzia e tutte le loro proprietà. Il caso non ramificato si generalizza in modo naturale con le azioni libere. Per qual che riguarda il caso moderato, al quale siamo particolarmente interessati in questa tesi, due generalizzazioni sono proposte nella letteratura: quella delle azioni moderate di schemi in gruppi affini introdotta da Chinburg, Erez, Pappas e Taylor nell'articolo [CEPT96] e quella di *stack* moderato introdotta da Abramovich, Olsson e Vistoli in [AOV08]. È stato quindi naturale cercare di confrontare queste due nozioni e capire come si generalizzano le proprietà classiche degli oggetti moderati ad azioni di schemi in gruppi affini.
Per cominciare, abbiamo tradotto algebricamente la proprietà di moderazione su un *stack* quoziente come l'esattezza del funtore degli invarianti. Ciò ha permesso d'ottenere agevolmente, usando [CEPT96], che un'azione moderata definisce sempre uno *stack* quoziente moderato. Quanto al viceversa, siamo riusciti ad ottenerlo solamente sotto l'ulteriore ipotesi che G sia finito e localmente libero su S e che X sia piatto su Y. Possiamo vedere che la nozione di moderazione per l'anello B dotato d'un'azione d'un gruppo finito astratto Γ è equivalente al fatto che tutti i gruppi d'inerzia sui punti topologici siano linearmente riduttivi se si considera l'azione dello schema in gruppi costante corrispondente a Γ su X. È stato quindi naturale domandarsi se questa proprietà sia vera in generale. In effetti, l'articolo [AOV08] caratterizza il fatto che lo *stack* quoziente $[X/G]$ è moderato tramite il fatto che i gruppi d'inerzia sui punti geometrici siano linearmente riduttivi.
Di nuovo, se consideriamo il caso degli anelli muniti d'un'azione d'un gruppo finito astratto, è ben noto che quest'azione può essere totalmente ricostruita a partire da un'azione in cui interviene un gruppo d'inerzia. Quando consideriamo il caso delle azioni degli schemi in gruppi costanti, questo si traduce come un teorema di *slices*, cioè una descrizione locale dell'azione di partenza (X, G) tramite un'azione in cui interviene un gruppo d'inerzia. Per esempio quando G è finito e localmente libero su S, stabiliamo che il fatto che un'azione è libera è una proprietà locale per la topologia fppf, ciò si può interpretare come un teorema di *slices*. Grazie a [AOV08] sappiamo già che uno *stack* quoziente moderato $[X/G]$ è localmente isomorfo per la topologia fppf a uno *stack* quoziente $[X/H]$, dove H è un'estensione d'un gruppo d'inerzia in un punto di Y. Quando G è finito su S è stato possibile dimostrare che H è un sottogruppo di G. Nella presente tesi non è stato possibile ottenere un teorema di *slices* in questa generalità. Tuttavia, quando G è commutativo e finito su S, è possibile dimostrare l'esistenza d'un torsore se si suppone che lo *stack* quoziente è moderato. Questo ci ha permesso di dimostrare un teorema di *slices* quando G è commutativo e finito su S e $[X/G]$ è moderato.

Parole chiave : Azioni moderate di schemi in gruppi, Stacks moderati, Spazi di moduli, Algebre di Hopf e comoduli, Teoria della ramificazione, Teoria della deformazione.

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Contents

Introduction

The question of prime decomposition in a finite extension of fields motivates classical ramification theory for field extensions. It takes a particularly interesting and important turn once we assume this extension to be Galois leading to Hilbert's ramification theory. Unramified extensions can be completely understood in terms of extension of the residue field. General ramification theory is a deep subject which is far from being completely understood. However, tamely ramified extensions are only slightly more complicated and well understood. This notion plays for instance an important role in Galois module theory.

Banard, in his article [Bar74], generalizes this well known theory for extensions of fields for commutative rings endowed with an action of a finite abstract group action. Most classical results have a good analogue permitting to extend ramification theory to this context. We can pass directly to a geometric language via the study of the action by a constant group scheme using extensions of rings yielding a ramification theory in arithmetic geometry. The purpose of this thesis is to study how to generalize the notions related with ramification, their properties and applications, in this algebro-geometric context. In this context, free actions take naturally the role of unramified extensions. Two candidates to play the role of tame extensions exist in the literature. One is the notion of tame actions by affine group schemes introduced in [CEPT96] and the other is the notion of tame quotient stacks introduced in [AOV08].

For this introduction, as the general context, we consider an action of an affine group scheme G finitely presented over S on an affine scheme X over S denoted by (X, G) with finite inertia group stack $\mathcal{I}_{[X/G]}$ in order to be able to define tameness for the quotient stack denoted by $[X/G]$ associated to this action.

Our contribution was first to translate algebraically the tameness defined in [AOV08] in the particular case of the quotient stack. More precisely, we show that tameness for quotient stacks induced by actions involving affine schemes is characterized by the exactness of the functor of invariants induced by the action (Theorem IX.8). Naturally, then we wanted to establish a connection between the two notions of tameness. Thanks to this algebraic characterization, we prove directly that under the general context in which tame quotient stacks are defined, a tame action by an affine group scheme always induces a tame quotient stack (Theorem IX.15). On the other hand, we manage to establish the converse only for actions by finite group schemes with a flat quotient map over a noetherian base (Theorem IX.10). In particular, finite flat linearly reductive group schemes (That is, group schemes inducing tame classifying spaces) induce tame trivial actions (Corollary IX.12).

Applying Theorem IX.20 taken from [AOV08], we obtain that inertia groups at any topological point under tame hypotheses are linearly reductive. More specifically, one can characterize the tameness of the quotient stack by requiring the inertia group to be linearly reductive at the geometric points. This answers partially to Question 3, §4 of [CEPT96]. The other interesting aspect of this inertia groups' description is that linearly reductive groups are fppf locally explicitely described as a semi-direct product of a tame constant group scheme by a diagonalizable group (Theorem VIII.13 recalling the classification of linearly reductive groups done in [AOV08]). Thanks to this local description of linearly reductive group schemes, it is possible to lift linearly reductive group schemes over the residue field $k(p)$ at the prime ideal p, fppf locally as a flat linearly reductive group scheme (Theorem VIII.17 in [AOV08]) which is a really interesting property. We prove that in particular, if we have that a linearly reductive group scheme H_0 over the residue field $k(p)$ at the prime ideal p is a subgroup of the pullback $G \times_S k(p)$, we can lift H_0 fppf locally as a subgroup of G (Theorem VIII.18). In particular, this can be applied for inertia groups of a tame action .

We are still investigating on the existence of a slice theorem for tame quotient stacks. Roughly speaking, fppf locally, an action which admits slices can be described by the action of a "stabilizer of a point" which in our case is linearly reductive. This would be interesting because under tame condition, we have seen that inertia groups are linearly reductive hence locally simple. From [AOV08] (Theorem IX.20), we already know that a tame quotient stack [X/G] is fppf locally isomorphic to a quotient stack [P/H] where H is a flat linearly reductive lifting of some inertia group, and this can be seen as a "weak" slice theorem. But, we didn't manage yet to have really a Slice theorem under this general context. However, we managed to establish some slice theorems, in some particular cases. For instance, we showed that freeness is a local property for the fppf topology for G finite locally free group scheme over S and this can be seen as a slice theorem (Theorem IX.18). Finally, we managed to prove also a slice theorem when the group scheme G is commutative and finite over S (Theorem IX.27), after showing the existence of a torsor (Proposition IX.23).

As starting point, we begin with recalling the ramification theory for commutative rings endowed with an action of a finite abstract group (Chapter I). This will permit to clearly state what we have been able or not to generalize. We took the opportunity to give an overview of the classical ramification theory for field extensions (Chapter II), as a specialization of notions and results of the first chapter.

The second part introduces the reader to the context that we are interested in. We generalize there the tools of ramification theory. After recalling the three equivalent ways to see actions of affine group schemes, we define important notions attached to them as quotient, freeness, induced actions or slices (Chapter III).

As it is well known, for a given action by a group scheme, categorical quotients in the category of schemes don't always exist, instead we can always define quotient stacks. Chapter IV permits to introduce what we will need to know about these objects. (For the readers' convenience, we have reviewed in Appendix B all the material on stacks required for understanding this thesis). The notion of tameness involves the important notion of inertia groups. We devote Chapter V to this purpose generalizing the classical properties of inertia groups in an algebro-geometric context that is not readily found in the literature. We will see that inertia groups can be viewed as automorphism groups of the quotient stack which explains why it is natural to define tameness on the quotient stack.

Having established the context, we can finally approach ramification theory (Part C). Chapter VI defines the two notions of tameness. Considering actions by constant group schemes (Chapter VII) permits to pass directly to the equivalent case of commutative rings endowed with an action of a finite abstract group and understand why all the notions defined in the algebro-geometric context generalize those of Chapter I. In Chapter VIII, we describe the interesting class of group schemes inducing tame classifying stacks, the linearly reductive group schemes. They will take an important role in our results thanks to their good properties. Finally, in Chapter X, we are able to state our main results mentioned before under a tameness condition.

Notations

Throughout the thesis, we fix the following notations.

We denote by R a commutative, unitary ring and $S := Spec(R)$ stands for an affine base scheme. Moreover all the considered algebras are commutative. All modules and algebras are over R . All the schemes are supposed to be over S.

We denote $\mathcal{M}_R$ (resp. ${}_R\mathcal{M}$) the category of the right (resp. left) R-modules. For M and N two R-modules, we denote $Hom_R(M, N)$ the sets of the morphisms of R-modules from M to N.

If D is a ring and $\mathfrak{p}$ is a prime ideal of D, $D_\mathfrak{p}$ will denote the localisation of R with respect to the multiplicative closed subset $R\backslash\mathfrak{p}$. The isomorphic fields $D_\mathfrak{p}/\mathfrak{p}D_\mathfrak{p}$ and the field of fractions of $D/\mathfrak{p}$ will always be identified and denoted by $k(\mathfrak{p})$ and called the residue field at $\mathfrak{p}$. If D is unitary, we denote by 1_D the unit. For B a ring or a scheme, we denote by Id_B the identity morphism. For T an R-algebra and D an R-algebra, we denote by A_T (reps. ${}_TA$) the base change $A \otimes_R T$ (resp. $T \otimes_R A$).

Moreover, for X a scheme over S and S' an S-scheme, $X_{S'}$ (resp. ${}_{S'}X$) stands for the base change $X \times_S S'$ (resp. $S' \times_S X$). In particular, for X a scheme over S and R' an R-algebra, we write $X_{R'}$ (resp. ${}_{R'}X$) instead of $X_{Spec(R')}$ (resp. ${}_{Spec(R')}X$) for the base change $X \times_S Spec(R')$ (resp. $Spec(R') \times_S X$). Given two S-morphisms of schemes $X \to Y$ and $Z \to Y$, we denote $pr_1 : X \times_Y Z \to X$ the first projection of the fiber product $X \times_Y Z$ and $pr_2 : X \times_Y Z \to Z$ the second.
For $\phi : X \to Y$ a morphism of S-schemes, we denote by $\phi^\sharp : \mathcal{O}_Y \to \mathcal{O}_X$ the corresponding morphism of sheaves of rings.

Part A

Classical ramification theory

Chapter I

Ramification theory for commutative rings endowed with an action by a finite, abstract group

Along this chapter, Γ denotes a finite abstract group and B a commutative ring. Denote (B,Γ) an action of Γ on B. For G a finite abstract group, we denote

$$B^G := \{b \in B | g.b = b, \forall\ g \in G\}$$

the ring of invariants by G and $C := B^\Gamma$.

In this chapter, we give an overview of the classical ramification theory for Γ-objects which is mainly the subject of the article of Barnard [Bar74]. Some results are taken from [Bou81, Chapitre 5]. In chapter VII, the reader can see how this theory can pass naturally to an algebro-geometric context, via the action of the constant group scheme.

1 Properties of Γ-objects and of the ring of the invariants

Definition I.1. *1. We denote (B,Γ) an action of Γ on B. We call such a B a* ***Γ-object*** *.*

2. *For Ω another finite abstract group, T an Ω-object and $\alpha : \Omega \to \Gamma$ a morphism of groups, a* ***morphism*** *$f : B \to T$ is a map such that $f(\alpha(\omega).b) = \omega.f(b)$, for all $b \in B$ and $\omega \in \Omega$. We often consider the particular case where $\Omega = \Gamma$ and $f = Id_\Gamma$.*

Remark I.2. *1. Any commutative ring B can be endowed with a trivial structure of Γ-object, considering the trivial action defined by $\gamma.b = b$ for any $b \in B$ and any $\gamma \in \Gamma$.*

2. ***(Quotient)*** *Consider I an ideal of B stable under a subgroup Ω of Γ together with the inclusion map $i : \Omega \to \Gamma$. Then, if $p : B \to B/I$ is the quotient map, Ω acts on the ring B/I by $\omega.p(b) = p(\omega.b)$, $\omega \in \Omega$ and $b \in B$, the canonical quotient $p : B \to B/I$ is a morphism between the Γ-object B and the Ω-object B/I. Moreover, this map is an initial object in the category of such morphisms $f : B \to U$ for which $I \subseteq Ker(f)$.*

3. ***(Localization)*** *Consider L a multiplicatively closed subset of B, stable under a subgroup Ω of Γ, then Ω acts on the ring of fractions $L^{-1}B$ by $\omega.(b/l) = (\omega.b)/(\omega.l)$ with $\omega \in \Omega$, $b \in B$, $l \in L$ and the canonical map $f : B \to L^{-1}B$ is a morphism between the Γ-object B and the Ω-object $L^{-1}B$.*

Proposition I.3. *The functor of invariants is compatible with flat base change. That is to say, if B is an R-algebra, for any flat morphism $f : R \to R'$,*

$$(B \otimes_R R')^\Gamma \simeq B^\Gamma \otimes_R R'$$

where Γ acts over $B \otimes_R R'$ via B (trivially on R').

Proof. Let E be the direct product of copies of B indexed by Γ. Define $\theta : B \to E$ by

$$\theta(b) = \{\gamma.b - b\}_{\gamma\in\Gamma} \quad (b \in B).$$

Then θ is a morphism of R-modules with kernel B^Γ and

$$\theta \otimes Id_{R'} : B \otimes_R R' \to E \otimes_R R'$$

is a morphism of R-modules with kernel $(B \otimes_R R')^\Gamma$. However, since $f : R \to R'$ is flat, the kernel of $\theta \otimes Id_{R'}$ must also be $B^\Gamma \otimes_R R'$. □

We start with some interesting properties of the ring of invariants of the action.

Proposition I.4. 1. *The R-algebra B is integral over B^Γ.*

2. *If B is an R-algebra of finite type then B is also a B^Γ-module of finite type. Moreover, if R is a noetherian ring, B^Γ is an R-algebra of finite type.*

Proof. 1. Let $b \in B$, we notice that b is a root of the unitary polynomial $\prod_{\gamma\in\Gamma}(X - \gamma.b)$. Moreover, the symmetric functions on $\gamma.b$ for $\gamma \in \Gamma$ are in B^Γ and this proves that b is integral over B^Γ.

2. Let $(b_i)_{1\geq i\geq m}$ be a system of generators of the R-algebra B. Since in particular, $B = B^\Gamma[b_1, ..., b_m]$ and the b_j are integral over B^Γ by 1., the result follows from Lemmas A.1 and A.2.

□

Proposition I.5. *Given two prime ideals $\mathfrak{P}$, $\mathfrak{P}'$ of B over the same prime ideal $\mathfrak{p}$ of C, there is $\gamma \in \Gamma$ such that $\mathfrak{P} = \gamma.\mathfrak{P}'$. In other words, Γ acts transitively on the set of prime ideals over $\mathfrak{p}$.*

Proof. If $x \in \mathfrak{P}'$, we have $\prod_{\gamma\in\Gamma} \gamma.x \in \mathfrak{P}' \cap C = \mathfrak{p} \subset \mathfrak{P}$, thus there is $\gamma \in \Gamma$ such that $\gamma.x \in \mathfrak{P}$, that is $x \in \gamma^{-1}.\mathfrak{P}$. We conclude that $\mathfrak{P}' \subset \cup_{\gamma\in\Gamma}\gamma.\mathfrak{P}$, so as Γ is finite and the $\gamma.\mathfrak{P}$'s are prime, there is $\gamma \in \Gamma$ such that $\mathfrak{P}' \subset \gamma.\mathfrak{P}$ (see [Bou81, Chapter II, §1, n°1, Proposition 2]); as $\mathfrak{P}'$ and $\gamma.\mathfrak{P}$ are both over $\mathfrak{p}$, we have $\mathfrak{P}' = \gamma.\mathfrak{P}$ (see [Bou81, Chapter VI, §2, n°1, Corollary 1 of Proposition 1]). □

Remark I.6. *For actions of finite locally free group schemes, all these properties are generalized in terms of quotients (see Theorem III.39). In a more general context, we find weaker analogue results under tame hypotheses, (see Theorem IX.6) but we could not prove transitivity, without finiteness the condition on G.*

2 Decomposition groups, inertia groups, and ramification groups

This section recalls the most important properties of the inertia groups, decomposition groups and ramification groups. We will dedicate Chapter V to the generalization and translation of the notion of inertia group and their properties in terms of actions of a group scheme. In Chapter VII, we will explain why this notion is the "natural" generalization of the inertia group. Unfortunately, we do not know any equivalent to decomposition group or ramification group in an algebro-geometric context.

Definition I.7. *Let $\mathfrak{P}$ be a prime ideal of B.*

1. *We call* ***group of decomposition at*** *$\mathfrak{P}$, denoted by $D_\Gamma(\mathfrak{P})$, the group defined by*

$$D_\Gamma(\mathfrak{P}) := \{\gamma \in \Gamma | \gamma.\mathfrak{P} = \mathfrak{P}\}.$$

   We call ***group of decomposition of*** *B, denoted by $D_\Gamma(B, \Gamma)$, the group generated by the groups of decomposition at $\mathfrak{P}$ for $\mathfrak{P}$ running over all the prime ideals of B and we call* ***ring of decomposition*** *denoted by $B^{D_\Gamma(\mathfrak{P})}$ the ring of the elements of B invariants by $D_\Gamma(\mathfrak{P})$.*

2. *For $\gamma \in D_\Gamma(\mathfrak{P})$, denote by $\bar{\gamma}$ the endomorphism of the ring $B/\mathfrak{P}$ induced by the endomorphism of B given by $z \mapsto \gamma.z$. We call* ***group of inertia at*** *$\mathfrak{P}$, denoted by $\Gamma_0(\mathfrak{P})$, the group defined by*

$$\Gamma_0(\mathfrak{P}) := \{\gamma \in D_\Gamma(\mathfrak{P}) | \bar{\gamma} = Id\}$$

   We call ***inertia group of*** *B, denoted by $\Gamma_0(B, \Gamma)$, the group generated by the inertia groups at $\mathfrak{P}$ for $\mathfrak{P}$ running over all the prime ideals of B and we call* ***inertia ring****, denoted by $B^{\Gamma_0(\mathfrak{P})}$, the ring of the elements of B invariants by $\Gamma_0((\mathfrak{P})$.*

3. *We call* ***ramification group at*** *$\mathfrak{P}$, denoted by $\Gamma_1(\mathfrak{P})$, the group equal to the trivial group if the characteristic is 0 and to the p-Sylow of $\Gamma_0(\mathfrak{P})$. We call* ***ramification group at*** *B, denoted by $\Gamma_1(B, \Gamma)$, the group generated by the ramification groups at $\mathfrak{P}$ for $\mathfrak{P}$ running over all the prime ideals of B and we call* ***ramification ring****, denoted by $B^{\Gamma_1(\mathfrak{P})}$, the ring of the elements of B invariants by $\Gamma_1(\mathfrak{P})$.*

4. *We call the* ***orbit at*** $\mathfrak{P}$ *denoted by* $\mathcal{O}(\mathfrak{P})$ *the set* $\mathcal{O}(\mathfrak{P}) := \{\gamma.\mathfrak{P},\ for\ \gamma \in \Gamma\}$.

Remark I.8. 1. *If* B *is local with maximal ideal* $\mathfrak{P}$*, then* $\Gamma_0 := \Gamma_0(B, \Gamma) = \Gamma_0(\mathfrak{P})$ *and* $\Gamma_1 := \Gamma_1(B, \Gamma) = \Gamma_1(\mathfrak{P})$

2. *For any* $\mathfrak{P}$ *prime ideal of* B*, we have* $\mathcal{O}(\mathfrak{P}) \simeq \Gamma/\Gamma_0(\mathfrak{P})$*. Indeed, the kernel of the natural map* $\gamma \to \mathcal{O}(\mathfrak{P})$ *sending* γ *to* $\gamma.\mathfrak{P}$ *is* $\Gamma_0(\mathfrak{P})$.

For $\mathfrak{P}$ prime ideal of B, it is clear that:

$$B^{\Gamma} \subset B^{D_{\Gamma}(\mathfrak{P})} \subset B^{\Gamma_0(\mathfrak{P})} \subset B^{\Gamma_1(\mathfrak{P})} \subset B$$

Corollary I.9. *The decomposition and inertia group at prime ideals of* B *above the same prime ideal of* C *are conjugated.*

Proof. In fact, from Proposition I.5, two prime ideals of B above the same prime ideal of C are conjugated and from the definitions, it is clear that for any $\gamma \in \Gamma$ and $\mathfrak{P}$ prime ideal of B, we have:

$$D_{\Gamma}(\gamma.\mathfrak{P}) = \gamma D_{\Gamma}(\mathfrak{P})\gamma^{-1},\ \Gamma_0(\gamma.\mathfrak{P}) = \gamma\Gamma_0(\mathfrak{P})\gamma^{-1}\ and\ \Gamma_1(\gamma.\mathfrak{P}) = \gamma\Gamma_1(\mathfrak{P})\gamma^{-1}$$

□

Proposition I.10. *Let* $\mathfrak{P}$ *be a prime ideal of* B*,* $\mathfrak{p} = \mathfrak{P} \cap C$*. Then,* $k(\mathfrak{P})$ *is a quasi-Galois extension of* $k(\mathfrak{p})$*, and the canonical homomorphism* $\gamma \mapsto \bar{\gamma}$ *of* $D_{\Gamma}(\mathfrak{P})$ *in the group* $\mathcal{G}$ *of the* $k(\mathfrak{p})$*-automorphisms of* $k(\mathfrak{P})$ *defined by passing to the quotient, an isomorphism from* $D_{\Gamma}(\mathfrak{P})/\Gamma_0(\mathfrak{P})$ *to* $\mathcal{G}$.

Proof. To see that $k(\mathfrak{P})$ is a quasi-Galois extension of $k(\mathfrak{p})$ it is enough to prove that any element $\bar{b} \in k(\mathfrak{P})$ is a root of a polynomial P of $k(\mathfrak{p})[X]$ such that its roots are in $k(\mathfrak{P})$. Let $b \in B$ be a representative of the class $\bar{b}$. The polynomial $Q(X) = \prod_{\gamma\in\Gamma}(X - \gamma.b)$ has all its coefficients in C. Let $P(X)$ be the polynomial of $k(\mathfrak{p})[X]$ whose coefficients are the images of the ones of Q by the canonical homomorphism $\pi_{\mathfrak{p}} : C \to k(\mathfrak{p})$. As $\pi_{\mathfrak{p}}$ can be considered as the restriction to C of the canonical homomorphism $\pi_{\mathfrak{P}} : B \to k(\mathfrak{P})$, we see that, in $k(\mathfrak{P})[X]$, P is product of the linear factors $X - \pi_{\mathfrak{P}}(\gamma.b)$, and answers then to the question, since $\bar{b} = \pi_{\mathfrak{P}}(b)$. It is clear that for all $\gamma \in D_{\Gamma}(\mathfrak{P})$, $\bar{\gamma}$ is a $k(\mathfrak{p})$-automorphism of $k(\mathfrak{P})$. Without loss of generality, we can replace B (resp. $\mathfrak{P}$) by $B_{\mathfrak{P}}$ (resp. $\mathfrak{P}B_{\mathfrak{P}}$), hence we can suppose that $\mathfrak{P}$ is maximal. Then $\mathfrak{p}$ is automatically also maximal and any element of $k(\mathfrak{p})$ is of the form $\pi_{\mathfrak{p}}(b)$ for some $b \in B$. As any finite separable extension of $k(\mathfrak{p})$ admits a primitive element, we can show that any finite separable extension of $k(\mathfrak{p})$ contained in $k(\mathfrak{P})$ has a degree at most $Card(\Gamma)$, thus the biggest separable extension $k(\mathfrak{P})_s$ of $k(\mathfrak{p})$ contained in $k(\mathfrak{P})$ has degree at most $Card(\Gamma)$. Let $y \in B$ a element such that $\pi_{\mathfrak{P}}(y)$ is a primitive element of $k(\mathfrak{P})_s$. The ideals $\gamma.\mathfrak{P}$ for $\gamma \in \Gamma - D_{\Gamma}(\mathfrak{P})$ are maximal and different from $\mathfrak{P}$ by definition; there is then $b \in B$ such that $b \equiv y\ (mod\ \mathfrak{P})$ and $b \in \gamma^{-1}\mathfrak{P}$, for $\gamma \in \Gamma - D_{\Gamma}(\mathfrak{P})$. Let now u be a $k(\mathfrak{p})$-automorphism of $k(\mathfrak{P})$

and let $P(X) = \prod_{\gamma\in\Gamma}(X - \pi_{\mathfrak{P}}(\gamma.b))$; as $\pi_{\mathfrak{P}}(b)$ is a root of P and $P \in k(\mathfrak{p})[X]$, $u(\pi_{\mathfrak{P}}(b))$ is also a root of P in $k(\mathfrak{P})$, hence there is $\tau \in \Gamma$ such that

$$u(\pi_{\mathfrak{P}}(b)) = \pi_{\mathfrak{P}}(\tau.b)$$

But, we have $u(\pi_{\mathfrak{P}}(b)) \neq 0$ and for $\gamma \in \Gamma - D_\Gamma(\mathfrak{P})$, we have $\gamma.b \in \mathfrak{P}$, therefore $\pi_{\mathfrak{P}}(\gamma.b) = 0$; we conclude that we must have $\tau \in D_\Gamma(\mathfrak{P})$. But, since u and $\bar{\tau}$ have the same value for the primitive element $\pi_{\mathfrak{P}}(y) = \pi_{\mathfrak{P}}(b)$ of $k(\mathfrak{P})_s$, they coincide in $k(\mathfrak{P})$. □

It is easily seen that the decomposition, inertia and ramification groups induce contravariant functors from $\mathcal{C}_\Gamma$ to the category of groups. In the following results, for G a finite group, Δ_G stands for any one of the symbols D_G, G_0 and G_1. The proofs are immediate and left to the reader.

Lemma I.11. *Let Ω be a finite abstract group together with a morphism $\alpha : \Omega \to \Gamma$, T a Ω-object and $f : B \to T$ an equivariant morphism.*

1. *For every prime $\mathfrak{q}$ of T, $\alpha(\Delta_\Omega\ (\mathfrak{q})) \subset \Delta_\Gamma(f^{-1}(\mathfrak{q}))$ with equality for Δ_G equal to G_0, so also G_1.*
2. *We have $\alpha(\Delta_\Omega(T, \Omega)) \subseteq \Delta_\Gamma(B, \Gamma)$*

We can obtain easily the following lemmas which show the behavior of these groups passing to quotients, localizations and tensor products.

Lemma I.12. *Let I be an ideal of B stable under the action of a subgroup Ω of Γ. Then for all prime ideals $\mathfrak{P}$ of B which contain I,*

$$\Delta_\Omega(\mathfrak{P}) = \Delta_\Gamma(\mathfrak{P}) \cap \Omega$$

Lemma I.13. *Let L be a multiplicatively closed subset of B stable under a subgroup Ω of Γ. Then for all prime ideals $\mathfrak{P}$ of B which do not meet L,*

$$\Delta_\Omega(L^{-1}\mathfrak{P}) = \Delta_\Gamma(\mathfrak{P}) \cap \Omega$$

As a direct consequence, we obtain:

Corollary I.14. *For any prime ideal $\mathfrak{P}$ of B and $\Delta = D_\Gamma$, Γ_0 or Γ_1, we have*

$$\Delta(B_{\mathfrak{P}}, D_\Gamma(\mathfrak{P})) = \Delta(\mathfrak{P})$$

Lemma I.15. *Let $C \to C'$ be a ring morphism and $f : B \to B \otimes_C C'$ be the extension map sending $b \in B$ to $b \otimes 1$. For all prime ideals $\mathfrak{P}$ of $B \otimes_C C'$,*

$$\Delta_\Gamma(\mathfrak{P}) = \Delta_\Gamma(f^{-1}(\mathfrak{P}))$$

The inertia and ramification groups have the following particular structures:

Theorem I.16. *Suppose that B is a local ring with maximal ideal $\mathfrak{P}$ and $k := k(\mathfrak{P})$. Write*

$$\Gamma_0 := \Gamma_0(B, \Gamma) = \Gamma_0(\mathfrak{P}), \ \Gamma_1 := \Gamma_1(B, \Gamma) = \Gamma_1(\mathfrak{P})$$

and suppose that Γ_0 acts faithfully on B and leaves every ideal of B fixed. Then

1. *The ramification group is equal to $\Gamma_1 = \{\gamma \in \Gamma_0 | \gamma.b \equiv b \ (mod\ b\mathfrak{P}), \ b \in B\}$*
2. *The quotient Γ_0/Γ_1 is abelian and the number of generators is bounded above by the number of generators of the monoid of principal ideals of B;*
3. *k contains all the n^{th} roots of unity, where n runs over the set of orders of the elements of Γ_0/Γ_1.*

Proof. Let b be a non-zero element of B. Then the principal ideal generated by b is stable under Γ_0 and so, for $\gamma \in \Gamma_0$, there exists $[\gamma, b] \in B$ such that $\gamma.b = b[\gamma, b]$. $[\gamma, b]$ is unique modulo $\mathfrak{P}$, since if u is another element with the same property $[\gamma, b] - u$ is a zero divisor mod B and hence lies in $\mathfrak{P}$. Moreover $[\gamma, b]$ is a unit in B. Again there exists $v \in B$ such that $b = (\gamma.b)v$, and then $1 - [\gamma, b]v$ is a zero divisor and $[\gamma, b]v$ must be a unit. Thus, if $< \gamma, b >$ denotes the image in k of $[\gamma, b]$ under the quotient map $B \to k$ and if k^* is the multiplicative group of non-zero elements of k, $\gamma \mapsto < \gamma, b >$ gives a well-defined function $\phi_b : \Gamma_0 \to k^*$. Further, for γ, $\delta \in \Gamma_0$,

$$\gamma.(\delta.b) = b[\gamma, b][\delta, b] \ (mod\ \mathfrak{P})$$

and therefore ϕ_b is a morphism of groups.
Let $\Omega = \{\gamma \in \Gamma_0 | \gamma.b = b \ (mod\ b\mathfrak{P}), \ b \in B\}$. (Here B could be replaced by $\mathfrak{P}$ since, if b is a unit, we have $\gamma(b) \equiv b \ (mod\ b\mathfrak{P})$ for any $\gamma \in \Gamma_0$). Then Ω is the intersection of the kernels of the maps ϕ_b above and is therefore normal. Moreover, since Γ_0 is finite, Ω must be the intersection of finitely many such kernels and therefore Γ_0/Ω is embedded in the direct product of a finite number of copies of k^*. It follows that Γ_0/Ω is abelian and moreover that, if k has characteristic $p \neq 0$, the order of Γ_0/Ω is prime to p.
We will prove that if k has characteristic zero, Ω is trivial, and if k has characteristic $p \neq 0$, Ω is a p-group. It follows that $\Omega = \Gamma_1$ is trivial if k has characteristic zero and is equal to the unique Sylow p-subgroup of $\Gamma_0(\mathfrak{P})$ if k has characteristic $p \neq 0$.
Let then γ be an element of Ω and suppose that $\gamma^r = 1$ where the integer r is a unit in B. Let $b \in B$. Then, since $\gamma \in \Omega$, $\gamma.b = b + bx$ for some $x \in \mathfrak{P}$, and again

$$\gamma.(bx) \equiv bx \ (mod\ bx\mathfrak{P}).$$

Therefore

$$\gamma^2(b) = \gamma.b + \gamma.(bx) = b + bx + \gamma.(bx) \equiv b + 2bx \ (mod\ bx\mathfrak{P}).$$

Thus,

$$\gamma^3(b) \equiv b + bx + 2\gamma(bx) \equiv b + 3bx \ (mod\ bx\mathfrak{P}),$$

and by repeating this we have

$$b = \gamma^r(b) \equiv b + rbx \ (mod\ bx\mathfrak{P})$$

Therefore $rbx \equiv 0 \ (mod\ bx\mathfrak{P})$ and so $bx \equiv 0 \ (mod\ bx\mathfrak{P})$, since r is a unit. It follows that bx annihilates a unit in B and is therefore zero. Thus, $\gamma.b = b$, for all $b \in B$ and since Γ_0 acts faithfully on B, $\gamma = 1$.
It follows that when k has characteristic zero, Ω is trivial, and when k has characteristic $p \neq 0$, Ω has no elements of order prime to p and is therefore a p-group. This completes the proof of 1.
Also now, Γ_0/Γ_1 is abelian. Moreover, if $b_1, ..., b_m$ are elements of B with $b = b_1...b_m \neq 0$, then clearly

$$< \gamma, b >= \prod_{i=1}^{m} < \gamma, b_i >, \ for \ \gamma \in \Gamma_0,$$

and so

$$\cap_{i=1}^{m} ker(\phi_{b_i}) \subseteq ker(\phi_b)$$

Hence the number of the cyclic components of Γ_0/Γ_1 is bounded above by the number of generators of the monoid of principal ideals of B.
Finally, let n be the order of an element of Γ_0/Γ_1. We have a direct product of finitely many copies of k^* which contains a subgroup isomorphic with Γ_0/Γ_1. Therefore at least one of the factors must contain a primitive n^{th} root of unity. □

Corollary I.17. *Let $\mathfrak{P}$ be a prime ideal of B. Suppose that $\Gamma_0(\mathfrak{P})$ acts faithfully on $B_{\mathfrak{P}}$ and leaves every ideal of $B_{\mathfrak{P}}$ fixed then*

1. *The ramification group is equal to $\Gamma_1(\mathfrak{P}) = \{\gamma \in \Gamma_0 | \gamma.b \equiv b \ (mod\ b\mathfrak{P}), \ b \in B_{\mathfrak{P}}\}$;*
2. *the quotient $\Gamma_0(\mathfrak{P})/\Gamma_1(\mathfrak{P})$ is abelian and the number of generators is bounded above by the number of generators of the monoid of principal ideals of B;*
3. *k contains all the n^{th} roots of unity, where n runs over the set of orders of $\Gamma_0(\mathfrak{P})/\Gamma_1(\mathfrak{P})$.*

Proof. It suffices to apply the above theorem to the $D_\Gamma(\mathfrak{P})$-object $B_{\mathfrak{P}}$ and Corollary I.14. □

3 Galois and tame objects

We can now define the basic notions for ramification theory of Galois and tame objects.

Definition I.18. *Let B be a Γ-object.*

1. *We call a prime ideal $\mathfrak{P}$ of B **ramified** if the inertia group $\Gamma_0(\mathfrak{P})$ is non trivial and **unramified** if the inertia group $\Gamma_0(\mathfrak{P})$ is trivial.*

2. *B is called a* ***Galois object*** *if all the prime ideals of B are unramified.*
3. *We call a prime ideal $\mathfrak{P}$ of B* ***tame*** *if the ramification group $\Gamma_1(\mathfrak{P})$ is trivial.*
4. *B is called a* ***tame object*** *if all the prime ideals of B are tame.*

Remark I.19. 1. *For $\mathfrak{p}$ a non-zero prime ideal of C, if the characteristic of its residue field is zero, any non-zero prime ideal above this prime ideal is tamely ramified.*

2. *The analogues of a Galois object in algebro-geometric terms are the free actions studied in Chapter VII, Section 3. Considering an action by a constant group scheme, we know that saying that a prime ideal $\mathfrak{P}$ of B is tamely ramified is equivalent to require the inertia group scheme at this prime ideal to be linearly reductive (see Corollary VII.5). We will give two notions of tameness generalizing the classical case, tame actions by an affine group scheme defined in Chapter VI, Section 1 and tame quotient stacks defined in Chapter VI, Section 2. We explain then why they are good candidates to generalize the tameness in the actual context (Chapter VIII), how they are related (Chapter X, Section 2) and under which hypotheses actions defining tame quotient stacks have linearly reductive inertias as expected.*
3. *If $D(B)$ is the product $\prod_{\gamma\in\Gamma,\gamma\neq 1} I_\gamma$ where I_γ is the ideal of B generated by the set $\{\gamma(b)-b\}_{b\in B}$, then the ramified prime ideals of B are precisely those which contain $D(B)$, and hence they form a closed subset in $Spec(B)$.*

Theorem I.20. *(see [CHR65, Theorem 1.3]) B is a Galois object if and only if Γ acts faithfully on B and B is a Galois extension of B^Γ with Galois group Γ.*

Both "Galois" and "tame" are local properties, in the sense of the next proposition.

Proposition I.21. *The following assertions are equivalent:*

1. *B is a Galois (tame) object;*
2. *$B\otimes_C C_\mathfrak{p}$ is a Galois (tame) object, for all prime/maximal ideals $\mathfrak{p}$ of C.*
3. *$B\otimes_C k(\mathfrak{p})$ is a Galois (tame) object, for all prime/maximal ideals $\mathfrak{p}$ of C.*
4. *$(B_\mathfrak{P}, D_\Gamma(\mathfrak{P}))$ is a Galois (tame) object, for all prime/maximal ideals $\mathfrak{P}$ of B.*
5. *$(k(\mathfrak{P}), D_\Gamma(\mathfrak{P}))$ is a Galois (tame) object, for all prime/maximal ideals $\mathfrak{P}$ of B.*

Proof. In the following, Γ stands for one of the symbols D_Γ, Γ_0 and Γ_1.
The equivalence of 1. and 3. is a consequence of Lemma I.12. Indeed if C is a local ring with maximal ideal $\mathfrak{p}$, then

$$\Delta(B/\mathfrak{p}B,\Gamma)=\Delta(B,\Gamma)$$

The equivalence of 1. and 2. is a consequence of lemma I.15. Indeed for all prime ideals $\mathfrak{p}$ of C, $\Delta(B\otimes_C C_\mathfrak{p},\Gamma)$ is the subgroup of Γ generated by the subgroups $\Delta(\mathfrak{P})$ where $\mathfrak{P}$ runs over the prime ideals of B lying above $\mathfrak{p}$.
The equivalences of 1., 4. and 5. is a consequence of lemma I.14. □

Remark I.22. *1. As a consequence of Lemma I.15, we can prove that if B is a Galois (tame) object then for any ring extension $C \to C'$, $B \otimes_C C'$ is also Galois. Moreover, the converse is true if for example C' is faithfully flat over C or if C' is integral over C.*

2. We will see later that freeness is a local property (see Property IX.19), also we have some localness results for tame actions (see Chapter VI, Section 1, 1.2) and for tame stacks (see Lemma VI.15 and Corollary IX.22).

4 Maximal Galois objects and maximal tame objects inside a given object

Let Ω be a normal subgroup of Γ. Denote by $i : B^\Omega \to B$ the inclusion map and by $\pi : \Gamma \to \Gamma/\Omega$ the quotient map. Γ/Ω acts on B^Ω, as follows. For any $\gamma \in \Gamma$, $t \in B^\Omega$ put $\pi(\gamma).t := \gamma.t$. This makes B^Ω a Γ/Ω-object and i a morphism $B \to B^\Omega$. Moreover, it is clear that this map is universal among the maps f from any Σ-object where Σ is finite group endowed with a morphism $\alpha : \Gamma \to \Sigma$ such that $\Omega \subseteq Ker(\alpha)$ to the Γ-object B.

Lemma I.23. *Let $\mathfrak{P}$ be a prime ideal of B. Then for Δ stands for one of the symbols D_Γ, Γ_0, and Γ_1,*

$$\Delta(\mathfrak{P}^\Omega) = \Delta(\mathfrak{P})/(\Delta(\mathfrak{P}) \cap \Omega)$$

where $\mathfrak{P}^\Omega := \mathfrak{P} \cap B^\Omega$.

Proof. The result for $\Delta = D_\Gamma$ follows from the fact that Ω acts transitively on the set of the prime ideals of B lying above the prime ideal $\mathfrak{P}^\Omega$ of B^Ω (see Proposition I.10, (i)). Now, let $\mathfrak{p} := \mathfrak{P} \cap C$. Then, the separable closure C of $k(\mathfrak{p})$ in $k(\mathfrak{P})$ is by Proposition I.10 a finite Galois extension of $k(\mathfrak{p})$ and the action of $\Delta(\mathfrak{P})$ on $k(\mathfrak{P})$ induces an exact sequence in the category of groups:

$$1 \to \Gamma_0(\mathfrak{P}) \to D_\Gamma(\mathfrak{P}) \to Gal(E/k(\mathfrak{p})) \to 1$$

Considering B as a Ω-object, we obtain:

$$1 \to \Gamma_0(\mathfrak{P}) \cap \Omega \to D_\Gamma(\mathfrak{P}) \cap \Omega \to Gal(E.k(\mathfrak{P}^\Omega)/k(\mathfrak{P}^\Omega)) \to 1$$

Considering B^Ω as a Γ/Ω-object, we have the sequence:

$$1 \to \Gamma_0(\mathfrak{P}^\Omega) \to D_\Gamma(\mathfrak{P}^\Omega) \to Gal(E \cap k(\mathfrak{P}^\Omega)/k(\mathfrak{p})) \to 1$$

and hence a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & D_\Gamma(\mathfrak{P})\cap\Omega & \longrightarrow & Gal(E/E\cap k(\mathfrak{P}^\Omega)) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma_0(\mathfrak{P}) & \longrightarrow & D_\Gamma(\mathfrak{P}) & \longrightarrow & Gal(E/k(\mathfrak{p})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma_0(\mathfrak{P}^\Omega) & \longrightarrow & D_\Gamma(\mathfrak{P}^\Omega) & \longrightarrow & Gal(E\cap k(\mathfrak{P}^\Omega)/k(\mathfrak{p})) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

with exact rows and columns. It follows, by diagram-chasing, that the map $\Gamma_0(\mathfrak{P}) \to \Gamma_0(\mathfrak{P}^\Omega)$ is onto, which gives the result for $\Delta = \Gamma_0, \Gamma_1$. □

From the previous lemma and the universality of the inclusion map i mentioned before, we obtain the following theorem:

Theorem I.24. *Write $\Delta = \Gamma_0(B, \Gamma)$ (respectively $\Delta = \Gamma_1(B, \Gamma)$). Then $(B^\Delta, \Gamma/\Delta)$ is a Galois (respectively tame) object; moreover, for any finite group Σ endowed with a map $\Gamma \to \Sigma$ and any Galois (respectively tame) Σ-object U, there is a unique equivariant morphism $f : U \to B^\Delta$ such that the diagram*

$$\begin{array}{ccc}
B^\Delta & \longrightarrow & B \\
\uparrow & \nearrow_{f} & \\
U & &
\end{array}$$

is commutative.

We can localize this theorem in the following sense:

Corollary I.25. *Let $\mathfrak{P}$ be a prime ideal of B. Write $\Delta = \Gamma_0(\mathfrak{P})$ (respectively $\Delta = \Gamma_1(\mathfrak{P})$) and consider $B_\mathfrak{P}$ as a $D_\Gamma(\mathfrak{P})$-object. Then $(B_\mathfrak{P}^\Delta, D_\Gamma(\mathfrak{P})/\Delta)$ is a Galois (respectively tame) object; moreover, for any finite group Σ endowed with a map $D_\Gamma(\mathfrak{P}) \to \Sigma$ and any Galois (respectively tame) Σ-object U, there is a unique equivariant morphism $f : U \to B_\mathfrak{P}^\Delta$ such that the diagram*

$$\begin{array}{ccc}
B_\mathfrak{P}^\Delta & \longrightarrow & B_\mathfrak{P} \\
\uparrow & \nearrow_{f} & \\
U & &
\end{array}$$

is commutative.

