Justification of and long-wave correction to Davey-Stewartson systems from quadratic hyperbolic systems

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Abstract

We prove that the Davey-Stewartson approximation (which degenerates into a cubic Schrödinger equation in $1D$) furnishes a good approximation for the exact solution of a wide class of quadratic hyperbolic systems. This approximation remains valid for large times of logarithmic order. We also consider the general case where the polarized component of the mean field needs not to be well-prepared. This is possible by adding to the Davey-Stewarston approximation a long-wave correction, which consists of a wave freely propagated by the long-wave operator associated to the original system.

1 Introduction

1.1 Setting up of the problem

The nonlinear Schrödinger equation and the Davey-Stewartson system are universal models in physics and mechanics, see for example [4], [5], [3] for water waves theory, [7] for internal gravity waves, [13, 14] for ferromagnetism, [17] for nonlinear optics and [16] for plasma theory. Some of the examples of the above references are nonlinear hyperbolic systems with quadratic nonlinearity (in ferromagnetism for instance); for the other examples, such quadratic hyperbolic systems appear as a crucial step in the derivation of the Davey-Stewartson system. This observation was the starting point of [2], where the obtention of the cubic Davey-Stewartson system (or of its degenerate version, the cubic Schrödinger equation) is studied for a wide class of nonlinear hyperbolic systems with quadratic nonlinearity. More precisely, the systems of this class are of the form

$$\partial_t u + \sum_{j=1}^{n} A_j \partial_{x_j} u + \frac{Eu}{\varepsilon} = f(u,u), \tag{1}$$

where ε is a small parameter, u is defined on $\mathbb{R}_t \times \mathbb{R}^n_x$ with values in $\mathbb{R}^q$, the $q \times q$ matrices A_j are real symmetric, E is real skew-symmetric and $f : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q$ is bilinear symmetric.
In order to observe Davey-Stewartson dynamics for u, one should consider oscillatory initial conditions of the form (see the discussion of [2])

$$u(0,x) = \Big(a(x) e^{i\frac{k\cdot x}{\varepsilon}} + \overline{a(x)} e^{-i\frac{k\cdot x}{\varepsilon}} \Big) + \varepsilon B(x, \frac{k \cdot x}{\varepsilon}) + \varepsilon^2 C(x, \frac{k\cdot x}{\varepsilon}) + \dots, \tag{2}$$

where $B(x,\theta)$ and $C(x,\theta)$ are real-valued and 2π-periodic in θ.

In fact, even at a formal level, an additional assumption is required on (1) to obtain the Davey-Stewartson system as a limit system for the Cauchy problem (1)-(2). Using the terminology of Joly, Métivier and Rauch [9], this assumption is called 'weak transparency', and is concerned with algebraic properties of the nonlinearity f.
Leaving the formal level yields many difficulties. One wants to prove: a) that the solution to (1)-(2) exists for large times of order $O(\frac{1}{\varepsilon})$ and b) that for such times, the Davey-Stewartson approximation converges to the exact solution u. There is little chance for such a result to be true, even under the 'weak transparency' assumption; indeed, the natural existence time for the solution u of (1)-(2) is $O(1)$. Making an additional assumption, namely that (1) has a Maxwell-Bloch structure (still using the terminology of [9]), T. Colin [2] was however able to prove the desired result. The solution u is constructed as a perturbation of the Davey-Stewartson approximation, and the Maxwell-Bloch structure is intensively used to obtain the needed estimates over times of order $O(\frac{1}{\varepsilon})$.
Unfortunately many of the physical systems evoked above do not have such a structure. A natural question is therefore: what is the relevance of the Davey-Stewartson approximation for these systems? This approximation is probably not as good as for Maxwell-Bloch systems (in the sense that it should diverge from the exact solution for smaller times). The first motivation of this paper is to give a justification of this approximation, though in a weaker sense: we prove that it remains a good approximation over times of order $O(|\ln \varepsilon|)$. We believe that this result is optimal in the sense that, in the general framework presented here, resonances may occur and force the approximation to diverge for times of order $O(\frac{1}{\varepsilon})$ (in [2], the Maxwell-Bloch structure controls such resonances). Note that for the logarithmic time scales considered here, our result justify in practical the use of the Davey-Stewartson approximation since it gives better error estimates than the mere geometric optics approximation.

Our second motivation is linked to the study of the rectification phenomenon, i.e. to the creation of a mean field by nonlinear interaction of the oscillating modes. The Davey-Stewartson model describes the evolution of the mean mode, but in a very particular case: indeed, the initial value of the polarized component of the mean mode must be equal to a certain quantity explicitly given in terms of the initial value of the oscillating modes. One wonders therefore what happens when the initial value is not 'well-prepared' in this sense, and in particular (which is physically the most relevant case) when the initial value of the mean mode is zero. Presence of a mean mode would thus fully be due to rectification effects. We show in this paper that one can take arbitrary initial values for the mean mode. In this case, a long-wave correction must be added to the Davey-Stewartson approximation. This long-wave correction corresponds to a wave propagating freely according to the long-wave operator associated to (1). Some secular effects also arise due to this long-wave correction.

1.2 Description of the results

Section 2 is devoted to the construction of an approximate solution to (1). This is done by multiscale BKW expansion for the *singular equation* associated to (1). The weak transparency assumption is introduced in Section 2.1 where the formal analysis is performed. Another assumption, the ellipticity assumption is needed in Section 2.2 for the asymptotic analysis of the equations and, further, to solve the Davey-Stewartson system. In Section 2.3 estimates on the components of the approximate solution are given, and it is proved in Section 2.4 that this approximate solution almost solves the singular equation, in the sense that the residual is small.

Section 3 is devoted to the convergence results. We first prove in Section 3.1 that the approximate solution constructed previously converges for large times of order $O(C_1|\ln\varepsilon|)$ to an exact solution. It must be noted that 'ill-prepared' initial data can be considered provided that the long-wave correction is taken into account. In Section 3.2 we deal with the convergence of the Davey-Stewartson approximation, corrected if necessary by the long-wave correction. The difference with what is done in Section 3.1 is that many terms of the approximation previously computed are proved to be small and therefore are dropped. The leading terms are the Davey-Stewartson approximation and the long-wave correction. Special attention is paid to the mean mode, since it reveals the presence of rectification effects.

Finally, in Section 4 an application of our theory is studied in ferromagetism. Rectification effects, i.e. creation of a mean field by non-linear interaction of oscillating modes is put in evidence. The long-wave correction is the tool which allows such a description.

Notations. i. For all $(\tau,\xi)\in\mathbb{R}^{1+n}$, we introduce the matrices $A(\xi) := \sum_{j=1}^n A_j\xi_j$ and $\mathcal{L}(\tau,\xi) := \tau Id + A(\xi) + E$.
ii. For all $n\in\mathbb{Z}$, we introduce the notation $\mathcal{L}_n := \mathcal{L}(-in\underline{\omega}, in\underline{k})$, denote by Π_n the orthogonal projector onto $\ker\mathcal{L}_n$ and by $\mathcal{L}_n^{(-1)}$ the inverse of $\mathcal{L}_n$ or, it this matrix is singular, its partial inverse defined as

$$\mathcal{L}_n\mathcal{L}_n^{(-1)} = \mathcal{L}_n^{(-1)}\mathcal{L}_n = Id - \Pi_n.$$

iii. In a neighborhood of a smooth point of the characteristic variety $\mathcal{C} := \{(-\omega,k)\in\mathbb{R}^{1+n}, \det\mathcal{L}(-\omega,k)=0\}$, $\mathcal{C}$ coincides with the graph of a smooth function $\omega(k)$ and the orthogonal projector onto $\ker\mathcal{L}(\omega(k),k)$ is a smooth function of k. We use the notations $\omega'(\underline{k})$, $\omega''(\underline{k})$ and $\partial_{k_j}\Pi_n$, whose meaning is obvious.
iv. For any $\varphi\in\mathcal{S}'(\mathbb{R}^n)$, we denote indifferently by $\widehat{\varphi}$ or $\mathcal{F}\varphi$ the Fourier transform of φ. If X is a functional space, $\mathcal{F}^{-1}X$ denotes the space of functions

whose Fourier transform is in X.
v. We denote by L^∞_s the weighted space $L^\infty_s := \{\varphi \in \mathcal{S}'(\mathbb{R}^n)^q, (1+|\cdot|^2)^{s/2}\varphi \in L^\infty(\mathbb{R}^n)^q\}$.

2 Construction of the approximate solution

The aim of this section is to construct an approximate solution v to (1)-(2). First remark that if a smooth function $U(t,x,\theta)$ solves the *singular equation*

$$\partial_t U + \sum_{j=1}^n A_j \partial_{x_j} U + \frac{1}{\varepsilon}\left(-i\underline{\omega}\partial_\theta + i\sum_{j=1}^n A_j \underline{k}_j \partial_\theta + E\right) = f(U,U), \qquad (3)$$

and satisfies the initial conditions

$$U(0,x,\theta) = \left(a(x)e^{i\theta} + \overline{a(x)}e^{-i\theta}\right) + \varepsilon B(x,\theta) + \varepsilon^2 C(x,\theta) + \dots, \qquad (4)$$

then $u(t,x) = U(t,x,\frac{\underline{k}\cdot x - \underline{\omega} t}{\varepsilon})$ solves the Cauchy problem (1)-(2). We therefore look for an approximate solution v of the form $v(t,x) = V(t,x,\frac{\underline{k}\cdot x - \underline{\omega} t}{\varepsilon})$ where $V(t,x,\theta)$ approximately solves (3)-(4).

2.1 The BKW expansion

Since the computations of this section follow those of [2], we will be quite sketchy, and refer to this latter reference for more details. One looks for V under the form

$$V(t,x,\theta) = \mathcal{V}(\tau,t,x,\theta)|_{\tau=\varepsilon t} := \sum_{j=0}^2 \varepsilon^j \mathcal{V}_j(\tau,t,x,\theta)|_{\tau=\varepsilon t}. \qquad (5)$$

Plugging (5) into (3) yields

$$(\partial_t + A(\partial_x))\,V + \frac{1}{\varepsilon}\mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta)V - f(V,V) = \sum_{j=-1}^4 \varepsilon^j \mathcal{R}_j(\tau,t,x,\theta)|_{\tau=\varepsilon t},$$

with

$$\begin{aligned}
\mathcal{R}_{-1} &= \mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta)\mathcal{V}_0 && (6)\\
\mathcal{R}_0 &= \mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta)\mathcal{V}_1 + (\partial_t + A(\partial_x))\,\mathcal{V}_0 - f(\mathcal{V}_0,\mathcal{V}_0) && (7)\\
\mathcal{R}_1 &= \mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta)\mathcal{V}_2 + (\partial_t + A(\partial_x))\,\mathcal{V}_1 + \partial_\tau\mathcal{V}_0 - 2f(\mathcal{V}_0,\mathcal{V}_1) && (8)\\
\mathcal{R}_2 &= (\partial_t + A(\partial_x))\,\mathcal{V}_2 + \partial_\tau\mathcal{V}_1 - 2f(\mathcal{V}_0,\mathcal{V}_2) - f(\mathcal{V}_1,\mathcal{V}_1) && (9)\\
\mathcal{R}_3 &= \partial_\tau\mathcal{V}_2 - 2f(\mathcal{V}_1,\mathcal{V}_2) && (10)\\
\mathcal{R}_4 &= -f(\mathcal{V}_2,\mathcal{V}_2). && (11)
\end{aligned}$$

The strategy consists in choosing $\mathcal{V}_0$, $\mathcal{V}_1$ and $\mathcal{V}_2$ in order to have $\mathcal{R}_{-1} = \mathcal{R}_0 = \mathcal{R}_1 = 0$.
Cancelling $\mathcal{R}_{-1}$. One seeks $\mathcal{V}_0$ under the form

$$\mathcal{V}_0(\tau,t,x,\theta) = \left(\mathcal{V}_{01}(\tau,t,x)e^{i\theta} + \text{c.c.}\right),$$

where c.c. denotes the complex conjugate of the preceding expression.
Cancelling $\mathcal{R}_{-1}$ is equivalent to choosing $(\underline{\omega}, \underline{k})$ such that the *dispersion relation*

$$\det \mathcal{L}(-i\underline{\omega}, i\underline{k}) = 0, \tag{12}$$

is satisfied and to taking $\mathcal{V}_{01}$ in the kernel of $\mathcal{L}(-i\underline{\omega}, i\underline{k})$ (*polarization condition*), or

$$\Pi_1 \mathcal{V}_{01} = \mathcal{V}_{01}, \tag{13}$$

according to our notations.
Cancelling $\mathcal{R}_0$. Since the term $f(\mathcal{V}_0, \mathcal{V}_0)$, which is present in the expression of $\mathcal{R}_{-1}$, creates the harmonics 0 and ± 2 in θ, we have to look for $\mathcal{V}_1$ under the form

$$\mathcal{V}_1(\tau, t, x, \theta) = \mathcal{V}_{10} + \left(\mathcal{V}_{11}(\tau, t, x)e^{i\theta} + \mathcal{V}_{12}(\tau, t, x)e^{2i\theta}\right) + \text{c.c.}.$$

The need for the weak transparency assumption appears when one tries to cancel the mean term of $\mathcal{R}_0$. The resulting equation reads

$$E\mathcal{V}_{10} = 2f\left(V_{01}, \overline{\mathcal{V}_{01}}\right),$$

which can only be solved if one assumes:

Assumption 2.1 (Weak transparency) *For all $V, W \in \mathbb{C}^q$, one has*

$$\Pi_0 f\left(\Pi_1 V, \overline{\Pi_1 W}\right) = 0.$$

Under such an assumption, one can choose $\mathcal{V}_0$ and $\mathcal{V}_1$ in order to have $\mathcal{R}_0 = 0$. Choosing $(\underline{\omega}, \underline{k})$ as a smooth point of the dispersion relation, and such that for all $n \notin \{0, \pm 1\}$, $(n\underline{\omega}, n\underline{k})$ does not satisfy the dispersion relation, the resulting equations (see [2] for details) are:

$$(\partial_t + \omega'(\underline{k}) \cdot \partial_x) \mathcal{V}_{01} = 0,$$

$$(1 - \Pi_0) \mathcal{V}_{10} = 2\mathcal{L}_0^{(-1)} f\left(V_{01}, \overline{\mathcal{V}_{01}}\right), \tag{14}$$

$$(1 - \Pi_1) \mathcal{V}_{11} = -\mathcal{L}_1^{(-1)} A(\partial_x)\mathcal{V}_{01}, \tag{15}$$

$$\mathcal{V}_{12} = \mathcal{L}_2^{(-1)} f(\mathcal{V}_{01}, \mathcal{V}_{01}). \tag{16}$$

Cancelling $\mathcal{R}_1$. Since at this stage the harmonics ± 3 also have been created by the nonlinearity, one seeks $\mathcal{V}_2$ under the form

$$\mathcal{V}_2(\tau, t, x, \theta) = \mathcal{V}_{20} + \left(\mathcal{V}_{21}(\tau, t, x)e^{i\theta} + \mathcal{V}_{22}(\tau, t, x)e^{2i\theta} + \mathcal{V}_{23}(\tau, t, x)e^{3i\theta}\right) + \text{c.c.}.$$

Without additional assumption, one can choose $\mathcal{V}_0$, $\mathcal{V}_1$ and $\mathcal{V}_2$ in order to have $\mathcal{R}_1 = 0$. The resulting equations are

$$\begin{aligned}\partial_\tau \mathcal{V}_{01} - \frac{i}{2}\omega''(\underline{k})(\partial_x, \partial_x)\mathcal{V}_{01} + (\partial_t + \omega'(\underline{k}) \cdot \partial_x) \Pi_1 \mathcal{V}_{11} = 2\Pi_1 f(\mathcal{V}_{01}, \Pi_0 \mathcal{V}_{10}) \\ +2\Pi_1 f\left(\mathcal{V}_{01}, \mathcal{L}_0^{(-1)} f\left(\mathcal{V}_{01}, \overline{\mathcal{V}_{01}}\right)\right) + \Pi_1 f\left(\overline{\mathcal{V}_{01}}, \mathcal{L}_2^{(-1)} f(\mathcal{V}_{01}, \mathcal{V}_{01})\right),\end{aligned} \tag{17}$$

$$\begin{aligned}\Pi_0 (\partial_t + A(\partial_x)) \Pi_0 \mathcal{V}_{10} = 2i\Pi_0 \sum_{j=1}^{n} \partial_{x_j} f\left(\partial_{\underline{k}_j} \Pi_1 \mathcal{V}_{01}, \overline{V_{01}}\right) \\ -2\Pi_0 (-\omega'(\underline{k}) \cdot \partial_x + A(\partial_x)) \mathcal{L}_0^{(-1)} f\left(\mathcal{V}_{01}, \overline{\mathcal{V}_{01}}\right),\end{aligned} \tag{18}$$

as well as other equations which give explicitly $(1 - \Pi_0)\mathcal{V}_{20}$, $(1 - \Pi_1)\mathcal{V}_{21}$, $\mathcal{V}_{22}$ and $\mathcal{V}_{23}$ in terms of $\mathcal{V}_1$ and $\mathcal{V}_0$.

2.2 Asymptotic analysis of the profile equations

In this section, we deduce the Davey-Stewartson system from Eqs. (17)-(18) using asymptotic arguments. We first make an assumption which ensures that the second equation of the Davey-Stewartson system is of elliptic type since this is the only case where the local in time Cauchy problem is well understood (see [6]).

ASSUMPTION 2.2 (ELLIPTICITY) *For all* $\sigma \in \mathbb{S}^{n-1}$*, the matrix* $-\omega'(\underline{k}) \cdot \sigma Id + \pi_0 A(\sigma)\pi_0$ *is invertible.*

The next lemma will be used to solve (18).