Proof. It is a direct application to the previous theorem to the $D_\Gamma(\mathfrak{P})$-object and Corollary I.14. □

Remark I.26. *As an analogue, we see how to get locally a free action from an action defining a tame quotient stack (see Corollary IX.23).*

5 Inertia group action inducing the initial action

One can describe the action Γ on B thanks to the action of $\Gamma_0(\mathfrak{P})$ on $B_\mathfrak{P}$, after an étale base change. More precisely,

Theorem I.27. *([Ray70, Chapitre X]) Let $\mathfrak{p}$ be a prime ideal of C and $\mathfrak{P}$ be a prime ideal of B above $\mathfrak{p}$. denote by ${C_\mathfrak{p}}^{sh}$ the strict henselization of C at the prime ideal $\mathfrak{p}$. Write $Map^{I_G(\mathfrak{P})}(\Gamma, B_\mathfrak{P} \otimes_{C_\mathfrak{p}} {C_\mathfrak{p}}^{sh})$ for the set of the applications u from Γ to $B_\mathfrak{P} \otimes_{C_\mathfrak{p}} {C_\mathfrak{p}}^{sh}$ such that $u(i^{-1}\gamma) = u(\gamma)i$, for any $\gamma \in \Gamma$ and $i \in \Gamma_0(\mathfrak{P})$. Then, Γ operates on $Map^{\Gamma_0(\mathfrak{P})}(\Gamma, B_\mathfrak{P} \otimes_{C_\mathfrak{p}} {C_\mathfrak{p}}^{sh})$ via ${}^\lambda u(\gamma) := \lambda.u(\gamma) = u(\lambda^{-1}\gamma)$, for any λ and $\gamma \in \Gamma$. We obtain a canonical isomorphism ϕ, compatible with the actions of Γ defined by*

$$\begin{array}{rccl} \phi: & B \otimes_C {C_\mathfrak{p}}^{sh} & \to & Map^{\Gamma_0(\mathfrak{P})}(\Gamma, B_\mathfrak{P} \otimes_{C_\mathfrak{p}} {C_\mathfrak{p}}^{sh}) \\ & b & \mapsto & u: \gamma \mapsto (\gamma^{-1}b)_\mathfrak{P} \end{array}$$

Moreover, we have $C \simeq B_\mathfrak{P}^{\Gamma_0(\mathfrak{P})}$

Proof. We can easily see via the integral morphism $C_\mathfrak{p}^{hs} \to B \otimes_C C_\mathfrak{p}^{hs}$ (base change of the integral morphism $C \to B$), that without loss of generality, it suffices to treat the case when C is a local strictly Henselian ring with maximal ideal $\mathfrak{p}$.
We want to show that the following morphism, compatible with the actions of Γ

$$\begin{array}{rccl} \phi: & B & \to & Map^{\Gamma_0(\mathfrak{P})}(\Gamma, B_\mathfrak{P}) \\ & b & \mapsto & u: \gamma \mapsto (\gamma^{-1}b)_\mathfrak{P} \end{array}$$

is an isomorphism.
Fix γ_1, ..., $\gamma_s \in \Gamma$ representatives of the classes of the quotient $\Gamma/\Gamma_0(\mathfrak{P})$ with $\gamma_1 = e$ (identity of Γ). If we put $\mathfrak{M}_j := \gamma_j^{-1}\mathfrak{P}$, then the maximal ideals $\mathfrak{M}_1$, ..., $\mathfrak{M}_s$ are exactly the prime ideals of B over $\mathfrak{p}$. So, since B is integral over C which is Henselian, we have $B \simeq \oplus_{j=1}^s B_{\mathfrak{M}_j}$. Moreover, $B_{\mathfrak{M}_j} = \gamma_j^{-1} B_\mathfrak{P}$ (this is a consequence of [AM69, cor. 3.2 chap. 3] applied to the composite $B \to B \to B_\mathfrak{P}$ of the multiplication by γ_j with the localization). Thus, $B_{\mathfrak{M}_j} \simeq \gamma_j^{-1}\gamma_i B_{\mathfrak{M}_i}$. As a consequence, any $b \in B$ can be written uniquely as $({\gamma_1}^{-1}b_1, {\gamma_2}^{-1}b_2, ..., {\gamma_s}^{-1}b_s)$ in $\oplus_{j=1}^s B_{\mathfrak{M}_j}$ with $b_i \in B_\mathfrak{P}$. For any i,j, from the isomorphism $B_{\mathfrak{M}_j} \simeq \gamma_j^{-1}\gamma_i B_{\mathfrak{M}_i}$ one obtains that $(\gamma_j^{-1}b)_\mathfrak{P} = \gamma_j^{-1}\gamma_j b_j = b_j$, and this allows to prove that ϕ is an isomorphism. Finally, the composite map

$$\begin{array}{ccccc} B & \simeq^\phi & Map^{\Gamma_0(\mathfrak{P})}(\Gamma, B_\mathfrak{P}) & \to & B_\mathfrak{P} \\ & & u & \mapsto & u(1_\Gamma) \end{array}$$

induces an isomorphism between C and $B_{\mathfrak{P}}^{\Gamma_0(\mathfrak{P})}$. □

Remark I.28. *The possibility to rebuild the initial action from the action of the inertia group motivates the notion of Slice (see Chapter III, Section 3, 3.3). As direct corollary, we obtain Theorem VII.9, for actions of a constant group scheme. But, we obtain just a weaker result for tame quotient stack (Theorem IX.20) and a slice theorem only when we suppose G to be commutative and finite over S (Theorem IX.27).*

6 Separability of the residue field of Γ-extension

Proposition I.29. *Let $\mathfrak{P}$ be a prime ideal of B, $\mathfrak{p} := \mathfrak{P} \cap C$, $D = B^{D_\Gamma(\mathfrak{P})}$ and $\mathfrak{P}^{D_\Gamma(\mathfrak{P})} = \mathfrak{P} \cap D$. Then, $k(\mathfrak{P}) = k(\mathfrak{P}^{D_\Gamma(\mathfrak{P})})$.*

Proof. Without loss of generality, we can suppose that $\mathfrak{P}$ is maximal. It is enough to prove that

$$(\square) \quad D = C + \mathfrak{P}^{D_\Gamma(\mathfrak{P})}$$

We know that there are only a finite number of prime ideals of D above $\mathfrak{p}$ that we denote by $\{\mathfrak{M}_i\}_{\{0 \le i \le r\}}$ putting $\mathfrak{M}_0 = \mathfrak{P}^{D_\Gamma(\mathfrak{P})}$. Let x be a element of D; as the ideals $\{\mathfrak{M}_i\}_{\{0 \le i \le r\}}$ are maximal, there is $y \in B$ such that $y \equiv x \ mod\ \mathfrak{M}_0$ and $y \in \mathfrak{M}_j$, for any $j \in \{1, ..., r\}$ (see [Bou81, Chapitre II, §1, n ° 2, Proposition 5]). Let $z := \sum_{\gamma \in \Gamma} \gamma.y$. Clearly, $z \in C$ and to prove (□), it is enough to prove that $\gamma.y \in \mathfrak{P}$ for any $\gamma \neq e$. Indeed, as a consequence, $z - y \in \mathfrak{P} \cap D = \mathfrak{P}^{D_\Gamma(\mathfrak{P})}$ and since $x \equiv y \ (mod\ \mathfrak{P}^{D_\Gamma(\mathfrak{P})})$ then $x \in A + \mathfrak{P}^{D_\Gamma(\mathfrak{P})}$. Let then $i \geq 2$ and $\sigma \in \Gamma$ be such that $\sigma.y = y_i$; we will see that $\sigma^{-1}.\mathfrak{P}$ is not above $\mathfrak{P}^{D_\Gamma(\mathfrak{P})}$. In fact, if it was, there would be $\tau \in D_\Gamma(\mathfrak{P})$ such that $\sigma^{-1}.\mathfrak{P} = \tau.\mathfrak{P}$ (see Theorem I.10), but then $(\tau^{-1}\sigma^{-1}).\mathfrak{P} = \mathfrak{P}$, in other words $\tau^{-1}\sigma^{-1} \in D_\Gamma(\mathfrak{P})$ and so also $\sigma \in D_\Gamma(\mathfrak{P})$, but this contradicts the fact that $y \in D$ and $\sigma.y \neq y$. We conclude from this that $\sigma^{-1}.\mathfrak{P}$ is above one of the ideals $\mathfrak{M}_j$ for $j \neq 0$ and since $y \in \mathfrak{M}_j$ by construction, we have $y \in \sigma^{-1}.\mathfrak{P}$ or $y_i = \sigma.y \in \mathfrak{P}$. □

Proposition I.30. *Let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p} := \mathfrak{P} \cap C$. The residue field $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ at $\mathfrak{P}^{I_\Gamma(\mathfrak{P})} := \mathfrak{P} \cap B^{I_\Gamma(\mathfrak{P})}$ is equal to the biggest separable extension $k(\mathfrak{P})_s$ of $k(\mathfrak{p})$ contained in $k(\mathfrak{P})$.*

Proof. Denote $\mathfrak{P}^{D_\Gamma(\mathfrak{P})} := \mathfrak{P} \cap B^{D_\Gamma(\mathfrak{P})}$. Without loss of generality, we can suppose that $\mathfrak{p}$ is a maximal ideal of C , then $\mathfrak{P}$, $\mathfrak{P}^{D_\Gamma(\mathfrak{P})}$ and $\mathfrak{P}^{I_\Gamma(\mathfrak{P})}$ are maximal in B, $B^{D_\Gamma(\mathfrak{P})}$ and $B^{I_\Gamma(\mathfrak{P})}$ respectively. For any $b \in B$, the polynomial $P(X) = \prod_{\gamma \in I_\Gamma(\mathfrak{P})}(X - \gamma.b)$ has its coefficients in $B^{I_\Gamma(\mathfrak{P})}$, and by definition of $I_\Gamma(\mathfrak{P})$, all its roots are congruent modulo $\mathfrak{P}$; the roots in $k(\mathfrak{P})$ of the polynomial $\pi_{\mathfrak{P}}(P)$ over $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ whose coefficients are the canonical images of the ones of P by the homomorphism $\pi_{\mathfrak{P}} : B \to k(\mathfrak{P})$ are equal to the image of b by $\pi_{\mathfrak{P}}$ and this shows that $k(\mathfrak{P})$ is an inseparable extension of $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$, therefore $k(\mathfrak{P})_s \subset k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$. By Theorem I.10, we know that $k(\mathfrak{P})_s$ is a Galois extension of $k(\mathfrak{p})$ and that its Galois group is isomorphic to $\mathcal{G} = D_\Gamma(\mathfrak{P})/I_\Gamma(\mathfrak{P})$. As $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ is an inseparable extension of $k(\mathfrak{P})_s$, $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$

is a quasi-Galois extension of $k(\mathfrak{p})$, and the separable factor of the degree of $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ over $k(\mathfrak{p})$ is $q = (D_\Gamma(\mathfrak{P}) : I_\Gamma(\mathfrak{P}))$. Now, it is enough to see that $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ is a separable extension of $k(\mathfrak{p})$. We can notice that $\mathcal{G}$ can be identified with the automorphism group of $B^{I_\Gamma(\mathfrak{P})}$ and that $B^{D_\Gamma(\mathfrak{P})}$ is the ring of invariants of $\mathcal{G}$. If $b \in B^{I_\Gamma(\mathfrak{P})}$, the polynomial $Q(X) = \prod_{g\in\mathcal{G}}(X - g.b)$ has its coefficients in $B^{D_\Gamma(\mathfrak{P})}$, the polynomial over $k(\mathfrak{P}^{D_\Gamma(\mathfrak{P})})$ whose coefficients are the images of the ones of Q by $\pi_{\mathfrak{P}}$ is of degree q and $\pi_{\mathfrak{P}}(b) \in k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$. As, by the previous proposition, $k(\mathfrak{P}^{D_\Gamma(\mathfrak{P})}) = k(\mathfrak{p})$, we can see that any element of $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ has degree at most q over $k(\mathfrak{p})$. Let k_1 be the subfield of invariants of the quasi-Galois extension $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ of $k(\mathfrak{p})$ by the group of $k(\mathfrak{p})$-automorphisms. Then we have $[k(\mathfrak{p}) : k_1] = q$. Let u be a primitive element of $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ over k_1; as it has degree q over k_1 and degree at most q over $k(\mathfrak{p})$ and its minimal polynomial over k_1 has its coefficients in $k(\mathfrak{p})$, u is separable over $k(\mathfrak{p})$. On the other hand, for any $v \in k_1$, there is a power p^f of the characteristic p of $k(\mathfrak{p})$ such that $v^{p^f} \in k(\mathfrak{p})$. We conclude that $k(\mathfrak{p})(u - v)$, which contains $(u - v)^{p^f} = u^{p^f} - v^{p^f}$, contains u^{p^f} and so $k(\mathfrak{p})(u^{p^f})$. But, as u is separable over $k(\mathfrak{p})$, we have $k(\mathfrak{p})(u) = k(\mathfrak{p})(u^{p^f})$. Since u is of degree q over $k(\mathfrak{p})$ and $u - v$ has degree at most q, it follows that $k(\mathfrak{p})(u) = k(\mathfrak{p})(u - v)$, so $v \in k(\mathfrak{p})(u)$. This shows that v is separable over $k(\mathfrak{p})$, therefore $k_1 = k(\mathfrak{p})$ and $k(\mathfrak{P}^{I_\Gamma(\mathfrak{P})})$ is separable over $k(\mathfrak{p})$. □

Corollary I.31. *If the order of the inertia $I_\Gamma(\mathfrak{P})$ is prime to the characteristic p of $k(\mathfrak{p})$, the field $k(\mathfrak{P})$ is a separable hence Galois extension of $k(\mathfrak{p})$.*

Proof. Using the notation of the previous proposition, the polynomial $\pi_{\mathfrak{P}}(P)$ has its coefficients in $k(\mathfrak{p})^{I_\Gamma(\mathfrak{P})} = k(\mathfrak{P})_s$ and all of its roots are equal to $\pi_{\mathfrak{P}}(P)$; we deduce then that $\pi_{\mathfrak{P}}(P)$ is a power of the minimal polynomial of $\pi_{\mathfrak{P}}(b)$ over $k(\mathfrak{P})_s$; but the degree of this polynomial equals the order of $I_\Gamma(\mathfrak{P})$, and $\pi_{\mathfrak{P}}(b) \in k(\mathfrak{P})_s$ as a consequence of the hypothesis on the order of $I_\Gamma(\mathfrak{P})$, therefore $k(\mathfrak{P})_s = k(\mathfrak{P})$ □

7 Surjectivity of the trace map

We can define a notion of trace map in the present case as the following morphism:

$$\begin{array}{rccc} tr: & B & \to & C \\ & b & \mapsto & \sum_{\gamma\in\Gamma}\gamma.b \end{array}$$

Theorem I.32. *The Γ-object B is tame if and only if tr is onto.*

Proof. Suppose tr is onto and let b be an element of B for which $tr(b) = 1$. Let $\mathfrak{P}$ be any prime ideal of B. Then, if $\{\gamma_1, ..., \gamma_m\}$ is a right transversal of $D_\Gamma(\mathfrak{P})$ in Γ,

$$1 = \sum_{\gamma\in\Gamma}\gamma.b = \sum_{D_\Gamma(\mathfrak{P})}\sum_{i=1}^{m}\delta\gamma_i.b = \sum_{\delta\in D_\Gamma(\mathfrak{P})}\delta.x$$

where $x = \sum_{i=1}^{m} \gamma_i.b$ and so $\{\delta_1, ..., \delta_n\}$ is a left transversal of $\Gamma_0(\mathfrak{P})$ in $D_\gamma(\mathfrak{P})$. We have:

$$1 = \sum_{\delta \in D_\Gamma(\mathfrak{P})} \delta.x = \sum_{i=1}^{n} \sum_{\gamma \in \Gamma_0(\mathfrak{P})} \delta_i \gamma.x \equiv |\Gamma_0(\mathfrak{P})| \sum_{i=1}^{n} \delta_i(x) \ (mod\ \mathfrak{P})$$

since $\delta\gamma.x \equiv \delta.x \ (mod\ \mathfrak{P})$, for all $\gamma \in \Gamma_0(\mathfrak{P})$ and $\delta \in D_\Gamma(\mathfrak{P})$. Therefore, $|\Gamma_0(\mathfrak{P})|$ is not divisible by the characteristic of $k(\mathfrak{P})$ and hence $\Gamma_1(\mathfrak{P}) = 1$. Therefore, B is tame.
Conversely, suppose that B is tame. For each prime ideal $\mathfrak{p}$ of C, the fixed subring $(B \otimes_C C_\mathfrak{p})^\Gamma$ is naturally isomorphic with $C_\mathfrak{p}$; so, to prove that tr is onto, we may assume that C is local. Let $\mathfrak{P}$ be a maximal ideal of B. Then $D_\Gamma(\mathfrak{P})/\Gamma_0(\mathfrak{P})$ acts faithfully on the field $k(\mathfrak{P})$ and therefore, if $\{\delta_1, ..., \delta_n\}$ is a left transversal of $\Gamma_0(\mathfrak{P})$ in $D_\Gamma(\mathfrak{P})$, there is an element b of B such that $\sum_{i=1}^{n} \delta_i.b \equiv 1 \ (mod\ \mathfrak{P})$, and hence

$$\sum_{\delta \in D_\Gamma(\mathfrak{P})} \delta.b \equiv |\Gamma_0(\mathfrak{P})| \sum_{i=1}^{n} \delta_i.b \equiv |\Gamma_0(\mathfrak{P})| \not\equiv 0 \ (mod\ \mathfrak{P})$$

Now, since the distinct conjugates of $\mathfrak{P}$ are pairwise coprime, there exists $a \in B$ such that

$$a \equiv b \ (mod\ \mathfrak{P}) \ and\ \gamma.a \equiv 0 \ (mod\ \mathfrak{P}) \ where\ \gamma \notin D_\Gamma(\mathfrak{P})$$

and then

$$\sum_{\gamma \in \Gamma} \gamma.a \equiv \sum_{\delta \in D_\Gamma(\mathfrak{P})} \delta.b \ (mod\ \mathfrak{P})$$

Therefore $tr(a)$ lies outside $\mathfrak{P}$ and hence outside all the conjugates of $\mathfrak{P}$. But, if C is local, all the maximal ideals of B are conjugate to each other. Therefore $tr(a)$ is a unit and tr is onto. □

Remark I.33. *We have the natural generalization of this result for tame actions by a constant group scheme (see Lemma VII.7). In a more general context, we obtain a Reynold operator which plays the role of the trace map (see Chapter VIII, Section 1, §1.1).*

Chapter II

Classical ramification in number theory

In this chapter, we present some classical results of ramification theory for Dedekind extensions that one can find for example in the book of Cassels and Fröhlich [Cas67, Chapter I, §5] as a specialization of what we discussed in the previous chapter.

First, we recall the classical context for number theory ramification. Thoughout this chapter, C is a Dedekind domain, K is its quotient field, L is a finite Galois extension field of K and B is the integral closure of C in L. We denote Γ the **Galois group** (i.e. the finite group of the K-embeddings of L into an algebraic closure $\overline{K}$ of K). For any $\gamma \in \Gamma$ and $b \in B$, one can prove that $\gamma.b \in B$. This gives B a structure of Γ-object.

For $\mathfrak{p}$ a non-zero prime ideal of C (which is then a maximal ideal) and $\mathfrak{P}$ a prime ideal of the integral closure B of C in L above $\mathfrak{p}$ i.e. $\mathfrak{P} \cap C = \mathfrak{p}$, we write $\chi_{\mathfrak{P}}$ for the characteristic of $k(\mathfrak{P})$.

Definition II.1. *1. We call the order of the inertia group at $\mathfrak{P}$, denoted by $e(\mathfrak{P}|\mathfrak{p})$, the **ramification index**. As a consequence of Corollary I.9, $e(\mathfrak{P}|\mathfrak{p})$ depends only on $\mathfrak{p}$, so we will denote it by $e_{\mathfrak{p}}$.*

*2. We define the **trace map** $tr_{B_{\mathfrak{P}}/C_{\mathfrak{p}}} : B_{\mathfrak{P}} \to C_{\mathfrak{p}}$ to be the map which sends $b \in B_{\mathfrak{P}}$ to the trace of the linear transformation $x \mapsto xb$ which is equivalent to sending b to $\sum_{\gamma \in D_\Gamma(\mathfrak{P})} \gamma.b \in C_{\mathfrak{p}}$. (This corresponds to the trace map introduced in the previous chapter for the $D_\Gamma(\mathfrak{P})$-object $B_{\mathfrak{P}}$.)*

Remark II.2. *Ramification in algebraic number theory means prime numbers factoring into some repeated prime ideal factors as follows. Indeed, by [Cas67, Chapter I, §8, Theorem 1], we know that for $\mathfrak{p}$ a prime ideal of C,*

$$\mathfrak{p}B = (\mathfrak{P}_1 ... \mathfrak{P}_n)^{e_{\mathfrak{p}}}$$

*where $\mathfrak{P}_i$ are distinct prime ideals of B called the **prime ideals over** $\mathfrak{p}$ and $e_{\mathfrak{p}}$ is equals to the index of ramification defined above. Considering number fields, ramification theory permits to understand with what complexity the arithmetic of $\mathbb{Q}$ pass to the arithmetic of the number field K.*

Corollary I.17 has as direct consequence the following theorem:

Theorem II.3. *([Cas67, Chapter I, §8, Theorem 1]) Let* $\mathfrak{p}$ *be a non-zero prime ideal of* C *and let* $\mathfrak{P}$ *be a non-zero prime ideal of* B *above* $\mathfrak{p}$*. If* $\chi_{\mathfrak{P}} = 0$*, the inertia group* $\Gamma_0(\mathfrak{P})$ *is cyclic and if* $\chi_{\mathfrak{P}} = p \neq 0$ *then* $\Gamma_0(\mathfrak{P})$ *is the extension of a* p*-group by a cyclic group.*

Proof. In fact, in the present context, it is well known that $\Gamma_0(\mathfrak{P})$ acts faithfully on $B_{\mathfrak{P}}$ and leaves every ideal of $B_{\mathfrak{P}}$ fixed. Indeed, for any $b \in B$ and $\gamma \in \Gamma$, either $\gamma.b \in bB$ or $b \in (\gamma.b)B$ and in either case $\gamma.(bB) = bB$ since γ has finite order. Hence Γ stabilizes all principal ideals and therefore all ideals of B. This permits to apply Corollary I.17. □

In this context, there is also the concept of tame and unramified prime ideals/extensions that we recall in the following definition.

Definition II.4. *Let* $\mathfrak{p}$ *be a non-zero prime ideal of* C*.*

1. *A non-zero prime ideal* $\mathfrak{P}$ *of* B *above* $\mathfrak{p}$ *is said to be* ***unramified*** *over* K *if* $e_{\mathfrak{p}} = 1$ *and* $k(\mathfrak{P})$ *is separable over* $k(\mathfrak{p})$*. The non-zero prime ideal* $\mathfrak{p}$ *of* C *is* ***unramified*** *(in* L*) if all (or equivalently one) prime ideals* $\mathfrak{P}$ *above* $\mathfrak{p}$ *in* B *are unramified over* K*. We will say that the extension* $L|K$ *is* ***unramified*** *if it is unramified at any non-zero prime ideal of* C*.*
2. *A non-zero prime ideal* $\mathfrak{P}$ *of* B *above* $\mathfrak{p}$ *is said to be* ***tamely ramified*** *over* K *if* $\chi_{\mathfrak{P}} \not| e_{\mathfrak{p}}$ *and* $k(\mathfrak{P})$ *is separable over* $k(\mathfrak{p})$*. The non-zero prime ideal* $\mathfrak{p}$ *of* C *is* ***tamely ramified*** *(in* L*) if all prime ideals* $\mathfrak{P}$ *above* $\mathfrak{p}$ *in* B *are tamely ramified over* K*. We will say that the extension* $L|K$ *is* ***tamely ramified*** *if it is tamely ramified at any non-zero prime ideal of* C*.*

This definition is a particular case of the one of the previous chapter, according to the Corollary I.31. As a consequence of the Theorem I.32 of the previous chapter:

Theorem II.5. *The following conditions are equivalent:*

1. L *is tamely ramified over* K*;*
2. $tr_{L/K}(B) = C$*.*

Using Theorem I.21 and Corollary I.14, one obtains the following consequence of the previous Theorem.

Corollary II.6. *Let* $\mathfrak{p}$ *be a prime ideal of* C*. The following conditions are equivalent:*

1. L *is tamely ramified over* K *at the prime* $\mathfrak{p}$*;*
2. $tr_{L_{\mathfrak{P}}/K_{\mathfrak{p}}}(B_{\mathfrak{p}}) = C_{\mathfrak{p}}$*.*

By Theorem I.24, we can deduce the following fact.

Proposition II.7. *Every finite extension of the field* K *has a maximal unramified extension which is the fixed field of the inertia group and a maximal tame extension which is the fixed field of the ramification group.*

Part B

Actions of affine group schemes and associated quotient stacks

Chapter III

Actions of affine group schemes

1 Affine group schemes

This section recalls the three equivalent ways to define an affine group scheme (representable group functor, Hopf algebra, functor of points). We shall omit the proofs referring to [Wat79, Chapter 1] instead.

1.1 As a representable group functor

Theorem III.1 (Definition)**.** *Let F be a functor from the category of R-algebras to the category of sets. For any T R-algebra, the elements in $F(T)$ are the solutions in T of some family of equations over R if and only if there are an R-algebra A and a bijective correspondence between $F(T)$ and $Hom_R(A, T)$. Such a functor F is called* ***representable****. We say also that A* ***represents*** *F.*

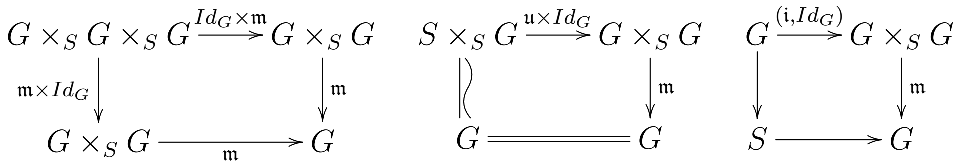
Definition III.2. *An* ***affine group scheme*** *over R is a representable functor from the category of commutative R-algebras to the category of groups.*

1.2 As a Hopf algebra

A group scheme G over S can be characterized as a functor endowed with three natural transformations:

- **a multiplication map** $\mathfrak{m} : G \times_S G \to G$,
- **an unit map** $\mathfrak{u} : S \to G$,
- **an inverse map** $\mathfrak{i} : G \to G$

satisfying associativity, left and right unit and left and right inverse properties. In other words, the following diagrams commute:



(Associativity) **(Left unit)** **(Left inverse)**

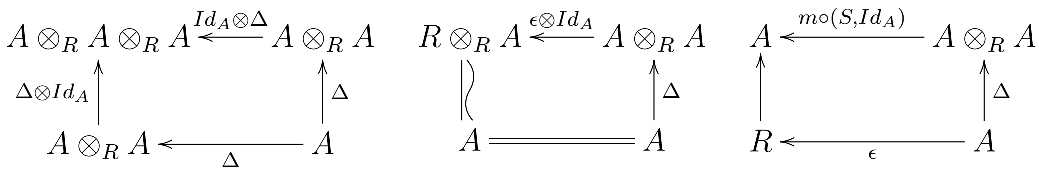
Given a representable functor G over S represented by an R-algebra A, the Yoneda lemma implies that the data of a group scheme G is equivalent to the data of three R-algebra morphisms:

– **a comultiplication map** $\Delta : A \to A \otimes_R A$,

– **a counit map** $\epsilon : A \to R$,

– **a coinverse (antipode) map** $S : A \to A$

satisfying coassociativity, left and right counity and left and right antipode properties. In other words, the following diagrams commute:



(Coassociativity) **(Left counit)** **(Left antipode)**

where $m : A \otimes_R A \to A$ is the algebra multiplication for A.

Remark III.3. *We will use in further proofs the useful* ***notation sigma*** *well-established in the Hopf algebra literature. Specifically, for any* $a \in A$*, we write* $\Delta(a) = \sum_{(a)} a_1 \otimes a_2$*. This presentation itself is purely symbolic; the terms* a_1 *and* a_2 *do not stand for particular elements of* A*. The comultiplication* Δ *takes values in* $A \otimes_R A$*, and so we know that:*

$$\Delta(a) = (a_{1,1} \otimes a_{1,2}) + (a_{2,1} \otimes a_{2,2}) + (a_{3,1} \otimes a_{3,2}) + ... + (a_{n,1} \otimes a_{n,2})$$

for some elements $a_{i,j}$ *of* A *and some integer* n*. The Sigma notation is just a way to separate the* $a_{i,1}$ *from the* $a_{j,2}$*. In other words, one can say that the notation* a_1 *stands for the generic* $a_{i,1}$ *and the notation* a_2 *stands for the generic* $a_{j,2}$*. According to this notation, we can rewrite for any* $a \in A$*,*

1. *the coassociativity diagram as* $\sum_{(a)} (a_1)_1 \otimes (a_1)_2 \otimes a_2 = \sum_{(a)} a_1 \otimes (a_2)_1 \otimes (a_2)_2$*,*
2. *the counit one as* $\sum_{(a)} \epsilon(a_1) a_2 = \sum_{(a)} a_1 \epsilon(a_2) = a$*,*
3. *the antipode one as* $\sum_{(a)} S(a_1) a_2 = \sum_{(a)} a_1 S(a_2) = \epsilon(a) 1_A$*.*

Definition III.4. *We keep the previous notations.*

1. *An R-algebra A endowed with the two R-algebra morphisms Δ and ϵ making the previous coassociativity and the counit diagrams commute is called a* ***coalgebra****. We denote such an R-algebra (A, Δ, ϵ) or simply A if there is no possible confusion.*
2. *An R-algebra A endowed with three R-algebra morphisms Δ, ϵ and S making the previous coassociativity, counit and antipode diagrams commute is called an* ***Hopf algebra****. We denote such an R-algebra (A, Δ, ϵ, S) or simply A if there is no possible confusion.*

Theorem III.5. *The category of the affine group schemes over S is anti-equivalent to the one of the commutative Hopf algebras over R.*

Remark III.6. *The antipode of a commutative algebra is involutive.*

1.3 As functor of points

Definition III.7. *Let T be a S-scheme and Z be a scheme over S. We denote by $Z(T)$ the* ***set of the T-points of*** *Z, that is, the set $Z(T) := Hom_{S\text{-}Sch}(T, Z)$. The arrow $T \mapsto Z(T)$ defines a contravariant functor called* ***the functor of points of*** *Z from the category of S-Schemes to the category of sets. In particular, for B an R-algebra, we write $Z(B)$ instead of $Z(Spec(B))$. If moreover $Z := Spec(D)$ is an affine scheme, then $Z(B) = Hom_{R\text{-}Alg}(D, B)$ and the arrow $B \mapsto Z(B)$ defines a covariant functor from the category of R-Algebras to the category of sets.*

By Yoneda lemma, the data of a scheme is equivalent to the data of its functor of points. In particular, a group functor G represented by the Hopf algebra A is the functor of points of the affine scheme $Spec(A)$, we write $G = Spec(A)$.

2 Comodules

In order to give the algebraic definition of an action by an affine group scheme, we need to define what a comodule is. The reader finds in this section the vocabulary around this notion and the properties that we will use later in the proofs. For more details or for the reader interested in the non-commutative case, he can refer to [BW03, Part 1]. Henceforth, A is a Hopf algebra (A, Δ, ϵ, S) over R (as above).

2.1 Definitions and examples

Definition III.8. *1. A* ***right comodule*** *consists in a pair (M, ρ_M) where M is an R-module and $\rho_M : M \to M \otimes_R A$ is the structural morphism for the comodule, that is a*

R-linear morphism making the following diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \otimes_R A \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle Id_M \otimes \Delta} \\ M \otimes_R A & \xrightarrow[\rho_M \otimes Id_A]{} & M \otimes_R A \otimes_R A \end{array} \qquad \begin{array}{ccc} M & = & M \otimes_R R\,. \\ {\scriptstyle \rho_M}\downarrow & \nearrow{\scriptstyle Id_M \otimes \epsilon} & \\ M \otimes_R A & & \end{array}$$

2. *We have also the* ***Sigma notation*** *as the one for Hopf algebras (Remark III.3). More precisely, for any* $m \in M$*, we write* $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$*. Following this notation, the previous diagrams become*

$$\sum (m_{(0)})_{(0)} \otimes (m_{(0)})_{(1)} \otimes m_{(1)} = \sum m_{(0)} \otimes (m_{(1)})_1 \otimes (m_{(1)})_2$$

and

$$\sum \epsilon(m_{(1)}) m_{(0)} = m.$$

3. *We denote by* $\mathcal{M}^A$ *(resp.* ${}^A\mathcal{M}$*) the category of the right (resp. left) A-comodules.*

Definition III.9. *We say that* (B, ρ_B) *is an* ***A-comodule algebra*** *if B is unitary and its stucture of right A-comodule is compatible with its algebraic structure (That is,* $\rho_B(1_B) = 1_B \otimes 1_A$ *and* $\rho_B(ab) = \rho_B(a)\rho_B(b) = \sum a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}$*, for any* $a, b \in B$*).*

Example III.10. *A coalgebra A is an A-comodule via its comultiplication map* $\Delta : A \to A \otimes_R A$*.*

Remark III.11. *We define similarly the notion of left comodule with a structure morphism* ${}_M\rho : M \to A \otimes_R M$*. We write* ${}_M\rho(m) = \sum m_{(-1)} \otimes m_{(0)}$*, for all* $m \in M$*, in sigma notation.*

Given a R-module M, the following lemma defines on it a structure of right comodule from a structure of left comodule and conversely.

Lemma III.12. *Let M be an R-module. If M is a left A-comodule via the morphism* ${}_M\rho : M \to A \otimes_R M$ *defined by* $m \mapsto \sum m_{(-1)} \otimes m_{(0)}$*, then M is also a right A-comodule via the morphism* $\rho_M : M \to M \otimes_R A$ *defined by* $m \mapsto \sum m_{(0)} \otimes S(m_{(-1)})$*. Similarly, if M is a right A-comodule via the morphism* $\rho_M : M \to M \otimes_R A$ *defined by* $m \mapsto \sum m_{(0)} \otimes m_{(1)}$*, it becomes a left A-comodule via the morphism* ${}_M\rho : M \to A \otimes_R M$ *defined by* $m \mapsto \sum S(m_{(1)}) \otimes m_{(0)}$*.*

Proof. By [DNR01, Proposition 4.2.6]), the antipode of an Hopf algebra is an antimorphism. That is, for any $a \in A$, $\Delta(S(a)) = \sum S(a_2) \otimes S(a_1)$ ($\sharp$) and $\epsilon(S(a)) = \epsilon(a)$ ($\square$). Let M be a left A-comodule. For any $m \in M$,

$$\begin{aligned} (Id_M \otimes \epsilon) \circ \rho_M(m) &= (Id_M \otimes \epsilon)(\textstyle\sum m_{(0)} \otimes S(m_{(-1)})) \\ &= \textstyle\sum m_{(0)} \otimes \epsilon(S(m_{(-1)})) \\ &= \textstyle\sum m_{(0)} \otimes \epsilon(m_{(-1)}) \\ &= m \otimes 1, \ \text{(by } (\square)\text{)}. \end{aligned}$$

Moreover,

$$\begin{aligned}(Id_M \otimes \Delta) \circ \rho_M(m) &= (Id_M \otimes \Delta)(\sum m_{(0)} \otimes S(m_{(-1)})) \\ &= \sum m_{(0)} \otimes S((m_{(-1)})_2) \otimes S((m_{(-1)})_1) \text{ (By } (\sharp)) \\ &= (Id_M \otimes S \otimes S) \circ T \circ (\Delta \otimes Id_M) \circ {}_M\rho(m) \text{ (with } T(a \otimes b \otimes c) = c \otimes a \otimes b) \\ &= (Id_M \otimes S \otimes S) \circ T \circ (A \otimes {}_M\rho) \circ {}_M\rho(m) \text{ (since } M \text{ is a left comodule)} \\ &= \sum (m_{(0)})_{(0)} \otimes S((m_{(0)})_{(-1)}) \otimes S(m_{(-1)}) \\ &= (\rho_M \otimes A) \circ \rho_M(m).\end{aligned}$$

Whence M is a right A-comodule. Using the same arguments, we can establish the rest of the proof. □

2.2 A-comodule morphisms

Definition III.13. *1. Let (M, ρ_M) and (N, ρ_N) be two right A-comodules. An R-linear map $g : M \to N$ is called A-**comodule morphism** if $\rho_N \circ g = (g \otimes Id_A) \circ \rho_M$.*

2. Denote by $Com_A(M, N)$ the R-module of the A-comodule morphisms from M to N. By the definition, we infer that $Com_A(M, N)$ is characterized by the exact sequence of $\mathcal{M}_R$:

$$0 \longrightarrow Com_A(M, N) \longrightarrow Hom_R(M, N) \xrightarrow{\gamma} Hom_R(M, N \otimes_R A)$$

where γ is defined by $\gamma(f) := \rho_N \circ f - (f \otimes Id_A) \circ \rho_M$, for any $f \in Hom_R(M, N)$.

Proposition III.14. *Let $g : M \to N$ be an A-comodule morphism.*

1. *The image of g is an A-subcomodule of N*
2. *The kernel of g is an A-subcomodule of M.*
3. *The composite of two A-comodule morphisms is an A-comodule morphism.*

Lemma III.15. *Let A be a flat R-coalgebra and $M \in \mathcal{M}^A$. Then,*

1. *The functor $Com_A(-, M) : \mathcal{M}^A \to \mathcal{M}_R$ is left exact.*
2. *The functor $Com_A(M, -) : \mathcal{M}^A \to \mathcal{M}_R$ is right exact.*

Proof. Given an exact sequence $X \to Y \to Z \to 0$ in $\mathcal{M}^A$, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Com_A(Z,M) & \longrightarrow & Com_A(Y,M) & \longrightarrow & Com_A(X,M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Hom_R(Z,M) & \longrightarrow & Hom_R(Y,M) & \longrightarrow & Hom_R(X,M) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Hom_R(Z,M\otimes_R A) & \longrightarrow & Hom_R(Y,M\otimes_R A) & \longrightarrow & Hom_R(X,M\otimes_R A)
\end{array}$$

where the vertical arrows are exact by the definition of Com_A. Moreover, the exactness of the functor Hom_R induces the exactness of the second and the third rows. Thanks to the flatness of A over R, we obtain the exactness for the third one. By the snake lemma, the first row is also exact. Using the same arguments, we prove the second part of the lemma. □

2.3 Duality

Given M an R-module, we denote by $M^* := Hom_R(M, R)$ its dual. In order to endow a natural comodule structure on the dual of a comodule, we will need these two following lemmas:

Lemma III.16. *Let M be an R-module of finite presentation and let A be a flat module over R. The map $v_M : A \otimes_R M^* \to Hom_R(M, A)$ defined by $a \otimes h \mapsto h(-)a$ is an isomorphism.*

Proof. The result is clear for $M = R$ and $M = R^k$ where $k \in \mathbb{N}$. Consider some exact sequence

$$R^k \to R^n \to M \to 0,$$

where k and $n \in \mathbb{N}$. Applying the functor $A \otimes_R Hom_R(-, R)$ and $Hom_R(-, A)$ to this exact sequence and using the flatness of A over R, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A\otimes_R Hom_R(M,R) & \longrightarrow & A\otimes_R Hom_R(R^n,R) & \longrightarrow & A\otimes_R Hom_R(R^k,R) \\
 & & \downarrow{\scriptstyle v_M} & & \downarrow{\scriptstyle v_{R^n}} & & \downarrow{\scriptstyle v_{S^k}} \\
0 & \longrightarrow & Hom_R(M,A) & \longrightarrow & Hom_R(R^n,A) & \longrightarrow & Hom_R(R^k,A)
\end{array}$$

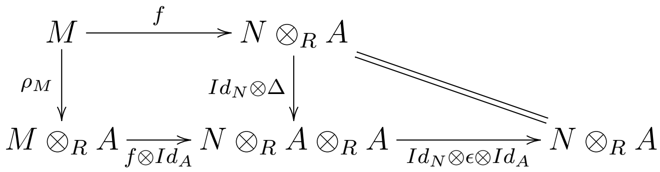
The result follows since v_{R^k} et v_{R^n} are isomorphisms. □

Lemma III.17. *Let N be an R-module. For any $M \in \mathcal{M}^A$, the R-linear map*

$$\begin{array}{rrcl}
\phi_N : & Com_A(M, N\otimes_R A) & \to & Hom_R(M,N) \\
 & f & \mapsto & (Id_N \otimes \epsilon)\circ f
\end{array}$$

is bijective and its inverse ψ_N *is defined by* $h \mapsto (h \otimes Id_A) \circ \rho_M$.

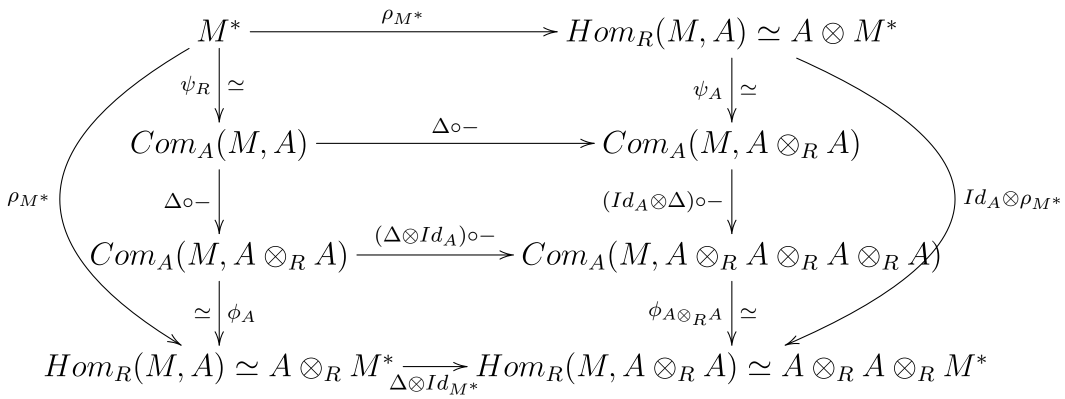
Proof. For $f \in Com_A(M, N \otimes_R A)$, the following diagram commutes:



In other words, $f = (Id_N \otimes \epsilon \otimes Id_A) \circ (f \otimes Id_A) \circ \rho_M = (\phi_N(f) \otimes Id_A) \circ \rho_M$. From this, we can deduce the injectivity of ϕ_N. Moreover, since ρ_M is an A-comodule map, $(h \otimes Id_A) \circ \rho_M$ also, for any $h \in Hom_R(M, N)$. Thus, $\phi_N((h \otimes Id_A) \circ \rho_M) = (Id_N \otimes \epsilon) \circ (h \otimes Id_A) \circ \rho_M = h \circ (Id_M \otimes \epsilon) \circ \rho_M = h$ hence ϕ is surjective. □

Lemma III.18. *Let* A *be a flat* R*-coalgebra and* M *be a right* A*-comodule finitely presented as an* R*-module. Then the dual* M^* *can be endowed with a structure of left* A*-comodule via the morphism* ${}_{M^*}\rho : M^* \to Hom_R(M, A) \simeq A \otimes_R M^*$ *defined by* $g \mapsto (g \otimes Id_A) \circ \rho_M$.

Proof. The following commutative diagram insures that ρ_{M^*} endows M^* with a structure of comodule:



The isomorphisms in the diagram were defined in the two previous lemmas. In this diagram, the Hopf algebra A (resp. $A \otimes_R A$, $A \otimes_R A \otimes_R A$) is seen as A-comodule via Δ (resp. $Id_A \otimes \Delta$, $Id_A \otimes Id_A \otimes \Delta$). The commutativity of the middle square follows from the coassociativity on A. The commutativity of the top square follows from the A-comodule property for M, $(Id_M \otimes \Delta) \circ \rho_M = (\rho_M \otimes Id_A) \circ \rho_M$. The commutativity of the bottom square is obvious. Thus, to conclude, it is enough to prove that $\rho_{M^*} = \phi_A \circ \Delta \circ \psi_R$ and $Id_A \otimes \rho_{M^*} = \phi_{A \otimes A} \circ (Id_A \otimes \Delta) \circ \psi_A$. Indeed, for any $g \in M^*$ and $a \in A$, we have:

$$\phi_A \circ \Delta \circ \psi_R(g) = (Id_A \otimes \epsilon) \circ \Delta \circ (g \otimes Id_A) \circ \rho_M = (g \otimes Id_A) \circ \rho_M = \rho_{M^*}(g)$$

and

$$\phi_{A\otimes A}\circ(Id_A\otimes\Delta)\circ\psi_A(a\otimes g) = (Id_A\otimes Id_A\otimes\epsilon)\circ(Id_A\otimes\Delta)\circ(g(-)a\otimes Id_A)\circ\rho_M = Id_A\otimes\rho_{M^*}(a\otimes g).$$

□

2.4 (B, A)-modules and modules of invariants

In this section, B is a right A-comodule algebra.