LEMMA 2.1 *Suppose that Assumption 2.2 is satisfied. Let* $s \geq 0$ *and* $g_0, h_0 \in H^s(\mathbb{R}^n)^q$ *and* $P(\partial_x)$ *be a homogeneous differential operator of first order. Then*
i. *The operator* Q *defined as*

$$Q := \left(-\omega'(\underline{k}) \cdot \partial_x + \Pi_0 A(\partial_x)\Pi_0\right)^{-1} P(\partial_x)$$

is a Fourier multiplier of order zero; in particular, one has $Q(h_0) \in H^s(\mathbb{R}^n)^q$.
ii. *The Cauchy problem*

$$\begin{aligned} &\left(\partial_t + \Pi_0 A(\partial_x)\Pi_0\right) g = \left[P(\partial_x)(h_0)\right](x - \omega'(\underline{k})t), \\ &g|_{t=0} = g_0 \end{aligned}$$

has a unique solution $g \in C(\mathbb{R}_+, H^s(\mathbb{R}^n)^q)$*. Moreover,* $g = \langle g \rangle + g^*$*, where* $\langle g \rangle$ *and* g^* *are the unique solutions in* $C(\mathbb{R}_+, H^s(\mathbb{R}^n)^q)$ *of*

$$\begin{aligned} &\left(\partial_t + \omega'(\underline{k})\partial_x\right) \langle g \rangle = 0 \\ &\langle g \rangle|_{t=0} = Q(h_0), \end{aligned} \qquad \textit{and} \qquad \begin{aligned} &\left(\partial_t + \Pi_0 A(\partial_x)\Pi_0\right) g^* = 0 \\ &g^*|_{t=0} = g_0 - Q(h_0). \end{aligned}$$

Since $\mathcal{V}_{01} = \Pi_1 \mathcal{V}_{01}$ is transported at the group velocity $\omega'(\underline{k})$, we write abusively $\mathcal{V}_{01}(\tau, t, x) = \mathcal{V}_{01}(\tau, x - \omega'(\underline{k})t)$. Hence, we can write Eq. (18) under the form

$$\left(\partial_t + \Pi_0 A(\partial_x)\Pi_0\right) \Pi_0 \mathcal{V}_{10} = \left[P(\partial_x)(\mathcal{V}_{01})\right](\tau, x - \omega'(\underline{k})t),$$

where $P(\partial_x)$, defined for all smooth function h as

$$P(\partial_x)(h) = 2i\Pi_0 \sum_{j=1}^{n} \partial_{x_j} f\left(\partial_{\underline{k}_j} \Pi_1 h, \overline{h}\right) - 2\Pi_0 \left(-\omega'(\underline{k}) \cdot \partial_x + A(\partial_x)\right) \mathcal{L}_0^{(-1)} f\left(h, \overline{h}\right), \tag{19}$$

is a homogeneous differential operator of first order.
By Lemma 2.1, we can therefore write $\Pi_0 \mathcal{V}_{10} = \langle \Pi_0 \mathcal{V}_{10} \rangle + (\Pi_0 \mathcal{V}_{10})^*$, with the same notations as in the lemma. As for $\mathcal{V}_{01}$, we denote abusively $\langle \Pi_0 \mathcal{V}_{10} \rangle(\tau, t, x) = \langle \Pi_0 \mathcal{V}_{10} \rangle(\tau, x - \omega'(\underline{k})t)$.
Using the average projectors introduced in [10], we know that in order for $\Pi_1 \mathcal{V}_{11}$ to remain a corrector term for large times of order $O(\frac{1}{\varepsilon})$, then Eq. (17) necessarily splits into

$$\begin{aligned} \partial_\tau \mathcal{V}_{01} - \frac{i}{2}\omega''(\underline{k})(\partial_\zeta, \partial_\zeta)\mathcal{V}_{01} &= 2\Pi_1 f\left(\mathcal{V}_{01}, \langle \Pi_0 \mathcal{V}_{10} \rangle\right) \\ +2\Pi_1 f\left(\mathcal{V}_{01}, \mathcal{L}_0^{(-1)} f\left(\mathcal{V}_{01}, \overline{\mathcal{V}_{01}}\right)\right) &+ \Pi_1 f\left(\overline{\mathcal{V}_{01}}, \mathcal{L}_2^{(-1)} f\left(\mathcal{V}_{01}, \mathcal{V}_{01}\right)\right), \end{aligned} \tag{20}$$

where $\zeta := x - \omega'(\underline{k})t$, and

$$(\partial_t + \omega'(\underline{k}) \cdot \partial_x)\, \Pi_1 \mathcal{V}_{11} = 2\Pi_1 f\left(\mathcal{V}_{01}, (\Pi_0 \mathcal{V}_{10})^*\right). \tag{21}$$

We recall that, by definition, $\langle \Pi_0 \mathcal{V}_{10} \rangle$ is given by

$$(-\omega'(\underline{k}) \cdot \partial_\zeta + \Pi_0 A(\partial_\zeta)\Pi_0)\, \langle \Pi_0 \mathcal{V}_{10} \rangle = P(\partial_\zeta)(\mathcal{V}_{01}), \tag{22}$$

were $P(\partial_x)$ is defined in (19).
The system formed by (20) and (22) is a Davey-Stewartson system of elliptic-elliptic or hyperbolic-elliptic configuration, according to the classification of [6].

2.3 Estimates of the correctors and of the residual

The Davey-Stewartson system (20), (22) is easy to solve in Sobolev spaces $H^s(\mathbb{R}^n)$. In the following proposition, we show that it is also well-posed in the space

$$\Sigma^s := \{\varphi \in H^{s+\frac{n}{2}^+}(\mathbb{R}^n)^q, \widehat{\varphi} \in L^\infty_s\},$$

where $\dfrac{n}{2}^+$ denotes any real number strictly larger than $\frac{n}{2}$.

PROPOSITION 2.1 *Suppose that Assumption 2.2 is satisfied. Let $s \geq 0$ and $a \in \Sigma^s$ such that $\Pi_1 a = a$.*
Then there exists a unique maximal solution $\mathcal{V}_{01} \in C([0, T_0); \Sigma^s)$, with $T_0 > 0$, to the Davey-Stewartson system (20), (22) and initial condition $\mathcal{V}_{01}(\tau = 0) = a$. Moreover, $\langle \Pi_0 \mathcal{V}_{10} \rangle$, as it is given by (22), belongs to $C([0, T_0); \mathcal{F}^{-1} L^\infty_s(\mathbb{R}^n)^q) \cap C([0, T_0); H^{s+\frac{n}{2}^+}(\mathbb{R}^n)^q)$.

Proof.
The integral version of (20), (22) with initial data a can be written

$$\mathcal{V}_{01}(t) = S(t)a + \int_0^t S(t-s) T(\mathcal{V}_{01})(s) ds,$$

where S denotes the solution group associated to the linear part of (20) and $T(\mathcal{V}_{01})$ is the right hand side of this same equation, where $\langle \Pi_1 \mathcal{V}_{10} \rangle$ has been expressed in terms of $\mathcal{V}_{01}$ thanks to (22).
Since for all $t \geq 0$, $S(t)$ is unitary on all the Sobolev spaces H^σ, $\sigma \in \mathbb{R}$, as well as on $\mathcal{F}^{-1} L^\infty_s$, it remains unitary on Σ^s.
Moreover, we have easily

$$|S(t-s)T(\mathcal{V}_{01})(s)|_{H^{s+\frac{n}{2}+}} \leq \mathrm{Cst}\, |\mathcal{V}_{01}(s)|^3_{H^{s+\frac{n}{2}+}},$$

and

$$\begin{aligned}
|S(t-s)T(\mathcal{V}_{01})(s)|_{\mathcal{F}^{-1}L^\infty_s} &= \left| (1 + |\cdot|^2)^{s/2} \widehat{T(\mathcal{V}_{01})} \right|_{L^\infty} \\
&\leq \mathrm{Cst} \left| (1 + |\cdot|^2)^{s/2} \widehat{\mathcal{V}_{01}} \right|_1^2 |\mathcal{V}_{01}|_{\mathcal{F}^{-1}L^\infty_s} \\
&\leq \mathrm{Cst}\, |\mathcal{V}_{01}|^2_{H^{s+\frac{n}{2}+}} |\mathcal{V}_{01}|_{\mathcal{F}^{-1}L^\infty_s}.
\end{aligned}$$

These estimates allow a standard fixed-point machinery which yields the local well-posedness result on $\mathcal{V}_{01}$ stated in the lemma.
The result concerning $\langle \Pi_0 \mathcal{V}_{10} \rangle$ is then a direct consequence of the representation formula (22).

□

We have already seen that $\Pi_0\mathcal{V}_{10}$ can be written $\Pi_0\mathcal{V}_{10} = \langle\Pi_0\mathcal{V}_{10}\rangle+(\Pi_0\mathcal{V}_{10})^*$. The previous proposition gives $\langle\Pi_0\mathcal{V}_{10}\rangle$. For the other component, various choices are possible because no equation determines the slow evolution (i.e. the evolution in τ) of $\Pi_0\mathcal{V}_{10}$. The following proposition gives the simplest of these choices.

PROPOSITION 2.2 *Suppose that Assumption 2.2 is satisfied. Let $s \geq 0$ and $a \in \Sigma^s$, $b \in H^s(\mathbb{R}^n)^q$ such that $\Pi_1 a = a$ and $\Pi_0 b = b$.*
In order to have $\Pi_0\mathcal{V}_{10}(\tau = t = 0, x) = b$, one can choose $(\Pi_0\mathcal{V}_{10})^* \in C([0,T_0)\times\mathbb{R}_t; H^s(\mathbb{R}^n)^q)$, given explicitly by

$$(\Pi_0\mathcal{V}_{10})^*(\tau,t,x) = e^{-it\Pi_0 A(\partial_x)\Pi_0}\,(b(x) - Q(a)(x))$$

with $Q = (-\omega'(\underline{k})\cdot\partial_x + \Pi_0 A(\partial_x)\Pi_0)^{-1}P(\partial_x)$ and $\langle\Pi_0\mathcal{V}_{10}\rangle$ is given by Prop. 2.2.

Proof.
From Lemma 2.1, we know that one must take

$$(\Pi_0\mathcal{V}_{10})^*(\tau,t,x) = e^{-it\Pi_0 A(\partial_x)\Pi_0}\left(\Pi_0\mathcal{V}_{10}(\tau,0,x) - Q(\mathcal{V}_{01})(\tau,x)\right).$$

However, at this stage, no condition has been given on the dependence on τ of $\Pi_0\mathcal{V}_{10}(\tau,0,x)$. The only condition we have is $\Pi_0\mathcal{V}_{10}(0,0,x) = b$. The choice of $\Pi_0\mathcal{V}_{10}(\tau,0,x)$ which gives the simplest final expression for $(\Pi_0\mathcal{V}_{10})^*$ and satisfies this condition is

$$\Pi_0\mathcal{V}_{10}(\tau,0,x) = b(x) + Q(\mathcal{V}_{01})(\tau,x) - Q(a)(x),$$

for which one obtains the expression given in the lemma. It is then straightforward to check that $(\Pi_0\mathcal{V}_{10})^* \in C([0,T_0)\times\mathbb{R}_t; H^s(\mathbb{R}^n)^q)$.

□

As shown in the previous section, $(1-\Pi_0)\mathcal{V}_{10}$, $\mathcal{V}_1$ and $\mathcal{V}_2$ can be explicitly given in terms of $\mathcal{V}_{01}$ and $\Pi_0\mathcal{V}_{10}$. The next proposition gives control of their Sobolev norms, and in particular of their secular growth.

PROPOSITION 2.3 *Suppose that Assumption 2.2 is satisfied. Let $s > n/2$ and $a \in \Sigma^{s+1}$, $b \in H^{s+1}(\mathbb{R}^n)^q$ such that $\Pi_1 a = a$ and $\Pi_0 b = b$.*
Then, with $\mathcal{V}_{01}$ and $\Pi_0\mathcal{V}_{10}$ as given by Props. 2.1 and 2.2, the following estimates hold for all $0 < T < T_0$ and $t \geq 0$:
i.

$$\sup_{\tau\in[0,T]}\left(\|(1-\Pi_0)\mathcal{V}_{10}(\tau,t)\|_{H^s} + \|(1-\Pi_1)\mathcal{V}_{11}(\tau,t)\|_{H^s} + \|\mathcal{V}_{12}(\tau,t)\|_{H^s}\right) \leq \mathrm{Cst}\,;$$

ii.

$$\sup_{\tau\in[0,T]}\|\Pi_1\mathcal{V}_{11}(\tau,t)\|_{H^s} \leq \mathrm{Cst}\,\chi_n(t),$$

where $\chi_n(t) = (1+\ln(1+t))$ if $n = 1,2$ and $\chi_n(t) = 1$ if $n \geq 3$;
iii. *For $j = 0,\dots,3$,*

$$\sup_{\tau\in[0,T]}\|\mathcal{V}_{2j}(\tau,t)\|_{H^s} \leq \mathrm{Cst}\,\chi_n(t);$$

iv. *If $b = Q(a)$, then $\Pi_1\mathcal{V}_{11} = 0$ and for $j = 0,\dots,3$, $\mathcal{V}_{2j}(\tau,t)$ is bounded in H^s-norm, uniformy in $(\tau,t) \in [0,T_0)\times\mathbb{R}$.*

Proof.
i. This point does not raise any difficulty, since all the components concerned here are polynomial expressions of $\mathcal{V}_{01}$ and $\overline{\mathcal{V}_{01}}$. The estimates therefore follows from Lemma 2.1.
ii. We know that $\Pi_1\mathcal{V}_{11}$ is given in terms of $\mathcal{V}_{01}$ and $(\Pi_0\mathcal{V}_{10})^*$ by (21). The difficulty here is the possible presence of secular growth. In general (see [8] [10]), we only know that the H^s-norm of $\Pi_1\mathcal{V}_{11}$ grows like $o(t)$. However, we can exploit here the fact that $(\Pi_0\mathcal{V}_{10})^* \in C([0,T_0);\mathcal{F}^{-1}L^\infty_s(\mathbb{R}^n)^q)$, as stated by Lemma 2.1, and the detailed secular growth estimates of [11].
When $n=1$, the result is a consequence of Prop. 3.4 of [11]. For $n\geq 2$, remark that

$$\Pi_1\mathcal{V}_{11}(\tau,t,x) = 2\Pi_1 f\left(\mathcal{V}_{01}(\tau,\zeta), \int_0^t \left(\Pi_0\mathcal{V}_{10}\right)^*(\tau,h,x-\omega'(\underline{k})(t-h))dh\right),$$

with $\zeta = x-\omega'(\underline{k})t$. Therefore,

$$\begin{aligned}\sup_{\tau\in[0,T]} \|\Pi_1\mathcal{V}_{11}(\tau,t)\|_{H^s} &\leq \mathrm{Cst} \sup_{\tau\in[0,T]} \|\mathcal{V}_{01}(\tau)\|_{H^s} \\ &\times \sup_{\tau\in[0,T]} \|\int_0^t \left(\Pi_0\mathcal{V}_{10}\right)^*(\tau,h,x-\omega'(\underline{k})(t-h))dh\|_{H^s},\end{aligned}$$

and the desired estimates follow from Lemma 2.1 and Prop. 2.3.i of [11] (note that the ellipticity assumption ensures that we are indeed in the first case of this latter proposition).
iii. All the Fourier components of $\mathcal{V}_2$ can be expressed in terms of $\mathcal{V}_{01}$, $\Pi_0\mathcal{V}_{10}$, $\Pi_1\mathcal{V}_{11}$, and of their derivatives of first order. Since the dependance on $\Pi_1\mathcal{V}_{11}$ is linear, the desired estimates are a consequence of the previous points of the proposition.
iv. When $b=Q(a)$, it follows from Prop. 2.2 that $(\Pi_0\mathcal{V}_{10})^*=0$ and therefore that the right hand side of (21) vanishes. Hence, we can take $\Pi_1\mathcal{V}_{11}=0$, and no secular growth phenomenon arises anymore.

□

From the estimates of Prop. 2.3 and the explicit expressions of the residual terms given by (6)-(11), one obtains easily the following estimates of these residual terms.

PROPOSITION 2.4 *Suppose that Assumption 2.2 is satisfied. Let $s>n/2$ and $a\in\Sigma^{s+3}$, $b\in H^{s+3}(\mathbb{R}^n)^q$ such that $\Pi_1 a=a$ and $\Pi_0 b=b$. Then, for all $0<T<T_0$ and $t\geq 0$,*
i.

$$\sup_{\tau\in[0,T]} \|\mathcal{R}_j(\tau,t)\|_{L^2(\mathbb{T}_\theta;H^s(\mathbb{R}^n)^q)} \leq \mathrm{Cst}\,\chi(t)^2,$$

where $\chi(t)$ is defined in Prop. 2.3.
ii. *If $b=Q(a)$, then*

$$\sup_{\tau\in[0,T]} \|\mathcal{R}_j(\tau,t)\|_{L^2(\mathbb{T}_\theta;H^s(\mathbb{R}^n)^q)} \leq \mathrm{Cst}\,.$$

2.4 The approximate solution

According to the previous sections, we have

$$\begin{aligned}
V(t,x,\theta) &= \mathcal{V}(\varepsilon t,t,x,\theta) \\
&= \big(\Pi_1\mathcal{V}_{01}(\varepsilon t, x-\omega'(\underline{k})t)e^{i\theta} + \text{c.c.}\big) \\
&+ \varepsilon\left(\langle \Pi_0\mathcal{V}_{10}\rangle(\varepsilon t, x-\omega'(\underline{k})t) + (\Pi_0\mathcal{V}_{10})^*(\varepsilon t,t,x)\right) \\
&+ \varepsilon\big((1-\Pi_0)\mathcal{V}_{10}(\varepsilon t,t,x) + \mathcal{V}_{11}(\varepsilon t,t,x)e^{i\theta} + \mathcal{V}_{12}(\varepsilon t,t,x)e^{2i\theta} + \text{c.c.}\big) \\
&+ \varepsilon^2\mathcal{V}_{sup}(\varepsilon t,t,x,\theta),
\end{aligned} \tag{23}$$

where $\mathcal{V}_{sup}(\tau,t,x,\theta)$ gathers all the terms of superior order, which are given in terms of $\mathcal{V}_{01}$ and $\Pi_0\mathcal{V}_{10}$ by the formulas of the two previous sections.
The components $\mathcal{V}_{01}(\tau,\zeta) = \Pi_1\mathcal{V}_{01}(\tau,\zeta)$ and $\langle\Pi_0\mathcal{V}_{10}\rangle(\tau,\zeta)$, which solve the Davey-Stewartson system (20), (22), are given by Prop. 2.1 while $(\Pi_0\mathcal{V}_{10})^*$ is given by Prop. 2.2. Finally, $(1-\Pi_0)\mathcal{V}_{10}$, $\mathcal{V}_{11}$ and $\mathcal{V}_{12}$ are given in terms of $\mathcal{V}_{01}$ and $\Pi_0\mathcal{V}_{10}$ by (14)-(16) and (21).