Definition III.19. *1. We denote by ${}_B\mathcal{M}^A$ the category of* ***(B,A)-modules****. An object of ${}_B\mathcal{M}^A$ is R-module N with a structure of left B-module and right A-comodule such that the A-comodule structural morphism ρ_B is B-linear, (that is, $\rho_N(bn) = \rho_B(b)\rho_N(n) = \sum b_{(0)}n_{(0)}\otimes b_{(1)}n_{(1)}$, for any $n\in N$ and any $b\in B$). A morphism of ${}_B\mathcal{M}^A$ is a morphism which is simultaneously a B-linear map and A-comodule morphism. For any M and any N in ${}_B\mathcal{M}^A$, we write ${}_BBim^A(M,N)$ for the set of these morphisms from M to N.*

2. Let M be a (B, A)-module. ***The module of invariants of*** *M, denoted by $(M)^A$, is defined by the following exact sequence:*

$$0\to(M)^A\to M \underset{M\otimes 1}{\overset{\rho_M}{\rightrightarrows}} M\otimes_R A$$

This defines a functor $(-)^A : {}_B\mathcal{M}^A\to {}_C\mathcal{M}$ (which is left exact by definition) called the ***functor of invariants for the action*** *(X, G).*

Lemma III.20. *For any $M\in {}_B\mathcal{M}^A$, there is an R-isomorphism $\mu_M : {}_BBim^A(B, M)\to M^A$ which maps f to $f(1_B)$. Its inverse map ω_M sends m to the map $b\mapsto b.m$.*

Proof. For any $f\in {}_BBim^A(B, M)$ and any $b\in B$, $\omega_M(\mu_M(f))(b) = bf(1_B) = f(b)$ since f is B-linear. Conversely, for any $m\in M$, $\mu_M(\omega_M(m)) = 1_B.m = m$. □

Lemma III.21. *Setting $C := B^A$ and $A_C := C\otimes_R A$, we have ${}_B\mathcal{M}^A = {}_B\mathcal{M}^{A_C}$.*

Proof. The coalgebra structure of A_C is given by $Id_C\otimes\Delta$. For some $M\in {}_B\mathcal{M}^A$, the structural morphism $\rho_M : M\to M\otimes_C A_C\simeq M\otimes_R A$ given M a comodule structure is B-linear by definition, so in particular, C-linear endowing a A_C-comodule structure on M. The converse is clear. □

We can prove readily the following lemmas:

Lemma III.22. *For any left A-comodule M, $B\otimes_R M\in {}_B\mathcal{M}^A$. More precisely, $B\otimes_R M$ is a left B-module via multiplication on the first factor and a right A-comodule via the morphism $\rho_{B\otimes_R M} := B\otimes_R M\to(B\otimes_R M)\otimes_R A$ defined by $b\otimes m\mapsto\sum b_{(0)}\otimes m_{(0)}\otimes b_{(1)}S(m_{(-1)})$.*

Lemma III.23. *The following map is an R-module isomorphism:*

$$\begin{array}{rlll}\psi: & Com_A(A,B) & \to & {}_B Bim^A(B\otimes_R A, B)\\ & \phi & \mapsto & \psi(\phi)=[b\otimes a\mapsto \sum b_{(0)}\phi(S(b_{(1)})a)]\end{array}$$

Its inverse map sends $F\in {}_B Bim^A(B\otimes_R A, B)$ *to* $[a\mapsto F(1_B\otimes a)]\in Com_A(A,B)$.

Lemma III.24. *Suppose that the Hopf algebra A is flat over R. Let* $B\in \mathcal{M}^A$ *and* $M\in {}_R\mathcal{M}$ *be finitely presented as R-modules. There is a natural (functorial on M) isomorphism:*

$$(B\otimes_R M^*)^A\simeq Com_A(M,B)$$

Proof. The commutativity of the right square of the diagram below insures the existence of λ which is an isomorphism and so α and γ as well.

$$\begin{array}{ccccccc}0\longrightarrow & (B\otimes_R M^*)^A & \longrightarrow & B\otimes_R M^* & \xrightarrow{\phi} & (B\otimes_R M^*)\otimes_R A\\ & \downarrow\lambda & & \downarrow\alpha & & \downarrow\gamma\\ 0\longrightarrow & Com_A(M,B) & \longrightarrow & Hom_R(M,B) & \xrightarrow[\psi]{} & Hom_R(M,B\otimes_R A)\end{array}$$

where $\phi:=\rho_{B\otimes_R M^*}-Id_B\otimes Id_{M^*}\otimes 1$, $\psi:=(\rho_B\circ -)-\rho_{M^*}$ and for any $b\otimes f\otimes a\in B\otimes_R M^*\otimes_R A$ and $m\in M$, $\gamma(b\otimes f\otimes a)(m)=\sum b.f(m_{(0)})\otimes a.m_{(-1)}$
Indeed, the map γ is an isomorphism as composite of the isomorphism $\gamma_1: B\otimes_R M^*\otimes_R A\to B\otimes_R A\otimes_R M^*$ defined for any $b\otimes f\otimes a\in B\otimes_R M^*\otimes_R A$ by $\gamma_1(b\otimes f\otimes a)=b\otimes a.m_{(-1)}\otimes f(m_{(0)})$ (Its inverse map is defined for any $b\otimes a\otimes f\in B\otimes_R A\otimes_R M^*$ by $\gamma_1^{-1}(b\otimes a\otimes f)=\sum b\otimes f(m_{(0)})\otimes a.S(m_{(-1)})$) with the canonical isomorphism $\gamma_2: B\otimes_R A\otimes_R M^*\to Hom_R(M, B\otimes_R A)$. For any $b\otimes f\in B\otimes_R M^*$, we have

$$\psi\circ\alpha(b\otimes f) \;=\; \psi([m\mapsto bf(m)])=[m\mapsto \rho_B(b)f(m)-\sum bf(m_{(0)})\otimes m_{(-1)}]$$

and

$$\begin{aligned}\gamma\circ\phi(b\otimes f) &= \gamma([m\mapsto \textstyle\sum b_{(0)}\otimes f(m_{(0)})\otimes b_{(1)}S(m_{(-1)})-b\otimes f(m)\otimes 1])\\ &= [m\mapsto (b_{(0)}\otimes b_{(1)}S(m_{(1)})(m_{(0)})_{(-1)})f((m_{(0)})_{(0)}-\textstyle\sum bf(m_{(0)}\otimes m_{(-1)})]\\ &= [m\mapsto \textstyle\sum(b_{(0)}\otimes b_{(1)}\epsilon(m_{(-1)}))f(m_{(0)})-\sum bf(m_{(0)})\otimes m_{(-1)}]\\ &= [m\mapsto \rho_B(b)f(m)-\textstyle\sum bf(m_{(0)})\otimes m_{(-1)}].\end{aligned}$$

□

2.5 Comodules and linear representations

The reader can refer to [Wat79, Chapter 3] for this section.

Definition III.25. *Let G be a R-group functor and M be a R-module. Consider the functor X which maps any R-algebra T to $X(T) = M \otimes_R T$ and let GL_M denote the R-group functor which sends an R-algebra T to $GL_M(T) = Aut_T(M \otimes_R T)$. A* ***linear representation of** G* ***on** M is the data (funtorial on T) of a T-linear action of $G(T)$ on $X(T)$, for any R-algebra T. In other words, a linear representation of G on M is characterized by the data of a homomorphism $\delta : G \to GL_M$. We denote such representation by (M, δ) or simply M if there is no confusion possible.*

Definition III.26. *Let (M, δ) be a linear representation of G.*

1. *A submodule N of M is said* ***G-invariant*** *if $\delta(T)(g)(N \otimes T) = N \otimes T$, for any R-algebra T and any $g \in G(T)$. In this case, $(N, \delta|_N)$ is a linear representation called* ***subrepresentation of*** *(M, δ).*
2. *A representation with no non trivial subrepresentations is called an* ***irreducible representation****.*
3. *Let (N, ϕ) be another representation of G. The* ***direct sum*** *of (M, δ) and (N, ϕ) is a representation defined by the action $g.(n, m) = (g.n, g.m)$, for any R-algebra T, any $g \in G(T)$ and any $m, n \in M \otimes_R T$.*
4. *A representation which is the direct sum of irreducible representations is called a* ***semisimple representation****.*

Theorem III.27. *Let G be an affine group scheme over R represented by the Hopf algebra A. The data of a linear representation of G on M is equivalent to the data of an A-comodule structure on M.*

Proof. Let (M, δ) be a representation. From $Id_A \in G(A)$, we obtain an A-linear map $\delta(A)(Id_A) : M \otimes_R A \to M \otimes_R A$. Take as ρ_M its restriction to $M \simeq M \otimes_R R$. By Yoneda lemma, for any $g : A \to T$ in $G(T)$ where T is an R-algebra, the following diagram is commutative:

$$\begin{array}{ccc} M \otimes_R A & \xrightarrow{\delta(A)(Id_A)} & M \otimes_R A \\ \downarrow{\scriptstyle Id_M \otimes g} & & \downarrow{\scriptstyle Id_M \otimes g} \\ M \otimes_R T & \xrightarrow[\delta(T)(g)]{} & M \otimes_R T \end{array}$$

Taking in this diagram, g equal to ϵ (resp. Δ) using Hopf algebra counity (resp. coassociativity) property for A, we prove that ρ_M gives M a structure of comodule.
Now, let M be an A-comodule with structural map $\rho_M : M \to M \otimes_R A$. We define naturally a map

$$\begin{array}{rccl} \delta(T) : & G(T) & \to & End_T(M \otimes_R T) \\ & g & \mapsto & (Id_M \otimes g) \circ \rho_M \otimes Id_T \end{array}$$

To insure that δ is a representation, it is enough to show that for any R-algebra T and $g, h \in G(T)$,

$$\delta(T)(g) \circ \delta(T)(h) = \delta(T)(gh)$$

Since gh is given by $\Delta \circ (g, h)$,

$$\delta(T)(gh) = (Id_M \otimes \Delta \circ (g, h)) \circ \rho_M \otimes Id_T$$

and

$$\delta(T)(g) \circ \delta(T)(h) = (Id_M \otimes g) \circ \rho_M \otimes Id_T) \circ ((Id_M \otimes h) \circ \rho_M \otimes Id_T)$$

Finally, the result follows from the equality $(Id_M \otimes \Delta) \circ \rho_M = (\rho_M \otimes Id_A) \circ \rho_M$ satisfied by the A-comodule M. □

Example III.28. *An important example is obtained by taking $M = A$ and $\rho_M = \Delta$. The corresponding representation is called* ***the regular representation of*** *G.*

Direct sums and tensor products of linear representations are also linear representations so the corresponding constructions are comodules.

Definition III.29. *Let M be an A-comodule.*

1. *An R-submodule N of M is a* ***subcomodule*** *if $\rho_M(N) \subseteq N \otimes_R A$.*
2. *If N is a subcomodule of M, the composite $M \to M \otimes A \to \big(M/N\big) \otimes_R A$ factors through M/N, and then M/N becomes a comodule called the* ***quotient comodule****.*
3. *A comodule with no non trivial subcomodules is called* ***simple*** *(The corresponding representation is irreducible).*
4. *A comodule which is the direct sum of simple comodules is called* ***semisimple****.*
5. *A coalgebra A such that any right A-comodule is semisimple is called* ***cosemisimple****.*

Lemma III.30. *Let M be an R-module, let $G = Spec(A)$ be an affine group scheme over R and let (M, δ) be a linear representation of G on M. Then (M, δ) is semisimple if and only if for any G-invariant R-submodule N of M, there is a G-invariant R-submodule of M which is a complement of N in M. Or, in other words, by Theorem III.27, an A-comodule M is semisimple if and only if any A-subcomodule of M is a direct factor of M as A-comodule.*

We end this section with Maschke theorem (see [Spr89, II, 1.2]) which will be useful for the following, but first we will establish a lemma needed in the proof:

Lemma III.31. *Let (M, δ) be a representation of a group G of finite order over a field K of characteristic zero or prime to the order of G. If N is a stable vector subspace of some representation (M, δ), then there is a complement subspace of N in M stable for this representation.*

Proof. Denote by N_q some complement of N and by p the projector over N along N_q. Then, consider the linear map $p_s : M \to N$ defined by $p_s = \frac{1}{g}\sum_{t\in G}\delta(t)\circ p\circ\delta(t^{-1})$ where g denotes the order of the group G. We will show that p_s is a projector. The restriction of p_s to N is equal to the identity, since the restriction of p to N is equal to the identity and N is stable under $\rho(t)$. Moreover, the image of p_s is equal to N. Now we will prove that, if N_s is the kernel of p_s, then it is stable by the representation. We first notice that if u is an element of the group, the endomorphism of G, which maps t to ut is a permutation. We have

$$\begin{array}{rcl} p_s & = & \frac{1}{g}\sum_{t\in G}\delta(u)\circ\delta(t)\circ p\circ\delta(t^{-1})\circ\delta(u^{-1}) \\ & = & \delta(u)\circ\left(\frac{1}{g}\sum_{t\in G}\delta(t)\circ p\circ\delta(t^{-1})\right)\circ\delta(u^{-1}) = \delta(u)\circ p_s\circ\delta(u^{-1}). \end{array}$$

Let k be in the kernel of p_s, then $\delta(u)(k)$ is also in this kernel. In fact,

$$\begin{array}{rcl} p_s(\delta(u)(k)) & = & (\delta(u)\circ p_s\circ\delta(u^{-1}))(\delta(u)(k)) \\ & = & (\delta(u)\circ p_s)(\delta(u^{-1})\circ\delta(u))(k)) = \delta(u)(p_s(k)) = \delta(u)(0) = 0. \end{array}$$

To conclude, N_s is a complement of N as a kernel of a projector map over N and it is stable by the representation therefore the lemma is proved. □

Theorem III.32 (Maschke)**.** 1. ***Representation of a finite group:*** *Let (M,δ) be a representation of a finite group G over a field K of characteristic zero or prime to the order of G, then M is the direct sum of irreducible subspaces. In other words, any G-module over a field K of characteristic zero or prime to the order of G is semisimple.*

2. ***Group algebra of a finite group:*** *If K is a field of characteristic zero or prime to the order of G with G finite group, then the group algebra $K[G]$ is semisimple.*

Proof. 1. We argue by induction on the dimension n. For $n = 1$ there is no non-trivial vector subspace, so the theorem is obviously verified. Suppose that the theorem is true for any vector space of dimension less than n. If the representation is irreducible, then the theorem is verified. Otherwise, there is a stable subspace N of dimension strictly less than n. The lemma guarantees the existence of a stable complement N_s of N in M, also of dimension strictly less than n. By the induction hypothesis, N and N_s are direct sums of stable subspaces for the representation. The fact that N and N_s are complements of each other permits to conclude. This proof works also for G-modules.

2. We want to prove that the kernel E of p_s is an ideal. Let $a \in E$ and $r \in K[G]$. Since p_s commutes with all the elements of G, p_s commutes also with all the elements of $K[G]$, we have $p_s(ar) = p_s ar = (p_s a)r = 0r = 0$ and $p_s(ra) = (p_s r)a = (rp_s)a = r(p_s a) = 0$. That concludes the proof.

□

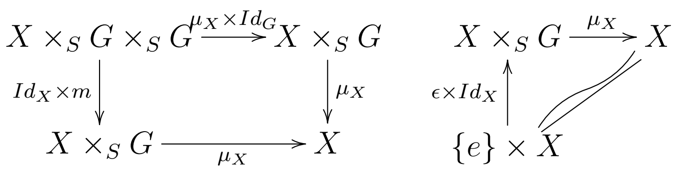
3 Action by an affine group scheme

In this section, we recall some basic notions connected to the notion of action by a group scheme and important results for our problem. The reader can refer mainly to [DG70], [MFK94] and [Wat79], for more details. We keep the same notations of the previous section; $G := Spec(A)$ is an affine flat group scheme over S and $X := Spec(B)$ is an affine scheme over S.

3.1 Definitions

Definition III.33. *An* ***action*** *of G on X over S, denoted by (X, G), can be defined following the three different points of view of seeing an affine group scheme (scheme, Hopf algebra, functor):*

1. ***(As schemes)*** *An action (X, G) is a morphism of S-schemes $\mu_X : X \times_S G \to X$ making the following diagrams commute:*



2. ***(As Hopf algebra)*** *An action (X, G) is the data of a structure of A-comodule algebra on B.*

3. ***(As functor of points)*** *An action (X, G) is a functorial data on T of actions of the abstract groups $G(T)$ on the sets $X(T)$, for any S-scheme T.*

This three points of view are equivalent. The equivalence between (1) and (2) follows from the Yoneda lemma. For the equivalence between (1) and (3), if $\mu_X : X \times_S G \to X$ is a S-scheme morphism, we take $\mu_X \circ -$ for the corresponding functor $\psi_X : X \times_S G \to X$. Conversely, if $\psi_X : X \times_S G \to X$ is a functor, we can consider the image of the identity via the isomorphism $(X \times_S G)(X \times_S G) \simeq X(X \times_S G) \times_S G(X \times_S G)$, and take $\psi_X(Id_{X \times_S G})$ to be the corresponding S-scheme morphism $\mu_X : X \times_S G \to X$.
We have naturally the notion of invariant and equivariant morphism.

Definition III.34. *Let X' be another affine scheme over S and G' be an affine group scheme over S together with an action (X', G'). We denote by $\mu_{X'} : X' \times_S G' \to X'$ its structural map.*

1. *A morphism $\phi : X \to X'$ is called G-**equivariant** if $\phi \circ \mu_X = \mu_{X'} \circ (\phi \times Id_G)$.*

2. *A morphism $\phi : X \to X'$ is called G-**invariant** if $\phi \circ \mu_X = \phi \circ p_1$.*

We take the opportunity to define the notion of orbit in this context.

Definition III.35. *Suppose that G is a flat, finitely presented affine group scheme over S. Let (X, G) be an action of G on X. For a T-point of X $f : T \to X$,* ***the orbit of*** *f, denoted by $O(f)$, is the scheme-theoretic image (see Definition A.4) of the S-scheme morphism*

$$(\sigma \circ (Id_G \times f), p_2) : G \times_S T \to X \times_S T$$

3.2 Quotients

3.2.1 Definitions

We will denote by $\mathcal{C}$ either the category of S-schemes or the category of S-algebraic spaces.

Definition III.36.

1. *Let (X, G) be an action and consider the pair of maps*

$$X \times_S G \underset{p_1}{\overset{\mu_X}{\rightrightarrows}} X$$

where μ_X is the action map and p_1 the projection. If the cokernel of this pair of maps exists in the category $\mathcal{C}$, then we say that (X, G) admits a ***categorical quotient in the category*** *$\mathcal{C}$, denoted X/G. The quotient X/G can be characterized up to isomorphism by the existence of a map $\phi : X \to X/G$, called the* ***quotient map****, such that:*
   - *(a) $\phi \circ \mu_X = \phi \circ p_1$ (In other terms, "ϕ is constant on the orbits")*
   - *(b) For any morphism of $\mathcal{C}$, $\psi : X \to Z$ with $\psi \circ \mu_X = \psi \circ p_1$, there is a unique morphism $\chi : X/G \to Z$ such that $\chi \circ \phi = \psi$.*
2. *A quotient Y is called* ***universal quotient in*** *$\mathcal{C}$ if, for any morphism of $\mathcal{C}$, $Y' \to X/G$, Y' is a categorical quotient in $\mathcal{C}$ for the action $(X_{Y'}, G_{Y'})$ and the projection $X_{Y'} := X \times_{X/G} Y' \to Y'$ is the associated quotient map.*
3. *A quotient X/G is called* ***geometric quotient*** *if the following properties are satisfied:*
   - *(a) the quotient map $\phi : X \to X/G$ is G-invariant;*
   - *(b) the geometric fibers of the quotient map $\phi : X \to X/G$ are precisely the orbits of the geometric points of X. (In particular, ϕ is surjective);*
   - *(c) the quotient map ϕ is submersive (that is, any subscheme M' of M such that $\phi^{-1}(M')$ is closed in X is closed in M).*
   - *(d) $\phi_*(\mathcal{O}_X)^G \simeq \mathcal{O}_{X/G}$.*
4. *A quotient Y is called* ***the fppf quotient*** *if it corresponds to the sheaf associated to the presheaf $T \mapsto X(T)/G(T)$, for the fppf topology.*

Proposition III.37. *[MFK94, proposition 0.1] A geometric quotient is a categorical quotient in the category of S-schemes. In particular, if $\phi : X \to X/G$ is a geometric quotient map and if $g : X \to Z$ is a G-invariant morphism, then there is a unique morphism $f : X/G \to Z$ such that $g = f \circ \phi$.*

Remark III.38. 1. *If a geometric quotient exists then it is unique in the category of schemes. Indeed, a categorical quotient in the category of S-schemes is unique in this category but not necessarily in the category of algebraic spaces.*

2. *Let (X, G) be an action and X/G be a categorical quotient for this action. The data of an action (X, G) over S is the same as the data of an action $(X, G_{X/G})$ over X/G where $G_{X/G} := G \times_S X/G$.*

3.2.2 Quotient by a finite locally free group scheme

Theorem III.39. *Suppose that G is finite and locally free over S and let (X, G) be an action over S. Denote by $C := B^A$ the ring of the invariants and by $q : X \to Spec(C)$ the natural morphism induced by the inclusion $C \subset B$. Then*

1. *The scheme $Spec(C)$ is a geometric quotient. Moreover, it is categorical in the category of algebraic spaces.*
2. *The inclusion morphism $C \to B$ is integral and the quotient map q is quasi-finite, closed and surjective. In particular, B is a finite C-algebra.*
3. *In addition, the Galois map $(\mu_X, p_1) : X \times_S G \to X \times_Y X$ is surjective. In particular, for any point $\xi : Spec(T) \to X$,*
$$O(\xi) = p^{-1}(\xi).$$
In other words, the action is ***transitive****.*

Proof. We can find the proof of this classical result for example in [DG70, III, §2, n°3]. Only the fact that the quotient is categorical in the category of algebraic spaces is a little less classical but the reader can find a detailed proof in [Con05, §3]. □

Corollary III.40. *Let H be an affine, finite, locally free normal subgroup scheme of G supposed flat.*

1. *In this case, the quotient G/H is the fppf quotient and representable by a group scheme over S.*
2. *Moreover, an action (X, G) over S induces naturally an action $(X/H, G/H)$ over S.*

3.3 Induced actions

Definition III.41. *We denote by H a flat closed subgroup of G. Suppose that H acts on a scheme Z over S. We define an action of the group H on the scheme $Z \times_S G$ on the right via $(z, g)h = (zh, h^{-1}g)$. Suppose that the categorical quotient for this action exists in the category of schemes. Denote it $Z \times^H G := (Z \times G)/H$, this scheme is called* ***balanced product of** Z **by** G. The group scheme G acts on this scheme via the factor G. We say that* ***the action** (X, G) **is induced by the action** (Z, H) if (X, G) is isomorphic to $(Z \times^H G, G)$. That is, there exists a G-equivariant isomorphism $Z \times^H G \simeq X$.*

We take the opportunity to mention the following lemma.

Lemma III.42. *For H a flat closed subgroup of G, there is a G-equivariant isomorphism*

$$H \times^H G \simeq G$$

where $H \times^H G$ denotes the balanced product.

Proof. We can show easily that the morphism $H \times_S G \to G$ that maps (h, g) to hg induces the required isomorphism. □

Lemma III.43. *[CEPT96, Lemma 1.2] If the quotient $Z \times^H G$ exists as an universal categorical quotient scheme, then it represents the fppf-sheaf, which, on an R-algebra T, takes the value $Z(T) \times G(T)/H(T)$.*

3.4 Free actions and torsors

Definition III.44.
1. *The morphism $(\mu_x, pr_1) : X \times_S G \to X \times_S X$ which maps (x, g) to $(g.x, x)$ is called the* ***Galois map****.*
2. *An action (X, G) is said to be* ***without fixed points or free*** *if the morphism (μ_x, pr_1) is a monomorphism.*

Definition III.45. *Let U be a scheme. A U-scheme Z with an action of G is a* ***torsor*** *over U under the action of G, if:*

1. *Z is faithfully flat and quasi-compact over U,*
2. *The morphism (μ_Z, pr_1) is an isomorphism.*

Torsors are often called ***principal homogeneous spaces****.*

Proposition III.46. *([DG70, III, §4, n°1, 1.3 and 1.9]) Let U be a S-scheme and Z a S-scheme. The following assertions are equivalent:*

1. *An U-scheme Z with an action of G is* ***torsor*** *over U under the action of G.*
2. (a) *$Z \to U$ is surjective as morphism of sheaves for the fppf topology.*
   (b) *The morphism $(\mu_Z, pr_1) : Z \times_S G \to Z \times_U Z$ is an isomorphism.*

We can relate free actions and quotients with torsors as follows:

Proposition III.47. *If an action (X, G) defines a torsor over Y, then:*

1. *the map $X \to Y$ is a quotient for the action (X, G);*
2. *the group scheme G acts freely on X.*

Proof. 1. See [DG70, III & IV].

2. This is part of condition *b.* of the definition of torsor.

□

Conversely, for actions of finite, locally free group schemes, free actions define torsors.

Theorem III.48. *(see [DG70, III, §2, n°3]) Suppose that G is finite, locally free over S and let (X, G) be an action over S. Denote by $q : X \to Spec(B^A)$ the canonical map associated with the inclusion $B^A \subset B$. If we suppose that the action is free then q is faithfully flat, finitely presented and the Galois map induces an isomorphism $X \times_S G \simeq X \times_Y X$, in other words, X is a G-torsor over Y.*

3.5 Slices

Definition III.49. *We say that the action (X, G) over S admits étale (respectively fppf) slices if:*

1. *There is a categorical quotient Y in the category of the algebraic spaces.*
2. *For any $y \in Y$, there are:*
   (a) *a scheme $Y' := Y'(y)$ and an étale (resp. fppf) S-morphism $Y' \to Y$ which contains y in its image.*
   (b) *a closed subgroup $G_0 := G_0(y)$ of $G_{(Y')}$ over Y' which stabilizes some point x of X above y, (this means that $G_{0k(x)} \simeq I_G(x)$).*
   (c) *a Y'-scheme $Z := Z(y)$ with a G_0-action such that $Y' = Z/G_0$ and the action $(X \times_Y Y', G_{Y'})$ is induced by (Z, G_0).*

   *The subgroup G_0 is called a **slice group**.*

Remark III.50. *1. Roughly speaking, for the étale (respectively fppf) topology, an action which admits slices can be described by the action of a stabilizer of a point.*

2. *The action $(X_{Y'}, G_{Y'})$ should be thought of as a G-stable neighborhood of an orbit. Such a neighborhood induced from an action (Z, H) where H is the stabilizer of a point is called a **"tubular neighborhood"***

3.6 Equivariant sheaves

Let $\mu_X : X \times_S G \to X$ be the structural map of the action and $m : G \times_S G \to G$ be the multiplication map for the action.

Definition III.51. *A G-**equivariant sheaf** $\mathcal{F}$ **on** X is a sheaf with an action of G. More precisely, it is a pair $(\mathcal{F}, \beta)$ where $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, and β is a $\mathcal{O}_{G \times_S X}$-module map $\beta : {\mu_X}^* \mathcal{F} \to {pr_1}^* \mathcal{F}$ such that:*

1. *the diagram*

$$\begin{array}{ccc} (Id_G \times \mu_X)^* {pr_2}^* \mathcal{F} & \xrightarrow{pr_{23}^* \beta} & pr_2^* \mathcal{F} \\ {\scriptstyle (Id_G \times \mu_X)^* \beta} \uparrow & & \uparrow {\scriptstyle (m \times Id_X)^* \alpha} \\ (Id_G \times \mu_X)^* \mu_X^* \mathcal{F} & = & (m \times Id_X)^* \mu_X^* \mathcal{F} \end{array}$$

   is commutative in the category of $\mathcal{O}_{G \times_S G \times_S X}$*-modules;*

2. *the pullback*

$$(\epsilon \times Id_X)^* \beta : \mathcal{F} \to \mathcal{F}$$

   is the identity map.

Remark III.52. *For an explanation, compare with the relevant diagrams of the definition of an action (Chapter III, Section 3, §3.1).*

There is naturally a definition of an equivariant morphism of equivariant sheaves.

Definition III.53. *Let* $(\mathcal{F}, \beta)$ *and* $(\mathcal{F}', \beta')$ *be* G*-equivariant sheaves on* X*. An* ***equivariant morphism*** $\alpha : \mathcal{F} \to \mathcal{F}'$ *is a morphism of sheaves such that the diagram:*

$$\begin{array}{ccc} {\mu_X}^* \mathcal{F} & \xrightarrow{{\mu_X}^* \alpha} & {\mu_X}^* \mathcal{F}' \\ {\scriptstyle \beta} \downarrow & & \downarrow {\scriptstyle \beta'} \\ {pr_1}^* \mathcal{F} & \xrightarrow[{pr_1}^* \alpha]{} & {pr_1}^* \mathcal{F}' \end{array}$$

is commutative in the category of $\mathcal{O}_{G \times_S X}$*-modules.*

We denote by $Qcoh^G(X)$ the category of G-equivariant quasi coherent sheaves on X.

Chapter IV

Quotient stacks

In this chapter, G denotes a flat group scheme over S and X denotes a scheme over S. We consider an action (X, G) over S whose structural map is denoted by $\mu_X : X \times_S G \to X$. For the readers' convenience, we have reviewed in Appendix B all the material on stacks required for understanding this thesis. This chapter introduces the notion of quotient stack and some basic properties that one can find in the literature (Project stacks, [Wan11]...). Some of the folklore results are stated without proof. Moreover, section 9 "Quasi-coherent sheaves over quotient stacks" is a generalization of [AOV08] §2.1.

1 Basic definitions

Definition IV.1. *We define **the groupoid quotient** $[X/G]$ **associated to the action** (X, G) in the following way. For U an S-scheme,*

1. *the sections (i.e. the objects) of $[X/G]$ over U are G-torsors $E \to U$ provided with G-equivariant maps $f : E \to X$;*
2. *a morphism from an object $E \to U$ provided with the G-equivariant map $f : E \to X$ to the object $E' \to U'$ provided with the G-equivariant map $f' : E' \to X$ is a cartesian diagram*

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array}$$

   where g is a G-equivariant map such that $g \circ f = f'$.

*In particular, if Z is a S-scheme, we write $B_Z G = [Z/G]$ for the quotient stack associated to the trivial action of G on Z and we call it the **classifying stack of** G **over** Z.*

Remark IV.2.

1. *Let Y be the quotient for the action (X, G). We have $[X/G] \simeq [X/G_Y]$ (see remark III.38).*
2. *A morphism from an object $E \to U$ provided with the G-equivariant map $f : E \to X$ to the object $E' \to U$ provided with the G-equivariant map $f' : E' \to X$ is always an isomorphism*

by [DG70, III, §4, n°4, proposition 1.4] and this shows that the fiber $[X/G](U)$ is a groupoid.

More precisely, the groupoid $[X/G]$ is a stack.

Proposition IV.3. *The groupoid $[X/G]$ defined above is a stack called* ***the quotient stack associated to the action*** *(X, G).*

Proof. Let e, e' be sections of $[X/G](U)$ corresponding respectively to a torsor $E \to U$ provided with the G-equivariant map $f : E \to X$ and a torsor $E' \to U'$ provided with the G-equivariant map $f' : E' \to X$. Then $\underline{Iso}_U(e, e')$ is the fppf sheaf given by the quotient $(X \times_{X \times_S X} E \times_U E')$ by the free product action of G. Moreover, descent theory shows that this sheaf is in fact a scheme.
When $E = E'$ and $f = f'$ the isomorphisms correspond to elements of G which preserve f. In other words $\underline{Iso}_U(e, e)$ is the stabilizer of the G-map $f : E \to X$.
Since any torsor $E \to U$ is locally trivial in the fppf topology it determines a descent data as follows: Let $\{p_i : U_i \to U\}$ be an étale cover on which $E \to U$ is trivial. Then we have an equivariant isomorphism $\phi_i : p_i^* E \to U_i \times_S G$, for any i. If $\phi_{i,j}$ is the pullback of ϕ_i to $U_i \times_U U_j$, then the $\phi_{i,j}$'s satisfy the cocycle condition i.e. $\phi_{i,j} \circ \phi_{i,k} = \phi_{j,k}$.
Descent theory gives us the opposite direction. Given torsors (not necessary trivial) $E_i \to U_i$ and isomorphisms $E_i|_{U_i \times_U U_j} \to E_j|_{U_i \times_U U_j}$ satisfying the cocycle condition, there is a torsor $E \to U$ such that $E_i = p_i^* E$, for any i. This is condition 2. in Definition B.9 of a stack. □

Definition IV.4. *We can naturally define a morphism $p : X \to [X/G]$, that we call* ***the quotient stack map****, to be the morphism corresponding, by the Yoneda lemma B.11, to the trivial G-torsor $pr_1 : X \times_S G \to X$ provided with the G-equivariant morphism $\mu_X : X \times_S G \to X$. More precisely,*

1. *given an object $u : U \to X$ of $X(U)$, we define $u^* p$ to be the trivial G-torsor $U \times_S G \to U$ provided with the equivariant map $\mu_X \circ (u \times Id_G) : U \times_S G \to X$;*

2. *given $f : U' \to U$ a morphism from an object $U' \to X$ of $X(U')$ to an object $U \to X$ of $X(U)$, we define $f^* p$ to be the cartesian diagram*

$$\begin{array}{ccc} U' \times_S G & \xrightarrow{f \times Id_G} & U \times_S G \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_1} \\ U' & \xrightarrow[f]{} & U \end{array}$$

2 The quotient stack map

Lemma IV.5. *Let $u : U \to [X/G]$ and let E be the associated G-torsor over B together with a G-equivariant morphism $f : E \to X$ (see Yoneda lemma B.11). Then the diagram*

$$\begin{array}{ccc} E & \xrightarrow{f} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow[u]{} & [X/G] \end{array}$$

is 2-cartesian.

Proof. Let U' be an S-scheme. Consider the category $U \times_{[X/G]} X(B')$. Its objects are triplet $(b : U' \to U, x : U' \to X, \psi)$ where $\psi : b^*u \simeq x^*p$ is an isomorphism or, in other words, (by the Yoneda lemma B.11) an isomorphism $\psi : b^*E \simeq x^*(X \times_S G)$, that is, an isomophism $\psi : b^*E \simeq U' \times_S G$. For a G-torsor, the data of a trivialisation ψ is equivalent to the data of a global section (see [DG70, III, §4, n°2, corollaire 1.5]). Thus, the data of the triplet $(b : U' \to U, x : U' \to X, \psi)$ is equivalent to the data of the pair $(b : U' \to U, s)$ where $s : U' \to b^*E$ is a section, and this is also equivalent to the data of a morphism $\tilde{b} : U' \to E$. Indeed, given $(b : U' \to U, s)$, we have the composite map $\tilde{b} : T \to b^*E \to E$ and given $\tilde{b} : U' \to E$, we have the composite map $b : U' \to E \to U$ and the section $s : U' \to b^*E$ given by the universal property of the pullback. This establishes the isomorphism $U \times_{[X/G]} X \simeq E$. □

Remark IV.6. *In particular, taking $U = X$ and $p = u$ in the previous lemma, we obtain that the diagram*

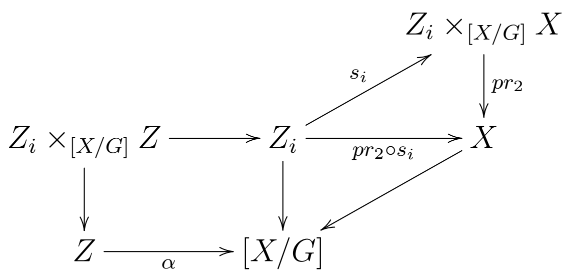
$$\begin{array}{ccc} X \times_S G & \xrightarrow{\mu_X} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & [X/G] \end{array}$$

is 2-cartesian.

Proposition IV.7. *The canonical morphism $p : X \to [X/G]$ is surjective, quasi-compact, flat and representable. Moreover, the diagonal $\Delta : [X/G] \to [X/G] \times_S [X/G]$ is representable.*

Proof. By the previous lemma, the morphism $X \to [X/G]$ is representable. For any $U \to [X/G]$, the corresponding G-torsor $E \to U$ is quasi-compact, surjective and flat. Since G is supposed flat, p has the same properties (see [DG70, III, §4, n°2, Corollaire 1.9]).
We want to show that Δ is representable, that is, for any two morphisms $Z \to [X/G] \times_S [X/G]$ where Z is a scheme, $[X/G] \times_{[X/G] \times_S [X/G]} Z$ is a scheme, or equivalently, for any two morphisms $\alpha : Z \to [X/G]$ and $\beta : Z \to [X/G]$, $Z \times_{[X/G]} Z$ is a scheme. Denote by $E \to Z$ the G-torsor associated to β. Choose a trivialization of the torsor $E \simeq Z \times_{[X/G]} X$, that is, the data of a covering $\{Z_i \to Z\}$ of Z and sections $s_i : Z_i \to Z_i \times_{[X/G]} X$ such that for every index i the

following diagram commutes:

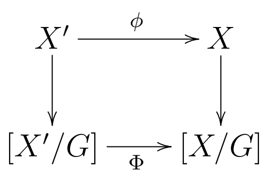


Then $F_i := Z_i \times_{[X/G]} Z \simeq Z_i \times_X (X \times_{[X/G]} Z) \simeq Z_i \times_X E$ and F_i is a scheme. Setting $F_{i,j} := (Z_i \times_Z Z_j) \times_{[X/G]} Z$, we obtain open immersions $F_{i,j} \to F_i$ which permit to glue the $F_i's$ into a scheme $F := F_i/\sim$ where $x \sim y$ if x and y are in the same intersection F_{jk} and their images under the map $F_{jk} \to F_j$ are the same. Since the $F_i's$ form a covering of $Z \times_{[X/G]} Z$, there is a map from F to $Z \times_{[X/G]} Z$, which is an isomorphism. This implies that $Z \times_{[X/G]} Z$ is represented by F. □

3 Change of spaces

In this section, we consider $\phi : X' \to X$ a G-equivariant morphism of schemes provided with a right G-action. Then, we define a natural morphism of stacks $\Phi : [X'/G] \to [X/G]$ as follows, for any S-scheme T, the morphism $\Phi(T) : [X'/G](T) \to [X/G](T)$ maps a G-torsor P over T provided with a G-equivariant morphism $f : P \to X'$ to the same G-torsor but provided with the G-equivariant morphism $\phi \circ f : P \to X$.

Lemma IV.8. *Under the previous hypotheses, the following diagram is 2-cartesian:*



Proof. See the proof of Lemma IV.5 □

Lemma IV.9. *The morphism $\Phi : [X'/G] \to [X/G]$ is representable. If $X' \to X$ has a property which is local for fppf topology on the target, then the associated quotient morphism has the same property.*

4 Quotient stacks as Artin stacks

The following lemma explains how we see the properties of the diagonal map of $[X/G]$ over S thanks to the Galois map and reciprocally.

Lemma IV.10. *If the diagonal map $\Delta : [X/G] \to [X/G] \times_S [X/G]$ has a property which is stable under base change, then the Galois map $X \times_S G \to X \times_S X$ has the same property. Conversely, if the Galois map has a property stable under base change and faithfully flat quasi compact descent then Δ has also this property.*

Proof. Write **R** for a property stable under base change. Suppose that the diagonal map Δ has property **R**. Without difficulties thanks to Remark IV.6, we get the following natural sequence of isomorphisms:

$$[X/G] \times_{[X/G]\times_S[X/G]} (X \times_S X) \simeq X \times_{[X/G]} X \simeq X \times_S G$$

Besides the projection $[X/G]\times_{[X/G]\times_S[X/G]} X\times_S X \to X\times_S X$ corresponds to the map $X\times_S G \to X \times_S X$ which has property **R**, by base change.
Conversely, we suppose that property **R** is stable under base change and faithfully flat, quasi compact descent and that the Galois map has this property **R**. As a consequence of what we say below, after the base change $X\times_S X \to [X/G]\times_S[X/G]$, the diagonal map has property **R** . Let $Z \to [X/G]\times_S[X/G]$ be a morphism where Z is a scheme, and set $W := [X/G]\times_{[X/G]\times_S[X/G]} Z$. Since by Proposition IV.7, the diagonal map is representable, it suffices to prove that $W \to Z$ has property **R**. Set $Z' := Z \times_{[X/G]\times_S[X/G]} X \times_S X$. Then the map $W' := W \times_Z Z' \to Z'$ has property **R**. As $X \to [X/G]$ is faithfully flat and quasi compact, the base change $Z' \to Z$ is also faithfully flat and quasi compact, thus by descent, $W \to Z$ has property **R**, hence also Δ. □

Proposition IV.11. *Suppose that G is affine, finitely presented over S and that X is an affine scheme. Then $[X/G]$ is an Artin stack.*

Proof. The diagonal map is representable by Lemma IV.7. As $X\times_S G \to X\times_S X$ is a morphism between two affine schemes, it is separated and quasi-compact, so it is the diagonal map Δ. Moreover, applying Artin's criterium B.14, since p is faithfully flat and finitely presented (cf. Lemma IV.7), it follows that $[X/G]$ is an Artin stack. □

5 G-invariant morphisms and G-torsors over stacks

Definition IV.12. *Let $\mathcal{Y}$ be an S-stack and $\pi : X \to \mathcal{Y}$ a morphism of stacks. We say that π is a G-**invariant morphism** if the following assertions are satisfied:*

1. *the diagram*

$$\begin{array}{ccc} X \times G & \xrightarrow{\mu_X} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\pi]{} & \mathcal{Y} \end{array}$$

is 2-commutative. Equivalently, there exists of a 2-morphism $\rho : pr_1^*\pi \to \mu_X^*\pi$;

2. *for a scheme* B, $x \in X(U)$ *and* $g \in G(U)$, *denote by* $\rho_{x,g}$ *the* 2*-morphisms:*

$$x^*\pi \simeq (x,g)^* pr_1^* \pi \xrightarrow{\rho_{x,g}} (x,g)^* \mu_X^* \pi \simeq (x.g)^* \pi$$

. For all $x \in Z(S)$ *and* $g_1, g_2 \in G(S)$, *the morphisms* $\rho_{x,g}$ *make the following diagram*

$$\begin{array}{ccc} x^*\pi & \xrightarrow{\rho_{x,g_1}} & (x.g_1)^*\pi \\ {\scriptstyle \rho_{x,g_1 g_2}}\downarrow & & \downarrow{\scriptstyle \rho_{x.g_1,g_2}} \\ (x.(g_1 g_2))^*\pi & = & ((x.g_1).g_2)^*\pi \end{array}$$

commutative.