We can now state the main proposition of this section, which states that V approximately solves (3). It is a straightforward consequence of the preceding propositions.

Proposition 2.5 *Suppose that Assumptions 2.1 and 2.2 are satisfied. Let $s > n/2$ and $a \in \Sigma^{s+3}$, $b \in H^{s+3}(\mathbb{R}^n)^q$ such that $\Pi_1 a = a$ and $\Pi_0 b = b$.*
Then there exists $T_0 > 0$ such that $V \in C^1([0,\frac{T_0}{\varepsilon}); L^2(\mathbb{T}_\theta; H^s(\mathbb{R}^n_x)^q))$, where V is given by (23).
Moreover, one has

$$\partial_t V + \sum_{j=1}^n A_j\partial_{x_j}V + \frac{1}{\varepsilon}\mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta) - f(V,V) = \varepsilon^2 R^\varepsilon,$$

where, for all $0 < T < T_0$,

$$\sup_{t\in[0,\frac{T}{\varepsilon}]} \|R^\varepsilon(t)\|_{L^2(\mathbb{T}_\theta;H^s(\mathbb{R}^n_x)^q)} \le \text{Cst}\,|\ln(\varepsilon)|^2.$$

If $n \ge 3$ or if $b = Q(a)$, then the better estimate holds:

$$\sup_{t\in[0,\frac{T}{\varepsilon}]} \|R^\varepsilon(t)\|_{L^2(\mathbb{T}_\theta;H^s(\mathbb{R}^n_x)^q)} \le \text{Cst}\,.$$

3 Convergence results

The aim of this section is to prove that the approximate profile V given by (23) converges to the exact solution of (3)-(4). The first thing to check is that the initial value of V is approximately the initial condition (4), i.e.

$$\begin{aligned}
V(0,x,\theta) &= U(0,x,\theta) + O(\varepsilon^2) \\
&= \Big(a(x)e^{i\theta} + \overline{a(x)}e^{-i\theta}\Big) + \varepsilon B(x,\theta) + O(\varepsilon^2).
\end{aligned} \tag{24}$$

However, if $a = \Pi_1 a$ and $b = \Pi_0 b$ are the initial values of $\mathcal{V}_{01}$ and $\Pi_0\mathcal{V}_{10}$ used in Props. 2.1 and 2.2 to determine these profiles, then $V(0,x,\theta)$, as given by

(23) reads

$$\begin{aligned} V(0,x,\theta) &= \left(a(x)e^{i\theta} + \overline{a(x)}e^{-i\theta}\right) \\ &+ \varepsilon\left(b + 2\mathcal{L}_0^{-1}f(a,\overline{a}) - \mathcal{L}_1^{(-1)}A(\partial_x)ae^{i\theta} + \mathcal{L}_2^{(-1)}f(a,a)e^{2i\theta} + \text{c.c.}\right) \\ &+ O(\varepsilon^2). \end{aligned} \tag{25}$$

Obviously, (24) and (25) can be both satisfied if and only if the initial condition (4) of U satisfies

$$\begin{array}{l} \Pi_1 a = a, \qquad \Pi_0 b = b, \\ B(\cdot,\theta) = b + 2\mathcal{L}_0^{-1}f(a,\overline{a}) - \mathcal{L}_1^{(-1)}A(\partial_x)ae^{i\theta} + \mathcal{L}_2^{(-1)}f(a,a)e^{2i\theta} + \text{c.c.}. \end{array} \tag{26}$$

In the next subsection, we prove that the full approximation V converges for large times towards an exact solution of (3) with initial condition (4) satisfying (26). Then, we consider the less precise (but easier to compute) approximation which consists in the classical Davey-Stewartson approximation corrected (if needed) by the long-wave correction.

3.1 Convergence of the full approximate solution

We state in this section our first main result, which proves that the approximate solution constructed in Section 2.4 converges to an exact solution to (1)-(2) over large times of order $O(|\ln\varepsilon|)$.

Proposition 3.1 *Suppose that Assumptions 2.1 and 2.2 are satisfied. Let s be an integer, $s > n/2$, $a \in \Sigma^{s+3}$, $b \in H^{s+3}(\mathbb{R}^n)^q$ and $B(\cdot,\cdot)$ satisfying condition (26).*
Then there exist a constant $\underline{C} > 0$ and a real polynomial q such that for all $0 < C_1 < \frac{1}{2\underline{C}}$ there exists $\varepsilon_0 > 0$ such that:
i. *The Cauchy problem (1) with initial condition*

$$u(0,x) = \left(a(x)e^{i\frac{\underline{k}\cdot x}{\varepsilon}} + \overline{a(x)}e^{-i\frac{\underline{k}\cdot x}{\varepsilon}}\right) + \varepsilon B\left(x, \frac{\underline{k}\cdot x}{\varepsilon}\right)$$

admits, for all $0 < \varepsilon < \varepsilon_0$, a unique solution $u \in C([0, C_1|\ln\varepsilon|], H^s(\mathbb{R}^n)^q)$.
Moreover, one can write u under the form $u(t,x) = U(t,x,\dfrac{\underline{k}\cdot x - \underline{\omega}t}{\varepsilon})$, with $U(t,x,\theta) \in C([0, C_1|\ln\varepsilon|], H^1(\mathbb{T}_\theta; H^s(\mathbb{R}^n_x)^q))$.
ii. *The function v defined as $v(t,x) = V(t,x,\dfrac{\underline{k}\cdot x - \ t}{\varepsilon})$, with V given by (23) is a good approximation of u in the sense that for all $0 \le t \le C_1|\ln\varepsilon|$,*

$$\sup_{\sigma\in[0,t]} \|U(\sigma) - V(\sigma)\|_{H^1(\mathbb{T};H^s(\mathbb{R}^n)^q)} \le \varepsilon\tan(\varepsilon t q(t)e^{2\underline{C}t}),$$

and

$$\|u - v\|_{L^\infty([0,t]\times\mathbb{R}^n)} \le \varepsilon\tan(\varepsilon t q(t)e^{2\underline{C}t}).$$

Proof.
The proof of the proposition is of the same vein as the proof of the main theorem

of [12]. First introduce the linearized operator $M(\partial)$, which is defined for all $0 \le t < \frac{T_0}{\varepsilon}$ (T_0 is given by Prop. 2.1) as

$$M(\partial) = (\partial_t + A(\partial_x)) + \frac{1}{\varepsilon}\mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta) - 2f(V_0 + \varepsilon V_1, \cdot),$$

where $V_j(t,x,\theta) := \mathcal{V}_j(\varepsilon t, t, x, \theta)$, $j = 0, \dots, 2$.

Step 1. Let $T < T_0$. Then there exists $\underline{C} > 0$ such that every solution Z of $M(\partial)Z = 0$ satisfies, for $0 \le \sigma \le t \le T/\varepsilon$,

$$\|Z(t)\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} \le e^{\underline{C}(t-\sigma)} \|Z(\sigma)\|_{L^2(\mathbb{T}\times\mathbb{R}^n)}. \tag{27}$$

Indeed, standard energy estimates yield

$$\begin{aligned} \|Z(t)\|^2_{L^2(\mathbb{T}\times\mathbb{R}^n)} &\le \|Z(\sigma)\|^2_{L^2(\mathbb{T}\times\mathbb{R}^n)} \\ &+ \mathrm{Cst} \int_\sigma^t \|(V_0 + \varepsilon V_1)(h)\|_{L^\infty(\mathbb{T}\times\mathbb{R}^n)} \|Z(h)\|^2_{L^2(\mathbb{T}\times\mathbb{R}^n)} dh. \end{aligned} \tag{28}$$

Moreover, one deduces from Prop. 2.3 that

$$\begin{aligned} \|(V_0 + \varepsilon V_1)(h)\|_{L^\infty(\mathbb{T}\times\mathbb{R}^n)} &\le \mathrm{Cst} \sup_{\tau\in[0,T]} \|(\mathcal{V}_0 + \varepsilon \mathcal{V}_1)(\tau, h)\|_{H^1(\mathbb{T}; H^s(\mathbb{R}^n))} \\ &\le \mathrm{Cst}\,(1 + \varepsilon \ln h). \end{aligned} \tag{29}$$

From (28), (29) and Gronwall's lemma, one then obtains

$$\|Z(t)\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} \le e^{\mathrm{Cst} \int_\sigma^t (1+\varepsilon \ln h) dh} \|Z(\sigma)\|_{L^2(\mathbb{T}\times\mathbb{R}^n)}.$$

Since $|(1 + \varepsilon \ln h)| \le \mathrm{Cst}$ for all $h \in [0, T/\varepsilon]$, (27) follows.

Step 2. From (27) one can obtain by induction on the Sobolev index -as in the proof of Th.2 of [12]- that for all integer $k, l \ge 0$, the following generalization of (27) holds for $0 \le \sigma \le t \le T/\varepsilon$:

$$\|Z(t)\|_{H^k(\mathbb{T}; H^l(\mathbb{R}^n)^q)} \le p_{k,l}(t) e^{\underline{C}(t-\sigma)} \|Z(\sigma)\|_{H^k(\mathbb{T}; H^l(\mathbb{R}^n)^q)}, \tag{30}$$

where $p_{k,l}$ is a real polynomial and $\underline{C}$ is the same constant as in (27).

Step 3. We look for the exact solution u as a perturbation of v, $u = v + w$ with $w(t,x) = W(t, x, \dfrac{\underline{k}\cdot x - \underline{\omega} t}{\varepsilon})$. The equation for $W(t,x,\theta)$ is therefore

$$(\partial_t + A(\partial_x))(V + W) + \frac{1}{\varepsilon}\mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta)(V + W) = f(V + W, V + W).$$

Recall that Prop 2.5 states that

$$(\partial_t + A(\partial_x))V + \frac{1}{\varepsilon}\mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta)V = f(V, V) + \varepsilon^2 R^\varepsilon;$$

hence, substracting these two equations yields

$$(\partial_t + A(\partial_x))W + \frac{1}{\varepsilon}\mathcal{L}(-\underline{\omega}\partial_\theta, \underline{k}\partial_\theta)W = f(V + W, V + W) - f(V, V) - \varepsilon^2 R^\varepsilon,$$

or equivalently, using the fact that $V = V_0 + \varepsilon V_1 + \varepsilon^2 V_2$,

$$M(\partial)W = f(W,W) + 2\varepsilon^2 f(V_2, W) - \varepsilon^2 R^\varepsilon.$$

This equation, together with the initial condition

$$W(t=0) = -\varepsilon^2 \mathcal{V}_{sup}(0,0,x,\theta) = O(\varepsilon^2)$$

defines a Cauchy problem for which it is known that there exists a unique solution in $C([0,T_\varepsilon), H^1(\mathbb{T}; H^s(\mathbb{R}^n)^q))$ for some $0 < T_\varepsilon \le T_0/\varepsilon$. Moreover, if $T_\varepsilon < T_0/\varepsilon$ then $\lim_{t\to T_\varepsilon} \|W(t)\|_{L^\infty(\mathbb{T}\times\mathbb{R}^n)} = \infty$.
From Duhamel's representation formula and (30), one has, for all $0 \le t < T_\varepsilon$, that

$$\begin{aligned}\|W(t)\| &\le p_{s+1}(t) e^{\underline{C}t} \|W(0)\| \\ &+ \int_0^t p_{s+1}(t-\sigma) e^{\underline{C}(t-\sigma)} \left[\|W(\sigma\|^2 + \varepsilon^2 (\|V_2(\sigma)\|\, \|W(\sigma)\| + \|R^\varepsilon(\sigma)\|)\right] d\sigma,\end{aligned}$$

where the norms $\|\cdot\|$ stand for $\|\cdot\|_{H^1(\mathbb{T};H^s(\mathbb{R}^n)^q)}$.
From Props. 2.3 and 2.4, we know that $\|V_2(\sigma)\| \le \mathrm{Cst}\,(1+\ln(1+\sigma))$ and that $\|R^\varepsilon(\sigma)\| \le \mathrm{Cst}\,(1+\ln(1+\sigma))^2$. Since both $e^{-\underline{C}\sigma}\ln(1+\sigma)$ and $e^{-\underline{C}\sigma}\ln(1+\sigma)^2$ are bounded for all $0 \le \sigma \le T/\varepsilon$ by a constant independent of ε, and since $W(0) = O(\varepsilon^2)$, it follows that

$$\|W(t)\|_{H^1(\mathbb{T};H^s(\mathbb{R}^n)^q)} \le e^{\underline{C}t} \int_0^t \widetilde{p_{s+1}}(t-\sigma) \left[\|W(\sigma)\|^2_{H^1(\mathbb{T};H^s(\mathbb{R}^n)^q)} + \varepsilon^2\right] d\sigma, \quad (31)$$

for some "big enough" polynomial $\widetilde{p_{s+1}}$.
Step 4. This steps consists in the proof of the following lemma.

LEMMA 3.1 *Let p be a real polynomial and $y \in C(\mathbb{R}_+)$ such that for all $t \ge 0$,*

$$y(t) \le e^{\underline{C}t} \int_0^t p(\sigma) \left[y^2(\sigma) + \varepsilon^2\right] d\sigma.$$

Then there exists a real polynomial q, such that for all $0 < C_1 < \frac{1}{2\underline{C}}$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $0 \le t \le C_1 |\ln \varepsilon|$,

$$y(t) \le \varepsilon \tan(\varepsilon t q(t) e^{2\underline{C}t}).$$

To prove this result, introduce

$$E(t) = \int_0^t p(\sigma) \left[y^2(\sigma) + \varepsilon^2\right] d\sigma.$$

Then one computes

$$\begin{aligned} E'(t) &= p(t)\left[y^2(t) + \varepsilon^2\right] \\ &\le p(t)\left[e^{2\underline{C}t} E(t)^2 + \varepsilon^2\right] \\ &\le e^{2\underline{C}t} p(t) \left[E(t)^2 + \varepsilon^2\right]. \end{aligned}$$

Since

$$\frac{d}{dx}\left[\frac{1}{\varepsilon}\arctan\left(\frac{x}{\varepsilon}\right)\right] = \frac{1}{x^2+\varepsilon^2},$$

it follows that

$$\arctan\left(\frac{E(t)}{\varepsilon}\right) \leq \varepsilon \int_0^t e^{2\underline{C}s} p(s)ds. \tag{32}$$

Now remark that for all $t \geq 0$, we can write

$$\int_0^t e^{2\underline{C}s} p(s)ds \leq tq(t)e^{2\underline{C}t}, \tag{33}$$

where q is a real valued polynomial.
Let $C_1 > 0$ be a constant; then, for all $0 \leq t \leq C_1|\ln\varepsilon|$, one has $e^{2\underline{C}t} \leq \frac{1}{\varepsilon^{2\underline{C}C_1}}$, and thus, if $C_1 < \frac{1}{2\underline{C}}$ it follows that

$$\sup_{t\in[0,C_1|\ln\varepsilon|]} \varepsilon \int_0^t e^{2\underline{C}s} p(s)ds \to 0 \quad \text{when } \varepsilon \to 0.$$

Therefore, for ε small enough and $0 \leq t \leq C_1|\ln\varepsilon|$, the right hand side of (32) is bounded by $\pi/2$. Since tan is increasing on $[0, \pi/2)$, the lemma follows from (32) and (33).
Step 5. Owing to Lemma 3.1 and (31), and thanks to the explosion condition $\|W(t)\|_\infty \to \infty$ as $t \to T_\varepsilon$, we can conclude that $T_\varepsilon \geq C_1|\ln\varepsilon|$ and that for all $0 \leq t \leq C_1|\ln\varepsilon|$,

$$\|W(t)\|_{H^1(\mathbb{T};H^s(\mathbb{R}^n)^q)} \leq \varepsilon \tan\left(\varepsilon tq(t)e^{2\underline{C}t}\right), \tag{34}$$

which achieves the proof of the existence part of point i. of the proposition since by definition $U = V + W$ solves the singular equation (3). The uniqueness part is standard.
Step 6. The estimate on the difference $U - V$ is of course exactly (34). The L^∞ estimate on $u - v$ is then obtained as a consequence of the Sobolev embedding $H^1(\mathbb{T}; H^s(\mathbb{R}^n)^q) \subset L^\infty(\mathbb{T} \times \mathbb{R}^n)$.