Remark IV.13. *Let* $\pi : X \to \mathcal{Y}$ *be a* G*-invariant morphism of stacks. For any scheme* U *provided with a morphism* $\alpha : U \to \mathcal{Y}$, *the fiber product* $X \times_{\mathcal{Y}} U$ *is a sheaf of sets defined for any scheme* T *by:*

$$(X \times_{\mathcal{Y}} U)(T) = \{(a,u,\phi) | a \in X(T),\ u \in U(T) \text{ and an isomorphism } \phi : a^*\pi \to u^*\alpha\}.$$

We define a right G*-action on* $X \times_{\mathcal{Y}} U$ *via* $(a,u,\phi).g = (a.g, u, \phi \circ \rho_{a,g}^{-1})$, *for* $g \in G(T)$. *Thus, by construction, the morphism* $X \times_{\mathcal{Y}} U \to U$ *is* G*-invariant and the morphism* $X \times_{\mathcal{Y}} U \to X$ *is* G*-equivariant.*

Definition IV.14. *Let* $\mathcal{Y}$ *be a stack and* $\pi : X \to \mathcal{Y}$ *be a* G*-invariant morphism of stacks. A morphism* $X \to \mathcal{Y}$ *is called a* G***-torsor*** *if for any scheme* U *provided with a morphism* $U \to \mathcal{Y}$, *the induced morphism* $X \times_{\mathcal{Y}} U \to U$ *is a* G*-torsor.*

6 Characterizing quotient stacks

Lemma IV.15. *The quotient stack map* $p : X \to [X/G]$ *is a* G*-invariant morphism.*

Proof. Consider the cartesian diagram of Remark IV.6

$$\begin{array}{ccc} X \times_S G & \xrightarrow{\mu_X} & X \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & [X/G] \end{array}$$

Thus, $Id_{X\times_S G}$ gives an isomorphism $\rho^{-1} : \mu_X^* p \to pr_1^* p$ corresponding to the cartesian diagram

$$\begin{array}{ccc} (X\times_S G)\times_S G & \xrightarrow{\phi} & (X\times_S G)\times_S G \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_1} \\ X\times_S G & \longrightarrow & X\times_S G \end{array}$$

where the morphism $\phi : X\times_S G\times_S G \to X\times_S G\times_S G$ maps (x, g_1, g_2) to $(x, g_1, g_1.g_2)$. So, for any scheme U, $x \in X(U)$ and $g \in G(U)$, the morphism $\rho_{x,g}$ corresponds to the cartesian diagram

$$\begin{array}{ccc} U\times_S G & \xrightarrow{\psi} & U\times_S G \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle pr_1} \\ U & \longrightarrow & U \end{array}$$

The morphism $\psi : U\times_S G \to U\times_S G$ maps (u, g_0) to $(u, g(u)^{-1}g_0)$. Therefore, since, for any g_1, $g_2 \in G$, $g_2^{-1}g_1^{-1} = (g_1g_2)^{-1}$, p is G-invariant. □

Remark IV.16. *Since p is a G-invariant morphism, we can provide $X\times_{[X/G]} X$ with a G-action (see Remark IV.13). The morphism $(\mu_X, pr_1) : X\times_S G \to X\times_{[X/G]} X$ maps (x, g) to $(\mu_X(x, g), x, \rho_{x,g}^{-1})$ and the associativity condition on ρ implies that the Galois group (μ_X, pr_1) is a G-equivariant morphism of sheaves of sets.*

Lemma IV.17. *[Wan11, Lemma 2.1.1.] Let $\mathcal{Y}$ be an S-stack and $\pi : X \to \mathcal{Y}$ be a G-torsor. Then, there is an isomorphism $\mathcal{Y} \to [X/G]$ of stacks making the following triangle commutes:*

$$\begin{array}{ccc} & X & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle \mu_X} \\ \mathcal{Y} & \xrightarrow{\sim} & [X/G] \end{array}$$

Remark IV.18. 1. *Directly by the lemma, in the particular case where Y is a scheme over S and X is a G-torsor over Y, $Y \simeq [Z/G]$, so the two notions of quotient as scheme and as stack coincide.*

2. *Let X be an S-scheme and give it the trivial G-action. Lemma IV.17 implies that $S \to B_SG$ is G-invariant, so by base change $X \to B_SG\times_S X$ is also G-invariant. Let $u : B \to X$ be a S-scheme morphism. Denote by P the G-torsor associated to $v : B \to X \to B_SG$. By Lemma IV.5, the fiber product $X\times_{B_SG\times_S X} B \simeq S\times_{B_SG} B$ is isomorphic to P. The morphism $P \to X$ is $P \to B$ composed with u. The previous lemma then gives an isomorphism $B_SG\times_S X \simeq B_XG$. This implies that for any G-torsor P over U, any G-invariant morphism $P \to X$ factorizes via an unique morphism $U \to X$.*

7 Induced actions

Proposition IV.19. *We denote by H a flat closed subgroup of G and $[X \times^H G/G]$ stands for the quotient stack associated to the induced action (see Definition III.41). We have a canonical isomorphism*

$$[X/H] \simeq [X \times^H G/G]$$

Proof. Given B an S-scheme, we define the morphism $\phi(B) : [X/H](B) \to [X \times^H G/G](B)$ by sending an H-torsor E over B provided with an equivariant map $E \to X$ to the G-torsor $E \times^H G$ over B provided with the induced equivariant map $E \times^H G \to X \times^H G$ (see [DG70, III §4 n°3, 3.1]). The inverse morphism maps a G-torsor F over B provided with a G-equivariant map $E \to X \times^H G$ to the H-torsor $F \times_{X \times^H G} X$ over B provided with the G-equivariant map $F \times_{X \times^H G} X \to X$. (To show that this defines a G-torsor, we remark that H acts freely on $X \times G$). One can check that these morphisms are inverses of each other. □

8 Change of groups and double quotient stacks

In this section, H denotes a flat, finite, locally free, closed subgroup of G. Denote by $\pi : G \to G/H$ the quotient map. We keep the notations of the previous section. Since the action of H on G is free, by Theorem III.48, G is H-torsor over G/H. Then, by Lemma IV.17, we have an isomorphism $G/H \simeq [G/H]$.

Definition IV.20. *We define a morphism $\lambda : B_S H \to B_S G$ called the **induction morphism** as follows. For any S-scheme T, $\lambda(T) : B_S H(T) \to B_S G(T)$ sends a G-torsor P over T to the balanced product $(P \times_S G)/H$ which is a G-torsor over S.*

Lemma IV.21. *[Wan11, Lemma 2.4.1.] The morphism $\lambda : B_S H \to B_S G$ is representable and finitely presented.*

Lemma IV.22. *[Wan11, Lemma 2.4.2.] The right H-torsor $G \to G/H$ induces a left G-torsor $G/H \to B_S H$. In particular,*

$$[(G/H)/G] \simeq B_S H$$

.

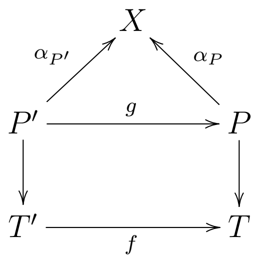
9 Quasi-coherent sheaves over quotient stacks

Proposition IV.23. *Giving a quasi-coherent sheaf $\mathcal{F}$ on $[X/G]$ (see Definition B.26) is equivalent to associating for any S-scheme T,*

1. *to each G-torsor $P \to T$ provided with a G-equivariant morphism $\alpha_P : P \to X$ an $\mathcal{O}(T)$-module $\mathcal{F}(P \to T, \alpha_P)$. This $\mathcal{O}(T)$-module defines a presheaf of $\mathcal{O}(T)$-modules by sending*

a Zariski-open subscheme U of T to the $\mathcal{O}(U)$-module $\mathcal{F}(P|_U \to U, \alpha_P|_U)$. We require this presheaf to be a quasi-coherent sheaf on $\mathcal{O}(T)$,

2. *to each commutative diagram*



where the columns are G-torsors and g is G-equivariant, a morphism

$$\mathcal{F}(P \to T, \alpha_P) \to \mathcal{F}(P' \to T', \alpha'_P)$$

that is linear with respect to the natural ring homomorphism $\mathcal{O}(T) \to \mathcal{O}(T')$. This morphism induces a homomorphism of quasi-coherent sheaves

$$\mathcal{F}(P \to T, \alpha_P) \to f_*\mathcal{F}(P' \to T', \alpha_{P'})$$

defined by the given homomorphism

$$\mathcal{F}(P|_U \to U, \alpha_P|_U) \to \mathcal{F}(P'|_{f^{-1}(U)} \to f^{-1}(U), \alpha_P|_U)$$

for each Zariski-open subscheme U of T. We require the corresponding homomorphism $f^*\mathcal{F}(P \to T, \alpha_P) \to \mathcal{F}(P' \to T', \alpha_{P'})$ to be an isomorphism.

Proof. This new definition implies easily Definition B.26, by the Yoneda lemma for stack B.11. Conversely, let $\mathcal{F}$ be a quasi coherent sheaf over $[X/G]$. Let $(P \to T, \alpha_P : P \to X)$ be an element of $[X/G](T)$. By the Yoneda lemma again, it corresponds to a morphism $t : T \to [X/G]$ and we have an $\mathcal{O}(T)$-module $\mathcal{F}(P \to T, \alpha_P) := t^*\mathcal{F}$. Applying [LMB00, Lemme 13.2.1 and Remarque 13.2.2], the module $\mathcal{F}$ is cartesian, so we obtain already 2. of the proposition. Fix an atlas $U \to [X/G]$. Denote by T' the fibered product $T \times_{[X/G]} U$. The morphism $T' \to T$ is fppf (by base change of the fppf morphism $U \to [X/G]$). By hypothesis, $\mathcal{F}|_U$ is a quasi-coherent $\mathcal{O}(U)$-module, hence also its inverse image via $T' \to U$ that we denote $\mathcal{F}_{T'}$. Moreover, by Definition B.26 of a quasi coherent sheaf, we have also that the inverse image via $T' \to T$ of the preimage $\mathcal{F}(P \to T, \alpha_P)$ of $\mathcal{F}$ via $T \to [X/G]$ is isomorphic to $\mathcal{F}_{T'}$.
Set $T'' := T' \times_T T'$ and let p, q be the two projections of T'' on T'. Then, as F is a cartesian module, we have a canonical isomorphism $\beta : p^*\mathcal{F}_{T'} \simeq q^*\mathcal{F}_{T'}$ as quasi-coherent $\mathcal{O}(T'')$-modules. The faithfully flatness descent theory of quasi-coherent modules says that the $\mathcal{O}(T')$-module $\mathcal{F}_{T'}$ provided with the descent datum β results by the base change $T' \to T$ from a unique $\mathcal{O}(T)$ (quasi-coherent) module L. But, as the base change $T' \to T$ applied to $\mathcal{F}(P \to T, \alpha_P)$ gives the same sheaf $\mathcal{F}_{T'}$ with the same descent datum, unicity implies that L is isomorphic to

$\mathcal{F}(P \to T, \alpha_P)$. Thus, $\mathcal{F}(P \to T, \alpha_P)$ is quasi-coherent.

□

Proposition IV.24. *The categories $Qcoh([X/G])$ and $Qcoh^G(X)$ are equivalent.*

Proof. We start by defining a functor $\Phi : QCoh^G(X) \to QCoh([X/G])$.
Suppose that $\mathcal{F}$ is an object of $QCoh^G(X)$. Let T be an S-scheme and let $P \to T$ be a G-torsor together with a G-equivariant morphism $\alpha_P : P \to X$. $\alpha_P^* \mathcal{F}$ is a quasi-coherent sheaf with a G-action since α_P is a G-equivariant map. On the other hand, by descent theory, we have an equivalence between the category of G-equivariant quasi-coherent sheaves on P and the category of quasi-coherent sheaves on T (see [FGI+05, Theorem 4.46]). Then, we define $\Phi(\mathcal{F})(P \to T, \alpha_P)$ to be a quasi-coherent sheaf on T whose pullback to P is isomorphic to $\alpha_P^* \mathcal{F}$. We can see that the map that sends $(P \to T, \alpha_P)$ to $\Phi(\mathcal{F})(P \to T, \alpha_P)$ has a natural structure of a quasi-coherent sheaf on $[X/G]$, and that a homomorphism $f : \mathcal{F} \to \mathcal{F}'$ of G-equivariant quasi-coherent sheaves on X induces a homomorphism $\phi(f) : \Phi(\mathcal{F}) \to \Phi(\mathcal{F}')$ of quasi-coherent sheaves on $[X/G]$. This defines the functor Φ.
Now, we define the inverse functor $\Psi : QCoh([X/G]) \to QCoh^G(X)$.
Given a quasi-coherent sheaf $\mathcal{F}$ on $[X/G]$, we define the quasi-coherent sheaf $\Psi(\mathcal{F})$ on X to be the sheaf $\mathcal{F}(G \times_S X \to X, \mu_X)$ associated to the trivial torsor $G \times_S X \to X$. Then, by 2. of the previous propositon, we get an isomophism

$$\beta : \mu_X^* \mathcal{F}(G \times_S X \to X, \mu_X) \to pr_1^* \mathcal{F}(G \times_S X \to X, \mu_X)$$

which satisfies the cocycle condition of Definition III.51, thus defining an action of G on $\mathcal{F}(G \times_S X \to X, \mu_X)$. We can extend then Ψ naturally to a functor $\Psi : QCoh([X/G]) \to QCoh^G(X)$.
One can prove that the composite $\Phi \circ \Psi$ equals $Id_{QCoh^G(X)}$. It is slightly less trivial to show that $\Psi \circ \Phi$ is isomorphic to $Id_{Qcoh([X/G])}$. The point is the following. Given a quasi-coherent sheaf $\mathcal{F}$ on $[X/G]$ and a G-torsor $\rho : P \to T$ over an S-scheme T together with a G-equivariant morphism $\alpha_P : P \to X$, the pullback $pr_2 : P \times_T P \to P$ of $P \to T$ to P has a canonical section, inducing a cartesian diagram:

$$\begin{array}{ccc} P \times_T P & \longrightarrow & G \times_S X \\ {\scriptstyle pr_2}\downarrow & & \downarrow \\ P & \xrightarrow[\alpha_P]{} & X \end{array} .$$

Since the pullback $\rho^* \mathcal{F}(P \to T, \alpha_P)$ is isomorphic to $\mathcal{F}(P \times_T P \to P, pr_2)$, this diagram induces an isomorphism of $\rho^* \mathcal{F}(P \to T, \alpha_P)$ with $\alpha_P^* \mathcal{F}(G \times_S X \to X, \mu_X)$. Moreover, by definition, one has that

$$\rho^* \Phi(\Psi(\mathcal{F}))(P \to T, \alpha_P) \simeq \alpha_P^* \mathcal{F}(G \times_S X \to X, \mu_X).$$

Hence, we get an isomorphism

$$\rho^* \Phi(\Psi(\mathcal{F})) \simeq \rho^* \mathcal{F}.$$

This isomorphism is G_T-equivariant; hence it descends to the expected isomorphism

$$\Phi(\Psi(\mathcal{F})) \simeq \mathcal{F}.$$

□

10 Moduli spaces for quotient stacks

Stacks are objects difficult to understand. In order to have more information about them we associate to them a more intuitive object (such as a scheme or an algebraic space) "sufficiently close" to it like a coarse moduli spaces. Here, we define moduli spaces in the particular case of quotients even though the definitions are the same for general Artin stacks. For more details, we can refer to the articles [AOV08] and [Alp08]. In this section, we suppose that G is affine, flat and finitely presented over S.

10.1 Good moduli spaces

Let $\phi : [X/G] \to M$ be a morphism where M is an algebraic space.

Definition IV.25. *We say that $\phi : [X/G] \to M$ is a **good moduli space** if the following properties are satisfied:*

1. *the map ϕ is quasi-compact, quasi-separated and the induced functor $\phi_* : Qcoh([X/G]) \to Qcoh(M)$ on the quasi-coherent sheaves is exact;*
2. *the natural map $\mathcal{O}_M \tilde{\to} \phi_* \mathcal{O}_{[X/G]}$ is an isomorphism.*

Remark IV.26. *The functor $\phi_* : Qcoh([X/G]) \to Qcoh(M)$ makes sense thanks to Proposition B.27, since the map ϕ is quasi-compact, quasi-separated.*

10.2 Coarse moduli spaces

Here, write $\mathcal{C}$ for the category of schemes or the category of algebraic spaces.

Definition IV.27. *We say that $\phi : [X/G] \to M$ is a **coarse moduli space in** $\mathcal{C}$ if the following properties are satisfied:*

1. *for any algebraically closed field k, $[[X/G](\Omega)] \to M(\Omega)$ is a bijection, where $\Omega = Spec(k)$ and $[[X/G](\Omega)]$ denotes the set of isomorphism classes of objects in the small category $[X/G](\Omega)$;*
2. *the map ϕ is universal for the maps taking value in $\mathcal{C}$ (This means that if N is an object of $\mathcal{C}$ then any morphism $[X/G] \to N$ factorizes via a morphism $M \to N$ of $\mathcal{C}$).*

10.3 Existence of a coarse moduli space

Theorem IV.28. *Suppose that the group scheme G and the scheme X are finitely presented over S and that the inertia group stack $\mathfrak{I}_{[X/G]}$ is finite (cf. Definition B.17). The quotient stack $[X/G]$ admits a coarse moduli space $\phi : [X/G] \to M$ such that ϕ is proper.*

Proof. Since G is finitely presented over S, so is $X \to [X/G]$ (X is a torsor over $[X/G]$). By Lemma IV.7, since $X \to [X/G]$ is surjective, flat of finite presentation and $X \to S$ is also of finite presentation, $[X/G] \to S$ is of finite presentation. By [Con05, Theorem 1.1], the hypotheses insure that the quotient stack admits a coarse moduli space that we denote $\phi : [X/G] \to M$ such that ϕ is proper. □

10.4 Relations with quotients

Lemma IV.29. *Let M be an algebraic space. The datum of a morphism $\phi : [X/G] \to M$ is equivalent to the datum of a G-invariant morphism of S-algebraic spaces $f : X \to M$.*

Proof. If ϕ is given, we take $f := \phi \circ p$. Conversely, let $f : X \to M$ be a morphism such that $f \circ p_1 = f \circ \mu_X$. We define a morphism $\phi : [X/G] \to M$, as follows. For any scheme U, the morphism $\phi : [X/G](U) \to M(U)$ sends a G-torsor P over U provided with the G-equivariant morphism $P \to X$ to the canonical morphism $U = P/G \to M$. We can easily check that $f = \phi \circ p$. □

In particular, we have the following description of coarse moduli spaces as quotients.

Lemma IV.30. *A geometric quotient $f : X \to Y$ which is categorical in the category of algebraic spaces induces a coarse moduli space $[X/G] \to Y$.*

Proof. Let k be an algebraically closed and let $f : Spec(k) \to X$ be a $Spec(k)$-point over S. Then as the geometric fibers of f are the orbits, $Hom_{S\text{-Sch}}(Spec(k), Y)$ is the set of the orbits of the k-points of X which is exactly the set $[[X/G](Spec(k))]$, which denotes the set of isomorphism classes of objects in the small category $[X/G](Spec(k))$. For any algebraic space N, any G-invariant morphism $[X/G] \to N$, Moreover, if we suppose that $[X/G] \to N$ is a morphism to an algebraic space induces by the previous lemma a G-invariant morphism $X \to N$. By the universal property of the quotient, the morphism $X \to N$ factorizes through Y. Thus, the morphism $[X/G] \to N$ factorizes through Y, so $[X/G] \to Y$ is a coarse moduli space. □

Remark IV.31. *If the morphism $f : X \to M$ defines a universal categorical quotient, if $[X/G] \to M$ is a coarse moduli space, then for any base change $M' \to M$, $[X/G] \times_M M' \to M'$ is a coarse moduli space.*

Chapter V

Inertia group schemes, orbits

For the definition of inertia groups, we can refer to [DG70, III, §2, n°2]. In this chapter, $G := Spec(A)$ is a flat, affine group scheme over S and $X := Spec(B)$ is a flat scheme over S. Let (X, G) be an action over S. We investigate the properties of inertia groups of an action by a group scheme that generalize the ones of the classical case, lot of times admitted but not readily found.

1 Definitions

Definition V.1. *1. The* ***inertia functor*** *I_G is the following fiber product of the Galois map (μ_X, p_1) and the diagonal map Δ_X:*

$$\begin{CD} I_G @>{pr_2}>> X \\ @V{pr_1}VV @VV{\Delta_X}V \\ X \times_S G @>>{(\mu_X, p_1)}> X \times_S X \end{CD}$$

2. Let $\zeta : Spec(T) \to X$ be a T-point. The ***inertia group at*** *ζ, denoted by $I_G(\zeta)$, is the group scheme over T defined as the fiber product:*

$$\begin{CD} I_G(\zeta) @>{p'_1}>> Spec(T) \\ @V{p'_2}VV @VV{\zeta}V \\ I_G @>>{pr_2}> X \end{CD}$$

In particular, for $x \in X$ a topological point, $T = k(x)$ the residual field at x, ζ the canonical morphism $Spec(k(x)) \to X$, we denote $I_G(\zeta)$ by $I_G(x)$ and we call it the ***inertia group at*** *x.*

Naturally from the definition, we obtain the following properties.

Proposition V.2. 1. ***(Description on the T-points)*** *Let $\xi : Spec(T) \to X$ be a T-point. For any T-algebra T',*

$$I_G(\xi)(T') = \{g \in G(T') | g.\xi_{T'} = \xi_{T'}\}$$

where $\xi_{T'} : Spec(T') \to Spec(T) \to X$.

2. ***(Trivial action)****Suppose that G is flat over S. Let (S, G) be the trivial action and $\xi : Spec(T) \to S$ be a T-point of S. Then*

$$I_G(\xi) = G_T.$$

3. ***(Base change)*** *Let $\xi : Spec(T) \to X$ be a T-point of X. For any base change $b : Spec(T') \to Spec(T)$, if we set $\xi' := \xi \circ b : Spec(T') \to X$ then*

$$I_G(\xi') = I_G(\xi) \times_{Spec(T)} Spec(T').$$

4. ***(Stabilizer)*** *Let $\xi : Spec(T) \to X$ be a T-point of X. We can define $I_G(\xi)$ as the fiber product:*

$$\begin{array}{ccc} I_G(\xi) & \xrightarrow{p_1} & G \times_S Spec(T) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \mu_X \circ (Id, \zeta)} \\ Spec(T) & \xrightarrow[(\zeta, Id) \circ \Delta]{} & X \times_S spec(T) \end{array}$$

Remark V.3. 1. *It follows from 4. of the proposition that the inertia group $I_G(\xi)$ of ξ is a closed subgroup of G_T.*

2. *The characterization of the inertia group via 4. is exactly the definition of* ***stabilizer of a point*** *in [MFK94, Definition 0.4]. For the definition of freeness, we obtain directly that the action (X, G) is free if and only if all the inertia groups are trivial.*

2 Conjugate inertia groups

Lemma V.4. *Suppose that G is finite over S where $S := Spec(B^A)$. Let $\mathfrak{q} \in S$, denote by $\mathfrak{p}_i$, $i \in \{1, ..., r\}$ the primes above $\mathfrak{q}$ and by $\overline{k}$ an algebraic closure of $k(\mathfrak{q})$. For any $i, j \in \{1, ..., r\}$, there exists $g_{i,j} \in G(\overline{k})$ such that for any $\overline{k}$-algebra T,*

$$(I_G(\mathfrak{p}_i) \times_{k(\mathfrak{p}_i)} \overline{k})(T) = (g_{i,j})_T (I_G(\mathfrak{p}_j) \times_{k(\mathfrak{p}_i)} \overline{k})(T)(g_{i,j})_T^{-1}.$$

Proof. Denote by $\xi_{\mathfrak{p}_i}$ the canonical morphism $Spec(k(\mathfrak{p}_i)) \to X$, for all $i \in \{1, ..., r\}$. Since we have the following exact sequence for geometric points :

$$X(\overline{k}) \times G(\overline{k}) \rightrightarrows X(\overline{k}) \to S(\overline{k}) \to 1$$

there exists $g_{i,j} \in G(\overline{k})$ such that $(\xi_{\mathfrak{p}_i})_{\overline{k}} = g_{i,j}(\xi_{\mathfrak{p}_j})_{\overline{k}}$.
Thus, for any $\overline{k}$-algebras T,

$$\begin{aligned}(I_G(\mathfrak{p}_i) \times_{k(\mathfrak{p}_i)} \overline{k})(T) &= \{g \in G(T) | g_T((\xi_{\mathfrak{p}_i})_{\overline{k}})_T = ((\xi_{\mathfrak{p}_i})_{\overline{k}})_T\} \\ &= \{g \in G(T) | g_T(g_{i,j}(\xi_{\mathfrak{p}_j})_{\overline{k}})_T = (g_{i,j}(\xi_{\mathfrak{p}_i})_{\overline{k}})_T\} \\ &= \{g \in G(T) | g_T(g_{i,j})_T((\xi_{\mathfrak{p}_j})_{\overline{k}})_T = (g_{i,j})_T((\xi_{\mathfrak{p}_i})_{\overline{k}})_T\} \\ &= (g_{i,j})_T(I_G(\mathfrak{p}_j) \times_{k(\mathfrak{p}_i)} \overline{k})(T)(g_{i,j})_T^{-1}.\end{aligned}$$

□

Corollary V.5. *Suppose G is finite over S where $S := Spec(B^A)$. Suppose that the inertia group is trivial at one point $\mathfrak{p}$ of X over $\mathfrak{q} \in S$. Then the inertia group is trivial at any point above $\mathfrak{q} \in S$.*

Proof. Denote by $\mathfrak{p}_0$ the prime where the inertia is trivial and by $\mathfrak{p}_i$, $i \in \{1,..,r\}$ the other primes over $\mathfrak{q}$. By the previous lemma, for any $i \neq 0$, there exist $g_{i,j} \in G(\overline{k})$ such that for all $\overline{k}$-algebras T,

$$(I_G(\mathfrak{p}_i) \times_{k(\mathfrak{p}_i)} \overline{k})(T) = (g_{i,1})_T(I_G(\mathfrak{p}_0) \times_{k(\mathfrak{p}_0)} \overline{k})(T)(g_{i,1})_T^{-1} = \{e\}.$$

As a consequence, $I_G(\mathfrak{p}_i)_{\overline{k}}$ is trivial, thus also $I_G(\mathfrak{p}_i)$. □

3 Inertia for actions induced by subgroups or quotients

Proposition V.6. *Let H be a subgroup scheme of G and f be a T-point of X. Then*

$$I_H(f) \simeq I_G(f) \cap H_T.$$

In particular, if H is a normal subgroup of G, $I_H(f)$ is a normal subgroup of $I_G(f)$.

Proof. We have

$$I_G(f) \cap H_T := I_G(f) \times_{G_T} H_T \simeq (G_T \times_{X_T} Spec(T)) \times_{G_T} H_T \simeq I_H(f).$$

□

Proposition V.7. *Let k be a field. Suppose that G is finite over k and let H be a normal subgroup scheme of G, f be a k'-point of X where k' is a field and $\bar{f} = \pi \circ f$ with $\pi : X \to X/H$ the quotient map. Then*

$$I_{G/H}(\bar{f}) \simeq I_G(f)/I_H(f)$$

Proof. We write $f_{k''} : Spec(k'') \to Spec(k') \to X$ and $\bar{f}_{k''} := \pi \circ f_{k''}$, where k'' is a k'-algebra. Since G/H is an universal fppf quotient, the following sequence is exact:

$$1 \to H_{k'} \to G_{k'} \to (G/H)_{k'} \to 1$$

The goal is to prove that this sequence is exact:

$$1 \to I_H(f) \to I_G(f) \to I_{G/H}(\bar{f}) \to 1$$

Left and middle exactness are clear. So, we have just to prove the surjectivity of the morphism $I_G(f) \to I_{G/H}(\bar{f})$. By definition, for any k'-algebra k'', we have:

$$I_{G/H}(\bar{f})(k'') = \{g \in G/H(k'') | g\bar{f}_{k''} = \bar{f}_{k''}\}$$

Let k'' be a k'-algebra and $g_0 \in I_{G/H}(\overline{f})(k'')$, in particular, $g_0 \in G/H(k'')$. Since $\pi_G : G_{k'} \to (G/H)_{k'}$ is surjective, there are a k''-algebra k_0 and $g \in G_{k'}(k_0)$ such that $\pi_G(g) = g_{0_{k_0}}$. By definition of the action $(X/H, G/H)$ induced by the action (X, G), $\pi_G(g)\pi(f_{k'}) = \overline{gf_{k_0}} = \overline{f_{k_0}}$. So, there is a k_0-algebra k_1 and $h \in H(k_1)$ such that $hgf_{k_1} = f_{k_1}$ and so $hg \in I_G(f)(k_1)$ and $\pi(hg) = g_{0_{k_1}}$. This proves the proposition. □

4 Examples

1. **Action** $(\mathbb{A}^1_k, \alpha_p)$**.**
   Let $R := k$ be a field of characteristic p, $\mathbb{A}^1_k := Spec(k[x])$ and let $\alpha_p := Spec(A_0)$ where $A_0 := k[v]/v^p$ is a Hopf algebra over k such that the comultiplication is given by the morphism $\Delta_0 : A_0 \to A_0 \otimes_R A_0$ which maps $\overline{v}$ to $\overline{v} \otimes 1 + 1 \otimes \overline{v}$ where $\overline{v}$ denotes the image of v in A_0. We consider the action $(\mathbb{A}^1_k, \alpha_p)$ defined by the k-linear map $\rho_{\mathbb{A}^1_k} : \mathbb{A}^1_k \to \mathbb{A}^1_k \otimes_k A$ which maps x to $x \otimes 1 + 1 \otimes \overline{v}$. Then, the Galois map $Gal_0 : \mathbb{A}^1_k \times_k \alpha_p \to \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ corresponds to the morphism $Gal_0^\sharp : k[x \otimes 1, 1 \otimes x] \to k[x] \otimes k[v]/v^p$ sending $x \otimes 1$ to $x \otimes 1 + 1 \otimes \overline{v}$ and $1 \otimes x$ to $x \otimes 1$. Moreover, we obtain that the ring of invariants is $k[x]^{\alpha_p} = k[x^p]$. Denote by $I_{A_0}(t)$ the Hopf algebra corresponding to the inertia group $I_{\alpha_p}(t)$ at a point t.

   Take $\xi_0 : Spec(k) \to \mathbb{A}^1_k$ the morphism corresponding to the morphism $z^\sharp : k[x] \to k$ mapping x to 0. So, the diagonal map $D_{\xi_0} : Spec(k) \to \mathbb{A}^1_k \otimes_k \mathbb{A}^1_k$ corresponds to the morphism $D^\sharp_{\xi_0} : k[x \otimes 1, 1 \otimes x] \to k$ which maps $x \otimes 1$ to 0 and $1 \otimes x$ to 0. By definition, the inertia group $I_{\alpha_p}(\xi_0)$ is defined as the fiber product $(\mathbb{A}^1_k \times_k \alpha_p) \times_{\mathbb{A}^1_k \times_k \mathbb{A}^1_k} Spec(k) = Spec((k[x] \otimes k[v]/v^p) \otimes_{k[x \otimes 1, 1 \otimes x]} k)$. We have $Gal_0^\sharp(1 \otimes x) = x \otimes 1$, $D^\sharp_{\xi_0}(1 \otimes x) = 0$, $Gal_0^\sharp(x \otimes 1 - 1 \otimes x) = 1 \otimes \overline{v}$ and $D^\sharp_{\xi_0}(x \otimes 1) = 0$, this implies that in the tensor product $(k[x] \otimes k[v]/v^p) \otimes_{k[x \otimes 1, 1 \otimes x]} k$, $x \otimes 1 \otimes 1 = 1 \otimes 1 \otimes 0 = 0$ and $1 \otimes \overline{v} \otimes 1 = 1 \otimes 1 \otimes 0 = 0$. Finally, since an element of $I_{A_0}(\xi_0)$ is of the form $\sum_{i=0}^n \sum_{j=0}^{p-1} a_{i,j} (x \otimes_k 1)^i (1 \otimes \overline{v})^j \otimes_{k[x \otimes 1, 1 \otimes x]} 1$, applying the previous equalities we see that the only non zero term is $a_{0,0}$. As a consequence, the inertia group at ξ_0 is trivial.
   Consider now the point $\xi : Spec(\widehat{k[x]_x}) \to \mathbb{A}^1_k$, where $\widehat{k[x]_x}$ is the completion of the localization B_x corresponding to the morphism $\xi^\sharp : k[x] \to k[[w]]$ sending x to w. The diagonal

map $D_\xi : Spec(k[[w]]) \to \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ corresponds to the morphism $D^\sharp_\xi : k[x\otimes 1, 1\otimes x] \to k[[w]]$ sending $x\otimes 1$ to w and $1\otimes x$ to w. We have $D^\sharp_\xi(1\otimes x) = w$, $Gal^\sharp_0(1\otimes x) = x\otimes 1$, $D^\sharp_\xi(x\otimes 1 - 1\otimes x) = w - w = 0$ and $Gal^\sharp_0(x\otimes 1 - 1\otimes x) = 1\otimes \bar{v}$. So, an element of $I_{A_0}(\xi)$ is of the form $\sum a_{i,j,h}(x\otimes 1)^i(1\otimes \bar{v})^j \otimes w^h$ which is equal to $\sum a_{i,0,h}\otimes w^{h+i}$ by the previous equalities. Then, the inertia group $I_{\alpha_p}(\xi)$ is trivial.

From now, $k := \mathbb{F}_p$ and $k_1 := k[\beta]$ for some β such that $\beta^p - \beta = 1$.

Consider the point $\phi : Spec(k_1) \to \mathbb{A}^1_k$ corresponding to the morphism $\phi^\sharp : k[x] \to k_1$ mapping x to β. The diagonal map $D_\phi : Spec(k_1) \to \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ corresponds then to the morphism $D^\sharp_\phi : k[x\otimes 1, 1\otimes x] \to k_1$ mapping $x\otimes 1$ to β and $1\otimes x$ to β. We have $D^\sharp_\phi(1\otimes x) = \beta$, $Gal^\sharp_0(1\otimes x) = x\otimes 1$, $D^\sharp_\phi(x\otimes 1 - 1\otimes x) = \beta - \beta = 0$ and $Gal^\sharp_0(x\otimes 1 - 1\otimes x) = 1\otimes\bar{v}$. An element of $I_{A_0}(\phi)$ is of the form $\sum a_{i,j,h}(x\otimes 1)^i(1\otimes\bar{v})^j\otimes\beta^h$, thus by the previous equalities, we obtain $\sum a_{i,0,h}\otimes\beta^{h+i}$. So, $I_G(\phi)$ is trivial.

By [Ser68, II §4 prop 5], any element of the completion at the localization at ϕ $\widehat{k[x]_\phi}$, can be written uniquely as $\sum_{n=0}^\infty s_n\pi^n$ where $\pi = x^p - x - 1$, $s_n \in \otimes_{i=0}^{p-1} x^i k$ $(k = \mathbb{F}_p)$. Consider the point $\theta : Spec(\widehat{k[x]_\phi}) \to \mathbb{A}^1_k$ corresponding to $\theta^\sharp : k[x] \to \widehat{k[x]_\phi}$ sending x to π. Then, $D^\sharp_\theta(x\otimes 1) = \pi$ and $D^\sharp_\theta(x\otimes 1 - 1\otimes x) = \pi - \pi = 0$. Since an element of $I_{A_0}(\theta)$ is of the form $\sum a_{i,j,l,h}(x\otimes 1)^i(1\otimes\bar{v})^j\otimes x^l\pi^h$, which is equal to $\sum a_{i,l,h}\otimes x^{l+i}\pi^h$ by the previous equalities. And so, $I_G(\theta)$ is trivial.

2. **Action** $(\mathbb{A}^1_k, \alpha_p \rtimes \mu_n)$**.**

Let k be an algebraically closed field of characteristic $p > 0$ and $S = Spec(k)$ be the basis. Take $G := Spec(A)$ with $A := k[u,v]/(u^n - 1, v^p)$. So $G = \alpha_p \rtimes \mu_n \subset GL_{2,k}$ and for any k-algebra R,

$$G(R) := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a, b \in R, a^n = 1, b^p = 0 \right\}$$

the multiplication on this group is given by the matrix multiplication.

So, the comultiplication $\Delta : A \to A\otimes_k A$ is defined by mapping u to $u\otimes u$ and v to $u\otimes v + v\otimes 1$. We consider an action of G on $\mathbb{A}^1_k$ defined by the comodule map $\rho_{k[x]} : k[x] \to k[x]\otimes_k A$ sending x to $x\otimes\bar{u} + 1\otimes\bar{v}$ giving to $k[x]$ its structure of A-comodule. Then, the Galois map $Gal : \mathbb{A}^1_k \times_k G \to \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ corresponds to the morphism $Gal^\sharp : k[x\otimes 1, 1\otimes x] \to k[x]\otimes k[u,v]/(u^n - 1, v^p)$ sending $1\otimes x$ to $x\otimes\bar{u} + 1\otimes\bar{v}$ and $1\otimes x$ to $x\otimes 1$. Moreover, we obtain that the ring of invariants is $k[x]^A \simeq k[x^{ppcm(n,p)}]$. Denote by $I_A(t)$ the Hopf algebra corresponding to the inertia group $I_G(t)$ at a point t. The action of the semi direct product G on $\mathbb{A}^1_k$ induces actions.

One is the action $(\mathbb{A}^1_k, \alpha_p)$ studied in Point 1.

An other induced action is the action $(\mathbb{A}^1_k/\alpha_p, G/\alpha_p)$ where $G/\alpha_p \simeq \mu_n$ defined by the comudule map $\rho''_{k[x]} : k[x^p] \to k[x^p]\otimes_k k[u]/(u^n - 1)$ sending to x^p to $x^p\otimes\bar{u}^p$ giving to $k[x^p]$

its structure of $k[u]/(u^n-1)$-comodule, so that the Galois map $Gal' : \mathbb{A}^1_k/\alpha_p \times_S \mu_n \to \mathbb{A}^1_k/\alpha_p \times_S \mathbb{A}^1_k/\alpha_p$ corresponding to the morphism $Gal'^\sharp : k[x^p \otimes 1, 1 \otimes x^p] \to k[x^p] \otimes k[u]/(u^n-1)$ sending $1 \otimes x^p$ to $x^p \otimes \bar{u}^p$ and $1 \otimes x^p$ to $x^p \otimes 1$. We notice that if $n = p$, this action $(Spec(k[x^p]), \mu_p)$ is trivial thus, all the inertia groups are equals to μ_p and $k[x]^A = (k[x]^{\alpha_p})^{\mu_n} = k[x^p]$, otherwise $k[x]^A = (k[x]^{\alpha_p})^{\mu_n} = k[x^{np}]$.

We have also the action $(\mathbb{A}^1_k, \mu_n)$ defined by the comudule map $\rho''_{k[x]} : k[x] \to k[x] \otimes_k k[u]/(u^n-1)$ sending to x to $x \otimes \bar{u}$ giving to $k[x]$ its structure of $k[u]/(u^n-1)$-comodule, so that the Galois map $Gal'' : \mathbb{A}^1_k \times_S \mu_n \to \mathbb{A}^1_k \times_S \mathbb{A}^1_k/\alpha_p$ corresponding to the morphism $Gal''^\sharp : k[x \otimes 1, 1 \otimes x] \to k[x] \otimes k[u]/(u^n-1)$ sending $1 \otimes x$ to $x \otimes \bar{u}$ and $1 \otimes x$ to $x \otimes 1$ and $B^{\mu_n} = k[x^n]$ in particular $k[x]^{\mu_n} \simeq k[x]^{\alpha_p}$, if $n = p$.

Let $\xi_0 : Spec(k) \to \mathbb{A}^1_k$ be the k-point defined by the k-algebra morphism $\xi^\sharp : k[x] \to k$ sending x to 0. The diagonal map $D_{\xi_0} : Spec(k) \to \mathbb{A}^1_k \otimes_k \mathbb{A}^1_k$ corresponds then to the morphism $D^\sharp_{\xi_0} : k[x \otimes 1, 1 \otimes x] \to k$ which maps $x \otimes 1$ to 0 and $1 \otimes x$ to 0. By definition, the inertia group $I_G(\xi_0)$ for the action $(\mathbb{A}^1_k, G)$ at the point ξ_0 is defined as the fiber product $(\mathbb{A}^1_k \times_k G) \times_{\mathbb{A}^1_k \times_k X} Spec(k) = Spec((k[x] \otimes k[u,v]/(u^n-1, v^p)) \otimes_{k[x \otimes 1, 1 \otimes x]} k)$. We remark that $Gal^\sharp(x \otimes 1) = x \otimes 1$, $D^\sharp_{\xi_0}(1 \otimes x) = 0$, $D^\sharp_{\xi_0}(x \otimes 1) = 0$, $Gal^\sharp(1 \otimes x) = x \otimes u + 1 \otimes v$ and $D^\sharp_{\xi_0}(x \otimes u + 1 \otimes v) = 0$, this implies that in the tensor product $(k[x] \otimes k[u,v]/(u^n-1, v^p)) \otimes_{k[x \otimes 1, 1 \otimes x]} k$, $x \otimes 1 \otimes 1 = 0$ and $1 \otimes \bar{v} \otimes 1 = 0$. Finally, since an element of $I_A(\xi_0)$ is of the form $\sum_{l=0}^{p-1} \sum_{j=0}^{n-1} \sum_{i \geq 0} a_{i,j,l} \otimes x^i \otimes u^j \bar{v}^l$, applying the previous equalities we see that the only non zero terms are $\sum_{l=0}^{p-1} \sum_{j=0}^{n-1} a_{0,j,0} \otimes 1 \otimes \bar{u}^j$. As a consequence, we prove that the inertia group at ξ_0 for the action $(\mathbb{A}^1_k, G)$ Using computations as the previous ones, the inertia group $I_{\alpha_p}(\xi_0)$ for the action $(\mathbb{A}^1_k, \alpha_p)$ at the point ξ_0 is trivial and the inertia group $I_{\alpha_p}(\overline{\xi_0})$ at the point $\overline{\xi_0} : Spec(k) \to \mathbb{A}^1_k \to \mathbb{A}^1_k/\alpha_p$ are isomorphic to μ_n over S.

Suppose that n is coprime to p. Let $\xi_a : Spec(k) \to \mathbb{A}^1_k$ be the k-point defined by the k-algebra morphism $\xi^\sharp_a : k[x] \to k$ sending x to $a \in k^\times$. The diagonal map $Spec(k) \to \mathbb{A}^1_k \otimes_k \mathbb{A}^1_k$ corresponds then to the morphism $D_{\xi_a} : k[x \otimes 1, 1 \otimes x] \to k$ which maps $x \otimes 1$ to a and $1 \otimes x$ to a. By definition, the inertia group $I_G(\xi_a)$ for the action $(\mathbb{A}^1_k, G)$ at the point ξ_a is defined as the fiber product $(\mathbb{A}^1_k \times_k G) \times_{\mathbb{A}^1_k \times_k \mathbb{A}^1_k} Spec(k) = Spec((k[x] \otimes k[u,v]/(u^n-1, v^p)) \otimes_{k[x \otimes 1, 1 \otimes x]} k)$. We remark that $Gal^\sharp(x \otimes 1) = x \otimes 1$, $D^\sharp_{\xi_0}(1 \otimes x) = a$, $D^\sharp_{\xi_0}(x \otimes 1) = a$, $Gal^\sharp(1 \otimes x) = x \otimes \bar{u} + 1 \otimes \bar{v}$ and $D_{\xi_0}(x \otimes \bar{u} + 1 \otimes \bar{v}) = a$, this implies that in the tensor product $(k[x] \otimes k[u,v]/(u^n-1, v^p)) \otimes_{k[x \otimes 1, 1 \otimes x]} k$, $x \otimes 1 \otimes 1 = 1 \otimes 1 \otimes a$ and $(x \otimes \bar{u} + 1 \otimes \bar{v}) \otimes 1 = 1 \otimes \bar{u} \otimes a + 1 \otimes \bar{v} \otimes 1 = 1 \otimes 1 \otimes a$ so, $1 \otimes \bar{u}^p \otimes a^p = 1 \otimes 1 \otimes a^p$ and $1 \otimes \bar{u}^p \otimes 1 = 1 \otimes 1 \otimes 1$ but since $(n,p) = 1$, there exist $l, h \in \mathbb{Z}$ such that $nl + ph = 1$ then $1 \otimes \bar{u} \otimes 1 = 1 \otimes 1 \otimes 1$ and $1 \otimes 1 \otimes \bar{v} = 0$. Finally, since an element of $I_A(\xi_a)$ is of the form $\sum_{l=0}^{p-1} \sum_{j=0}^{n-1} \sum_{i \geq 0} a_{i,j,l} \otimes x^i \otimes \bar{u}^j \bar{v}^l$, applying the previous equalities we see that the only non zero terms is $a_{0,0,0}$. As a consequence, the inertia group at ξ_0 for the action $(\mathbb{A}^1_k, G)$ is trivial. Using computations as before, we show that the inertia group at ξ_a for $(\mathbb{A}^1_k, \alpha_p)$ and the inertia group $I_{\alpha_p}(\overline{\xi_a})$ at the point $\overline{\xi_a} : Spec(k) \to \mathbb{A}^1_k \to \mathbb{A}^1_k/\alpha_p$ are trivial.

5 Inertia group and orbit via the quotient stack

Suppose that G is flat. Consider an action (X, G) over S and the quotient stack $[X/G]$ for this action. We denote by $p : X \to [X/G]$ the quotient stack morphism.