□

REMARK 1 **i.** *For times of logarithmic order, the error estimates given by the proposition are $o(\varepsilon)$. For times of order one, the error terms can be bounded by the better estimate $O(\varepsilon^2 t)$.*
ii. *In this proposition, the error estimates are not better when one considers initial data ill-prepared in the sense that $b \neq Q(a)$. Such ill-prepared initial conditions are made possible (while it is not the case in previous studies) thanks to the introduction of the long-wave correction. The fact that the error estimates are not affected by the secular growth which arises in the ill-prepared case is due to the fact that the secular term $\Pi_1 V_{11}$ is included in the approximate profile V.*

3.2 Convergence of the (corrected) Davey-Stewartson approximation

We proved above that the approximate solution $v(t,x) = V(t, x, \frac{\underline{k}\cdot x - \underline{\omega}t}{\varepsilon})$, with V given in (23) converges to an exact solution of (1)-(2). The profile V defines what we called the full approximate solution. This full approximation is the sum of the Davey-Stewartson approximation, the long-wave correction and of higher order corrector terms. More precisely, one has

$$V(t,x,\theta) = V_{DS}(t,x,\theta) + V_{LW}(t,x,\theta) + V_{cor}(t,x,\theta) \tag{35}$$

where

$$\begin{aligned}
V_{DS}(t,x,\theta) &= \mathcal{V}_{01}(\varepsilon t, x-\omega'(\underline{k})t)e^{i\theta} + \text{c.c.} + \varepsilon\langle \Pi_0 \mathcal{V}_{10}\rangle(\varepsilon t, x-\omega'(\underline{k})t) \\
&\quad + \varepsilon 2\mathcal{L}_0^{(-1)} f(\mathcal{V}_{01}, \overline{\mathcal{V}_{01}}) \\
&\quad - \varepsilon \mathcal{L}_1^{(-1)} A(\partial_x)\mathcal{V}_{01} e^{i\theta} + \varepsilon \mathcal{L}_2^{(-1)} f(\mathcal{V}_{01},\mathcal{V}_{01})e^{2i\theta} + \text{c.c.} &(36)\\
V_{LW}(t,x,\theta) &= \varepsilon \left(\Pi_0 \mathcal{V}_{10}\right)^* (\varepsilon t, t, x) &(37)\\
V_{cor}(t,x,\theta) &= \varepsilon \Pi_1 \mathcal{V}_{11}(\varepsilon, t, t, x)e^{i\theta} + \text{c.c.} + \varepsilon^2 \mathcal{V}_{sup}(\varepsilon t, t, x, \theta), &(38)
\end{aligned}$$

are respectively the Davey-Stewartson approximate profile, the long-wave correction and the higher order corrector.
The leading terms of V_{DS}, namely $\mathcal{V}_{01}$ and $\langle \mathcal{V}_{10}\rangle$, solve the Davey-Stewartson system (20), (22); the other terms are given by algebraic explicit expressions of $\mathcal{V}_{01}$. The long-wave correction $(\Pi_0\mathcal{V}_{10})^*$ is given by Prop. 2.2. Finally, the corrective term V_{cor} is the sum of the secular term $\varepsilon\Pi_1\mathcal{V}_{11}$ given by (21) and of higher order terms gathered in $\mathcal{V}_{sup}$.
By corrected Davey-Stewartson approximation, we mean the sum of the Davey-Stewartson approximation and of the long-wave correction.

Remark 2 *Note that in space dimension $n = 1$, the Davey-Stewartson system degenerates into a cubic Schrödinger equation.*

The following theorem states that if one replaces the full approximate solution in Prop. 3.1 by the corrected Davey-Stewartson approximation, we still have convergence, though with a worse convergence rate.

Theorem 3.1 *Under the same assumptions as in Prop. 3.1 and with the same notations, one has:*
i. *If $n = 1$ or $n = 2$,*

$$\sup_{t\in[0,C_1|\ln\varepsilon|]} \|U(t) - (V_{DS}(t) + V_{LW}(t))\|_{H^1(\mathbb{T};H^s(\mathbb{R}^n)^q)} \leq \mathrm{Cst}\,\varepsilon\ln|\ln\varepsilon|.$$

and

$$\|u - (v_{DS} + v_{LW})\|_{L^\infty([0,C_1|\ln\varepsilon|]\times\mathbb{R}^n)} \leq \mathrm{Cst}\,\varepsilon\ln|\ln\varepsilon|,$$

with $v_{DS}(t,x) := V_{DS}(t,x,\frac{\underline{k}\cdot x - \underline{\omega} t}{\varepsilon})$ and $v_{LW} := V_{LW}(t,x,\frac{\underline{k}\cdot x - \underline{\omega} t}{\varepsilon})$.
ii. *If $n \geq 3$, then $\mathrm{Cst}\,\varepsilon\ln|\ln\varepsilon|$ can be replaced by $\mathrm{Cst}\,\varepsilon$ in the right hand side of the previous estimates.*
iii. *If the initial conditions are such that $b = Q(a)$ then $V_{LW} = 0$ and for all $0 \geq t \geq C_1|\ln\varepsilon|$,*

$$\sup_{\sigma\in[0,t]} \|U(\sigma) - V_{DS}(\sigma)\|_{H^1(\mathbb{T};H^s(\mathbb{R}^n)^q)} \leq \mathrm{Cst}\,\varepsilon\tan\left(\varepsilon t q(t) e^{2\underline{C}t}\right)$$

and

$$\|u - v_{DS}\|_{L^\infty([0,t]\times\mathbb{R}^n} \leq \mathrm{Cst}\,\varepsilon\tan\left(\varepsilon t q(t) e^{2\underline{C}t}\right).$$

Proof.
From Prop. 2.3 we can estimate $\|\varepsilon\Pi_1\mathcal{V}_{11}(\varepsilon t,t)+\varepsilon^2\mathcal{V}_{sup}(\varepsilon t,t)\|_{H^s}\leq \mathrm{Cst}\,\chi_n(t)$, where we recall that $\chi_n(t)=(1+\ln(1+t))$ if $n=1,2$ and $\chi_n(t)=1$ if $n\geq 3$. Since

$$\|U(t)-(V_{DS}(t)+V_{LW}(t))\|\leq\|U(t)-V(t)\|+\|V_{cor}(t)\|,$$

the first two points of the proposition follow easily from Prop. 3.1.
As already seen, the secular term $\Pi_1\mathcal{V}_{11}$ and the long-wave correction $(\Pi_0\mathcal{V}_{10})^*$ vanish when $b=Q(a)$ and the fourth point of Prop. 2.3 allows us to conclude the proof.

□

Remark 3 *Contrary to what happened in Prop. 3.1, the error estimates are better in Th. 3.1 in the case of well-prepared initial data (i.e. $b=Q(a)$). This is not surprising since the secular terms created when initial data are ill-prepared are not taken into account in the corrected Davey-Stewartson approximation while they are in the full approximation (23).*

We end this section with a theorem concerning the mean mode created by the nonlinear interaction of the oscillating modes. Precise description of such rectification effects is of great practical interest, but Th. 3.1 does not tell anything about it (except in the case of well-prepared inital data $b=Q(a)$). Indeed, while the leading terms of the oscillating mode are of order $O(1)$, the leading order of the mean mode is $O(\varepsilon)$. Hence, nothing can be deduced on it from the $O(\varepsilon\ln|\ln\varepsilon|)$ or even $O(\varepsilon)$ error estimates of the first two points of Th. 3.1.
We show in the next theorem that even in the case of ill-prepared initial data, the (corrected) Davey-stewartson approximation furnishes a good description of the rectification effects (i.e. of the mean mode). The mean part of the corrected Davey-Stewartson approximation, denoted by $\underline{V}_{DS}+V_{LW}$ reads, according to (36)-(37),

$$\begin{aligned}\underline{V}_{DS}(t,x)+V_{LW}(t,x) &= \varepsilon\langle\Pi_0\mathcal{V}_{10}\rangle(\varepsilon t,x-\omega'(\underline{k})t)+\varepsilon 2\mathcal{L}_0^{(-1)}f(\mathcal{V}_{01},\overline{\mathcal{V}_{01}})\\ &+ \varepsilon\left(\Pi_0\mathcal{V}_{10}\right)^*(\varepsilon t,t,x).\end{aligned}$$

The theorem says that it converges to the mean part $\underline{U}$ of an exact solution U to (3) with a convergence rate $O\left(\varepsilon\tan\left(\varepsilon t q(t)e^{2\underline{C}t}\right)\right)$. The relative error is therefore $O\left(\tan\left(\varepsilon t q(t)e^{2\underline{C}t}\right)\right)$, and hence $o(1)$ for large times of logarithmic order.

Theorem 3.2 *Under the same assumptions as in Prop. 3.1 and Th. 3.1 and with the same notations, one has, for all $0\leq t\leq C_1|\ln\varepsilon|$,*

$$\sup_{\sigma\in[0,t]}\|\underline{U}(t)-(\underline{V}_{DS}(t)+V_{LW}(t))\|_{H^s(\mathbb{R}^n)^q}\leq\mathrm{Cst}\,\varepsilon\tan\left(\varepsilon t q(t)e^{2\underline{C}t}\right).$$

and

$$\|\underline{U}-(\underline{V}_{DS}+V_{LW})\|_{L^\infty([0,t]\times\mathbb{R}^n)}\leq\mathrm{Cst}\,\varepsilon\tan\left(\varepsilon t q(t)e^{2\underline{C}t}\right).$$

Proof.
As for Th. 3.1, the proof is a simple consequence of Prop. 3.1. One just has to notice that the mean part of V_{cor} is $O(\varepsilon^2|\ln\varepsilon|)$ and hence $O\left(\varepsilon\tan\left(\varepsilon t q(t)e^{2\underline{C}t}\right)\right)$ for the time scale considered.