Lemma V.8. *Let $f : T \to X$ be a T-point of X. Set $\overline{f} := \pi \circ f$. Then, the automorphism group $\underline{Aut}_T(\overline{f})$ of the T-point $\overline{f}$ corresponds to the inertia group $I_G(f)$ at the T-point f.*

Proof. By Definition B.17,

$$
\begin{aligned}
Aut_T(\overline{f}) &\simeq ([X/G] \times_{[X/G]\times_S[X/G],\Delta,\Delta} [X/G]) \times_{[X/G]} T \\
&\simeq ([X/G] \times_{[X/G]\times_S[X/G],\Delta,(p,p)} (X \times_S X)) \times_{X\times_S X, pr_2, \Delta\circ f} T \\
&\simeq (X \times_{[X/G]} X) \times_{X\times_S X} T \\
&\simeq (G \times_S X) \times_{X\times_S X,(\mu_X, pr_1),\Delta\circ f} T \\
&\simeq I_G(f)
\end{aligned}
$$

□

Remark V.9. *Take $f : Spec(k) \to [X/G]$ a geometric point of $[X/G]$. From the surjectivity of $p : X \to [X/G]$, we know that there is $f_0 : Spec(k) \to X$ such that $f = f_0 \circ p$. Thus, for any such point above f, the automorphism group of f can be written as the inertia group of this point. In particular, all the inertia groups at geometric points above f are isomorphic.*

Lemma V.10. *Suppose that G is finitely presented over S. Let $f : T \to X$ be a T-point of X. Moreover, suppose that $I_G(f)$ is a flat group scheme over T.*

1. *We have a canonical isomorphism $[G_T/I_G(f)] \simeq O(f)$.*
2. *We have a canonical isomorphism $B_T I_G(f) \simeq [O(f)/G_T]$.*
3. *The orbit $O(f)$ can be characterized as the fiber product*

$$
\begin{array}{ccc}
O(f) & \longrightarrow & X_T \\
\downarrow & & \downarrow \\
B_T I_G(f) & \longrightarrow & [X_T/G_T]
\end{array}
$$

4. *We have a canonical decomposition $T \twoheadrightarrow B_T I_G(f) \hookrightarrow [X_T/G_T]$*

Proof.
1. It is easy to prove that the Galois map $(m_{G_T}, pr_1) : G_T \times I_G(f) \to G_T \times_{O(f)} G_T$ is well defined and that it is an isomorphism. Thus, the map $G_T \to O(f)$ is a G-torsor. Applying Lemma IV.17, we obtain the isomorphism $[G_T/I_G(f)] \simeq O(f)$.
2. Thanks to 1., applying Lemma IV.22 to $[(G_T/I_G(f))/G_T]$, we get the isomorphism $B_T I_G(f) \simeq [O(f)/G_T]$.
3. The 2-cartesian diagram comes from the previous isomorphism combined with the fact that the morphism $O(f) \to X_T$ is G-equivariant.

4. The last decomposition results from the decomposition in G-equivariant morphisms

$$G_T \twoheadrightarrow O(f) \hookrightarrow X_T$$

□

Part C

Ramification in arithmetic geometry

Standing hypotheses

All along Part C, A denotes a flat commutative Hopf algebra (A, Δ, ϵ, S) over R, G the affine flat group scheme associated to A over S and $X := Spec(B)$ an affine scheme over S.
We fix an action (X, G) over S. We denote by $\mu_X : X \times_S G \to X$ its structural map.
We write $C := B^A$ for the ring of invariants for this action, $Y := Spec(C)$ and $\pi : X \to Y$ for the morphism induced by the inclusion $C \subset B$.
We write $[X/G]$ for the quotient stack associated to this action, $p : X \to [X/G]$ for the quotient stack morphism and $\rho : [X/G] \to Y$ for the canonical morphism induced by the morphism $\pi : X \to Y$ (see Lemma IV.29). Recall that $\mathcal{I}_{[X/G]}$ stands for the inertia group stack (see Definition B.17).

Chapter VI

Different notions of tameness

1 Tame actions of an affine group scheme: definitions and examples

We recall the notion of tameness for actions by affine group schemes defined by Chinburg, Erez, Pappas and Taylor in the article [CEPT96, §2]. Our contributions in this section are §1.2 stating that tameness is preserved by induced actions and Lemma VI.4. All the other results are taken from [CEPT96].

1.1 Definition

Definition VI.1. *An action $(X, G) = (SpecB, SpecA)$ is called* ***tame*** *if there is a A-comodule map $\alpha \in Com_A(A, B)$ which is unitary (that is, $\alpha(1_A) = 1_B$). This map is called a* ***total integral****.*

1.2 Induced actions

The following proposition establishes the behavior of tameness for induced actions.

Proposition VI.2. *We denote by $H = Spec(A')$ a flat affine closed subgroup of G. Suppose that H acts on an affine scheme $Z := Spec(D)$ over S. If the action (Z, H) is tame, then the action $(Z \times^H G, G)$ of G on the balanced product is also tame.*

Proof. Denote by $\alpha_H : A' \to D$ the total integral for the action (Z, H). This naturally induces a unitary A-comodule map $\alpha_H \otimes Id_A : (A' \otimes_R A)^{A'} \to (B \otimes_R A)^{A'}$. By lemma III.42, we have an A-comodule isomorphism $\beta : A \simeq (A' \otimes_R A)^{A'}$. The composite $\alpha_H \circ \beta$ is a total integral and the action $(Z \times^H G, G)$ is tame. □

1.3 Base changes

Tame actions are stable under base change.

Lemma VI.3. *If the action (X, G) is tame, then after an affine base change $R' \to R$ the action $(X_{R'}, G_{R'})$ is also tame.*

Proof. Since the action (X, G) is tame, there is a A_C-comodule map $A \to B$, which induces naturally a comodule map $A_{R'} \to B_{R'}$. □

The next lemma will allow us to assume that the base is equal to the quotient, if the structural map $X \to S$ has the same properties as the quotient morphism $X \to Y$.

Lemma VI.4. *The following assertions are equivalent:*

1. *The action $(X, {}_CG)$ over C is tame .*
2. *The action (X, G) over S is tame.*
3. *The action $({}_CX, {}_CG)$ over C is tame.*

Proof. (1) $\Rightarrow$ (2) Let $\alpha : {}_CA \to B$ be a total integral for the tame action $(X, {}_CG)$. Since B is a ${}_CA$-comodule via $\rho_B : B \to B \otimes_R A \simeq B \otimes_C {}_CA$, the composite $\alpha' := \alpha \circ i_A : A \to B$, where $i_A : A \to C \otimes_R A$ maps a to $1 \otimes a$, is a unitary A-comodule map so a total integral for the action (X, G).
(2) $\Rightarrow$ (3) Follows from base change, by Lemma VI.3.
(3) $\Rightarrow$ (1) Denote by $\beta : {}_CA \to {}_CB$ the total integral for the action $({}_CX, {}_CG)$. Recall that $Id_C \otimes_R A$ is a ${}_CA$-comodule via $Id_C \otimes \Delta$ and ${}_CB$ is a ${}_CA$-comodule map via $Id_C \otimes \rho_B$. Consider the composite $\beta' := \mu_C \circ \beta : C \otimes_R A \to B$ where $\mu_C : C \otimes_R B \to B$ comes from the algebra multiplication of B. For any $b \in B$ and $c \in C$,

$$\begin{aligned}(\mu_C \otimes Id_A)((Id_C \otimes \rho_B)(c \otimes b)) &= (\mu_C \otimes Id_A)(\textstyle\sum c \otimes b_{(0)} \otimes b_{(1)}) \\ &= \textstyle\sum cb_{(0)} \otimes b_{(1)} = \rho_B(c)\rho_B(b) \text{ (since } C = B^A) \\ &= \rho_B(\mu_C(b \otimes c)) \text{ (since } B \text{ is an } A\text{-comodule algebra).}\end{aligned}$$

Thus, μ_C is an A_C-comodule map and β' also being compositions of A_C-comodule maps. □

We take the opportunity to indicate without proofs some more results about the local nature of tameness.

Proposition VI.5. *(see [CEPT96, Proposition 2.11]) Suppose that X is finite over S and G is finite and locally free over S. If $R \to R'$ is a faithfully flat base change and the action $(X_{R'}, G_{R'})$ is tame, then (X, G) is also tame.*

Corollary VI.6. *(see [CEPT96, Proposition 2.15]) Suppose that X is finite over S and G is finite and locally free over S.*

1. *The action (X, G) is tame if and only if for all prime ideals $\mathfrak{p}$ of R the action $(X_{R_\mathfrak{p}}, G_{R_\mathfrak{p}})$ over the local ring $R_\mathfrak{p}$ is tame. The analogous statement obtained by replacing the localization $R_\mathfrak{p}$ by the strict local henselizations of R at $\mathfrak{p}$ is also true.*

2. *If $\{Spec(R_i)\}$ is a cover of $Spec(R)$ by affine open subschemes, then (X, G) is tame if and only if all (X_{R_i}, G_{R_i}) are tame.*

We have more generally the following result.

Proposition VI.7. *(see [CEPT96, Corollary 2.19.]) For actions over Noetherian rings, tameness descends from finite, faithfully flat extensions.*

1.4 Cosemisimple Hopf algebras induce tame trivial actions

According to [CEPT96, 2.d.], the following result shows that the trivial action of a group scheme associated to a cosemisimple Hopf algebra is tame (see Definition III.29).

Proposition VI.8. *If A is cosemisimple Hopf algebra then the trivial action $(Spec(R), G)$ is tame.*

Proof. Suppose that A is cosemisimple. Since R is an A-comodule, by lemma III.30 R admits a complement H in A, which is an A-comodule. The projection $\alpha : A = H \oplus R \to R$ is R-linear and such that $\alpha(1_A) = 1_R$. For $a \in A$, write $a = (\lambda, h)$ where $\lambda \in R$ and $h \in H$. Then

$$(\alpha \otimes A) \circ \Delta_A(a) = (\alpha \otimes A)(\Delta_A(\lambda), \Delta_A(h)) = \lambda \otimes 1 = \rho_A \circ \alpha(a).$$

Therefore, α is a total integral and $(Spec(R), G)$ is tame. □

Lemma VI.9. *If the trivial action $(Spec(R), G)$ is tame, then any action by the group scheme G is tame.*

Proof. This follows from the inclusion $Com_A(A, R) \subseteq Com_A(A, B)$. □

As an immediate consequence, we obtain the following corollary.

Corollary VI.10. *Any action by group schemes associated to cosemisimple Hopf algebras is tame.*

More precisely, tame trivial actions correspond exactly to the relatively cosemisimple Hopf algebras.

Definition VI.11. *We say that a Hopf algebra A is **relatively cosemisimple** if for all $M \in \mathcal{M}^A$ the submodules which are direct summands in $\mathcal{M}_R$ are also direct summands in $\mathcal{M}^A$.*

Proposition VI.12. *(see [BW03, 16.10]) The Hopf algebra A is relatively cosemisimple if and only if the trivial action $(Spec(R), G)$ is tame.*

2 Tame quotient stacks

In this section, suppose that the group scheme G and the scheme X are finitely presented over S and that the inertia group stack $\mathcal{I}_{[X/G]}$ is finite. By Theorem IV.28, this insures that the quotient stack admits a coarse moduli space $\phi : [X/G] \to M$ which is proper.

2.1 Definition

We first recall the definition of tame quotient stack introduced in [AOV08].

Definition VI.13. *Under the previous hypotheses, we say that $[X/G]$ is a **tame quotient stack** if the functor $\phi_* : Qcoh([X/G]) \to Qcoh(M)$ is exact.*

Remark VI.14. *We denote by H a flat closed subgroup of G. Suppose that H acts on a scheme Z over S. Directly from Proposition IV.19, we have that if the quotient stack $[X/H]$ is tame, then the quotient stack associated to the action of G on the balanced product $[X \times^H G/G]$ is also tame.*

2.2 Base change

The following lemma studies the local nature of tameness for quotient stacks.

Lemma VI.15. *([Alp08, Proposition 3.10]) For any morphism of S-schemes $g : M' \to M$, we consider the following 2-cartesian diagram:*

$$\begin{array}{ccc} [X/G]_{M'} & \xrightarrow{g'} & [X/G] \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ M' & \xrightarrow[g]{} & M \end{array}$$

Suppose that ϕ (resp. ϕ') is the coarse moduli space for $[X/G]$ (resp. $[X/G]_{M'}$).

1. *If g is faithfully flat and the quotient stack $[X/G]_{M'}$ is tame then the quotient stack $[X/G]$ is tame.*
2. *If the quotient stack $[X/G]$ is tame then the quotient stack $[X/G]_{M'}$ is also tame.*

Proof. 1. From the 2-cartesian diagram we deduce that the functors $g^*\phi_*$ and $\phi'_*g'^*$ are isomorphic. Since g is flat, g' is flat as well and g'^* is exact; also ϕ'_* is exact by assumption, so the composite $\phi'_*g'^*$ is exact, hence so is $g^*\phi_*$. But, since g is faithfully flat, we have that ϕ_* is also exact as required.

2. First suppose that g is an open, quasi-compact immersion. Let $0 \to \mathcal{F}'_1 \to \mathcal{F}'_2 \to \mathcal{F}'_3 \to 0$ be an exact sequence of $\mathcal{O}_{[X/G]_{M'}}$-modules. Set $\mathcal{F}_3 := g'_*\mathcal{F}_2/g'_*\mathcal{F}_1$. Then

$$0 \to g'_*\mathcal{F}'_1 \to g'_*\mathcal{F}'_2 \to \mathcal{F}_3 \to 0$$

is exact. Moreover, $g'^*\mathcal{F}_3 \simeq \mathcal{F}'_3$ since the adjunction morphism $g'^*g'_* \to id$ is an isomorphism. Since ϕ_* is exact by assumption, $\phi_*g'_*\mathcal{F}'_2 \to \phi_*\mathcal{F}_3$ is also surjective, but then $g_*\phi'_*\mathcal{F}'_2 \to \phi_*\mathcal{F}_3$ is surjective as well. Since g is an open immersion, $\phi'_*\mathcal{F}'_2 \to g^*\phi_*\mathcal{F}_3$ is surjective. Finally, since $g^*\phi_*$ and $\phi'_*g'_*$ are isomorphic functors, $\phi'_*\mathcal{F}'_2 \to \phi'_*\mathcal{F}'_3$ is surjective.
We consider now any morphism of schemes $g : M' \to M$. Since the tameness property on a stack is Zariski local, we can assume M' and M affine. Then g' is also affine, so the functor g'_* is exact. By assumption ϕ_* is exact, therefore $\phi_*g'_* = g_*\phi'_*$ is exact. But the functor g_* has the property that a sequence $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is exact if and only if $g_*\mathcal{F}_1 \to g_*\mathcal{F}_2 \to g_*\mathcal{F}_3$ is exact. It follows that ϕ'_* is exact as required.

□

Chapter VII

Tame actions by a constant group scheme as generalization of the arithmetic case

In this chapter, we find the full study of the particular case of actions by constant group schemes. Some of this study was began in [CEPT96]. This is important to understand how to pass from an algebraic to a geometric context. In particular, Lemma VII.4 explains the usual confusion, that is, the definition of inertia groups for action of group schemes we generalize not the abstract decomposition group but really the abstract inertia group (See Chapter I). Through this personal detailed proof, the reader will understand also the difficulty of defining a decomposition group in this context.

In the following, B is a commutative R-algebra and Γ is an abstract finite group. We consider an action (B,Γ) of Γ on B by automorphisms of R-algebras. We denote by $B^\Gamma := \{b \in B | \gamma.b = b\}$ the ring of invariants for this action.

1 Actions by a constant group scheme

Definition VII.1. *Given Γ an abstract finite group, we can consider the R-algebra $A := Map(\Gamma, R)$ of all the maps from Γ to R. This is a free R-algebra, a basis is given by the maps f_γ such that $f_\gamma(\sigma) = \delta_{\gamma,\sigma}$, for all γ, $\sigma \in \Gamma$. Notice that $f_\gamma^2 = f_\gamma$, $f_\gamma f_\tau = 0$ if $\tau \neq \gamma$ and $\sum_\gamma f_\gamma = 1_A$. The R-algebra A is a Hopf algebra where*

1. *the comultiplication $\Delta : A \to A \otimes_R A$ is defined by $\Delta(f_\rho) = \sum_{\rho=\sigma\tau}(f_\sigma \otimes f_\tau)$;*
2. *the counity $\epsilon : A \to R$ is defined by $\epsilon(f_\sigma) = \begin{cases} 1 & si\ \sigma = e \\ 0 & otherwise. \end{cases}$*
3. *the coinverse $S : A \to A$ is defined by $S(f_\sigma) = f_{\sigma^{-1}}$.*

The associated affine group scheme G over S is called ***the constant group scheme attached to*** *Γ.*

The following lemma explains the terminology.

Lemma VII.2. *Let Γ be an abstract finite group, G be the constant group scheme associated to Γ over S and T a connected R-algebra. Then, for $\phi \in Hom_{R\text{-}Alg}(R^\Gamma, T)$, there is a unique*

$\gamma_0 \in \Gamma$ *such that* $\phi(f_{\gamma_0}) = 1$ *and* $\phi(\gamma) = 0$, *for* $\gamma \neq \gamma_0$. *This defines an isomorphism* $G(T) \simeq \Gamma$ *sending* ϕ *to this unique* γ_0.

Proof. Given $\gamma \in \Gamma$, since $f_\gamma^2 = f_\gamma$, an R-morphism $R^\Gamma \to T$ maps f_γ to an idempotents of T, whose only indempotents are 0 and 1. So, since $\sum_{\gamma\in\Gamma} f_\gamma = 1$, there exists $\gamma_0 \in \Gamma$ such that f_{γ_0} maps to 1. But, since $f_\gamma f_\beta = 0$ for $\gamma \neq \beta$ in Γ, f_γ maps to 0 whenever $\gamma \neq \gamma_0 \in \Gamma$. Thus, the morphism $G(T) \to \Gamma$ which maps $\phi \in Hom_{R\text{-Alg}}(R^\Gamma, T)$ to this unique γ_0 such that $\phi(f_{\gamma_0}) = 1$ is an isomorphism. □

The following lemma permits to see the action by constant group schemes attached to an abstract group Γ on X as an action of Γ on B defined in Chapter I.

Lemma VII.3. *Let G be the constant group scheme attached to Γ. An application $\rho_B : B \to B \otimes_R A$ endowes B with a structure of A-comodule algebra if and only if the map $r : \Gamma \times B \to B$ given by $\rho_B(b) = \sum_\gamma r(\gamma, b) \otimes f_\gamma$ defines an action of Γ on B by automorphisms of R-algebras. In other words, the data of an action of G on X is equivalent to the data of an action of Γ on B. Moreover, the ring of invariants B^A is equal to the ring of invariants B^Γ.*

Proof.

1. Since, for any $b \in B$, $\rho_B(b) \in B \otimes_R B$, so, we can write it uniquely thanks to the basis $\{f_\gamma\}$ as $\rho_B(b) = \sum_\gamma r(\gamma, b) \otimes f_\gamma$. Then,

$$\begin{aligned}(Id_B \otimes \Delta)(\rho_B(b)) &= (Id_B \otimes \Delta)(\textstyle\sum_\gamma r(\gamma, b) \otimes f_\gamma) \\ &= \textstyle\sum_\gamma r(\gamma, b) \otimes \Delta(f_\gamma) = \sum_\gamma \sum_{\gamma=\sigma\tau} r(\gamma, b) \otimes f_\sigma \otimes f_\tau \ (1)\end{aligned}$$

and,

$$\begin{aligned}(\rho_B \otimes Id_B)(\rho_B(b)) &= (\rho_B \otimes Id_B)(\textstyle\sum_\beta r(\beta, b) \otimes f_\beta) \\ &= \textstyle\sum_\beta \rho_B(r(\beta, b)) \otimes f_\beta = \sum_\beta \sum_\lambda r(\lambda, r(\beta, b)) \otimes f_\lambda \otimes f_\beta \ (2)\end{aligned}$$

applying equalities (1) and (2) to $(1 \otimes g \otimes g')$, we obtain:

$$(Id_B \otimes \Delta_A) \circ \rho_B = (\rho_B \otimes Id_B) \circ \rho_B \Leftrightarrow \forall b \in B,\ g,\ g' \in \Gamma, r(g, r(g', b)) = r(gg', b).$$

Moreover, by definition of the counity ϵ, we have:

$$(Id_B \otimes \epsilon)(\rho_B(b)) = (Id_B \otimes \epsilon)(\sum_\gamma r(\gamma, b) \otimes f_\gamma) = \sum_\gamma r(\gamma, b) \otimes \epsilon(f_\gamma) = r(e, b) \otimes 1.$$

So,

$$(Id_B \otimes \epsilon) \circ \rho_B = Id_B \otimes 1 \Leftrightarrow \forall b \in B, r(b, e) = b.$$

2. For any $b \in B$, we have

$$\begin{aligned}b \in B^A \Leftrightarrow\ & \rho_B(b) = b \otimes 1_A \Leftrightarrow \textstyle\sum_\gamma r(\gamma, b) \otimes f_\gamma = b \otimes \sum_\gamma f_\gamma \\ \Leftrightarrow\ & \textstyle\sum_\gamma r(\gamma, b) \otimes f_\gamma = \sum_\gamma b \otimes f_\gamma \Leftrightarrow r(\gamma, b) = b,\ \forall \gamma \in \Gamma.\end{aligned}$$

The direct direction of the last equivalence is obtained by applying the equality to $1 \otimes \gamma$.
□

2 Inertia groups for actions by a constant group scheme

Proposition VII.4. *Let Γ be an abstract finite group, G be the associated constant group scheme and $\mathfrak{p} \in X$. The inertia group $I_G(\mathfrak{p})$ of G at $\mathfrak{p}$ is the constant group scheme associated to the abstract inertia group $\Gamma_0(\mathfrak{p})$ at the prime ideal $\mathfrak{p}$ for the action of Γ on B (cf. Lemma VII.3).*

Proof. In the following, we follow the notations of Definition VII.1. Write $\zeta : Spec(k(\mathfrak{p})) \to X$ corresponding to the morphism $\zeta^\sharp : B \to k(\mathfrak{p})$. The inertia group at $\mathfrak{p}$ is by Proposition V.2, defined as the fiber product:

$$\begin{array}{ccc} I_G(\mathfrak{p}) & \xrightarrow{p_1} & G \times_S Spec(k(\mathfrak{p})) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \mu_X \circ (Id,\zeta)} \\ Spec(k(\mathfrak{p})) & \xrightarrow{(\zeta,Id)\circ\Delta} & X \times_S Spec(k(\mathfrak{p})) \end{array}$$

Since $I_G(\mathfrak{p})$ is a closed subgroup scheme of a constant group over the field $k(\mathfrak{p})$, so, it is also constant. Thus, it is enough to prove that $I_G(\mathfrak{p})(Spec(k(\mathfrak{p})) = \Gamma_0(\mathfrak{p})$.
We consider the following diagram at the $k(\mathfrak{p})$-points:

$$\begin{array}{ccc} I_G(\mathfrak{p})(k(\mathfrak{p})) & \xrightarrow{p_1} & G \times_S Spec(k(\mathfrak{p}))(k(\mathfrak{p})) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \mu_X \circ (Id,\zeta)} \\ Spec(k(\mathfrak{p}))(k(\mathfrak{p})) & \xrightarrow{(\zeta,Id)\circ\Delta} & X \times_S Spec(k(\mathfrak{p}))(k(\mathfrak{p})) \end{array}$$

Since G is the constant group scheme associated to Γ, $G_{k(\mathfrak{p})}(Spec(k(\mathfrak{p})) \simeq \Gamma$ where the isomorphism is described in lemma VII.2. Moreover, $X_{k(\mathfrak{p})}(k(\mathfrak{p})) = Hom_{k(\mathfrak{p})}(B, k(\mathfrak{p}))$. Recall that the Galois morphism $\Lambda : B \otimes_R k(\mathfrak{p}) \to R^\Gamma \otimes_R k(\mathfrak{p})$ maps $b \otimes t \in B \otimes_R k(\mathfrak{p})$ to $(t \otimes 1).(\zeta \otimes Id_A) \circ \rho_B(b)$. And so the morphism $G \times_S Spec(k(\mathfrak{p}))(Spec(k(\mathfrak{p})) \to X \times_S Spec(k(\mathfrak{p}))(k(\mathfrak{p}))$ or equivalently, the morphism $Hom_{k(\mathfrak{p})}(k(\mathfrak{p})^\Gamma, k(\mathfrak{p})) \to Hom_{k(\mathfrak{p})}(B \otimes_R k(\mathfrak{p}), k(\mathfrak{p}))$ corresponds to the composition $\Lambda \circ -$. By the description of the isomorphism $G_{k(\mathfrak{p})}(Spec(k(\mathfrak{p})) \simeq \Gamma$, we obtain that the morphism $\theta : \Gamma \to Hom_{k(\mathfrak{p})}(B, k(\mathfrak{p}))$ maps $\gamma \in \Gamma$ to the $k(\mathfrak{p})$-morphism $\zeta^\sharp(r(\gamma, -)) : B \to k(\mathfrak{p})$. Finally, we notice that $Spec(k(\mathfrak{p}))(k(\mathfrak{p})) = \{Id\}$ which implies that the $k(\mathfrak{p})$-morphism $Spec(k(\mathfrak{p}))(k(\mathfrak{p})) \to X \times_S Spec(k(\mathfrak{p}))(k(\mathfrak{p}))$ or equivalently the morphism $\{Id_{k(\mathfrak{p})}\} \to Hom_{k(\mathfrak{p})}(B, k(\mathfrak{p}))$ maps $Id_{k(\mathfrak{p})}$ to $\zeta^\sharp$.
After making all these identifications, we obtain that the inertia group can be defined as the

fiber product:

$$\begin{array}{ccc} I_G(\mathfrak{p})(k(\mathfrak{p})) & \xrightarrow{p_1} & \Gamma \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \theta} \\ \{Id_{k(\mathfrak{p})}\} & \xrightarrow[\zeta^\sharp]{} & Hom_{k(\mathfrak{p})}(B, k(\mathfrak{p})) \end{array}$$

This proves the proposition, since here we have a description of the inertia group corresponding to the action of Γ on B at the prime ideal $\mathfrak{p}$. □

We obtain naturally the following corollary, giving a hint on how to generalize the notion of tameness defined in I.18 to this context.

Corollary VII.5. *Let Γ be an abstract finite group, G be the associated constant group scheme and $\mathfrak{p} \in X$. The inertia group $I_G(\mathfrak{p})$ of G at $\mathfrak{p}$ is associated to a cosemisimple Hopf algebra if and only if the order of the abstract inertia group at the prime ideal $\mathfrak{p}$ for the action of Γ on B $\Gamma_0(\mathfrak{p})$ is prime to the characteristic of the residue field $k(\mathfrak{p})$.*

Proof. This is a direct consequence of Maschke's Theorem III.32. □

3 Tame actions by a constant group scheme and trace surjectivity

Recall that for an action of Γ on B by automorphisms, the tameness is characterized by the surjectivity of the trace map (see Theorem I.32). We will see that this can be generalized for the notion of tameness on actions by constant group schemes which explains the terminology. Here, we suppose that G is a constant group scheme associated to Γ. Keeping the notations of Lemma VII.3, define the trace map t_Γ by:

$$\begin{array}{cccc} t_\Gamma : & B & \to & C \\ & b & \mapsto & \sum_{\gamma' \in \Gamma} r(\gamma', b) \end{array}$$

Indeed, for any $b \in B$ and $\gamma \in \Gamma$, we have:

$$r(\gamma, t_\Gamma(b)) \quad = \quad r(\gamma, \textstyle\sum_{\gamma' \in \Gamma} r(\gamma', b)) = \sum_{\gamma' \in \Gamma} r(\gamma, r(\gamma', b)) = \sum_{\gamma' \in \Gamma} r(\gamma^{-1}\gamma', b) = t_\Gamma(b).$$

So, $tr(b) \in C$.

Lemma VII.6. *1. The map t_Γ is surjective if and only if there is $b \in B$ such that $t_\Gamma(b) = 1_C$*

2. A map α is an A-comodule map if and only if for any γ, $\sigma \in \Gamma$, $\alpha(f_{\gamma\sigma^{-1}}) = r(\sigma, \alpha(f_\gamma))$.

Proof. 1. The direct sense is clear. Conversely, suppose that there is $b \in B$ such that $t_\Gamma(b) = 1_C$. Let $c \in C$. For any $\gamma \in \Gamma$, $r(\gamma, c) = c$ ($\square$). Thus,

$$\begin{aligned} c &= c1_C = ct_\Gamma(b) = c\sum_{\gamma' \in \Gamma} r(\gamma', b) = \sum_{\gamma' \in \Gamma} cr(\gamma', b) \\ &= \sum_{\gamma' \in \Gamma} r(\gamma, c) r(\gamma', b) \ (by\ (\square)) \\ &= \sum_{\gamma' \in \Gamma} r(\gamma', bc) = tr(bc) \end{aligned}$$

This means that t_Γ is surjective.

2. α is an A-comodule map. That is, for any $\theta \in \Gamma$,

$$\begin{aligned} & \rho \circ \alpha(f_\theta) = (\alpha \otimes B)\Delta(f_\theta) \\ \Leftrightarrow\ & \rho(\alpha(f_\theta)) = (\alpha \otimes B)(\textstyle\sum_{ab=\theta} f_a \otimes f_b) \\ \Leftrightarrow\ & \textstyle\sum_{\lambda \in \Gamma} r(\lambda, \alpha(f_\theta)) \otimes f_\lambda = \sum_{ab=\theta} \alpha(f_a) \otimes f_b \end{aligned}$$

As $(f_\lambda)_{\lambda \in \Gamma}$ is a base for A, by identification, we obtain the equivalence that we want.

$\square$

Lemma VII.7. *The action $(X, G) = (Spec(A), Spec(B))$ is tame if and only if t_Γ is surjective.*

Proof. Suppose that the action (X, G) is tame, that is, there is $\alpha : A \to B$ a unitary A-comodule map. Thus,

$$\begin{aligned} 1_B &= \alpha(1_A) \text{ (because } \alpha \text{ is unitary)} \\ &= \alpha(\textstyle\sum_\gamma f_\gamma) \text{ (since } \sum_{\gamma \in \Gamma} f_\gamma = 1_A) \\ &= \textstyle\sum_\gamma \alpha(f_\gamma) = \sum_\gamma \gamma^{-1}\alpha(f_1) \text{ (by the previous lemma)} \\ &= \textstyle\sum_\gamma \gamma\alpha(f_1)) = tr(\alpha(f_1)) \end{aligned}$$

Then, for $b = \alpha(f_1)$, we have $t_\Gamma(b) = 1_C$ and the result follows by Lemma VII.6. Conversely, suppose there is $b \in B$ such that $t_\Gamma(b) = 1_C$. Define α taking $\alpha(f_1) := b$. So that, by the previous lemma, we put $\alpha(f_\gamma) := \gamma^{-1}b$, for any $\gamma \in \Gamma$ and this defines a total integral. $\square$

Example VII.8. *Let $B = \mathbb{Z}[X]/(X^2 + 1)$ and $C_2 =< g >$ be a cyclic group of order 2. Let G be the constant group scheme associated to C_2 over $\mathbb{Z}$. Denote by x the class of X in B. The action (X, G) is induced by the action (B, Γ) (cf. lemma VII.3)*

$$\begin{aligned} \Gamma \times B &\to B \\ (g, x) &\mapsto r(g, x) = \begin{cases} x & \text{if } g \text{ trivial} \\ -x & \text{if } g \text{ non trivial.} \end{cases} \end{aligned}$$

We have $B^A = B^\Gamma = \mathbb{Z}$ (we can use that B is integral since $X^2 + 1$ is irreducible over $\mathbb{Z}$ of characteristic zero). We have that the trace map $t_{C_2} : B \to \mathbb{Z}$ is defined by mapping an element $\alpha_1 + \alpha_2 x$ (where $\alpha_i \in \mathbb{Z}$) to $2\alpha_1$. So, $1_\mathbb{Z}$ has no antecedent by the trace map thus this action is not tame.

4 Slice theorem for actions by a constant group scheme

Let Γ be an abstract finite group. Suppose that G is the constant group scheme associated to Γ over S.

Theorem VII.9. *An action (X, G) over S admits slices at each closed point with étale slice groups.*

Proof. This theorem is a direct consequence of Theorem I.27 together with Proposition VII.4. Indeed, take $\mathfrak{p} \in Y$ and $\mathfrak{P} \in X$ over $\mathfrak{p}$. Write $C_{\mathfrak{p}}^{hs}$ for the strict Henselization of C at $\mathfrak{p}$. In Definition III.49, we can take $Y' = Spec(C')$ as an étale subextension of $Spec(C_{\mathfrak{p}}^{hs})$ over Y (since $X \to Y$ is finite) with C' a local ring, G_0 the constant group scheme associated to $\Gamma_0(\mathfrak{P})$ over Y' and $Z := Spec((B \otimes_C C')_{\mathfrak{m}})$ where $\mathfrak{m}$ is a prime ideal above the maximal ideal of the local ring C' corresponding to $\mathfrak{p}$. □

Chapter VIII

Linearly reductive group schemes

In this chapter, the group scheme G is finitely presented over S. We recall interesting properties of linearly reductive group schemes established in [AOV08, §2.3], adding some new result (Theorem VIII.18), the most important being about lifting of linearly reductive groups as subgroups. Also, we find Proposition VIII.14 about the structure of inertia groups for some particular linearly reductive group schemes. We dedicate §3 to the study of the particular case of diagonalizable group schemes, giving the structure of inertia groups for action of diagonalizable groups.

1 Definition

Definition VIII.1. *A group scheme $G \to S$ is called* ***linearly reductive*** *if the functor $Qcoh^G(S) \to Qcoh(S)$ sending F to F^G is exact.*

Remark VIII.2.
1. *If $G \to S$ is a finite, flat group scheme, then the coarse moduli space of $B_S G$ is S. Thus, $B_S G$ is tame if and only if G is linearly reductive.*
2. *If G is linearly reductive, then the quotient stack $[X/G]$ is tame. Every quotient stack defined by an action of a linearly reductive group is tame.*

Lemma VIII.3. *(see [Alp08, Proposition 12.6]) If $R = k$ is a field and $G := Spec(A)$ is finite over S, the group scheme G is linearly reductive if and only if the Hopf algebra A is cosemisimple.*

Remark VIII.4. *Corollary VII.5 says that the tameness condition for extensions of rings can be translated, for an action by a constant group scheme, as requiring that the inertia group schemes at any topological point of X are linearly reductive.*

2 Base changes

As a direct consequence of Lemma VI.15, we obtain the following proposition.

Proposition VIII.5. *Let $S' \to S$ be a morphism of schemes. Then,*

1. *If $G \to S$ is linearly reductive, then $G_{S'} \to S'$ is linearly reductive.*
2. *If $S' \to S$ is faithfully flat and $G_{S'} \to S'$ is linearly reductive, then $G \to S$ is linearly reductive.*

3 Particular case: Diagonalizable group schemes

3.1 Actions by a diagonalizable group scheme

Definition VIII.6. *Let M be an abelian group and $A := R[M]$ the* ***group algebra of*** *M* ***over*** *R i.e. the free R-module such that the elements of M form a basis (the multiplication map giving the structure of algebra on A is induced by the multiplication on M). The R-algebra has a structure of Hopf algebra over R taking $\Delta(m) = m \otimes m$ as comultiplication, $\epsilon(m) = 1$ as counitity and $S(m) = m^{-1}$ as antipode, for any $m \in M$. The corresponding group scheme G is called a* ***diagonalizable group scheme****.*

Lemma VIII.7. *Let M be an abelian group. The datum of an action by a diagonalizable group $G = Spec(R[M])$ on X is equivalent to the datum of a graduation for the algebra B that means $B = \oplus_{m \in M} B_m$ and $B_{m'} B_m \subseteq B_{m+m'}$, for any m, $m' \in M$. Moreover, $B_0 \simeq C$ is the ring of invariants.*

Proof. We have seen that the datum of this action is equivalent to the datum of a R-linear map $\rho_B : B \to B \otimes_R R[M] = \oplus_{m \in M} B \otimes_R mR$ giving B a structure of A-comodule algebra which is here equivalent to the data of R-linear maps of B on itself $(\rho_m)_{m \in M}$ defined uniquely (as M is a basis of $R[M]$) by setting for any $b \in B$,

$$\rho_B(b) = \sum_{m \in M} \rho_m(b) \otimes m.$$

Since B is an A-comodule, $\sum_{m \in M} \rho_m(b) \otimes 1 = b \otimes 1$, for any $b \in B$. By identification (since M is a basis of $R[M]$), this is equivalent to have $\sum_{m',\, m \in M} \rho_m(b) = b$, for any $b \in B$. Then, $B = \oplus_{m \in M} B_m$ where $B_m = \rho_m(B)$. Moreover, for any $b, b' \in B$, $\rho_B(bb') = \rho_B(b)\rho_B(b')$, thus $\sum_{n \in N} \rho_n(bb') \otimes n = \sum_{m,m' \in M} \rho_m(b)\rho'_m(b) \otimes (m + m')$. Finally, we obtain by identification on the basis, $\rho_{m+m'}(bb') = \rho_m(b)\rho_{m'}(b')$, hence $B_{m'} B_m \subseteq B_{m+m'}$, for any m and $m' \in M$. $\square$

Remark VIII.8. *Since $B_0 = C$, we have a natural projector $p : B \to C$. This result will be generalized for tame actions (see Lemma IX.1).*

3.2 Inertia groups for actions by a diagonalizable group scheme

Proposition VIII.9. *Let M be an abelian group, $R[M]$ be the group algebra over M and G the diagonalizable group scheme over R. The inertia group of G at $x \in X$ is the diagonalizable group scheme over $k(x)$ associated to the group algebra $k(x)[M']$ for some M' quotient of M.*

Proof. Since the inertia group is a closed subgroup scheme of $G \times_S Spec(k(x)) \simeq Spec(k(x)[M])$ over the field $k(x)$, [SGA64, exposé IX , §8] proves the theorem. □

3.3 Actions by a diagonalizable group scheme are always tame

Proposition VIII.10. *Let M be an abelian group and $G = Spec\ R[M]$ be a diagonalizable group scheme over R. Then, the action (X, G) is tame. In other words, the action by a diagonalizable group scheme is always tame.*

Proof. We define the total integral α by $\alpha(m) = 0_B$ if $m \neq 1_M$ and $\alpha(1_M) = 1_B$. For all $m \neq 1_M \in M$, $(\alpha \otimes 1) \circ \Delta(m) = 0_B \otimes 1_B = \rho_B \otimes \alpha(m)$ and for $m = 1_M$, $(\alpha \otimes 1) \circ \Delta(m) = 1_B \otimes 1_M = \rho_B \otimes \alpha(m)$. Thus, (X, G) is tame. □

4 Classification of linearly reductive group schemes

In this section, all groups schemes will be flat, finite and finitely presented over S. We recall here without proof the classification of linearly reductive group schemes established in [AOV08].

Definition VIII.11. 1. *An étale finite group scheme $H \to S$ is called **tame** if its degree is prime to all the residual characteristics.*

2. *A group scheme is called **well split** if it is isomorphic with a semi-direct product $H \ltimes \Delta$ where H is a tame constant group scheme and Δ is diagonalizable.*

Lemma VIII.12. *[AOV08, Lemma 2.11] Let G be a locally well split group scheme over a field k. Let Δ_0 be the connected component of the identity and $H = G/\Delta_0$.*

1. *The group scheme Δ_0 is locally diagonalizable and H is an étale and tame group scheme.*
2. *There is a finite purely inseparable extension k' of k, such that $G_{k'}$ is a semi-direct product $H_{k'} \ltimes \Delta_{0k'}$. In particular, if k is perfect, then G is the unique semi-direct product $H \ltimes \Delta_0$ corresponding to the connected étale sequence.*
3. *There is a finite extension k' of k such that $G_{k'}$ is locally well split. In particular, if k is algebraically closed, then G is well split.*

The following theorem gives the structure of linearly reductive group schemes.

Theorem VIII.13. *[AOV08, Theorem 2.16] Let $G \to S$ be a finite flat group scheme of finite presentation. The following conditions are equivalent.*

1. *$G \to S$ is linearly reductive.*
2. *$G \to S$ is locally well-split for the faithfully flat, quasi compact topology.*
3. *The fibers of $G \to S$ are linearly reductive.*

4. The geometric fibers of $G \to S$ are well-split.

Furthermore, if S is noetherian these conditions are equivalent to either of the following two conditions.

5. The closed fibers of $G \to S$ are linearly reductive.

6. The geometric closed fibers of $G \to S$ are well-split.

5 Inertia groups for an action of a finite flat linearly reductive group scheme

Proposition VIII.14. *Suppose that G is a finite flat linearly reductive group scheme over S. Let $x : Spec(\Omega) \to X$ be an Ω-point where Ω is a perfect field. Write the well split group scheme G_Ω as a semi-direct product $\Delta \rtimes \Gamma$. The inertia group at x is $I_G(x) = I_\Delta(x) \rtimes I_\Gamma(\bar{x})$ where $I_\Delta(x)$ is the inertia group of the induced action (X, Δ) and $I_\Gamma(\bar{x})$ is the inertia group of the induced action $(X/\Delta, \Gamma)$, $\bar{x} : Spec(\Omega) \to X \to X/\Delta$.*

Proof. This proposition is a consequence of Proposition V.6 and of the fact that the semi-direct products correspond to the connected étale sequence over a perfect field (see Lemma VIII.12) and that the sections defining these semi-direct products are given by the fact that the étale component is isomorphic to G_{red}. □

6 Cohomology for lineary reductive group schemes

As we will see, the cohomology of linearly reductive group schemes has some very interesting cohomological vanishing properties that are useful for deformation involving such group schemes.

Lemma VIII.15. *Let k be a field, $\mathcal{F}$ be a coherent sheaf on B_kG and $L \in D^{[0,1]}_{coh}(\mathcal{O}_{B_kG})$. If $G \to Spec(k)$ is an fppf linearly reductive group scheme, then $Ext^i(L, \mathcal{F}) = 0$, for $i \neq -1,\ 0$.*

Proof. Since k is a field, any coherent sheaf on B_kG is locally free, and therefore for any coherent sheaf $\mathcal{F}$ on B_kG, we have:

$$RHom(L, \mathcal{F}) \in D^{[-1,0]}_{coh}(\mathcal{O}_{B_kG})$$

Since the global section functor is exact on the category $Coh(\mathcal{O}_{B_kG})$ (since G is linearly reductive), we obtain that

$$Ext^i(L, \mathcal{F}) = 0,\ \ for\ i \neq -1, 0$$

□

Remark VIII.16. *In particular, by Lemma B.30, the cotangent complex $L_{B_kG/k}$ of the structural morphism $B_kG \to k$ (where k is a field) belongs to $D^{[-1,0]}_{coh}(\mathcal{O}_{B_kG})$. If $G \to Spec(k)$ is an* *fppf linearly reductive group scheme and* $\mathcal{F}$ *is a coherent sheaf on* B_kG *then* $Ext^i(L_{B_kG/k}, \mathcal{F}) = 0$*, for* $i \neq -1,\ 0$*. Moreover, if* G *is smooth,* $L_{B_kG/k} \in D^{[0]}_{coh}(\mathcal{O}_{B_kG})$ *and* $Ext^i(L_{B_kG/k}, \mathcal{F}) = 0$ *also for* $i = 0$*.*

7 Etale local liftings of linearly reductive group schemes

First, we just extend a linearly reductive group on a point to a linearly reductive group on an étale neighborhood of the point.

Theorem VIII.17. *([AOV08, Proposition 2.18]) Let* $\mathfrak{p} \in S$ *be a point,* G *be a linearly reductive group over* $Spec(k(\mathfrak{p}))$*. Then there are an étale morphism* $U \to S$*, a point* $\mathfrak{q} \in U$ *mapping to* $\mathfrak{p}$ *and a flat linearly reductive group scheme* Γ *over* U *such that the pullback* $\Gamma_{k(\mathfrak{q})}$ *is isomorphic to the pullback* $G_{k(\mathfrak{q})}$*.*

Proof. Let $\overline{k(\mathfrak{p})}$ be the algebraic closure of $k(\mathfrak{p})$. By [AOV08, proposition 2.10], $G_{\overline{k(\mathfrak{p})}}$ is well split. This means that it can be written as a semi direct product $H_{\overline{k(\mathfrak{p})}} \ltimes \Delta_{\overline{k(\mathfrak{p})}}$ where $H_{\overline{k(\mathfrak{p})}}$ is étale, thus constant being finite and $\Delta_{\overline{k(\mathfrak{p})}}$ is connected and diagonalizable. Up to passing to a Zariski open neighborhood $S' = Spec(R_{\mathfrak{p}})$ of the point $\mathfrak{p}$ in S, $G_{\overline{k(\mathfrak{p})}}$ is the pullback of $\Gamma = H \ltimes \Delta \to S'$, where H is constant and also tame over S' and then Δ is diagonalizable over S'.
The group G is a twisted form of the fiber $\Gamma_{\mathfrak{p}}$. So, we have to show that all twisted forms of $\Gamma_{\mathfrak{p}} \to Spec(k(\mathfrak{p}))$ can be extended to an étale neighborhood of $\mathfrak{p}$. By [Mil80, III §Twisted form], we know that the twisted forms are classified by an element of the non-abelian cohomological group $H^1_{fppf}(Spec(k(\mathfrak{p})), \underline{Aut}_{k(\mathfrak{p})}(\Gamma_{\mathfrak{p}}))$. Denote by $c(G)$ the class representing G and $\Delta' := \Delta/\Delta^H$. By Lemma A.14, $\underline{Aut}_{k(\mathfrak{p})}(\Gamma)/\Delta'_{\mathfrak{p}}$ is étale, but since all torsors over an étale group are locally trivial for the étale topology then the image of $c(G)$ in

$$H^1_{fppf}(Spec(k(\mathfrak{p})), \underline{Aut}_{k(\mathfrak{p})}(\Gamma_{\mathfrak{p}})/\Delta'_{\mathfrak{p}}) \simeq H^1_{\acute{e}t}(Spec(k(\mathfrak{p})), \underline{Aut}_{k(\mathfrak{p})}(\Gamma_{\mathfrak{p}})/\Delta'_{\mathfrak{p}})$$

is trivial after a finite separable extension k' of $k(\mathfrak{p})$. Thus, denote by $c''(G)$ is the image of $c(G)$ in $H^1_{fppf}(Spec(k(\mathfrak{q})), \underline{Aut}_{k(\mathfrak{q})}(\Gamma_{\mathfrak{p}}))$, the image of $c''(G)$ is trivial in

$$H^1_{fppf}(Spec(k(\mathfrak{q})), \underline{Aut}_{k(\mathfrak{q})}(\Gamma_{\mathfrak{q}})/\Delta'_{\mathfrak{q}}).$$

Since the following sequence is exact

$$H^1_{fppf}(k(\mathfrak{q}), \Delta'_{\mathfrak{q}}) \to H^1_{fppf}(k(\mathfrak{q}), \underline{Aut}_{k(\mathfrak{q})}(\Gamma)) \to H^1_{fppf}(k(\mathfrak{q}), \underline{Aut}_{k(\mathfrak{q})}(\Gamma_{\mathfrak{q}})/\Delta'_{\mathfrak{q}})$$

$c(G)$ comes from a class of $H^1_{fppf}(Spec(k(\mathfrak{q})), \Delta'_{\mathfrak{q}})$. Since Δ' is diagonalizable as a quotient of a diagonalizable group scheme, by the structure of finite diagonalizable groups, it is enough to prove that any element of $H^1_{fppf}(Spec(k(\mathfrak{q})), \mu_n)$ comes from $H^1_{fppf}(U, \mu_n)$ where U is an étale

neighborhood of S''.
By Kummer theory (see [Mil80, III §4 Kummer theory]), we know that all μ_n-torsors over $k(\mathfrak{q})$ are of the form

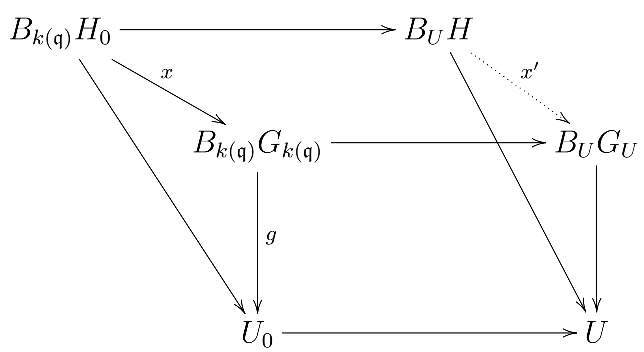
$$Spec(k(\mathfrak{q})[t]/(t^n - a)) \to Spec(k(\mathfrak{q}))$$

for some $a \in k(\mathfrak{p})^*$. Up to taking an affine Zariski open $Spec(D)$ of S'' containing $\mathfrak{q}$ instead of S'', we can assume that $f \in D$ maps on a via the canonical morphism $D \to k(\mathfrak{q})$. Take then $U = Spec(D_f)$ which is étale over S'' and such that $f \in \mathcal{O}_U^*(U)$. But then, the μ_n-torsor $Spec(D_f[t]/(t^n - f)) \to U$ becomes $Spec(k(\mathfrak{q})[t]/(t^n - a)) \to Spec(k(\mathfrak{q}))$ after base change, and this concludes the proof. □

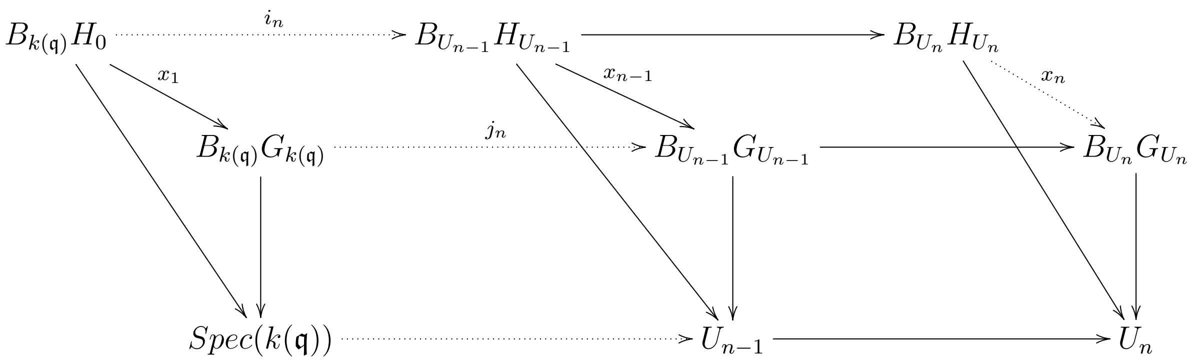
The previous lemma allows us to prove that linearly reductive groups can be extended as subgroups fppf locally, in the following sense.

Theorem VIII.18. *Let $\mathfrak{p}$ be a point of S, G be a finite, flat group scheme over S and H_0 be a finite, flat, linearly reductive closed subgroup scheme of $G_{k(\mathfrak{p})}$ over $Spec(k(\mathfrak{p}))$. Then, there exists a fppf morphism $U \to S$ with a point $\mathfrak{q} \in U$ mapping to $\mathfrak{p}$ and a flat linearly reductive closed subgroup scheme H of G_U over U whose pullback $H_{k(\mathfrak{q})}$ is isomorphic to the pullback of $H_{0k(\mathfrak{q})}$.*

Proof. Let $\mathfrak{p} \in S$. By the previous theorem, there exists an étale morphism $U \to S$ with a point $\mathfrak{q} \in U$ mapping to $\mathfrak{p}$ and a linearly reductive group scheme H over U such that $H_{k(\mathfrak{q})} \simeq H_{0k(\mathfrak{q})}$. Set $U_n := Spec(R/\mathfrak{q}^{n+1})$. One has that H_0 is a subgroup scheme of $G_{k(\mathfrak{p})}$, and this defines a representable morphism of algebraic stacks $x_0 : B_{k(\mathfrak{q})}H_0 \to B_{k(\mathfrak{q})}G_{k(\mathfrak{q})}$ over $k(\mathfrak{q})$. We want to prove the existence a representable morphism of algebraic stacks $x : B_U H \to B_U G_U$ filling in the following 2-commutative diagram



which is the deformation situation treated in Annexe C, Section 5. We can prove, thanks to Grothendieck 's Existence theorem for algebraic stacks B.31 and Artin's approximation theorem B.35, that the existence of such x only depends on the existence of a formal deformation, that is morphisms $x_n : B_{U_n}H_{U_n} \to B_{U_n}G_{U_n}$ filling in the following 2-commutative diagram:



By Theorem B.32, the obstruction to extend the morphism x_{n-1} to x_n lies in

$$Ext^1(Lx^*L_{B_{k(\mathfrak{q})}G_{k(\mathfrak{q})}/k(\mathfrak{q})}, \mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{k(\mathfrak{q})} \mathcal{O}_{B_{k(\mathfrak{q})}G_{k(\mathfrak{q})}})$$

which is trivial for any $n \in \mathbb{N}$ by Corollary VIII.15. It follows that there exists an arrow x_n filling the previous diagram. This leads to the existence of a representable morphism $F : B_U H_U \to B_U G_U$. Take $Q \to U$ the G_U-torsor which is the image via F of the trivial H_U-torsor $H_U \to U$. Furthermore, the functor F induces a homomorphism from $\underline{Aut}_S(H_Q \to Q) = H_Q$ to the automorphism group scheme of the image of the H_Q-torsor $H_Q \to Q$ in $B_Q G_Q$. Since this image is the pullback of Q to Q over $B_Q G_Q$, which is canonically a trivial torsor, its automorphism group is H_Q. So, this defines a group morphism $H_Q \to G_Q$, with $Q \to U$ an fppf morphism as G_U-torsor. Since F is representable, by Lemma B.22, this morphism is injective. Finally, since H is proper and G is separated, H_V is closed in G_V. □

Chapter IX

Main results under tameness conditions in a more general context

1 Properties of tame actions by an affine group scheme

In this section, we start with proving the existence of a projector for tame actions which insures the exactness of the functor of invariants. This property characterizes the tameness of the quotient stack (see Theorem IX.8). We give also another characterization of the tameness for actions by affine group schemes and exactness of the functor of invariants. We end the section with Proposition IX.4 which will permit the connection between the two notions of tameness.

1.1 Existence of projectors

In the general case, we can define a projector which plays the role of the trace map that we can construct only for actions by constant group schemes.

Lemma IX.1. *([Doi90, §1]) From a total integral map $\alpha : A \to B$ and $M \in {}_B\mathcal{M}^A$, we can define a R-linear projector called **Reynold operator***

$$\begin{array}{rccl} pr_M : & M & \to & M^A \\ & m & \mapsto & pr_M(m) := pr_{M,\alpha}(m) = \sum_{(m)} \alpha(S(m_{(1)}))m_{(0)}. \end{array}$$

Proof. For any $m \in M$ and $a \in A$, we have

$$\rho_M(\alpha(a)) = (\alpha \otimes A)\Delta(a) = (\alpha \otimes A)(\sum a_1 \otimes a_2) = \sum \alpha(a_1) \otimes a_2.$$

Thus,

$$\rho_M(m.\alpha(a)) = \sum m_{(0)}\alpha(a_1) \otimes m_{(1)}a_2 \ (\square).$$

We obtain:

$$\begin{aligned}
\rho_M(pr_M(m)) &= \rho_M(\sum m_{(0)}\alpha(S(m_{(1)}))\\
&= \sum (m_{(0)})_{(0)}\alpha(S(m_{(1)})_1 \otimes (m_{(0)})_1 S(m_{(1)})_1) \text{ (by } (\square))\\
&= \sum (m_{(0)})_{(0)}\alpha(S((m_{(1)})_2) \otimes (m_{(0)})_1 S((m_{(1)})_1) \text{ (since } S \text{ is an antimorphism)}\\
&= \sum m_{(0)}\alpha(S(m_{(3)})) \otimes m_{(1)}S(m_{(2)}) = \sum m_{(0)}\alpha(S(m_{(2)})) \otimes (m_{(1)})_1 S(m_{(1)})_2\\
&= \sum m_{(0)}S(m_{(2)}) \otimes \epsilon(m_{(1)})1 \text{ (by the definition of an antipode)}\\
&= \sum m_{(0)}S(m_{(1)}\epsilon(m_{(1)})) \otimes 1 = \sum m_{(0)}S((m_{(1)})_1\epsilon(m_{(1)})_1) \otimes 1\\
&= \sum m_{(0)}S(m_{(1)}) \otimes 1 \text{(by the properties of the counity).}\\
&= pr_M(m) \otimes 1
\end{aligned}$$

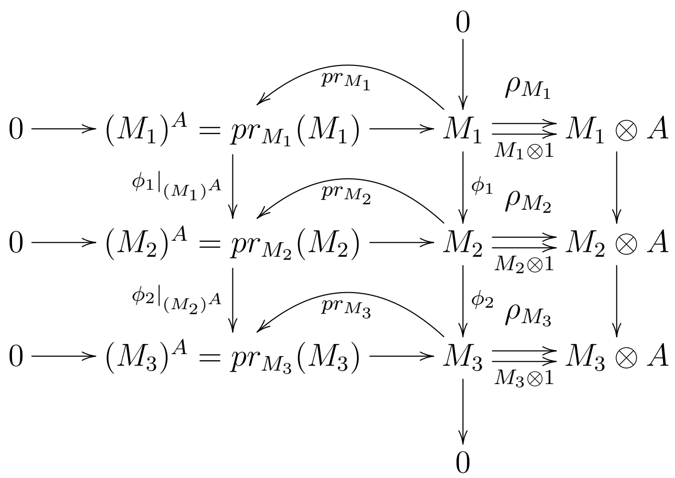
So, $pr_M(m) \in (M)^A$ and ${pr_M}^2 = pr_M$. Moreover, for any $m \in (M)^A$, $\rho_M(m) = m \otimes 1$, Hence, $pr_M(m) = m\alpha(s(1)) = m\alpha(1) = m$ since $\alpha(1) = 1$. This proves that pr_M is a Reynold operator. $\square$

1.2 Exactness of the functor of invariants

The existence of a Reynold operator permits to prove that tame actions by affine group schemes always induce the exactness of the functor of invariants.

Lemma IX.2. *([CEPT96, Lemma 2.3]) If the action (X, G) is tame then the functor of invariants $(-)^A : {}_B\mathcal{M}^A \to {}_C\mathcal{M}$ is exact.*

Proof. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of ${}_B\mathcal{M}^A$. We have the following commutative diagram:



Left exactness is automatic, right exactness follows from the previous diagram. $\square$

We have also a characterization of tameness of an action by the existence of a section for the structural morphism ρ_B.

Lemma IX.3. *([Doi90, §1]) The following assertions are equivalent:*

1. *The action (X, G) is tame.*
2. *There is an A-comodule map $\lambda_B : B \otimes_R A \to B$ such that $\lambda_B \circ \rho_B = Id_B$ where $B \otimes_R A$ is an A-comodule via $(Id_B \otimes \Delta)$.*

Proof. (1) $\Rightarrow$ (2) Let $\lambda_B : B \otimes_R A \to B$ be the map sending ϕ to $[b \otimes a \mapsto \sum b_{(0)}\alpha(S(b_{(1)})a)]$. For any $b \otimes a \in B \otimes_R A$,

$$\begin{aligned}(\rho_B \circ \lambda_B)(b \otimes a) &= \rho_B(\textstyle\sum b_{(0)}\alpha(S(b_{(1)})a)) \\ &= \textstyle\sum (b_{(0)})_{(0)}\alpha(S((b_{(1)})_2)a_1) \otimes (b_{(0)})_{(1)}S((b_{(1)})_1)a_2 \\ &= \textstyle\sum (b_{(0)})_{(0)}\alpha(S(b_{(1)})a_1) \otimes ((b_{(0)})_{(1)})_2 S(((b_{(0)})_{(1)})_1)a_2 \\ &= \textstyle\sum b_{(0)}\alpha(S(b_{(1)})a_1) \otimes a_2 \\ &= ((\lambda_B \otimes Id_A) \circ (Id_B \otimes \Delta))(b \otimes a)\end{aligned}$$

This means that λ_B is an A-comodule map. Moreover, for any $b \in B$,

$$\begin{aligned}\lambda_B \circ \rho_B(b) &= \lambda_B(\textstyle\sum b_{(0)} \otimes b_{(1)}) = \sum (b_{(0)})_{(0)}\alpha(S((b_{(0)})_{(1)})b_{(1)}) \\ &= \textstyle\sum b_{(0)}\alpha(S((b_{(1)})_1)(b_{(1)})_2) = \sum b_{(0)}\alpha(\epsilon(b_{(1)})1_A) \\ &= (\textstyle\sum \epsilon(b_{(1)})b_{(0)})1_B = b\end{aligned}$$

Thus, $\lambda_B \circ \rho_B = Id_B$.
(2) $\Rightarrow$ (1) We define $\alpha : A \to B$ as the composite of two unitary A-comodule maps $1 \otimes Id_A : A \to B \otimes_R A$ and $\lambda_B : B \otimes_R A \to B$. □

The next proposition is one of our results which is essential for the following as it, compares the exactness of the functor of invariants with the tameness of the action.

Proposition IX.4. *Suppose that C is locally noetherian, that B is flat over C and that A is finite, locally free over R. Then, the following assertions are equivalent:*

(1) The action (X, G) is tame.

(2) The functor $(-)^A : {}_B\mathcal{M}^A \to {}_C\mathcal{M}$, $N \mapsto (N)^A$ is exact.

(1') The action $(X, {}_CG)$ is tame.

(2') The functor $(-)^{{}_CA} : {}_B\mathcal{M}^{{}_CA} \to {}_C\mathcal{M}$, $N \mapsto (N)^{{}_CA}$ is exact.

Proof. (1) $\Leftrightarrow$ (1′) follows from lemma VI.4.
(2) $\Leftrightarrow$ (2′) is a consequence of lemma III.21.
(1) $\Rightarrow$ (2) follows from lemma IX.1
(2′) $\Rightarrow$ (1′) Suppose that $(-)^A$ is exact. Since ${}_CA$ is finite over C (base change of A which is finite over R) and B is finite over C (cf. Theorem III.39), $B \otimes_C {}_CA$ is finite over C. Moreover, as C is locally noetherian, B and $B \otimes_R A$ are also of finite presentation as algebras over C (see

Lemma A.3, 3.) so, in particular, as C-modules (see Lemma A.3, 4.). By lemma III.24 since we suppose B flat over C, we have the following isomorphism

$$(B \otimes_R B^*)^{cA} \simeq Com_{cA}(B, B) \textit{ and } (B \otimes_R (B \otimes_R A)^*)^{cA} \simeq Com_{cA}(B \otimes_C {}_CA, B)$$

Since $(B \otimes \epsilon)$ is a C-linear section of $\rho_B : B \to B \otimes_R A$, $(B \otimes_C {}_CA)^* \to B^*$ is surjective. Thus, from the exactness of $(-)^{cA}$, we obtain the surjectivity of

$$(B \otimes_R (B \otimes_R A)^*)^{cA} \to (B \otimes_R B^*)^{cA}$$

Finally, the isomorphisms above imply the surjectivity of the natural map

$$Com_{cA}(B \otimes_C {}_CA, B) \to Com_{cA}(B, B)$$

This insures the existence of a ${}_CA$-comodule map $\lambda_B : B \otimes_C {}_CA \to B$ such that $\lambda_B \circ \rho_B = B$. The previous lemma permits to conclude the proof. □

2 Exactness of the functor of invariants and nature of the quotient

2.1 The adjunction map defines an isomorphism for tame actions

For any C-module M, $M \otimes_C B$ is a right A-comodule via $Id_M \otimes \rho_B$ and B-module via $(m \otimes b)b' = m \otimes bb'$, for any $m \in M$ and $b, b' \in B$. For $M \in \mathcal{M}_C$ and $N \in \mathcal{M}_B^A$, the **counit map** $\mu_N : N^A \otimes_C B \to N$ sends $n \otimes a$ to na and the **unit map** $\eta_M : M \to (M \otimes_C B)^A$ sends m to $m \otimes 1$.

Lemma IX.5. *([Obe77, Section 4]) Suppose that the action (X, G) is tame. Then, for any B^A-module M, the unit map $\eta_M : M \to (M \otimes_C B)^A$ is an isomorphism.*

Proof. Choose a free presentation of M

$$C^{(J)} \longrightarrow C^I \longrightarrow M \longrightarrow 0$$

The functor $- \otimes_C B$ is right exact, so

$$C^{(J)} \otimes_C B \longrightarrow C^I \otimes_C B \longrightarrow M \otimes_C B \longrightarrow 0$$

is again exact. Since $(-)^A$ is (right) exact, we get that

$$(C^{(J)} \otimes_C B)^A \longrightarrow (C^I \otimes_C B)^A \longrightarrow (M \otimes_C B)^A \longrightarrow 0$$

is exact. Because $(-)^A$ is exact, it commutes with arbitrary direct sums and it preserves cokernels. The same goes for the left adjoint of $(-)^A$, which sends M to $M \otimes_C B$, and hence also for the adjunction homomorphism η_M. Note that $\eta_C : C \simeq (C \otimes_C B)^A$, and so, by the above, $\eta_{C^{(J)}}$ and $\eta_{C^{(I)}}$ are isomorphisms. It follows that η_M is an isomorphism. □

2.2 Nature of the quotient

If we suppose the exactness of the functor of invariants, the quotient has very nice properties.

Theorem IX.6. *Suppose that the functor of invariants $(-)^A$ is exact. Then, the map $\pi : X \to Y$ is a categorical quotient in the category of schemes. Moreover, the underlying topological map $\pi^\sharp$ is surjective and the image of any closed G-equivariant subscheme of X is closed. That is, $\pi^\sharp$ is submersive. Moreover, if X is noetherian, the map $\pi : X \to Y$ is a categorical quotient in the category of algebraic spaces.*

Proof. By the previous lemma and Lemma IX.2, for any ideal $\mathfrak{c}$ of C, we have

$$C/\mathfrak{c} \simeq (C/\mathfrak{c} \otimes_C B)^A \simeq (B/B\mathfrak{c})^A \simeq C/(B\mathfrak{c})^A \simeq C/B\mathfrak{c} \cap C$$

Thus, $B\mathfrak{c} \cap C = \mathfrak{c}$. Let $\mathfrak{p} \in Spec(C)$. Since $C_\mathfrak{p} \subset B_\mathfrak{p}$ and $B\mathfrak{p} \cap C = \mathfrak{p}$, $\mathfrak{m} := \mathfrak{p}_\mathfrak{p} B_\mathfrak{p} \neq B_\mathfrak{p}$ is contained in the maximal ideal of $B_\mathfrak{p}$. We have $\mathfrak{p}C_\mathfrak{p} \subseteq \mathfrak{m} \cap C_\mathfrak{p}$ and we obtain by maximality, $\mathfrak{p}C_\mathfrak{p} = \mathfrak{m} \cap C_\mathfrak{p}$. Set $\mathfrak{q} = \mathfrak{m} \cap B$, so $\mathfrak{q} \cap C = (\mathfrak{m} \cap B) \cap C = \mathfrak{m} \cap C = (\mathfrak{m} \cap C_\mathfrak{q}) \cap C = \mathfrak{q}C_q \cap C = \mathfrak{q}$ which shows the surjectivity of $\pi^\sharp$.
Let W be a closed G-invariant subscheme of X. W has the form $Spec(B/\mathfrak{b})$, with $\mathfrak{b} \in Spec(B)$ G-invariant. By the exactness of the functor of invariants (Lemma IX.2), we have:

$$(B/\mathfrak{b})^A \simeq B^A/\mathfrak{b}^A \simeq C/\mathfrak{b} \cap C$$

Thus, $\pi(W) = Spec(C/(\mathfrak{b} \cap C))$ is closed in Y.
By the previous lemma, for any ring morphism $C \to C'$, we have $C' = (C' \otimes_C B)^A$. This means that the invariants commute with affine base change. To show that the quotient is universal in the category of algebraic spaces, we have to prove that for any base change $Y' \to Y$ where Y' is an algebraic space, $X_{Y'} \to Y'$ is a quotient. Considering a presentation $Y'_1 \rightrightarrows Y'_2$ of Y' (see [sta05, Algebraic spaces, lemme 9.1]), it is enough to deal with the case where Y' is a scheme. Finally, using [MFK94, Remark 6, p 8], we are reduced to the case where Y' is affine.
Consider a morphism of schemes $\psi : X \to Z$ such that $\mu_X \circ \psi = pr_1 \circ \psi$. We have to construct a morphism $\chi : Y \to Z$ such that $\psi = \chi \circ \pi$ and show that it is unique. Consider an affine covering $\{Z_i\}$ of Z. We want to find a covering $\{Y_i\}$ of Y such that $\pi^{-1}(Y_i) \subset \psi^{-1}(Z_i)$ (□) in order to define χ_i locally on the $Y_i's$. Indeed, if we find such a covering $\{Y_i\}$, since Z_i is supposed affine, it is enough to find a morphism $h_i : \Gamma(Z_i, \mathcal{O}_Z) \to \Gamma(Y_i, \mathcal{O}_Y)$. For this, consider

$$res_i \circ \psi_* : \Gamma(Z_i, \mathcal{O}_Z) \to \Gamma(\psi^{-1}(Z_i), \mathcal{O}_X) \to \Gamma(\pi^{-1}(Y_i), \mathcal{O}_X)$$

where res_i is the restriction map. Moreover, $\Gamma(\pi^{-1}(Y_i), \mathcal{O}_X) = \Gamma(Y_i, \pi_*(\mathcal{O}_X) = \Gamma(Y_i, \mathcal{O}_Y)$. So, for any i, we can take $h_i := res_i \circ \psi_*$ which defines $\chi_i : Y_i \to Z_i$. Finally, since $\chi_i = \chi_j$ on $Y_i \cap Y_j$ we can glue and obtain $\psi = \chi \circ \pi$. The unicity of χ comes from the unicity of the $\chi_i's$ such that $\psi = \chi_i \circ \pi$.
So, we have to construct the covering $\{Y_i\}$ satisfying ($\Box$). We notice that for any family of ideals $\{b_i\}$ of B which are A-subcomodules of B, we have $(\sum b_i) \cap C = \sum(b_i \cap C)$ (The existence of a projector $pr_B : B \to C$ (cf. Lemma IX.1) shows that if $b = \sum b_i$ is in $(\sum b_i) \cap C$ then $b = pr_B(b) = \sum pr_B(b_i)$ is also in $\sum(b_i \cap C)$. The converse is clear.) Since π is closed on the closed invariant subschemes, the previous equality can be written in geometric terms as $\pi(\cap W_i) = \cap_i \pi(W_i)$ $(*)$ for any family $\{W_i\}$ of closed invariant subschemes. Set $X_i := X \backslash \psi^{-1}(Z_i)$ and $Y_i := Y \backslash \pi(X_i)$. As ψ is invariant on the orbits, X_i is closed and invariant so π is closed, hence Y_i is open. Thus, $\pi^{-1}(Y_i) \subset \psi^{-1}(Z_i)$. Moreover, since $\{Z_i\}$ is a covering of Z, $\cap_i X_i$ is empty and by $(*)$, $\cap_i \pi(X_i)$ is also empty. So, $\{Y_i\}$ is a covering of Y. The fact that π is a categorical quotient in the category of algebraic spaces when X is noetherian is a consequence of [Alp08, Theorem 6.6]. Indeed, since X is noetherian and $X \to [X/G]$ is faithfully flat quasi-compact then $[X/G]$ is noetherian. $\Box$

Corollary IX.7. *Suppose that X is noetherian, finitely presented over S, that G is finitely presented over S and that the inertia group stack $\mathcal{I}_{[X/G]}$ is finite. If the action (X, G) is tame, the map $\rho : [X/G] \to Y$ is a coarse moduli space of $[X/G]$ and ρ is proper.*

Proof. By the previous theorem, the quotient $\pi : X \to Y$ is categorical in the category of algebraic spaces. By Theorem IV.28, this insures that the quotient stack admits a coarse moduli space that we denote $\rho : [X/G] \to M$ such that ρ is proper. Via Lemma IV.30, since $\rho \circ p : X \to [X/G] \to M$ and $\pi : X \to Y$ are categorical quotients in the category of algebraic spaces, we obtain that $M \simeq Y$ and that $\rho : [X/G] \to Y$ is a coarse moduli space. $\Box$

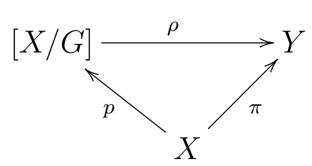
3 Comparing the two notions of tameness

Using all the previous studies, we can finally compare the two notions of tameness.

3.1 Tame actions by an affine group scheme vs good moduli spaces

Theorem IX.8. *Suppose that the functor $\rho_* : Qcoh([X/G]) \to Qcoh(Y)$ is well defined. Then, the functor ρ_* is exact if and only if the functor of invariants $(-)^A$ is exact.*

Proof. The quotient map $\pi : X \to Y$ is such that $\pi \circ p_1 = f \circ \mu_X$. Lemma IV.29 insures the existence of a map ρ making the diagram below commutative:



Thus, as all the morphisms of the diagram above are quasi-compact and quasi-separated, the following diagram is well defined and commutes:

$$\begin{array}{ccccc} Qcoh([X/G]) & & \xrightarrow{\rho_*} & & Qcoh(Y) \\ & \nwarrow^{p_*} & & \nearrow_{\pi_*} & \\ & & Qcoh^G(X) & & \end{array}$$

By Proposition IV.24, the functor p_* is an equivalence of categories. Moreover, by Proposition A.8 and Proposition A.11, we have the following commutative diagram:

$$\begin{array}{ccc} Qcoh^G(X) & \xrightarrow{\pi_*} & Qcoh(Y) \\ {\scriptstyle \Gamma(X,\_)}\downarrow & & \uparrow{\scriptstyle \tilde{}} \\ (B,A)\text{-modules} & \xrightarrow[(\_)^A]{} & B^A\text{-modules} \end{array}$$

where $\Gamma(X, \_)$ and $\tilde{}$ are equivalences of categories. This proves the lemma. □

Proposition IX.9. *Suppose that (X, G) is a tame action. Then, $\rho : [X/G] \to Y$ is a good moduli space.*

Proof. Since $\pi : X \to Y$ satisfies $\pi \circ p_1 = \pi \circ \mu_X$, Lemma IV.29 insures the existence of the application ρ making the following diagram commute:

$$\begin{array}{ccccc} [X/G] & & \xrightarrow{\rho} & & Y \\ & \nwarrow^{p} & & \nearrow_{\pi} & \\ & & X & & \end{array}$$

Thus, we have $\rho_* \mathcal{O}_{[X/G]} \simeq \pi_*(\mathcal{O}_X)^G \simeq \mathcal{O}_Y$.
The map $\rho_* : Qcoh([X/G]) \to Qcoh(Y)$ is well defined, by Lemma IV.10. The proposition is now a direct consequence of the previous lemma. □

3.2 Comparing tameness notions for actions of a finite group scheme

Theorem IX.10. *Suppose that G is finite and locally free over S. If the action (X, G) is tame then the quotient stack $[X/G]$ is tame. When C is locally noetherian and B flat over C, the converse is true.*

Proof. By Theorem III.39, we know that under the hypothesis of the proposition the map $\pi : X \to Y$ is a geometric categorical quotient in the category of algebraic spaces, thus by Corollary IV.30, $\rho : [X/G] \to Y$ is a coarse moduli space. Moreover, $\rho_* : Qcoh([X/G]) \to Qcoh(Y)$ is exact by Proposition IX.4, thus $[X/G]$ is tame. The converse follows from Proposition IX.4. □

Remark IX.11. *We can replace the hypothesis Y noetherian by X of finite type over S if we suppose the base S noetherian. In fact, by [Con05, Theorem 3.1, (2)], Y is of finite type over S so also noetherian.*

We obtain easily the following corollary which gives also an interesting correlation between linearly reductive groups and cosemisimplicity thanks to Proposition VI.12.

Corollary IX.12. *Suppose that S is noetherian and G is finite and flat over S. Then, G is linearly reductive over S if and only if the trivial action (S, G) is tame over S.*

We thought that it might be interesting to prove that the tameness can be characterized by the existence of projectors.

Corollary IX.13. *Suppose that G is finite, locally free over S, C is locally noetherian and B flat over C. The following assertions are equivalent:*

1. *The action (X, G) is tame.*
2. *The quotient stack $[X/G]$ is tame.*
3. *There is a Reynold operator $pr_M : M \to M^A$ for any $M \in {}_B\mathcal{M}^A$.*

Proof. 1. $\Leftrightarrow$ 2. Follows from the previous theorem.
1. $\Rightarrow$ 3. follows from Lemma IX.1.
3. $\Rightarrow$ 2. As the functor $(-)^A$ is left exact, it is enough to prove exactness on the right. So, let $\xi : M \to N \in {}_BBim^A(M, N)$ be an epimorphism. It induces a morphism $\xi^A : M^A \to N^A$. For $n \in N^A$, by the surjectivity of ξ, there is $m \in M$ such that $\xi(m) = n$. Moreover, $pr_M(m) \in M^A$, so $\xi(pr_M(m)) = pr_N(\xi(m)) = n$. Therefore ξ is surjective.

□

Remark IX.14. *By the previous lemma, we characterize tameness by the existence of projectors which can be an analogue of the trace surjectivity that we have proved in the constant case.*

3.3 Comparing tameness notions for actions with finite inertia groups

Theorem IX.15. *Suppose that X is noetherian, finitely presented over S, that G is finitely presented over S and and that the inertia group stack $\mathcal{I}_{[X/G]}$ is finite. If the action (X, G) is tame then the quotient stack $[X/G]$ is tame.*

Proof. This follows from Proposition IX.9 together with Corollary IX.7.

□

4 Free actions

4.1 Tameness and freeness

Theorem IX.16. *The following assertions are equivalent:*

1. *The functors $B \otimes_C -$ and $(-)^A$ form a pair of equivalences inverse of each other between the category $\mathcal{M}_C$ and the category ${}_B\mathcal{M}^A$.*
2. *The Y-scheme X is a G-torsor over Y for the fppf topology.*
3. *The map $X \to Y$ is a categorical quotient for the action (X, G) and the functor $\rho_* : Qcoh[X/G] \to Qcoh(Y)$ is an equivalence of categories.*

Moreover, requiring that the action (X, G) is tame and free implies all the previous assertions. In particular, if we suppose G finite and flat over S, all the previous assertions are equivalent to requiring that the action (X, G) is free.

Proof. 1. $\Rightarrow$ 2. Consider $M \to M'$ an injective map of right C-modules. Since the categories $\mathcal{M}_C$ and ${}_B\mathcal{M}^A$ are equivalent via the functor $B \otimes_C -$, then $B \otimes_C M \to B \otimes_C M'$ is a monomorphism in ${}_B\mathcal{M}^A$, in particular injective. So, B is flat over C. The faithful flatness is obtained similarly. Let $0 \to M \to M' \to M'' \to 0$ be a short sequence such that

$$0 \to B \otimes_C M \to B \otimes_C M' \to B \otimes_C M\text{''} \to 0$$

is exact in $\mathcal{M}_B$. As the modules of the exact sequence are in ${}_B\mathcal{M}^A$, the exact sequence remains exact after applying the functor $(-)^A$ (it is an equivalence of category) and the initial sequence is exact. Consider $B \otimes_R A \in {}_B\mathcal{M}^A$ which is a right A-comodule via $Id_B \otimes \Delta$ and a left B-module via $(b \otimes a) * b' = (b \otimes a)\rho_B(b')$, for any $b \otimes a \in B \otimes_R A$ and $b' \in B$. The counit map applied to $B \otimes_R A$ and defined by

$$\begin{array}{rccc} \mu_{B \otimes_R A} : & (B \otimes_R A)^A \otimes_C B & \to & B \otimes_R A \\ & (b \otimes a) \otimes b' & \mapsto & (b \otimes a).b' \end{array}$$

is then an isomorphim. Via the canonical isomorphism $B \simeq (B \otimes_R A)^A$, we obtain the Galois map defined by:

$$\begin{array}{rccc} Gal : & B \otimes_C B & \to & B \otimes_R A \\ & b \otimes b' & \mapsto & (b \otimes 1)\rho_B(b') \end{array}$$

1. $\Leftrightarrow$ 3. Using the same arguments as in the proof of Theorem IX.8, we can show that requiring the functor ρ_* to be an equivalence of categories is equivalent to requiring that the functor $(-)^A$ is an equivalence of categories.
2. $\Rightarrow$ 3. As Y is a torsor, by Lemma IV.17, $[X/G] \simeq Y$ and the result follows immediately. In order to show that requiring that the action (X, G) is tame and free implies all the previous assertions, we can refer to [Doi90, Theorem 3]. Finally, if we suppose G finite, flat over S,

Theorem III.48 shows that all the previous assertions are equivalent to requiring that the action (X, G) is free. □

4.2 Slices theorem for free actions

Lemma IX.17. *Suppose that $S := Y$ is local, with maximal ideal $\mathfrak{q}$. The following assertions are equivalent:*

1. *(X, G) is a free action.*
2. *$I_G(\mathfrak{p})$ is trivial for some $\mathfrak{p}$ over $\mathfrak{q}$*

Proof. The direct implication follows from the definition. Let us prove 2. $\Rightarrow$ 1.. Let $f : X \to S$ be the quotient morphism: since G is finite, f is finite. Set $S_0 := Spec(R/\mathfrak{q})$ and $X_0 := f^{-1}(\mathfrak{q}) = \{\mathfrak{p}_1, .., \mathfrak{p}_r\} = X \times_S S_0$ where for any $i \in 1, ..., r$, $\mathfrak{p}_i$ are the primes above $\mathfrak{q}$. Since the inertia is trivial at some prime over $\mathfrak{q}$, it is trivial at any prime over $\mathfrak{q}$ since all the prime ideals over $\mathfrak{q}$ are conjugate. The action (X_{S_0}, G_{S_0}) is free, this mean thats $X_0 \times_{S_0} G_{S_0} \to X_0 \times_{S_0} X_0$ is a closed immersion (see [DG70, III, §2, n°2, Proposition 2.2]). In other words, we have the surjection $(B \otimes_R B) \otimes_R R/\mathfrak{q} \to (B \otimes_R A) \otimes_R R/\mathfrak{q}$. But, since $B \otimes_R B$ and $B \otimes_R A$ are finite R-algebras, by Nakayama lemma, $B \otimes_R B \to B \otimes_R A$ is a surjection hence the action is free. □

From this lemma, we can deduce easily the following proposition which is a fppf slice theorem for free action, more precisely , freeness is local for the étale topology:

Theorem IX.18. *Suppose that G is finite, flat over S. Let $x \in X$ and $y \in Y$ its image via the morphism $\pi : X \to Y$. The following assertions are equivalent:*

1. *The inertia group scheme is trivial at x.*
2. *There is an fppf morphism $Y' \to Y$ containing y in its image such that the action $(X \times_Y Y', G_{Y'})$ is free. Thus, $X \times_Y Y'$ is a $G_{Y'}$-torsor over Y'.*
3. *There is an fppf morphism $Y'' \to Y$ containing y in its image and Z a scheme over Y'' such that the action $(X \times_Y Y'', G_{Y''})$ is induced by the action (Z, e) where e denote the trivial group scheme.*

Proof. 1. $\Leftrightarrow$ 2. Follows from the previous lemma.
2. $\Rightarrow$ 3. By Theorem III.48, $X \times_Y Y'$ is a $G_{Y'}$-torsor over Y' hence locally trivial for the fppf topology.
3. $\Rightarrow$ 1. It follows from 3. that after a fppf base change the inertia group scheme is trivial, hence it is also trivial itself. □

As a global version of the previous result, we have the following theorem.

Theorem IX.19. *([DG70, III, §2, n° 2, 2.1] Suppose that G is finite over S. The following assertions are equivalent:*

1. *(X,G) is an action with trivial inertia at any $x \in X$.*
2. *The action (X,G) is free.*
3. *X is a G-torsor over Y.*

5 Existence of slices for tame quotient stacks

In this section, we suppose that X is noetherian, finitely presented over S, that G is finitely presented over S and that the inertia group stack $\mathcal{I}_{[X/G]}$ is finite.

5.1 "Weak" slice theorem for tame quotient stacks

We can now characterize the tameness of the quotient stack via the structure of the inertia group schemes and give locally the structure of a tame quotient stack which can be seen as a "weak variant" of a slice theorem.

Theorem IX.20. *[AOV08, Theorem 3.2] The following assertions are equivalent:*

1. *The quotient stack $[X/G]$ is tame.*
2. *The inertia groups $I_G(\xi) \to Spec(k)$ are linearly reductive groups, for any $\xi : Spec(k) \to X$, where k is a field.*
3. *The inertia groups $I_G(\xi) \to Spec(k)$ are linearly reductive groups, for any geometric point $\xi : Spec(k) \to X$, where k is an algebraically closed field.*
4. *For any point $x \in X$, there exist an fppf cover $Y' \to Y$ containing x in its image, a linearly reductive group scheme $H \to Y'$, such that $H_{k(x)} \simeq I_G(x)$, acting on a finite and finitely presented scheme $P \to Y'$ and an isomorphism of algebraic stacks over Y'*

$$[X/G] \times_Y Y' \simeq [P/H].$$

Proof. 1. $\Rightarrow$ 2. Let $\xi : Spec(k) \to X$ be a k-point where k is a field, and $I_G(\xi)$ is the inertia group in ξ. The quotient stack $[G_k/I_G(\xi)]$ is a scheme, denote it $G_k/I_G(\xi)$. Since the square

$$\begin{array}{ccc} G_k/I_G(\xi) & \longrightarrow & X \times_S Spec(k) \\ \downarrow & & \downarrow \\ B_k I_G(\xi) & \longrightarrow & [X/G] \times_S Spec(k) \end{array}$$

is 2-cartesian, $B_k I_G(\xi) \to [X/G] \times_S Spec(k)$ is affine since $G_k/I_G(\xi) \to X \times_S Spec(k)$ is affine.

Now, let us consider the following commutative diagram:

$$\begin{array}{ccc} B_kI_G(\xi) & \xrightarrow{\ g\ } & [X/G] \\ {\scriptstyle (-)^{I_G(\xi)}}\downarrow & & \downarrow{\scriptstyle \rho} \\ Spec(k) & \xrightarrow[\ f\ ]{} & Y \end{array} \tag{IX.1}$$

Since we have seen that g is affine, $g_* : Qcoh(B_kI_G(\xi)) \to Qcoh([X/G])$ is an exact functor and $\rho_* : Qcoh([X/G]) \to Qcoh(Y)$ is exact by definition of tameness. Since $f_*\rho'_* = \rho_*g_*$, if

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

is an exact sequence of G-representations, considered as exact sequence of quasi coherent sheaves over $B_kI_G(\xi)$, we have the following exact sequence:

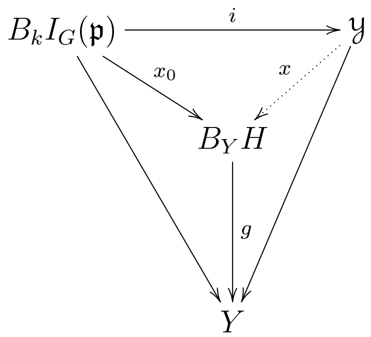
$$0 \to f_*(V_1)^{I_G(\xi)} \to f_*(V_2)^{I_G(\xi)} \to f_*(V_3)^{I_G(\xi)} \to 0$$

Moreover $(-)^{I_G(\xi)}$ is left exact and this implies that

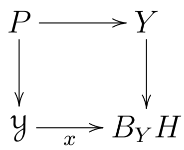
$$0 \to (V_1)^{I_G(\xi)} \to (V_2)^{I_G(\xi)} \to (V_3)^{I_G(\xi)} \to 0$$

is exact. So, $I_G(\xi)$ is linearly reductive.
2. $\Rightarrow$ 3. Immediate.
3. $\Rightarrow$ 2. Let $\xi : Spec(k) \to X$ be a k-point where k is a field. By Theorem VIII.13, in order to prove that a group scheme is linearly reductive, it is enough to prove that all the geometric fibers are linearly reductive. Moreover, by Proposition V.2, for any $\bar{x} : Spec(\bar{k}) \to Spec(k)$ where $\bar{k}$ is an algebraically closed field, we have $(I_G(\xi))_{\bar{x}} = I_G(\xi \circ \bar{x})$ where $\xi \circ \bar{x} : Spec(\bar{k}) \to X$ is a geometric point of X. So, $(I_G(\xi))_{\bar{x}}$ is linearly reductive hence the inertia group $I_G(\zeta)$ as well.
2. $\Rightarrow$ 4. Set $\mathcal{Y} := [X/G]$. Let $\mathfrak{p} \in X$. Write $k := k(\mathfrak{p})$ for the residue field at $\mathfrak{p}$. Without loss of generality, one can assume that Y is the base. Following Lemma VIII.17, after passing to an fppf cover of Y, we may also assume that $I_G(\mathfrak{p})$ extends to a linearly reductive group scheme H over Y. By standard limit argument, after an étale base change, we may assume that Y is the spectrum of a local Henselian ring R with residue field k and maximal ideal $\mathfrak{p}$. Set $Y_n := Spec(R/\mathfrak{p}^{n+1})$. We have that $B_kI_G(\mathfrak{p})$ is a closed substack of $\mathcal{Y}$ (we can see this thanks to the commutative diagram (IX.1) remembering that the maps $(-)^{I_G(\xi)}$ and ρ are proper). Denote by $x_0 : B_kI_G(\mathfrak{p}) \to \mathcal{Y}$ this closed immersion, by $\mathcal{I} \subset \mathcal{O}_{\mathcal{Y}}$ the sheaf of ideals defining this closed substack and by $\mathcal{Y}_n$ the closed substack of $\mathcal{Y}$ defined by the sheaf of ideals $\mathcal{I}^{n+1}$. Our goal is to find a Y-scheme P such that P is an H-torsor over $[X/G]$ which is equivalent to find

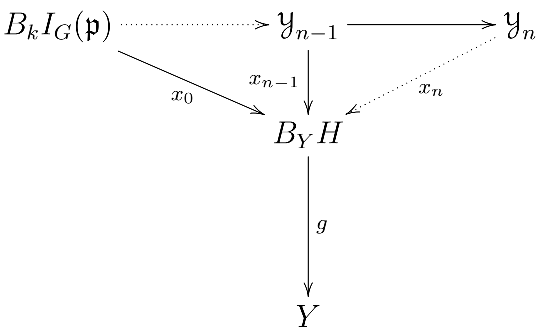
a representable morphism $x : \mathcal{Y} \to B_Y H$ filling in the following commutative diagram:



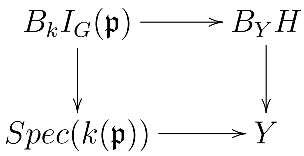
Indeed, this morphism defines the H-torsor required as the fiber product:



- **Formal deformation of x_0:** We want to find a formal deformation datum of x_0 via the sheaf of ideals $\mathcal{I}$, i.e. morphisms x_n filling in the following commutative diagram:



Applying Theorem B.32 and using the cartesian diagram:



we know that the obstruction to extend x_{n-1} to x_n lies in $Ext^1(Lx_0^* L_{B_Y H}, \mathcal{I}^n/\mathcal{I}^{n+1})$ which is trivial for any $n \in \mathbb{N}$ by Corollary VIII.15 and this shows the existence of a formal deformation.