$\square$

REMARK 4 **i.** *Since $\underline{U}(t,x)$ is left unchanged by the substitution $\theta = \underline{k} \cdot x - \underline{\omega} t$ which makes the link between (3) and (1), it may be seen as the mean part of the solution u of the Cauchy problem (1)-(2).*
ii. *Ill-prepared initial data do not affect the validity of the corrected Davey - Stewartson approximation because all the subsequent secular effects do not occur on the mean mode.*
iii. *In general, the long wave correction consists in a standing wave and of waves propagated by wave equations. These latter decay in L^∞ norm as $O(t^{-(n-1)/2})$ as $t \to \infty$, when $n \geq 2$.*

4 Application to ferromagnetism

In a ferromagnetic medium, Maxwell equation read, according to the Landau-Lifshitz model,

$$\begin{cases} \partial_t E - \operatorname{curl} H = 0, \\ \partial_t H + \operatorname{curl} E + \partial_t M = 0, \\ \partial_t M = M \wedge H, \end{cases} \tag{39}$$

where E, H and M denote respectively the electric field, the magnetic field and the magnetization density. We consider here the cases $n = 1, 2, 3$.
We want to study carefully the rectification effects which arise in a situation investigated by H. Leblond [14] where one considers solutions to (39) which are small, slowly variable perturbations of a constant solution $(\mathbf{E}, \mathbf{H}, \mathbf{M})$. It is clear that $\mathbf{H}$ and $\mathbf{M}$ must be colinear; let therefore $\alpha > 0$ be such that $\mathbf{H} = \alpha \mathbf{M}$. Hence, the solutions we look for are of the form $(\mathbf{E} + \varepsilon e(\varepsilon t, \varepsilon x), \alpha \mathbf{M} + \varepsilon h(\varepsilon t, \varepsilon x), \mathbf{M} + \varepsilon m(\varepsilon t, \varepsilon x))$, ε being a small parameter. It follows that $u = (e, h, \sqrt{\alpha} m)$ must satisfy the equation

$$\partial_t u + \begin{pmatrix} 0 & -\operatorname{curl} & 0 \\ \operatorname{curl} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u + \frac{1}{\varepsilon} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{M}\wedge & -\sqrt{\alpha}\mathbf{M}\wedge \\ 0 & -\sqrt{\alpha}\mathbf{M}\wedge & \alpha \mathbf{M}\wedge \end{pmatrix} u = f(u,u), \tag{40}$$

where f is the bilinear mapping associated to the quadratic form $q(u) = (0, \frac{-1}{\sqrt{\alpha}} u_3 \wedge u_2, u_3 \wedge u_2)^T$, where $u = (u_1, u_2, u_3)^T \in \mathbb{R}^9$.
As in [14], we consider the case where $\underline{k}$ and $\mathbf{M}$ are colinear. Up to a change of coordinates, one has $k = k e_1$, $\mathbf{M} = m e_1$, $k, m > 0$ and (e_1, e_2, e_3) being an orthonormal basis of $\mathbb{R}^3$. The dispersion relation associated to our problem reads (see [15]),

$$(\omega^2 + \alpha \omega^2 - \alpha k^2)^2 m^2 = (\omega^2 - k^2)^2 \omega^2.$$

The characteristic variety associated to this dispersion relation has three branches in the quarter plane $\{\omega \geq 0, k \geq 0\}$. It is easy to check (see [14]) that if $(\underline{\omega}, \underline{k})$ belongs to the lowest branch, then Assumptions 2.1 and 2.2 are satisfied.
It is therefore possible to construct a formal approximate solution to (40), following the lines of the previous sections. In the case of well-prepared initial data ($b = Q(a)$ with the notation of the previous section, i.e. the initial value of the polarized mean field is exactly equal to the initial value of the polarized mean component of V_{DS}), the results of course coincide with the formal computations of Leblond [14]. The leading oscillating and mean terms can be easily expressed

in terms of $\phi(\varepsilon t, x-\omega'(\underline{k})t)$ and $\psi(\varepsilon t, x-\omega'(\underline{k})t)$ respectively, where $\phi(\tau,\zeta)$ and $\psi(\tau,\zeta)$ satisfy the Davey-Stewartson system

$$i\partial_\tau\phi + \frac{1}{2}\omega''(\underline{k})(\partial_\zeta,\partial_\zeta) + A\phi|\phi|^2 + B\phi\psi = 0 \tag{41}$$

$$C\partial_{\zeta_1}^2\psi + \sum_{j=2}^{n}\partial_{\zeta_j}^2\psi = D\sum_{j=2}^{n}\partial_{\zeta_j}^2\psi \tag{42}$$

where A, B, C, D are real constants. Moreprecisely, $\Pi_0\underline{V}_{DS}$ is proportionnal to $\psi + E|\phi|^2$, for some constant E, and the leading oscillating term of V_{DS} is proportionnal to ϕ (see [14] for details).
Justification of this approximation does not fall into the range of [2] but perfectly correspond to the framework of this paper. By Prop. 3.1 and Th. 3.2 we know that there exists an exact solution to (40) for large times of logarithmic order and that the Davey-Stewartson approximation furnishes a good approximation for such times.
Another interest of the present study is that it allows us to treat the case of ill-prepared initial data, and in particular, the case when the polarized part of the mean component of the initial data is zero. Apparition of a mean polarized field puts in evidence the rectification effects.
According to our results, one has to add a long-wave correction $V_{LW}(t,x)$ to the mean part polarized part of the Davey-Stewartson approximation $\Pi_0\underline{V}_{DS}(\varepsilon t, x-\omega' t)$. If the initial polarized mean field is zero, then it follows from (37) and Prop. 2.2 that V_{LW} solves

$$\Pi_0 A(\partial_x)\Pi_0 V_{LW} = 0,$$
$$V_{LW}(t=0) = -\Pi_0\underline{V}_{DS}(0,x).$$

The characteristical variety of the operator $\Pi_0 A(\partial_x)\Pi_0$ is the tangent cone at the origin to the characteristical variety associated to (1) (see [10]). Here it consists of two cones which coincide along two generatrix. In an appropriate basis, the symbol $\Pi_0 A(\xi)\Pi_0$ reads

$$\Pi_0 A(\xi)\Pi_0 = \mathrm{diag}\left(0,0,0,\sqrt{\tfrac{\alpha}{1+\alpha}}|\xi|, -\sqrt{\tfrac{\alpha}{1+\alpha}}|\xi|, \sqrt{\tfrac{\alpha|\xi|^2+|e_1\wedge\xi|^2}{1+\alpha}}, -\sqrt{\tfrac{\alpha|\xi|^2+|e_1\wedge\xi|^2}{1+\alpha}}\right). \tag{43}$$

The rectification effects are therefore described qualitatively as follows:
Case $\mathbf{n=1}$ (see Fig. 1) From (43) the long-wave correction has three components. A standing wave and two waves travelling at speed $\pm\sqrt{\frac{\alpha}{1+\alpha}}$. The polarized mean field of the Davey-Stewartson approximation travels at the group speed, which is (by our choice of $(\underline{\omega},\underline{k})$) smaller in absolute value than $\sqrt{\frac{\alpha}{1+\alpha}}$. In Fig. 1, this latter component is the positive one.
Case $\mathbf{n\geq 2}$. The polarized mean field of the Davey-Stewartson approximation still travels at the group speed, as in the case $n=1$. Except for its standing wave component, the long-wave correction does no longer consist in travelling waves. There is a component which is governed by the wave operator $\partial_t^2 - \frac{\alpha}{1+\alpha}\Delta$ and another one which is governed by the anisotropic wave operator $\partial_t^2 - \frac{\alpha\Delta+\Delta_T}{1+\alpha}$, where Δ_T denotes the transverse Laplacian $\Delta_T = \partial_2^2 + \ldots + \partial_n^2$.
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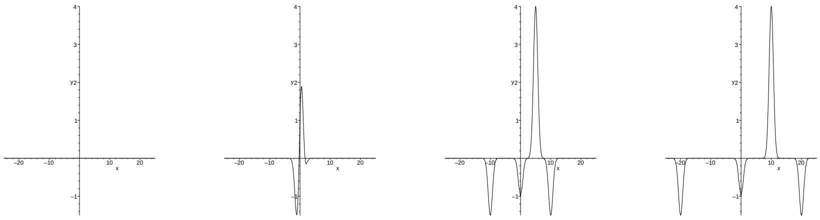

Figure 1: Evolution of the polarized mean field in dimension $n = 1$.

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