- **Algebraization of the formal deformation** Since $\mathcal{Y}_0 \to Y_0$ is a coarse moduli space, $|\mathcal{Y}_0(\Omega)| \simeq Y_0(\Omega)$ where $|\mathcal{Y}_0(\Omega)|$ is the equivalence class of the geometric points of $\mathcal{Y}_0$

with values in an algebraically closed field Ω. This implies that the systems of ideals $\{\mathcal{J}^n\}$ and $\{\mathfrak{p}^n\mathcal{O}_{\mathcal{Y}}\}$ are cofinal, i.e. there is an index j_n such that there is a morphism $\alpha_n : [X/G] \times_S Y_n \to \mathcal{Y}_{j_n}$.
Considering the composition $x'_n = x_{i_n} \circ \alpha_n : [X/G] \times_S Y_n \to B_Y H$ we get a formal deformation, for our problem. Now, one can apply Artin approximation theorem B.35 and get the morphism $x : \mathcal{Y} \to B_Y G$ that induces the closed immersion x_0. This defines the required H-torsor $P \to [X/G]$.

4. $\Rightarrow$ 1. Follows from Lemma VI.15, 1. □

Remark IX.21. *1. This theorem shows that the action $(\mathbb{A}^1_k, \alpha_p)$ and the action $(\mathbb{A}^1_k, G)$, for n coprime to p of Chapter V, §4 are tame since all the inertia groups at the geometric point are linearly reductive.*

2. In order to have really a slice theorem, it would be enough to have an G-equivariant morphism between X after a fppf base change and $(P \times_S G)/H$. It seems difficult to do it by using for example deformation theory.

We obtain the following corollary.

Corollary IX.22. *The stack $[X/G] \to S$ is tame if and only if for any morphism $Spec(k) \to S$, where k is an algebraically closed field, the geometric fiber $[X/G] \times_S Spec(k) \to Spec(k)$ is tame.*

5.2 Slice theorem for tame quotient stacks defined by actions of finite commutative group schemes

We state now an interesting consequence of the previous theorem which permits to define a torsor for a tame quotient stack for actions of finite commutative group schemes.

Proposition IX.23. *Suppose that G is commutative over S. If the quotient stack $[X/G]$ is tame, then for any point $x \in X$, denoting by $y \in Y$ its image by the quotient morphism $\pi : X \to Y$, there exist an fppf morphism $Y' \to Y$ containing y in its image and a subgroup H of $G_{Y'}$ over Y' lifting the inertia group at x such that the action $((X \times_Y Y')/H, G_{Y'}/H)$ is free. If we suppose moreover that G is finite, $(X \times_Y Y')/H$ is a torsor over Y'.*

Proof. By the previous proposition, we know that the inertia at x is linearly reductive. So, by Lemma VIII.18, there are an fppf cover $Y' \to Y$ containing y in its image and a linearly reductive group $H \to Y'$ lifting this inertia group as a subgroup of $G_{Y'}$. Moreover, by Lemma V.7, the inertia group at $\bar{x}$, image of x by the quotient morphism $(X \times_Y Y') \to (X \times_Y Y')/H$ for the action $((X \times_Y Y')/H, G_{Y'}/H)$, is equal to

$$I_{G_{Y'}/H}(\mathfrak{p}) = I_G(\mathfrak{p})/(I_G(\mathfrak{p}) \cap H_{k(\mathfrak{p})}) = \{e\}$$

By Theorem IX.19, up to passing to an fppf extension of Y', the action $((X \times_Y Y')/H, G_{Y'}/H)$ is free. □

Remark IX.24. *We cannot establish the previous proposition in general if G is not commutative. Indeed, in this case G/H is not necessarily a group scheme and we cannot even define the notion of action or of torsor. But, if the lifting H is a normal subgroup of G, we can establish the same result.*

We manage to prove a slice theorem for actions of a finite commutative group scheme, using the following lemma.

Lemma IX.25. *[CEPT96, Proposition 6.5] Let H be a subgroup of G such that the quotient for the natural translation action exists and is universal. Let $\psi : X \to G/H$ be a morphism of schemes preserving the G-actions. Let*

$$Z = X \times_{G/H} e$$

be defined as the fibered product of the two maps ψ and the inclusion $e = H/H \to G/H$. Assume that the balanced product $Z \times^H G$ exists and is a universal quotient. Then we have an isomorphism of G-actions

$$(X, G) \simeq (Z \times^H G, G).$$

Proof. Let us begin by constructing a map $Z \times^H G \to X$. Consider the composition of the projection maps $Z \times_S G \to Z$ and $Z \to X$. This is an H-invariant morphism, and hence by the universal property of the quotient, there is a map $f : Z \times^H G \to X$ through which the above composition factors. We will show that this map is an isomorphism of sheaves for the fppf-topology. In fact, we will show that the underlying morphism of functors is an isomorphism (so this is a proof by "reduction au cas ensembliste", see [DG70, III, Section 1, N°2]). From Lemma III.43, $Z \times^H G$ is the sheaf associated to the functor which, on an R-algebra T, takes the value

$$(Z \times_S G)(T)/H(T).$$

Let X_0 be the functor which, on the R-algebra T, takes the value

$$X_0(T) = X(T) \times_{(G/H)(T)} G(T)/H(T)$$

Since passing to associated sheaves commutes with taking fibered products, we see that the sheaf associated to X_0 is X. We now show that the map analogous to f gives a bijection

$$(Z \times_S G)(T)/H(T) \simeq X_0(T)$$

Let $Z_0(T) = X_0(T) \times_{G(T)/H(T)} e(T)$. Then, for the action of $H(T)$ on $Z_0(T) \times G(T)$ given by $(z, g)h = (zh, h^{-1}g)$, we have the bijection

$$(Z_0(T) \times G(T))/H(T) \simeq X_0(T).$$

Indeed, let Γ be an abstract group, and let $\Delta \subset \Gamma$ be a subgroup. Suppose E is a Γ-set with a Γ-equivariant map $\phi : E \to \Gamma/\Delta$. Then, the map ϕ is surjective, so E is the disjoint union of the fibers of ϕ. If $F = \phi^{-1}(\Delta/\Delta) = E \times_{\Gamma/\Delta} e$, then a computation shows that $E = F \times^{\Delta} \Gamma$. Now by definition $Z_0(T) \times G(T) = Z(T) \times G(T)$, so the lemma follows. □

Remark IX.26. *If there is a subgroup H of G such that $(X/H, G/H)$ defines a torsor over Y, then $X/H \times_Y X/H \simeq X/H \times_Y G/H$. This gives us a G-equivariant morphism $\psi : X \to G/H$ after the fppf base change $X/H \to Y$. So up to making a fppf base change, by the previous lemma, (X, G) is induced by the action (Z, H) defined in the previous proof.*

As direct consequence of this remark, Theorem IX.20 and Proposition IX.23, we have the following slice theorem which extends [CEPT96, Theorem 6.4]:

Theorem IX.27. *Suppose that G is commutative and finite over S. The quotient stack $[X/G]$ is tame if and only if the action (X, G) admits fppf slices such that the slice group at $x \in X$ are linearly reductive.*

Appendices

Appendix A

Reminder about schemes

1 Finiteness

We can obtain easily by induction the following result :

Lemma A.1. *([Bou81, Chap V, n° 1, Proposition 4]) Let B be an R-algebra and $(b_i)_{1\geq i\geq n}$ be a finite family of elements of B. If for any $i \in \{1, ..., n\}$, b_i is integral over $R[b_1, ..., b_{i-1}]$ (in particular, if any b_i is integral over R) then the sub-algebra $R[b_1, ..., b_n]$ of B is an R-module of finite type.*

We can also prove the following lemma :

Lemma A.2. *([Bou81, Chap V, n° 9, Lemma 5]) Let R be a noetherian ring, B an R-algebra of finite type, C an R-subalgebra of B such that B is integral over C. Then C is an R-algebra of finite type.*

Moreover, we have the following properties in algebro-geometric terms :

Lemma A.3. *(cf. [Gro64, §1 particularly §1.6, §1.7...])*

1. *A group scheme is finite and locally free over S if and only if it is flat, finite and of finite presentation over S.*
2. *If S is noetherian then the group scheme is finite and flat over S if and only if it is finite and locally free over S.*
3. *If S is locally noetherian, a scheme is of finite presentation over S if and only if it is of finite type over S.*
4. *If B is a finite algebra over R, then B is an R-algebra of finite presentation if and only if B is an R-module of finite presentation.*

2 Scheme-theoretic image

Definition A.4. *If $f : Y \to X$ is any morphism of schemes, the **scheme-theoretic image of** f is the unique closed subscheme $i : Z \to X$ which satisfies the following universal property:*

1. *f factors through i,*
2. *if $j : Z' \to X$ is any closed subscheme of X such that f factors through j, then i also factors through j.*

Remark A.5. *This notion is different from that of the usual set-theoretic image of f, $f(Y)$. For example, the underlying space of Z always contains (but is not necessarily equal to) the Zariski closure of $f(Y)$ in X, so if Y is any open (and not closed) subscheme of X and f is the inclusion map, Z is different from $f(Y)$. When Y is reduced, Z is the Zariski closure of $f(Y)$ endowed with the structure of reduced closed subscheme. But in general, unless f is quasi-compact, the construction of Z is not local on X.*

3 Quasi-coherent sheaves over a scheme

In this section, we find very classical results around the notion of quasi-coherent scheme. The reader can refer to [Har77, §5] for more details.

Definition A.6. *Let B be a ring and let M be a B-module. We define the sheaf associated to M over Spec(B), denote by $\widetilde{M}$, as follows. For any prime ideal $\mathfrak{p} \subset B$, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$. For any open set $U \subset$ Spec(B), we define the group $\widetilde{M}(U)$ as the set of functions $s : U \to \bigsqcup_{\mathfrak{p}\in U} M_\mathfrak{p}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_\mathfrak{p}$, and such that s is locally of the form m/f with $m \in M$ and $f \in B$. More precisely, for any $\mathfrak{p} \in U$, there are a neighborhood V of $\mathfrak{p} \in U$ and elements $m \in M$ and $f \in B$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = m/f$ in $M_\mathfrak{q}$. $\widetilde{M}$ is a sheaf via the natural restriction maps.*

Proposition A.7. *Let B be a ring, M be a B-module and $\widetilde{M}$ be a sheaf over $X :=$ Spec(B) associated to M. Then:*

1. *$\widetilde{M}$ is an B-module;*
2. *For any $\mathfrak{p} \in X$, the stalk $(\widetilde{M})_\mathfrak{p}$ of the sheaf $\widetilde{M}$ at $\mathfrak{p}$ is isomorphic to the localization $M_\mathfrak{p}$;*
3. *For any $f \in B$, the B_f-module $\widetilde{M}(D(f))$ is isomorphic to the localization M_f;*
4. *In particular, $\Gamma(X, \widetilde{M}) = M$.*

Proposition A.8. *Let A be a ring, $X =$ Spec(A) and let $A \to B$ be a ring homomorphism inducing the morphism $f :$ Spec $B \to$ Spec A. Then:*

1. *The map $M \to \widetilde{M}$ defines an exact faithful flat functor from the category of A-modules to the category of $\mathcal{O}_X$-modules;*
2. *If M and N are two A-modules, then $(M \times_A N)\widetilde{\ } \simeq \widetilde{M} \times_{\mathcal{O}_X} \widetilde{N}$;*
3. *If $\{M_i\}$ is a family of A-modules, then $(\oplus M_i)\widetilde{\ } \simeq \oplus \widetilde{M}_i$;*
4. *For any B-module N, $f_*(\widetilde{N}) \simeq ({}_AN)\widetilde{\ }$, where ${}_AN$ means that N is seen as A-module;*
5. *For any A-module M, $f^*(\widetilde{M}) \simeq (M \otimes_A B)\widetilde{\ }$*

Definition A.9. *Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasi-coherent if there is a covering of X by affine, open subspaces $U_i = Spec(A_i)$, such that for any i there is an A_i-module M_i with $\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$. Denote by $Qcoh(X)$ the set of quasi-coherent $\mathcal{O}_X$-modules.*

Example A.10. *For any scheme X, the structural sheaf $\mathcal{O}_X$ is quasi-coherent.*

Proposition A.11. *Let B be a ring and $X := Spec(B)$. The functor $M \mapsto \widetilde{M}$ defines an equivalence of categories between the category of B-modules and the category of quasi-coherent $\mathcal{O}_X$-modules. Its inverse functor is $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$. In particular, if $G = Spec(A)$ is a group scheme and (X, G) is an action, we obtain an equivalence between the (B, A)-modules and the quasi-coherent G-equivariant $\mathcal{O}_X$-modules. Denote by $Qcoh^G(X)$ the set of G-equivariant quasi-coherent modules.*

Proposition A.12. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. If $F \in Qcoh(X)$, then $f_*F \in Qcoh(Y)$.*

4 Group schemes

4.1 Cohomology

Lemma A.13. *([AOV08, §2.3]) Consider $1 \to A \to E \to G \to 1$ an extension of groups, with A an abelian group. If $H^1(G, A) = 0$, the unique automorphism of E which induces the identity on A and G is obtained by conjugation by an element of A.*

Proof. Consider the induced action of G on A by conjugation. The conjugation by an element of A gives an automorphism of E, which induces the identity on A and G. Let $\phi : E \to E$ be an automorphism of E which induces the identity on A and G. We can consider the map $E \to A$ defined by $u \to \phi(u)u^{-1}$; one shows that it is induced by a map $\psi : G \to A$ such that $\phi(u) = \psi([u])u$ where $[u] \in G$ is the image of u. One can show that ψ is a crossed morphism and we obtain an isomorphism between the automorphisms of E which induce the identity on A and G and the group $Z^1(G, A)$ of the crossed homomorphisms mapping ϕ to ψ. Finally, the result follows noticing that ϕ is given by conjugation by an element of A if and only if ψ is a boundary. □

4.2 Connected component of the identity on the automorphism group

Let k be a field and $G := Spec(A)$ an affine well split group scheme over k. Denote by Δ the connected component of the identity of G, $H = G/\Delta$ and by $\underline{Aut}_k(G)$ the group scheme which represents the automorphisms of G as a group scheme. Thus there is a homomorphism $\Delta \to \underline{Aut}_k(G)$ mapping every section of Δ to the inner automorphism of G which induces the embedding $\Delta/\Delta^H \subseteq \underline{Aut}_k(G)$, where Δ^H represents the H-invariants of Δ.

Lemma A.14. *([AOV08, Lemma 2.19]) The connected component of identity of $\underline{Aut}_k(G)$ is Δ/Δ^H.*

Proof. We know that Δ is a characteristic subgroup of G meaning that all automorphisms of $G_D \to Spec(D)$, where D is a k-algebra, stabilize Δ_D. This defines an homomorphism of group schemes $\underline{Aut}_k(G) \to \underline{Aut}_k(\Delta)$ and thus, $\underline{Aut}_k(G) \to \underline{Aut}_k(H)$, inducing a homomorphism

$$\underline{Aut}_k(G) \to \underline{Aut}_k(\Delta) \times \underline{Aut}_k(H)$$

Denote by E the kernel of this homomorphism. We notice that it contains Δ/Δ^H which is connected. As $\underline{Aut}_k(\Delta) \times \underline{Aut}_k(H)$ is étale over $Spec(k)$ by [Wat79, 7.6], if we consider the connected-étale sequence of $\underline{Aut}_k(G)$, it is enough to prove that E coincides with Δ/Δ^H, so that the inclusion $\Delta/\Delta^H \to E$ is surjective, or in other words, for all k-algebra D and for all $\alpha \in E(D)$, there is an extension D_0 of D faithfully flat such that the image of α in $E(D_0)$ comes from $\Delta/\Delta^H(D_0)$.
As H is constant, one can find a faithfully flat extension D_0 of D such that $G(D') \to H(D')$ is surjective for all D_0-algebra D', which allows us to write the following exact sequence

$$1 \to \Delta(D') \to G(D') \to H(D') \to 1$$

Moreover, since H is constant we can show that for any D_0-algebra D', $\Delta^H(D') = \Delta(D')^{H(D')}$. This defines the injective morphism $\Delta(D')/\Delta(D')^{H(D')} \to (\Delta/\Delta^H)(D')$. As the group $\Delta(A_{D_0})$ has an exponent which is a power of the characteristic of k while the one of $H(A_{D_0})$ is prime to the characteristic of k, we obtain that $H^1(H(A_{D_0}), \Delta(A_{D_0})) = 0$. But, the image of α in $Aut(G(A_{D_0}))$ by the following natural diagram

$$\begin{array}{ccccc} 1 \to E(A_{D_0}) & \longrightarrow & Aut_{A_{D_0}}(G_{A_{D_0}}) & \longrightarrow & Aut_{A_{D_0}}(\Delta_{A_{D_0}}) \times Aut_{A_{D_0}}(H_{A_{D_0}}) \\ & & \downarrow & & \downarrow \\ & & Aut(G(A_{D_0})) & \longrightarrow & Aut(\Delta(A_{D_0})) \times Aut(H(A_{D_0})) \end{array}$$

is an automorphism of $G(A_{D_0})$ which becomes trivial over $\Delta(A_{D_0})$ and $H(A_{D_0})$. The previous lemma insures the existence of an element $\delta_{A_{D_0}}$ of $\Delta(A_{D_0})$ such that the automorphism of $G(A_{D_0})$ induced by conjugation by $\delta_{A_{D_0}}$ coincides with the image of α in $Aut(G(A_{D_0}))$.
Write $\overline{\delta_{A_{D_0}}}$ for the image of $\delta_{A_{D_0}}$ in $(\Delta/\Delta^H)(D_0)$. It is enough to prove that $\overline{\delta_{A_{D_0}}}$ is the image of one element $\overline{\delta}$ of $(\Delta/\Delta^H)(D_0)$, then the image of $\overline{\delta}$ in $E(D_0)$ coincides with the image of α in $E(D_0)$, because one can prove that the natural restriction morphism $Aut_A(G_A) \to Aut(G(A_{D_0}))$ is injective. Since $(\Delta/\Delta^H)(D_0)$ is the equalizer of the two natural maps

$$(\Delta/\Delta^H)(A_{D_0}) \rightrightarrows (\Delta/\Delta^H)(A_{D_0} \otimes_D A_{D_0})$$

we have to prove that the two images of $\overline{\delta_{A_{D_0}}}$ in $(\Delta/\Delta^H)(A_{D_0} \otimes_D A_{D_0})$ coincide.

If we consider the following commutative diagram where the horizontal sequences are exact,

$$
\begin{array}{ccccccc}
1 & \longrightarrow & E(D_0) & \longrightarrow & E(A_{D_0}) & \rightrightarrows & E(A_{D_0} \otimes_D A_{D_0}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & Aut_{D_0}(G_{D_0}) & \longrightarrow & Aut_{A_{D_0}}(G_{A_{D_0}}) & \rightrightarrows & Aut_{A_{D_0} \otimes_D A_{D_0}}(G_{A_{D_0} \otimes_D A_{D_0}})
\end{array}
$$

we show that the two images of $\delta_{A_{D_0}}$ coincide in $Aut(G(A_{D_0} \otimes_{D_0} A_{D_0}))$ (recall that the image of $\delta_{A_{D_0}}$ in $Aut(G(A_{D_0}))$ coincides with the one of α). The result follows from the following diagram:

$$
\begin{array}{ccc}
\Delta(A_{D_0} \otimes_{D_0} A_{D_0}) & \longrightarrow & \Delta/\Delta^H(A_{D_0} \otimes_{D_0} A_{D_0}) \\
\downarrow & \searrow \quad \circlearrowleft & \uparrow \\
Aut(G(A_{D_0} \otimes_{D_0} A_{D_0})) & \hookleftarrow & \Delta(A_{D_0} \otimes_{D_0} A_{D_0})/\Delta(A_{D_0} \otimes_{D_0} A_{D_0})^{H(A_{D_0} \otimes_{D_0} A_{D_0})}
\end{array}
$$

□

Appendix B

Generality about stacks

For precisions, the reader can refer to [sta05] or [LMB00]. This chapter tries to give all the ingredients around stacks sufficient to understand this thesis. Let S be a scheme, and let $\mathcal{S} = (sch/S)$ be the category of schemes over S.

1 Algebraic spaces

Definition B.1. *An* ***algebraic space*** *over S is a sheaf of sets $\mathcal{X}$ over $(Sch/S)_{Et}$ such that:*

1. *The diagonal $\Delta_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by a quasi-compact scheme;*
2. *There is an étale and surjective map $U \to \mathcal{X}$ called an* ***atlas*** *where U is a scheme.*

2 Groupoids

2.1 Definitions

Definition B.2. *A* ***category over*** *S, denoted by $(\mathcal{X}, p_{\mathcal{X}})$, is a category $\mathcal{X}$ together with a covariant functor $p_{\mathcal{X}} : \mathcal{X} \to \mathcal{S}$. If B is an object of $\mathcal{S}$, we say that X* ***lies over*** *B if $p_{\mathcal{X}}(X) = B$.*

Definition B.3. *A category $(\mathcal{X}, p_{\mathcal{X}})$ over S is* ***a groupoid*** *over S if the following conditions are satisfied:*

1. *If $f : B' \to B$ is a morphism in $\mathcal{S}$, and X is an object of $\mathcal{X}$ lying over B, then there are an object X' over B' and a morphism $\phi : X' \to X$ such that $p_{\mathcal{X}}(\phi) = f$.*
2. *Let X, X', X'' be objects of $\mathcal{X}$ lying over B, B', B'' respectively. If $\phi : X' \to X$ and $\psi : X'' \to X$ are morphisms in $\mathcal{X}$, and $h : B' \to B''$ is a morphism such that $p_{\mathcal{X}}(\psi) \circ h = p_{\mathcal{X}}(\phi)$ then there is a unique morphism $\lambda : X' \to X''$ such that $\psi \circ \lambda = \phi$ and $p_{\mathcal{X}}(\lambda) = h$.*

Definition B.4. 1. *For some $B \in \mathcal{S}$, define the subcategory $\mathcal{X}(B)$ whose object X are such that $p_{\mathcal{X}}(X) = B$ and whose morphisms f are such that $p_{\mathcal{X}}(f) = id_B$. $\mathcal{X}(B)$ is called the* ***fiber over*** *B.*

2. *By 2., the object X' over B' of condition (1) of the previous definition is unique up to isomorphism. This object is called the* ***pull-back of** X **via** f and denoted by f^*X. Moreover, if $s : X \to Y$ is a morphism in $\mathcal{X}(B)$ then there is a canonical morphism $f^*s : f^*X \to f^*Y$. In other words, given a morphism $f : B' \to B$ of S-schemes, there is an induced covariant functor $f^* : \mathcal{X}(B) \to \mathcal{X}(B')$.*

Remark B.5. *We show thanks to condition 2. that a morphism $\phi : X' \to X$ (where X and X' are respectively over B and B') is an isomorphism if and only if $p_{\mathcal{X}}(\phi) : B' \to B$ is an isomorphism. We can then notice that the fiber $\mathcal{X}(B)$ is a groupoid (i.e. a category where all the morphisms are isomorphisms), which explains the terminology.*

2.2 Morphisms of groupoids

Definition B.6. 1. *If $(\mathcal{X}_1, p_{\mathcal{X}_1})$ and $(\mathcal{X}_2, p_{\mathcal{X}_2})$ are groupoids over S then a* ***morphism of groupoids*** *$\mathcal{X}_1 \to \mathcal{X}_2$ is a functor $p : \mathcal{X}_1 \to \mathcal{X}_2$ such that $p \circ p_{\mathcal{X}_2} = p_{\mathcal{X}_1}$.*

2. *A morphism of groupoids p is called an* ***isomorphism of groupoids*** *if it is an equivalence of categories.*

Remark B.7. *An isomorphism of groupoids has no inverse but has a quasi-inverse. The groupoids over $\mathcal{S}$ form a 2-category whose objects are the groupoids, 1-morphisms are the functors and 2-morphisms are the natural isomorphisms of functors. (The category of groupoids contains extra information about isomorphisms between morphisms.)*

2.3 Fibered products and cartesian diagrams

Definition B.8. 1. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be groupoids over S. If $f : \mathcal{X} \to \mathcal{Z}$ and $h : \mathcal{Y} \to \mathcal{Z}$ are morphisms of groupoids then we define the* ***fibered product*** *$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ as the S-groupoid whose:*

   (a) *objects are the triples (x, y, ψ) where $(x, y) \in \mathcal{X}(B) \times \mathcal{Y}(B)$ and $\psi : f(x) \to h(y)$ is an isomorphism in $\mathcal{Z}(B)$, with B fixed in $\mathcal{S}$,*

   (b) *the morphisms between two objects (X, Y, ψ) where $(X, Y) \in \mathcal{X}(B) \times \mathcal{Y}(B)$ and (X', Y', ψ') where $(X', Y') \in \mathcal{X}(B') \times \mathcal{Y}(B')$ is a pair of morphisms $\alpha : X' \to X$, $\beta : Y' \to Y$ over a same morphism $B \to B'$ such that $\psi \circ f(\alpha) = g(\beta) \circ \psi'$.*

   By construction, there are functors $p : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}$ and $q : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{Y}$. Notice however that the following diagram is non-commutative:

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Z} \end{array}$$

*In fact, $f \circ p(X,Y,\psi) = f(X)$, $g \circ q(X,Y,\psi) = g(Y)$ and the objects $f(X)$ and $g(Y)$ are isomorphic but not necessarily equal. Such a diagram is called 2-**commutative**.*
More generally, given a 2-commutative diagram of groupoids

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Z} \end{array}$$

*there is a morphism $T \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ unique up to isomorphism. If this morphism is an isomorphism then the diagram is called 2-**cartesian**.*

3 Stacks

3.1 The functor of isomorphisms

Let $(\mathcal{X}, P_{\mathcal{X}})$ be an S-groupoid. Let B be an S-scheme and let X and Y be two objects of $\mathcal{X}(B)$. We defined the contravariant functor $\underline{Iso}_B(X,Y) : (Sch/B) \to (Sets)$. Let (B', f) be a B-scheme $f : B' \to B$. The set $\underline{Iso}_B(X,Y)(B', f)$ is the set of the isomorphisms in $\mathcal{X}(B')$ between f^*X and f^*Y. Let now $f : B' \to B$ and $g : B'' \to B$ be two B-schemes and $h : B'' \to B'$ be a morphism of B-schemes (i.e. $g = f \circ h$), we need to define a map $\underline{Iso}_B(X,Y)(h) : \underline{Iso}_B(X,Y)(B', f) \to \underline{Iso}_B(X,Y)(B'', g)$. First, notice that by construction of the pullback, we have the canonical isomorphisms $\psi_X : g^*X \to h^*f^*X$ and $\psi_Y : g^*Y \to h^*f^*Y$. Let $\phi : f^*X \to f^*Y$ be an isomorphism in $\underline{Iso}_B(X,Y)(B', f)$ and consider its image by the functor $h^* : \mathcal{X}(B') \to \mathcal{X}(B'')$. We obtain an isomorphism $h^*\phi : h^*f^*X \to h^*f^*Y$. Define the image of ϕ via $\underline{Iso}_B(X,Y)(h)$ in $\underline{Iso}_B(X,Y)(B'', g)$ to be the composite $\psi_Y^{-1} \circ h^*\phi \circ \psi_X$. In particular, if $X = Y$ then $\underline{Iso}_B(X,X)$ is the functor whose sections over B' mapping on B are the automorphisms of the pull-back of X on B'.

3.2 Definition

Definition B.9. *A groupoid $(\mathcal{X}, p_{\mathcal{X}})$ over S is a **stack** if :*

1. *$\underline{Iso}_B(X,Y)$ is a sheaf for the étale topology for any S-scheme B and X, Y in $\mathcal{X}(B)$.*
2. *If $\{B_i \to B\}$ is a covering of B for the étale topology and X_i is a collection of objects in $\mathcal{X}(B_i)$ with isomorphisms*
$$\phi_{i,j} : \mathcal{X}_j|_{B_i \times_B B_j} \to \mathcal{X}_i|_{B_i \times_B Bj}$$
in $\mathcal{X}(B_i \times_B B_j)$ satisfying the cocycle condition, then there is an object $X \in \mathcal{X}(B)$ with an isomorphism $\mathcal{X}|_{B_i} \simeq \mathcal{X}_i$ inducing the isomorphisms $\phi_{i,j}$ above.

Remark B.10.
1. *The algebraic spaces give an example of stacks.*
2. *If $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}$ are stacks over S then the fibered product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is also a stack.*

3.3 Yoneda lemma for stacks

Lemma B.11. *Let $\mathcal{X}$ be a stack and X be an algebraic space. Then, there is an equivalence of categories*

$$\phi_X : \mathcal{X}(X) \simeq Mor_{Stacks}(X, \mathcal{X})$$

Proof. Let $P \in \mathcal{X}(X)$ and define a map $F_P : X \to \mathcal{X}$ by

$$\begin{array}{rccc} F_P(Y) : & X(Y) & \to & \mathcal{X}(Y) \\ & f : Y \to X & \mapsto & f^*P \end{array}$$

For any isomorphism $\phi : P' \to P \in \mathcal{X}(X)$, define a natural transformation $A_\phi : F_P \to F_{P'}$ by $f^*\phi : f^*P \to f^*P'$. Conversely, consider a morphism $F : X \to \mathcal{X}$, we obtain an object $P_F := F(X)(Id_X) \in \mathcal{X}(X)$. An automorphism of F defines an isomorphism of P_F. We can then show that these two maps are inverses of each other. □

3.4 Schemes as stacks

If X is an S-scheme then its functor of points gives a groupoid $\underline{X}$ whose objects are the X-schemes (B, ϕ) where $\phi : B \to X$ and a morphism from an object (B', ϕ') to an object (B, ϕ) is a morphism $f : B' \to B$ such that $\phi \circ f = \phi'$. The functor $p_{\underline{X}}$ is the forgetful functor which passes from the X-structure to the S-structure.
Take $f : X \to Y$ a scheme morphism, it induces a groupoid functor $F : \underline{X} \to \underline{Y}$ in the following way. The objects of $\underline{X}$ are X-schemes, which via the morphism f, can be seen as Y-schemes, that means the objects of $\underline{Y}$. Thus, $F((U, u)) = (U, f \circ u)$. If $s : B \to B'$ is a morphism of X-schemes, then $F(s)$ is the morphism s seen as a morphism of Y-schemes. Conversely, for $F : \underline{X} \to \underline{Y}$ a groupoid morphism over $\mathcal{S}$, we set $f := p(id_X) : X \to Y$. The Yoneda lemma implies that $F = f \circ -$.
The functor of points for a scheme is then a stack since condition 1. of the definition of stack is trivially satisfied and condition 2. is equivalent to say that the functor of points is a sheaf for the étale topology (see [FGI+05, theorem 2.55]).

Proposition B.12. *Let X and Y be S-schemes. Then, there is an isomorphism $f : X \to Y$ if and only if there is an equivalence of $\mathcal{S}$-groupoids $F : \underline{X} \to \underline{Y}$.*

Proof. If f is an isomorphism, denote by $F : \underline{X} \to \underline{Y}$ the induced functor as defined previously and by $F_0 : \underline{Y} \to \underline{X}$ the functor induced by f^{-1}. We can easily show that $F \circ F_0 = id_Y$ and $F_0 \circ F = Id_X$. Conversely, suppose that $F : \underline{X} \to \underline{Y}$ is an isomorphism of groupoids, set $F^{-1} : \underline{Y} \to \underline{X}$ its quasi-inverse and $f : X \to Y$ (resp. $f_0 : Y \to X$) the S-morphism inducing F (resp. F_0). Then, $F_0 \circ F(Id_X) = g \circ f$. Since $F \circ F_0$ and $Id_{\overline{X}}$ are naturally isomorphic, $Id_X : X \to X$ and $g \circ f : X \to X$ are also isomorphic as X-schemes. So, $g \circ f : X \to X$ is an automorphism. Using the same argument, we show that $f \circ g : X \to X$ is an automorphism. Finally, $f : X \to Y$ is an isomorphism. □

Remark B.13. *1. In the following, when B is a scheme, we write simply $B \to \mathcal{X}$ (resp. $\mathcal{X} \to B$) instead of $\underline{B} \to \mathcal{X}$ (resp. $\mathcal{X} \to \underline{B}$).*

2. If X, Y and Z are schemes then $\underline{X} \times_{\underline{Z}} \underline{Y}$ is isomorphic to $\underline{X \times_Z Y}$ so this notion of the fiber product is an extension of the usual one for schemes.

3. The notion of fiber product allows us to talk about base change. Indeed, suppose that $\mathcal{X}$ is an S-groupoid and $T \to S$ a morphism of schemes. Then if $U \to T$ is a T-scheme, one can check that the groupoids $\mathcal{X}(U)$ and $(\mathcal{X} \times_S T)(U)$ are equivalent; i.e. $\mathcal{X} \times_S T$ is the T-groupoid obtained by base change to T.

3.5 Artin stacks

Definition B.14. *An **Artin stack** over S is a stack $\mathcal{X}$ over $(Sch/S)_{Et}$ such that:*

1. *the diagonal $\Delta_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, separated and quasi-compact;*
2. *there is a smooth and surjective map $U \to \mathcal{X}$ where U is a scheme called **atlas**.*

Remark B.15. *In particular, an Artin stack is quasi-separated.*

We end this section with the Artin criterion giving a sufficient condition for a stack to be an Artin stack.

Theorem B.16. *([LMB00, Théorème 10.1]) Let $\mathcal{X}$ be a S-stack satisfying the following axioms:*

1. *the diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, separated and quasi-compact;*
2. *there is an algebraic space $\mathcal{Y}$ over S with a morphism of S-stacks $\mathcal{Y} \to \mathcal{X}$ which is representable, faithfully flat and locally finitely presented.*

Then $\mathcal{X}$ is a Artin stack over S.

3.6 Group of automorphisms

Definition B.17. *Given an Artin stack $\mathcal{X}$ and a morphism $f : T \to \mathcal{X}$ where T is a scheme, we define the group of automorphism of f, denoted $\underline{Aut}_T(f)$, as the fiber product:*

$$\begin{array}{ccc} \underline{Aut}_T(f) & \longrightarrow & T \\ \downarrow & & \downarrow f \\ \mathcal{I}_{\mathcal{X}} & \xrightarrow{p_2} & \mathcal{X} \end{array}$$

*where $\mathcal{I}_{\mathcal{X}}$, called **inertia stack**, is defined as the fiber product*

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \Delta_{\mathcal{X}/S} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}/S}} & \mathcal{X} \times_S \mathcal{X} \end{array}$$

Remark B.18. *1. The inertia stack represents the functor of isomorphisms (i.e. the fibered category over S whose objects over some S-scheme U are the pairs (ξ, α) where $\xi \in \mathcal{X}(U)$ and α is an automorphism of ξ).*

2. Let $Spec(T') \to Spec(T)$ be a morphism of schemes, let $f : Spec(T) \to \mathcal{X}$ be a T-point of X and let $f' : Spec(T') \to Spec(T) \to \mathcal{X}$ be associated T'-point. Then

$$\underline{Aut}_T(f') = \underline{Aut}_T(f) \times_{Spec(T)} Spec(T').$$

3.7 Characterization of algebraic spaces via stacks

Theorem B.19. *([LMB00, Corollaire 8.1.1]) Let $\mathcal{X}$ be an Artin stack and $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ be the 1-diagonal map. The following assertions are equivalent:*

1. *The Artin stack $\mathcal{X}$ is an algebraic space.*
2. *For any affine scheme U over S and any morphism $x : U \to \mathcal{X}$, $\underline{Aut}_{\mathcal{X}(U)}(x) = \{Id_x\}$.*
3. *The morphism Δ is a monomorphism of stacks.*

3.8 Representable morphisms of algebraic stacks

In the following, $\mathcal{C}$ denotes the category of schemes or the one of algebraic spaces.

Definition B.20. *A morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks is said to be* ***representable in*** *$\mathcal{C}$ if for any morphism $B \to \mathcal{Y}$ where B is an object of $\mathcal{C}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} B$ is isomorphic to an object of $\mathcal{C}$.*

Definition B.21. *1. We say that a morphism of Artin stacks $f : \mathcal{X} \to \mathcal{Y}$ representable in $\mathcal{C}$ has a property P if for any map $B \to \mathcal{Y}$, where B is an object of $\mathcal{C}$, the corresponding morphism $\mathcal{X} \times_{\mathcal{Y}} B \to B$ of $\mathcal{C}$ has property P.*

2. Let $\mathcal{X}$ be a stack. For a property of schemes local for the smooth topology (for example, locally noetherian, quasi-compact, (quasi)-separated, surjective...), we say that the stack $\mathcal{X}$ has this property if there is a scheme U with this same property and a smooth and surjective morphism $U \to \mathcal{X}$.

Lemma B.22. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks. The following are equivalent:*

1. *The morphism f is representable;*
2. *For any $T \to S$ and any object ξ of $\mathcal{X}(T)$, the induced homomorphism $\underline{Aut}_T(\xi) \to \underline{Aut}_T(f(\xi))$ is injective.*

Proof. By Theorem B.19, the morphism $f : \mathcal{X} \to \mathcal{Y}$ is representable if and only if for any algebraic space V and any morphism $g : V \to \mathcal{Y}$ for $\mathcal{F} := \mathcal{X} \times_{\mathcal{Y}} V$, we have that the diagonal map $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is a monomorphism.

Let T be an S-scheme, $q \in \mathcal{F} \times_S \mathcal{F}(T)$ and take two elements $p_i \in \mathcal{F}(T)$ (where $i = 1, 2$), with isomorphisms $\beta_i : \Delta(p_i) \to q$. There exists a unique isomorphism $\phi : p_1 \to p_2$ such that $\beta_1 = \beta_2 \circ \Delta(\phi)$.
We can write

$$q = ((x_1, v_1, \alpha_1), (x_2, v_2, \alpha_2))$$

where $x_i \in \mathcal{X}$, $v_1 \in V$ and $\alpha_i : f(x_i) \simeq g(v_i)$. Similarly $p_i = (x'_i, v'_i, \alpha'_i)$ as above. The existence of β_i implies that $v_1 = v_2 = v'_1 = v'_2$. Also, composing with the given isomorphisms we may reduce to the case in which in fact, the same is true for x_i, x'_i, α_i and α'_i. Thus, the condition above is equivalent to saying that for $x \in \mathcal{X}(T)$, there is a unique $\beta \in \underline{Aut}_T(x)$ such that $f(\beta) = Id_{f(x)}$, which is what we wanted.

□

4 Sheaves on Artin stacks

The goal of this section is to give a brief overview about the notion of sheaves on Artin Stacks and required knowledge about simplicial topoi and derived categories. The reader can find details and proof in the article of Olsson [Ols07].
If $\mathcal{A}$ is an abelian category, we denote by $D^+(\mathcal{A})$ (respectively $D^-(\mathcal{A})$, $D^b(\mathcal{A})$) the derived category of complexes bounded below (respectively bounded above, bounded below and above). For $n \in \mathbb{Z}$, we denote $\tau_{\geq n}$ the "canonical truncation in degree $\geq n$" functor (see [SGA73, VII. Définition 1.1.13]). Let $D'(\mathcal{A})$ denote the category of projective systems

$$K = (... \to K_{\geq -n-1} \to K_{\geq -n} \to ... \to K_{\geq 0})$$

where each $K_{\geq -n} \in D^+(\mathcal{A})$ and the maps

$$K_{\geq -n} \to \tau_{\geq -n} K_{\geq -n}, \ \tau_{\geq -n} K_{\geq -n-1} \to \tau_{\geq -n} K_{\geq -n}$$

are all isomorphisms.

4.1 The lisse-étale site on an algebraic stack

Let $\mathcal{X}$ be an algebraic stack. We will define a site on $\mathcal{X}$ which will permit to have a notion of Cech cohomology for this site.

Definition B.23. *The* ***lisse-étale site of*** $\mathcal{X}$*, denoted Lis-ét*$(\mathcal{X})$*, is the site with underlying category the full subcategory of* $\mathcal{X}$*-schemes whose objects are the smooth* $\mathcal{X}$*-schemes and whose covering families* $\{U_i \to U\}_{i \in I}$ *are families of étale morphisms such that the amalgamation*

$$\coprod_{i \in I} U_i \to U$$

is surjective . We denote by $\mathcal{X}_{Lis\text{-}\acute{e}t}$ the associated topos. The topos $\mathcal{X}_{Lis\text{-}\acute{e}t}$ is naturally ringed with structure sheaf $\mathcal{O}_{\mathcal{X}_{Lis\text{-}\acute{e}t}}$ which associates to any $U \in$Lis-ét$(\mathcal{X})$ the ring $\Gamma(U, \mathcal{O}_U)$.

Remark B.24. *A sheaf $F \in \mathcal{X}_{Lis\text{-}\acute{e}t}$ defines for every object $U \in$ Lis-ét$(\mathcal{X})$ a sheaf F_U on U_{et} by restriction. As proved in [LMB00, 12.2.1], this gives an equivalence of categories between the category of sheaves $\mathcal{X}_{Lis\text{-}\acute{e}t}$ and the category of systems $\{F_U, \theta_\phi\}$ consisting of the following things: a sheaf $F_U \in U_{\acute{e}t}$ for every $U \in$ Lis-ét$(\mathcal{X})$ and a morphism $\theta_\phi : \phi^{-1}F_V \to F_U$ for every morphism $\phi : U \to V$ in Lis-ét$(\mathcal{X})$, such that*

1. *θ_ϕ is an isomorphism if ϕ is étale.*
2. *For a composite*

$$U \xrightarrow{\phi} V \xrightarrow{\psi} W$$

We have $\theta_\phi \circ \phi^*(\theta_\psi) = \theta_{\psi\circ\phi}$.

Definition B.25. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks, and define the* ***direct image functor*** *f_* to be the functor defined by sending a sheaf $\mathcal{M}$ to the sheaf which to any $U \in$ Lis-ét$(\mathcal{X})$ associates $\mathcal{M}(U \times_{\mathcal{X}} \mathcal{Y})$, and the* ***inverse image functor*** *f^{-1} to be the functor defined by sending a sheaf $\mathcal{N}$ to the sheaf which to any $V \in$ Lis-ét$(\mathcal{X})$ associates the limit $\varinjlim_{V \to U} \mathcal{M}(U)$. Here the limit is taken over the category of morphisms over f from V to the objects $U \in$ Lis-ét$(\mathcal{X})$.*

4.2 Quasi-coherent sheaves on stacks

The theory for schemes is recalled in appendix B.

Definition B.26. 1. *A quasi-coherent sheaf $\mathcal{M}$ on $\mathcal{X}$ is given by:*

   (a) *for each atlas $U \to \mathcal{X}$, a quasi-coherent sheaf $\mathcal{M}_U$ over U;*

   (b) *for each commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\phi} & V \\ & \searrow \quad \swarrow & \\ & \mathcal{X} & \end{array}$$

   in Lis-ét$(\mathcal{X})$ where U are V atlases, an isomorphism $\alpha_\phi : \mathcal{M}_U \to \phi^*\mathcal{M}_V$.

2. *If $\mathcal{X}$ is locally noetherian, a quasi-coherent sheaf $\mathcal{M}$ is called* ***coherent*** *if for every $U \in$ Lis-ét$(\mathcal{X})$ the restriction $\mathcal{M}_U$ of $\mathcal{M}$ to U is a coherent sheaf.*

We write $Qcoh(\mathcal{X})$ (respectively $Coh(\mathcal{X})$) for the category of quasi-coherent (respectively coherent) sheaves on $\mathcal{X}$.

Proposition B.27. *([Ols07, Lem. 6.5(i)]) Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks and denote by f^* the functor $f^{-1}(-) \otimes_{f^{-1}\mathcal{O}_\mathcal{Y}} \mathcal{O}_\mathcal{X}$*

1. *For any quasi-coherent sheaf $\mathcal{M} \in Qcoh(\mathcal{X})$, the sheaf $f^*\mathcal{M}$ is a quasi-coherent sheaf on $\mathcal{Y}$.*
2. *For any quasi-coherent sheaf $\mathcal{N}$ on $\mathcal{Y}$ the sheaf $f^*\mathcal{N}$ is quasi-coherent on $\mathcal{X}$. The induced functor $f^* : Qcoh(\mathcal{Y}) \to Qcoh(\mathcal{X})$ is left adjoint to the functor $f_* : Qcoh(\mathcal{X}) \to Qcoh(\mathcal{Y})$.*

For $* \in \{b, +, -, [a,b]\}$, we write $D^*_{qcoh}(\mathcal{X}) \subset D^*_{qcoh}(\mathcal{X}_{\text{Lis-ét}}, \mathcal{O}_{\mathcal{X}_{\text{Lis-ét}}})$ for the full subcategory of objects whose cohomology sheaves are quasi-coherent sheaves. We will consider the subcategory $D'_{qcoh}(\mathcal{X}) \subset D'(\mathcal{X})$ consisting of systems K for which $K \geq -n$ is in $D^+_{qcoh}(\mathcal{X})$ for all n. If $\mathcal{X}$ is locally noetherian, we also define $D^*_{coh}(\mathcal{X}) \subset D^*_{qcoh}(\mathcal{X})$ to be the full subcategory of objects with coherent cohomology sheaves.

Proposition B.28. *([Ols07, Proposition 6.4])*

1. *If $\mathcal{M}$ and $\mathcal{N}$ are quasi-coherent sheaves on $\mathcal{X}$, then $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{N}$ is also a quasi-coherent sheaf. More generally, $(-) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{X}} (-)$ induces a functor:*
$$(-) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{X}} (-) : D^-_{qcoh}(\mathcal{X}) \times D^-_{qcoh}(\mathcal{X}) \to D^-_{qcoh}(\mathcal{X})$$
2. *If $\mathcal{X}$ is locally noetherian, then $(-) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{X}} (-)$ induces a functor*
$$(-) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{X}} (-) : D^-_{coh}(\mathcal{X}) \times D^-_{coh}(\mathcal{X}) \to D^-_{coh}(\mathcal{X})$$
3. *If $\mathcal{X}$ is locally noetherian, then $RHom_{\mathcal{O}_\mathcal{X}}(-,-)$ induces functors*
$$RHom_{\mathcal{O}_\mathcal{X}}(-,-) : D^-_{coh}(\mathcal{X}) \times D^+_{qcoh}(\mathcal{X}) \to D^+_{qcoh}(\mathcal{X})$$
$$RHom_{\mathcal{O}_\mathcal{X}}(-,-) : D^-_{coh}(\mathcal{X}) \times D^+_{coh}(\mathcal{X}) \to D^+_{coh}(\mathcal{X})$$

4.3 The cotangent complex of a morphism of algebraic stacks

Theorem B.29. *([Ols07, Theorem 8.1]) Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Then one can associate to f an object $L_{\mathcal{X}/\mathcal{Y}} \in D'_{qcoh}(\mathcal{X}_{Lis\text{-}ét})$ called **cotangent complex of** f such that the following holds:*

1. *If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic spaces, then the complex $L_{\mathcal{X}/\mathcal{Y}}$ is canonically isomorphic to the object $\{\tau_{\geq -n}\pi^* L_{\mathcal{X}/\mathcal{Y},ét}\}_n$, where $\pi : \mathcal{X}_{Lis\text{-}ét} \to \mathcal{X}_{ét}$ denotes the projection to the étale topos and $L_{\mathcal{X}/\mathcal{Y},ét}$ denotes the usual cotangent complex of the morphism of ringed topoi $\mathcal{X}_{ét} \to \mathcal{Y}_{ét}$.*
2. *For any 2-commutative square of algebraic stacks*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{A} & \mathcal{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{B} & \mathcal{Y} \end{array}$$

there is a natural functoriality morphism

$$LA^*L_{\mathcal{X}/\mathcal{Y}} \to L_{\mathcal{X}'/\mathcal{Y}'}$$

If the square is cartesian and either f or B is flat then this functoriality morphism is an isomorphism, and the sum map

$$LA^*L_{\mathcal{X}/\mathcal{Y}} \oplus Lf'^*L_{\mathcal{Y}'/\mathcal{Y}} \to L_{\mathcal{X}'/\mathcal{Y}}$$

is an isomorphism.

3. *If $g : \mathcal{Y} \to \mathcal{Z}$ is another morphism of algebraic stacks, then there is a natural map*

$$L_{\mathcal{X}/\mathcal{Y}} \to Lf^*(L_{\mathcal{Y}/\mathcal{Z}})[1]$$

such that

$$Lf^*L_{\mathcal{Y}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Y}} \to Lf^*L_{\mathcal{Y}/\mathcal{Z}}[1]$$

is a distinguished triangle in $D'_{qcoh}(\mathcal{X}_{Lis\text{-}ét})$.

In particular, the cotangent complex of the canonical morphism $B_SG \to S$ has the following property, for any fppf group scheme.

Lemma B.30. *Let $G \to S$ be an fppf group scheme. Denote by $L_{B_SG/S}$ the cotangent complex (see Theorem B.29) of the structural morphism $B_SG \to S$ and by $L_{G/S}$ the cotangent complex L_G of the group scheme G. We consider the following 2-cartesian diagram:*

$$\begin{array}{ccc} G & \xrightarrow{q} & S \\ \downarrow & & \downarrow p \\ S & \xrightarrow[p]{} & B_SG \end{array}$$

where q is the structural morphism of G and p is the canonical morphism.
We have $q^*p^*L_{B_SG/S} \simeq L_G$ and thus $L_{B_SG/S} \in D^{[0,1]}_{coh}(\mathcal{O}_{B_SG})$. If we suppose that G is smooth, $L_G \simeq \mathfrak{g}*$ where $\mathfrak{g}*$ is the dual of the Lie algebra of the group scheme G, hence $L_{B_SG/S} \in D^{[0]}_{coh}(\mathcal{O}_{B_SG})$.

Proof. Since the map p is faithfully flat, it suffices to show that $p^*L_{B_SG/S}$ has cohomology concentrated in degree 0 and 1. Applying Theorem B.29 3. to the composite map

$$S \xrightarrow{p} B_SG \longrightarrow S$$

one obtains the following distinguished triangle

$$p^*L_{B_SG/S} \longrightarrow L_{S/S} \longrightarrow L_{S/B_SG} \longrightarrow p^*L_{B_SG/S}[1]$$

From the fact that $L_{S/S} = 0$, we see that $p^*L_{B_SG/S} \simeq p^*L_{S/B_SG}[-1]$. Therefore it suffices to show that L_{S/B_SG} is concentrated in degrees -1 and 0. Considering the cartesian diagram of the proposition and applying Theorem B.29 2., we obtain that $q^*L_{S/B_SG} \simeq L_{G/S}$ thus $q^*p^*L_{B_SG/S} \simeq L_G$. Moreover, since q is faithfully flat, it suffices to prove that $L_{G/S}$ is concentrated in degrees -1 and 0. This follows from the fact that any fppf group is a local complete intersection and we complete the proof thanks to [Ill71, III.3.2.6]. □

4.4 The Grothendieck existence theorem for stacks

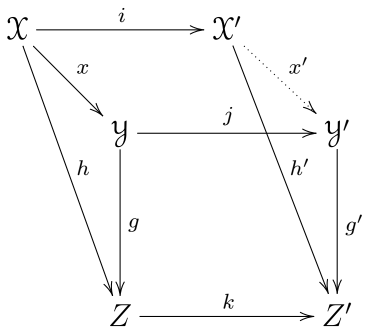
Theorem B.31. *([Ols07, Theorem 11.1]) Let A be a noetherian adic ring and let $\mathfrak{a} \subset A$ be an ideal of definition. Let $\mathcal{X}$ be a proper algebraic stack over A, and for every $n \geq 0$, let $\mathcal{X}_n$ denote the stack $\mathcal{X} \times_{Spec(A)} Spec(A/\mathfrak{a}^{n+1})$.*

1. *The functor sending a coherent sheaf to its reductions over $Spec(A/\mathfrak{a}^{n+1})$ for any n, defines an equivalence of categories between the category of coherent sheaves on $\mathcal{X}$ and the category of compatible system $\{\mathcal{M}_n\}_{n\geq 0}$ of coherent sheaves on the $\mathcal{X}_n'$s.*
2. *If $\mathcal{M}$ is a coherent sheaf on $\mathcal{X}$ with reduction $\{\mathcal{M}_n\}_{n\geq 0}$ then for every $i \geq 0$ the natural map*
$$H^i(\mathcal{X}, \mathcal{M}) \to \varprojlim_n H^i(\mathcal{X}_n, \mathcal{M}_n)$$
is an isomorphism. If the finitely generated A-module is viewed as a topological A-module with the $\mathfrak{a}$-adic topology and $\varprojlim_n H^i(\mathcal{X}_n, \mathcal{M}_n)$ is viewed as a topological A-module with the inverse limit topology, then this is an isomorphism of topological A-modules.

5 Deformation for representable morphisms of algebraic stacks

5.1 Deformation via closed immersions defined by a square zero ideal

Let $x : \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks. Suppose that x fits into a 2-commutative diagram of solid arrows between algebraic stacks



where Z and Z' are schemes, and i (respectively j, k) is a closed immersion defined by a square-zero sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}'}$ (respectively $\mathcal{J} \subset \mathcal{O}_{\mathcal{Y}'}$, $K \subset \mathcal{O}'_Z$).

Theorem B.32. *(see [Ols06, Theorem 1.5]) Let $L_{\mathcal{Y}/Z}$ denote the cotangent complex of $g : \mathcal{Y} \to Z$.*

1. *There is a canonical class $o(x, i, j) \in Ext^1(Lx^* L_{\mathcal{Y}/Z}, I)$ whose vanishing is necessary and sufficient for the existence of an arrow $x' : \mathcal{X}' \to \mathcal{Y}'$ filling the previous diagram.*
2. *If $o(x, i, j) = 0$, then the set of isomorphism classes of maps $x' : \mathcal{X}' \to \mathcal{Y}'$ filling the previous diagram is naturally a torsor under $Ext^0(Lx^* L_{\mathcal{Y}/Z}, I)$.*
3. *For any morphism $x' : \mathcal{X}' \to \mathcal{Y}'$ the group of automorphisms of x' (as a deformation of x) is canonically isomorphic to $Ext^{-1}(Lx^* L_{\mathcal{Y}/Z}, I)$.*

Let $P \to \mathcal{X}$ be a G-torsor, $i : \mathcal{X} \to \mathcal{X}'$ be a closed immersion of stacks defined by a square-zero sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathcal{Y}'}$ and let $k : Z \to Z'$ be a closed immersion of schemes defined by a square-zero ideal $K \subset \mathcal{O}'_Z$.

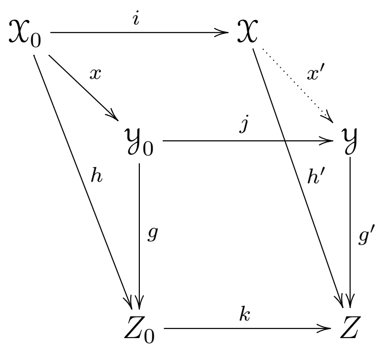
Corollary B.33. *Let $L_{B_Z G/Z}$ denote the cotangent complex of the structural map $g : B_Z G \to Z$.*

1. *There is a canonical class $o(x, i, j) \in Ext^1(L_{B_Z G/Z}, I/I^2)$ whose vanishing is necessary and sufficient for the existence of a G-torsor $P' \to \mathcal{X}'$ which lifts the G-torsor $P \to \mathcal{X}$.*
2. *If $o(x, i, j) = 0$, then the set of isomorphism classes of the G-torsor $P' \to \mathcal{X}'$ extending the G-torsor $P \to \mathcal{X}$ is naturally a torsor under $Ext^0(L_{B_Z G/Z}, I/I^2)$.*
3. *For any morphism $P' \to \mathcal{X}'$, the group of automorphisms of the G-torsor $P' \to \mathcal{X}'$ (as a deformation of the G-torsor $P \to \mathcal{X}$) is canonically isomorphic to $Ext^{-1}(L_{B_Z G/Z}, I/I^2)$.*

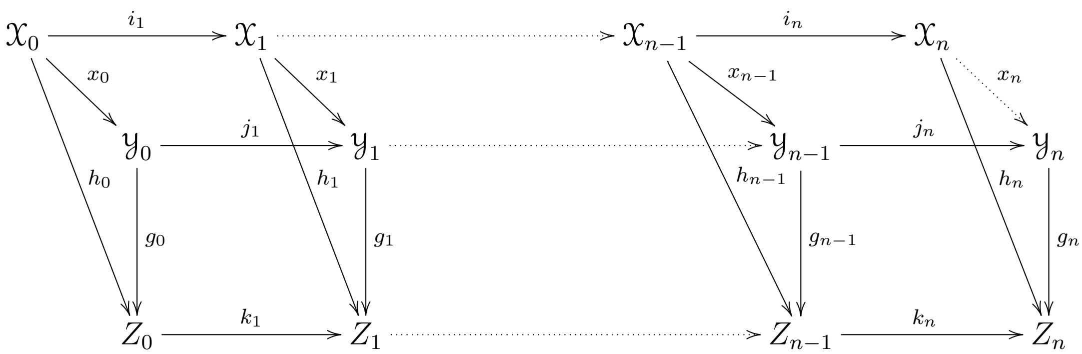
Proof. It is enough to notice that the data of a G-torsor $P \to \mathcal{X}$ is equivalent to the data of a representable morphism of stacks $\mathcal{X} \to B_Z G$. Indeed, if one has a G-torsor $P \to \mathcal{X}$ then $[P/G] \simeq \mathcal{X}$, which defines naturally a representable morphism $\mathcal{X} \to B_Z G$. Conversely, a representable morphism of stacks $x : \mathcal{X} \to B_Z G$ defines a G-torsor as the pullback of the trivial G-torsor $Z \to B_Z G$ to $\mathcal{X}$ via x. □

5.2 Formal deformations

Let now $x_0 : \mathcal{X}_0 \to \mathcal{Y}_0$ be a representable morphism of algebraic stacks. Suppose that x_0 fits into a 2-commutative diagram of solid arrows between algebraic stacks



where Z_0 and Z are schemes, and i (respectively j, k) is a closed immersion defined by the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ (respectively $\mathcal{J} \subset \mathcal{O}_{\mathcal{Y}}$, $K \subset \mathcal{O}_Z$). One can show that if x' exists then it is necessarily representable. Denote by $\mathcal{X}_n$ the closed substack of $\mathcal{X}$ (respectively $\mathcal{Y}_n$, Z_n) whose sheaf of ideals is $\mathcal{I}^{n+1}$ (respectively $\mathcal{J}^{n+1}$, K^{n+1}). One obtains the following diagram



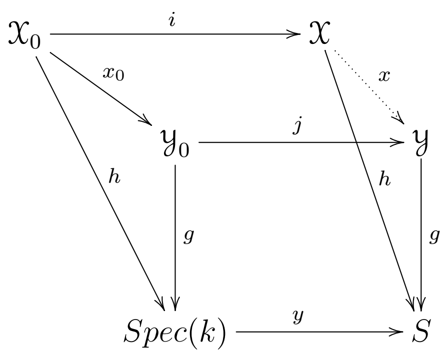
where each cube verifies the condition of the previous section.

Definition B.34. *We call a **formal deformation of** x_0 the data of morphisms x_n filling in the previous diagram for any $n \in \mathbb{N}$.*

So, one can use step after step Theorem B.32 for the existence of a formal deformation.

5.3 Algebraization of formal deformations

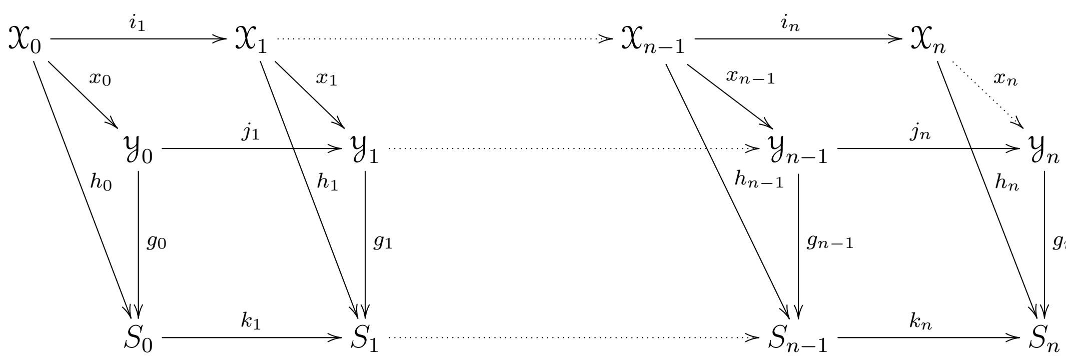
Denote by $S = Spec(R)$ a scheme where R is a strict local henselian ring of maximal ideal $\mathfrak{m}$, by $k = R/\mathfrak{m}$ its residue field, $S_n = Spec(R/\mathfrak{m}^{n+1})$ and $k : Spec(k) \to S$ the natural closed immersion.
Let now $x_0 : \mathcal{X}_0 \to \mathcal{Y}_0$ be a representable morphism of algebraic stacks. Suppose that x_0 fits into a 2-commutative diagram of solid arrows between algebraic stacks



where i (respectively j) is a closed immersion defined by the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ (respectively $\mathcal{J} \subset \mathcal{O}_{\mathcal{Y}}$). We consider the (contravariant) functor

$$F : (Algebras/R) \to (Sets)$$

sending an R-algebra A to the set of morphisms of algebraic stacks $\mathcal{X} \times_S Spec(A) \to \mathcal{Y} \times_S Spec(A)$ over $Spec(A)$. Thanks to the Existence Theorem of Grothendieck for stacks B.31, One can show that this functor is **locally of finite presentation**, (that is, for every filtering inductive system of R-algebras $\{B_i\}$, the canonical map $\varinjlim F(B_i) \to F(\varinjlim B_i)$ is bijective).
Suppose that one has a formal deformation x_n filling in the following diagram:



The purpose is to see if one can can obtain the existence of the morphism $x : \mathcal{X} \to \mathcal{Y}$ filling in the first diagram, algebraizing the formal deformation. One of the tools which can lead to such a result is Artin's Approximation theorem which is valid when R is the henselization of a finite type algebra over an excellent discrete valuation ring or a field.

Theorem B.35. *(see [Art69, Théorème 1.12]) Let K be a field or an excellent discrete valuation ring, and let R be the henselization of a K-algebra of finite type at a prime ideal. Let $\mathfrak{m}$ a proper ideal of A and $F : (R\text{-algebras}) \to (Sets)$ be a functor locally of finite presentation and so that for every A-algebra B, $F(B)$ is the set of isomorphism classes of structures over B. Given $\bar{\xi} \in F(\hat{R})$ there is a $\xi \in F(R)$ such that*

$$\xi \equiv \bar{\xi} \ (mod\ \mathfrak{m}^n)$$

for any $n \in \mathbb{N}$.

In order to be able to apply this theorem to obtain an algebraization of some formal data, we can see R as limit of rings R_i which are henselizations of $\mathbb{Z}$-algebras of finite type. Since the functor F is locally of finite presentation, one can see that it suffices to prove the theorem in the case where R is the henselization of a $\mathbb{Z}$-algebra of finite type. Using again the Existence Theorem of Grothendieck for stacks B.31 to go to the completion, one can then obtain the morphism x via the Artin's Approximation theorem.

Abstracts in French and Italian (long versions)

Résumé en Français (Version longue):

Ramification modérée pour des actions par des schémas en groupes affines et pour des champs quotients

L'objet de cette thèse est de comprendre comment se généralise la théorie de la ramification pour des actions par des schémas en groupes affines avec un intérêt particulier pour la notion de modération.

Dans la suite, les schémas considérés seront affines sur une base affine. Plus précisément, la base sera $S := Spec(R)$ où R est un anneau commutatif unitaire, $G = Spec(A)$ sera un schéma en groupes affine plat sur S et $X := Spec(B)$ sera un schéma affine sur S. La donnée d'une action de G sur X définie par un morphisme $\mu_X : X \times_S G \to X$ est équivalente à la donnée d'un B-comodule défini par le morphisme structural, que l'on notera $\rho_B : B \to B \otimes_R A$. On notera cette action (X, G). On notera $C := B^A = \{b \in B \mid \rho_B(b) = b \otimes 1\}$ l'anneau des invariants pour l'action, $(\mu_X, p_1) : X \times_S G \to X \times_S X$ l'application de Galois, $[X/G]$ le champ quotient associé à l'action (X, G) et $Y := Spec(C)$.

Comme point de référence, nous prenons la théorie classique de la ramification pour des anneaux munis d'une action d'un groupe fini abstrait qui est l'objet de l'article [Bar74], qui généralise la théorie de la ramification pour des extensions de corps. Afin de comprendre comment généraliser cette théorie pour des actions de schémas en groupes, nous considérons les actions de schémas en groupes constants en se rappelant que la donnée de telles actions est équivalente à celle d'un anneau muni d'une action par un groupe fini abstrait, nous ramenant au cas classique. Nous obtenons ainsi dans ce nouveau contexte des notions généralisant l'anneau des invariants en tant que quotient, les groupes d'inertie et toutes leurs propriétés. Rappelons simplement ici la définition indispensable pour la suite de groupe d'inertie dans ce contexte.

Pour ζ un T-point de X où T est une R-algèbre, on notera $I_G(\zeta)$ le groupe d'inertie de l'action au point $\zeta : Spec(T) \to X$, défini comme le produit fibré

$$\begin{array}{ccc} I_G(\zeta) & \xrightarrow{pr_2} & T \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \Delta_X \circ \zeta} \\ X \times_S G & \xrightarrow[(\mu_X, p_1)]{} & X \times_S X \end{array}$$

où p_1 est la première projection. Nous montrons que les propriétés classiques des groupes d'inertie se généralisent bien dans ce nouveau contexte.

Ayant rappelé le contexte propice pour pouvoir parler de ramification, nous pouvons chercher à distinguer les différents types de ramification. Le cas non ramifié se généralise naturellement avec les actions libres puisque caractérisées comme dans le cas classique par le fait que tous leurs groupes d'inertie sont triviaux. Ceci est équivalent, dans le cas où le schéma en groupes agissant est fini et localement libre à l'existence de torseurs, qui sont des objets localement simples, bien compris et très importants lorsque l'on considère des actions par des schémas en groupes. En ce qui concerne le cas de la ramification modérée qui nous intéresse particulièrement, deux généralisations sont proposées dans la littérature. La première est celle d'actions modérées de schémas en groupes affines introduite dans l'article [CEPT96] par Chinburg, Erez, Pappas et Taylor, définies *via* l'existence d'une intégrale totale. Plus précisément, on dit qu'une *action* (X, μ_X) *est modérée* s'il existe un morphisme de A-comodules $\alpha : A \to B$, qui est unitaire, i.e. $\alpha(1_A) = 1_B$, un tel α est appelé *intégrale totale*. (On rappelle qu'un morphisme $\alpha : (B, \rho_B) \to (C, \rho_C)$ de A-comodules est une application R-linéaire telle que $\rho_C \circ \alpha = (\alpha \otimes 1_B) \circ \rho_B$. La R-algèbre A peut être vue comme un A-comodule via la comultiplication $\Delta : A \to A \otimes_R A$.) La deuxième est celle de champ modéré introduite dans [AOV08] par Abramovich, Olsson et Vistoli. Considérons le champ quotient $[X/G]$ associé à l'action (X, μ_X). La définition de champ quotient modéré requiert l'existence d'un espace de modules grossier pour l'action (On rappelle qu'un *espace de modules grossier* pour $[X/G]$ est un couple (M, ρ) où M est un espace algébrique et $\rho : [X/G] \to M$ est un morphisme universel pour les morphismes de $[X/G]$ vers un espace algébrique tel que pour tout corps Ω algébriquement clos, $|[X/G](\Omega)| \simeq M(\Omega)$ où $|[X/G](\Omega)|$ est l'ensemble des classes d'isomorphismes du groupoïde). Pour assurer cette existence, nous supposons dans la suite que le champ quotient $[X/G]$ est un champ algébrique localement de présentation finie et que tous les groupes d'inertie sont finis. En effet, sous ces hypothèses, on peut montrer qu'il existe un espace de modules grossier $\rho : [X/G] \to M$ et que le morphisme ρ est propre. On dit alors que le *champ* $[X/G]$ *est modéré* si le foncteur entre les catégories de faisceaux quasi-cohérents $\rho_* : Qcoh([X/G]) \to Qcoh(M)$ est exact. Il a été alors naturel d'essayer de comparer ces deux notions de modération et de comprendre comment se généralisent les propriétés classiques d'objets modérés à des actions de schémas en groupes affines.

Tout d'abord, nous avons traduit algébriquement la propriété de modération sur un champ quotient comme l'exactitude du foncteurs des invariants.

Théorème IX.10. *Supposons que le foncteur $\rho_* : Qcoh([X/G]) \to Qcoh(Y)$ soit bien défini. Alors le foncteur ρ_* est exact si et seulement si le foncteur des invariants $(-)^A$ est exact.*

Ce qui nous a permis d'obtenir aisément à l'aide de [CEPT96] qu'une action modérée définit toujours un champ quotient modéré.

Théorème IX.15. *Supposons que X est noethérien, de présentation finie sur S, que G est de présentation finie sur S et que tous les groupes d'inertie soient finis. Si l'action est modérée alors le champ quotient est modéré.*

Quant à la réciproque, grâce à une étude algébrique, nous avons réussi à l'obtenir seulement lorsque nous supposons de plus que G est fini et localement libre sur S et que X est plat sur Y.

Théorème IX.12. *Supposons que G soit fini et localement libre sur S. Si l'action (X, G) est modérée alors le champ quotient $[X/G]$ est modéré. Si l'on suppose de plus que C soit localement noethérien et B soit plat sur C, la réciproque est vraie.*

Si l'on considère l'action triviale de G sur S avec G fini et localement libre sur S, la notion de modération introduite par Abramovich, Olsson et Vistoli appliquée au champ classifiant $[S/G]$ définit une classe de schémas en groupes, on dit dans ce cas que G est *linéairement réductif*. De tels schémas en groupes sont pourvus de propriétés intéressantes. L'une d'entre elles est qu'ils sont localement simples pour la topologie fppf, plus précisément ils sont, après un changement de base fppf, isomorphes à un produit semi-direct d'un schéma en groupes constant modéré (c'est-à-dire ayant un ordre premier aux caractéristiques résiduelles) avec un schéma en groupes diagonalisable. Cette description locale permet d'étendre tout groupe linéaire réductif au dessus d'un point de S en un groupe linéairement réductif au dessus d'un recouvrement fppf de S contenant ce point. De ces études faites aussi dans l'article [AOV08], nous avons aussi réussi à établir que tout schéma en groupes linéairement réductif se relève aussi en tant que sous-schéma en groupes linéairement réductif dans le sens du théorème suivant:

Théorème VIII.18. *Soit $\mathfrak{p}$ un point de S, G un schéma en groupes fini et plat sur S et H_0 un sous-schéma en groupes fermé, fini, plat et linéairement réductif de $G_{k(\mathfrak{p})}$ sur $Spec(k(\mathfrak{p}))$. Alors il existe un morphisme fppf $U \to S$ avec un point $\mathfrak{q} \in U$ s'envoyant sur $\mathfrak{p}$ et un sous-schéma en groupes fermé, plat et linéairement réductif H de G_U sur U tel que le* pull-back *$H_{k(\mathfrak{q})}$ soit isomorphe au* pull-back *$H_{0k(\mathfrak{q})}$.*

La notion de modération pour l'anneau B muni d'une action d'un groupe fini abstrait Γ est équivalente, si l'on considère l'action du schéma en groupes constant correspondant à Γ sur X, au fait que tous les groupes d'inertie aux points topologiques soient linéairement réductifs. Il a été donc naturel de se demander si cette propriété est encore vraie en général. Effectivement, l'article [AOV08] caractérise le fait que le champ quotient $[X/G]$ soit modéré par le fait que les groupes d'inertie aux points géométriques soient linéairement réductifs.

À nouveau, si l'on considère le cas des anneaux munis d'une action d'un groupe fini abstrait, il est bien connu que l'action peut être totalement reconstruite à partir d'une action faisant intervenir un groupe d'inertie. Lorsque l'on considère le cas des actions de schémas en groupes constants, cela se traduit comme un théorème de *slices*, c'est-à-dire une description locale de l'action de départ (X, G) par une action faisant intervenir un groupe d'inertie. Par exemple, lorsque G est fini, localement libre sur S, nous établissons que le fait qu'une action soit libre est une propriété locale pour la topologie fppf, ce qui peut se traduire comme un théorème de *slices* « local ». Plus précisément,

Théorème IX.18. *Supposons que G soit fini et plat sur S. Soit $x \in Y$ et $y \in Y$ son image* via *le morphisme $\pi : X \to Y$. Les assertions suivantes sont équivalentes:*

1. *Le groupe d'inertie au point x est trivial.*
2. *Il existe un morphisme fppf $Y' \to Y$ contenant y dans son image tel que l'action $(X \times_Y Y', G_{Y'})$ soit libre.*
3. *Il existe un morphisme fppf $Y'' \to Y$ contenant y dans son image et un schéma Z sur Y'' tels que l'action $(X \times_Y Y'', G_{Y''})$ soit induite par l'action triviale (Z, e) où e dénote le schéma en groupes trivial au dessus de Y''.*

Grâce à [AOV08], nous savons déjà qu'un champ quotient modéré $[X/G]$ est localement isomorphe pour la topologie fppf à un champ quotient $[X/H]$ où H est une extension d'un groupe d'inertie pour l'action en un point de Y. Lorsque G est fini sur S, il nous a été possible de montrer grâce au théorème VIII.18 que H est aussi un sous-schéma en groupes de G. Dans la présente thèse, il n'a pas été possible d'obtenir un théorème de *slices* dans cette généralité. Cependant, lorsque G est commutatif, fini sur S, si l'on suppose que le champ quotient soit modéré, il est possible de montrer l'existence d'un torseur.

Proposition IX.23 *Supposons que X soit noethérien, de présentation finie sur S, que G soit commutatif, de présentation finie sur S et que tous les groupes d'inertie soient finis. Si le champ quotient $[X/G]$ est modéré, alors pour tout point $x \in X$, en notant par $y \in Y$ son image par l'application quotient $\pi : X \to Y$, il existe un morphisme fppf $Y' \to Y$ contenant y dans son image et un sous-schéma en groupes H de $G_{Y'}$ sur Y' relevant le groupe d'inertie en x tels que l'action $(X \times_Y Y'/H,\ G_{Y'}/H)$ soit libre. Si l'on suppose de plus que G soit fini, $X \times_Y Y'/H$*

est un torseur sur Y'.

Ceci nous a permis de prouver un théorème de *slices* lorsque G est commutatif, fini sur S et $[X/G]$ modéré.

Théorème IX.27 *Supposons que G soit commutatif et fini sur S. Le champ quotient $[X/G]$ est modéré si et seulement si pour tout $x \in X$, en tout point $x \in X$, l'action (X, G) est induite après un changement de base fppf $S' \to S$ par une action (Z, H) où Z est un schéma sur S' et H est un sous-schéma en groupes de G linéairement réductif relevant le groupe d'inertie en x*

Riassunto in Italiano (Longa versione):

Ramificazione moderata per delle azioni di schemi in gruppi affini e per *stacks* quoziente.

Lo scopo di questa tesi è capire come si generalizza la teoria della ramificazione per azioni di schemi in gruppi affini, con un interesse particolare per la nozione di moderazione.

Nel seguito, gli schemi considerati saranno affini su una base affine. Più precisamente, la base sarà $S := Spec(R)$, dove R è un anello commutativo unitario, $G = Spec(A)$ sarà uno schema in gruppi affine piatto su S e $X := Spec(B)$ sarà uno schema affine su S. Il dato di un'azione di G su X definita da un morfismo $\mu_X : X \times_S G \to X$ è equivalente al dato d'un B-comodulo definito dal morfismo strutturale, che denoteremo $\rho_B : B \to B \otimes_R A$. Denoteremo questa azione con (X, G). Denoteremo con $C := B^A := \{b \in B \mid \rho_B(b) = b \otimes 1\}$ l'anello degl'invarianti per l'azione, $(\mu_X, p_1) : X \times_S G \to X \times_S X$ l'applicazione di Galois, e $[X/G]$ il campo quoziente associato all'azione (X, G), e $Y := Spec(C)$.

Come punto di riferimento prendiamo la teoria classica della ramificazione per anelli muniti d'un'azione d'un gruppo finito astratto, la quale è l'oggetto dell'articolo [Bar74], che generalizza la teoria della ramificazione per estensioni di campi. Al fine di capire come generalizzare questa teoria per azioni di schemi in gruppi, consideriamo le azioni di schemi in gruppi costanti ricordando che il dato di tali azioni è equivalente al dato d'un anello dotato d'un'azione d'un gruppo finito astratto, il che ci riporta al caso classico. Otteniamo quindi in questo nuovo contesto delle nozioni che generalizzano l'anello degl'invarianti in quanto quoziente, i gruppi d'inerzia e tutte le loro proprietà. Ricordiamo qui semplicemente la definizione indispensabile per la sequenza di gruppi d'inerzia in questo contesto. Dato ζ un T-punto di X dove T è una R-algebra, denoteremo con $I_G(\zeta)$ il gruppo d'inerzia dell'azione sul punto $\zeta : Spec(T) \to X$, definito come il prodotto fibrato

$$\begin{array}{ccc} I_G(\zeta) & \xrightarrow{pr_2} & T \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \Delta_X \circ \zeta} \\ X \times_S G & \xrightarrow[(\mu_X, p_1)]{} & X \times_S X \end{array},$$

dove p_1 è la prima proiezione. Mostriamo che in effetti le proprietà classiche dei gruppi d'inerzia si generalizzano in questo nuovo contesto.

Avendo ricordato il contesto più adatto per poter parlare di ramificazione, possiamo cercare di distinguere i diversi tipi di ramificazione. Il caso non ramificato si generalizza in modo naturale con le azioni libere essendo queste caratterizzate come nel caso classico dal fatto che tutti i loro gruppi d'inerzia sono banali. Ciò equivale, nel caso in cui lo schema in gruppi che agisce è finito e localmente libero, all'esistenza di torsori, che sono oggetti localmente semplici, conosciuto in dettaglio e molto importanti nell'ambito delle azioni di schemi in gruppi. Per quanto riguarda il caso della ramificazione moderata, al quale siamo particolarmente interessati, due generalizzazioni sono proposte nella letteratura. La prima è quella delle azioni moderate di schemi in gruppi affini introdotta nell'articolo [CEPT96] da Chinburg, Erez, Pappas e Taylor, definite tramite l'esistenza d'un integrale totale. Più precisamente, diremo che un'*azione* (X, μ_X) *è moderata* se esiste un morfismo di A-comoduli $\alpha : A \to B$ che sia unitario, cioè tale che $\alpha(1_A) = 1_B$. Un tale α è chiamato *integrale totale*. (Ricordiamo che un morfismo $\alpha : (B, \rho_B) \to (C, \rho_C)$ di A-comoduli è un'applicazione R-lineare tale che $\rho_C \circ \alpha = (\alpha \otimes 1_B) \circ \rho_B$. La R-algebra A può essere vista come un A-comodulo attraverso la comoltiplicazione $\Delta : A \to A \otimes_R A$.) La seconda generalizzazione è quella di stack moderato introdotta in [AOV08] da Abramovich, Olsson e Vistoli. Consideriamo lo stack quoziente $[X/G]$ associato all'azione (X, μ_X). La definizione di stack quoziente moderato richiede l'esistenza d'uno spazio di moduli grossolano per l'azione (Ricordiamo che uno *spazio di moduli grossolano* per $[X/G]$ è una coppia (M, ρ) dove M è uno spazio algebrico e $\rho : [X/G] \to M$ è un morfismo universale per i morfismi di $[X/G]$ verso uno spazio algebrico tale che per ogni campo Ω algebricamente chiuso, $|[X/G](\Omega)| \simeq M(\Omega)$ dove $|[X/G](\Omega)|$ è l'insieme delle classi d'isomorfismo del gruppoide). Per garantire questa esistenza, supponiamo nel seguito che lo stack quoziente $[X/G]$ sia un campo algebrico localmente di presentazione finita e che tutti i gruppi di inerzia siano finiti. Infatti, sotto queste ipotesi, possiamo mostrare che esiste uno spazio di moduli grossolano $\rho : [X/G] \to M$ e che il morfismo ρ è proprio. Diciamo allora che lo *stack* $[X/G]$ *è moderato* se il funtore tra le categorie di fasci quasi coerenti $\rho_* : Qcoh([X/G]) \to Qcoh(M)$ è esatto. È stato quindi naturale cercare di confrontare queste due nozioni di moderazione e capire come si generalizzano le proprietà classiche d'oggetti moderati ad azioni di schemi in gruppi affini.

Per cominciare, abbiamo interpretato algebricamente la proprietà di moderazione su uno stack quoziente come l'esattezza del funtore degl'invarianti.

Teorema IX.10. *Supponiamo che il funtore $\rho_* : Qcoh([X/G]) \to Qcoh(Y)$ sia ben definito. Allora il funtore ρ_* è esatto se e solo se il funtore degl'invarianti $(-)^A$ è esatto.*

Ciò ha permesso d'ottenere agevolmente, grazie a [CEPT96], che un'azione moderata definisce sempre uno stack quoziente moderato.

Teorema IX.15. *Supponiamo che X sia noetheriano, di presentazione finita su S, che G sia di presentazione finita su S e che tutti i gruppi d'inerzia siano finiti. Se l'azione è moderata allora lo stack quoziente è moderato.*

Per quale che riguarda il viceversa, grazie ad uno studio algebrico, siamo riusciti a ottenerlo solo nel caso in cui facciamo l'ulteriore ipotesi che G sia finito e localmente libero su S e che X sia piatto su Y.

Teorema IX.12. *Supponiamo che G sia finito e localmente libero su S. Se l'azione (X, G) è moderata allora lo stack quoziente $[X/G]$ è moderato. Se supponiamo inoltre che C sia localmente noetheriano e che B sia piatto su C allora il viceversa è vero.*

Se consideriamo l'azione banale di G su S con G finito e localmente libero su S, la nozione di moderazione introdotta da Abramovich, Olsson e Vistoli applicata allo stack classificante $[S/G]$ definisce una classe di schemi in gruppi, e in questo caso diciamo che G è linearmente riduttivo. Tali schemi in gruppi sono dotati di proprietà interessanti. Una di queste è che essi localmente semplici per la topologia fppf, più precisamente sono isomorfi, dopo un cambiamento di base fppf, ad un prodotto semidiretto d'uno schema in gruppi costante moderato (cioè il cui ordine è coprimo con le caratteristiche residuali) e d'uno schema in gruppi diagonalizzabile. Questa descrizione locale permette d'estendere ogni gruppo lineare riduttivo su un punto di S ad un gruppo linearmente riduttivo su un ricoprimento fppf di S contenente questo punto. Attraverso questo studio, fatto anche nell'articolo [AOV08], siamo riusciti a stabilire che ogni schema in gruppi linearmente riduttivo si rialza anche in quanto sottoschema in gruppi linearmente riduttivo, nel senso del teorema seguente:

Teorema VIII.18. *Siano $\mathfrak{p}$ un punto di S, G uno schema in gruppi finito e piatto su S e H_0 un sottoschema in gruppi chiuso, finito, piatto e linearmente riduttivo di $G_{k(\mathfrak{p})}$ su $Spec(k(\mathfrak{p}))$. Allora esiste un morfismo fppf $U \to S$ con un punto $\mathfrak{q} \in U$ mandato su $\mathfrak{p}$ e un sottoschema in gruppi chiuso, piatto e linearmente riduttivo H di G_U su U tale che il* pull-back *$H_{k(\mathfrak{q})}$ sia isomorfo al* pull-back *$H_{0k(\mathfrak{q})}$.*

La nozione di moderazione per l'anello B munito d'un'azione di un gruppo finito astratto Γ è equivalente, se consideriamo l'azione dello schema in gruppi costante corrispondente a Γ su X, al fatto che tutti i gruppi d'inerzia sui punti topologici siano linearmente riduttivi. È stato quindi naturale domandarsi se questa proprietà è ancora vera in generale. In effetti, l'articolo [AOV08] caratterizza il fatto che lo stack quoziente $[X/G]$ sia moderato tramite il fatto che i gruppi d'inerzia sui punti geometrici siano linearmente riduttivi.

Di nuovo, se consideriamo il caso degli anelli muniti d'un'azione d'un gruppo finito astratto, è ben noto che l'azione può essere totalmente ricostruita a partire da un'azione che fa intervenire un gruppo d'inerzia. Nel caso delle azioni di schemi in gruppi costanti, ciò s'interprete come un teorema di *slice*, cioè una descrizione locale dell'azione di partenza (X, G) tramite un'azione che fa intervenire un gruppo d'inerzia. Per esempio, quando G è finito e localmente libero su S, stabiliamo che il fatto che un'azione sia libera è una proprietà locale per la topologia fppf, il che si può interpretare come un teorema di *slice* « locale ». Più precisamente,

Teorema IX.18. *Supponiamo che G sia finito e piatto su S. Sia $x \in Y$ e sia $y \in Y$ l'immagine di x tramite il morfismo $\pi : X \to Y$. Le seguenti affermazioni sono equivalenti:*

1. *Il gruppo d'inerzia al punto x è banale.*
2. *Esiste un morfismo fppf $Y' \to Y$ contenente y nella sua immagine e tale che l'azione $(X \times_Y Y', G_{Y'})$ sia libera.*
3. *Esistono un morfismo fppf $Y'' \to Y$ contenente y nella sua immagine e uno schema Z su Y'' tali che l'azione $(X \times_Y Y'', G_{Y''})$ sia indotta dall'azione banale (Z, e) dove e denota lo schema in gruppi banale su Y''.*

Grazie a [AOV08], sappiamo già che uno stack quoziente moderato $[X/G]$ è localmente isomorfo per la topologia fppf a uno stack quoziente $[X/H]$ dove H è un'estensione d'un gruppo d'inerzia per l'azione in un punto di Y. Quando G è finito su S, ci è stato possibile dimostrare grazie al teorema VIII.18 che H è altresì un sottoschema in gruppi di G. In questa tesi non è stato possibile ottenere un teorema di *slice* in questa generalità. Tuttavia, quando G è commutativo e finito su S, se supponiamo che lo stack quoziente sia moderato, è possibile dimostrare l'esistenza d'un torsore.

Proposizione IX.23 *Supponiamo che X sia noetheriano, di presentazione finita su S, che G sia commutativo, di presentazione finita su S e che tutti i gruppi di inerzia siano finiti. Se lo stack quoziente $[X/G]$ è moderato allora per ogni punto $x \in X$, detta $y \in Y$ la sua immagine tramite l'applicazione quoziente $\pi : X \to Y$, esiste un morfismo fppf $Y' \to Y$ contenente y nella sua immagine e un sottoschema in gruppi H di $G_{Y'}$ su Y' che rialza il gruppo di inerzia in x, tali che l'azione $(X \times_Y Y'/H,\ G_{Y'}/H)$ sia libera. Se supponiamo inoltre che G sia finito, $X \times_Y Y'/H$ è un torsore su Y'.*

Questo ci ha permesso di dimostrare un teorema di *slice* quando G è commutativo e finito su S e $[X/G]$ è moderato.

Teorema IX.27 *Supponiamo che G sia commutativo e finito su X. Lo stack quoziente $[X/G]$ è moderato se e solo se per ogni $x \in X$, in ogni punto $x \in X$, l'azione (X, G) è indotta, dopo un cambiamento di base fppf $S' \to S$, da un'azione (Z, H), dove Z è uno schema su S' e H è un sottoschema in gruppi di G linearmente riduttivo che rialza il gruppo d'inerzia in x.*

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