

Instabilities in Zakharov Equations for Laser Propagation in a Plasma

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Abstract

In [LPS], F.Linares, G.Ponce, J-C.Saut have proved that a non-fully dispersive Zakharov system arising in the study of Laser-plasma interaction, is locally well posed in the whole space, for fields vanishing at infinity. Here we show that in the periodic case, seen as a model for fields non-vanishing at infinity, the system develops strong instabilities of Hadamard's type, implying that the Cauchy problem is strongly ill-posed.

1 Introduction

1.1 Physical context

The construction of powerful lasers allows new experiments where hot plasmas are created in which laser beams can propagate. The main goal is to simulate in a laboratory nuclear fusion by inertial confinement. This requires precise and reliable models for laser-plasma interactions which can be used to produce numerical simulations that are usable to predict and illustrate the experiments. The kinetic-type models are the more precise ones but their cost in term of computations is exorbitant and, so far, no physically relevant situation for nuclear fusion can be simulated using these models. Another approach uses a bi-fluid model for the plasma, coupling two compressible Euler systems with Maxwell equations. Even in this form, it is not possible to perform direct computations because of the high frequency motions and of the small wavelength involved in the problem. At the beginning of the 70's, Zakharov and his collaborators introduced the so-called Zakharov's equations [ZMR] to describe electronic plasma waves. These systems couple the slowly varying envelope of the electric field and the low-frequency variation of the density of the ions. A typical non-dimensional form of such a system is:

$$\begin{aligned}i\partial_t u + \Delta u &= nu \\ \partial_t^2 n - \Delta n &= \Delta |u|^2\end{aligned}$$

When considering the propagation of a laser beam in a plasma, several such systems have to be coupled in order to take into account the laser beam, the Raman component and the electronic plasma waves (see [CC1, CC2] for example). The laser beam and the Raman component correspond to transverse electromagnetic waves, while the electronic plasma waves are longitudinal waves. In the latter case, the electric field is a gradient $E = \nabla\psi$ which is linked to low-frequency variations of the density of the ions δn by the following equations

$$\begin{cases} i\partial_t \nabla\psi + \frac{3v_{th}^2}{2\omega_{pe}} \Delta(\nabla\psi) = \frac{\omega_{pe}}{2n_0} \nabla \Delta^{-1} \text{div}(\delta n \nabla\psi), \\ \partial_t^2 \delta n - c_s^2 \Delta \delta n = \frac{1}{4\pi m_i} \Delta(|\nabla\psi|^2), \end{cases} \quad (1)$$

where v_{th} is the thermal velocity of electrons, ω_{pe} the plasma electronic frequency, n_0 the mean density of the plasma, m_i the mass of the ions and c_s the sound velocity in the plasma. Typical values of v_{th} is $0.1c$ where c is the speed of light and $\omega_{pe} \sim 10^{15} s^{-1}$. The underlying wavelength is of order of the micro-meter. For laser propagation or for the Raman component, one often uses the paraxial approximation and the Zakharov system that couples the vector potential A of the electromagnetic field to the low-frequency variation of the density of the ions reads

$$\begin{cases} i(\partial_t + \frac{k_0}{c^2} \omega_0 \partial_z) A + \frac{c^2}{2\omega_0} \Delta_x A = \frac{\omega_{pe}^2}{2n_0 \omega_0} n A, \\ (\partial_t^2 - c_s^2 \Delta_x) n = \frac{\omega_{pe}^2}{4\pi m_i c^2} \Delta_x |A|^2, \end{cases} \quad (2)$$

where ω_0 is the frequency of the laser and k_0 its wave number; they are linked by the dispersion relation $\omega_0^2 = \omega_{pe}^2 + k^2 c^2$. The space variables are (z, x) , $z \in \mathbb{R}$ and $x \in \mathbb{R}^2$: z is the component in the direction of propagation of the laser beam and x denotes the components in directions that are transversal to the propagation. See [R] or [S] for a symmetric use of this kind of model. In numerical simulations, systems (1) or (2) have to be used in various situations. Usually one considers that the unit of space is $\frac{1}{k_0}$ while the relevant unit for time is $\frac{1}{\omega_{pe}}$ and the space and time steps have to be respectively of the order of magnitude of $\frac{1}{k_0}$ and $\frac{1}{\omega_{pe}}$. For experiments concerning fusion by inertial confinement, one has to consider domains with spatial dimension of order of the centimeter and over several millions of $\frac{1}{\omega_0}$. In the 3-D configuration, this is often out of reach of computational capacities and one restricts attention to a small piece of the spatial domain. Moreover, in such scalings,

it is not realistic to consider that the fields are localized and, to do numerics, one usually considers that the plasma as well as the laser have a locally periodic structure at least at the scales that are considered here. The systems are then endowed with periodic boundary conditions, since in such situations. On the contrary, for propagation of lasers in the air or in crystals, one uses propagation in the whole space $\mathbb{R}^n$ with functions tending to zero at infinity. In this paper, we will focus on the former case, that is periodic boundary conditions that are useful in the physical framework explained above. The propagation in the whole space studied in [LPS] would rather correspond to the latter case.

1.2 The mathematical framework and the main result

The goal of this paper is to prove an illposedness result for a non-dimensional form of system (2):

$$\begin{cases} i(\partial_t + \partial_z)E + \Delta_x E = nE, \\ (\partial_t^2 - \Delta_x)n = \Delta_x |E|^2. \end{cases} \quad (3)$$

We consider the Cauchy problem for (3) with initial data

$$\begin{cases} E|_{t=0} = E_0, \\ n|_{t=0} = n_0, \quad \partial_t n|_{t=0} = n_1. \end{cases} \quad (4)$$

The existence theorem (see [S, GTV, OT] and references therein) for the classical Zakharov system, that is when Δ_x is replaced by $\Delta_{(z,x)}$, does not apply. In [LPS], it is proved that the Cauchy problem for (3) is well posed, locally in time, for data in suitable Sobolev spaces. The proof is based on dispersion estimates. For periodic data, these dispersion estimates are not valid. This is a well known phenomenon, even in the simpler case of Schrödinger equations. However, the new phenomenon here is that the consequences of this lack of dispersive effects are much more dramatic since it implies strong instabilities of Hadamard's type, so that the Cauchy problem for periodic data is strongly ill-posed in Sobolev spaces.

For the applications that we have described in the preceding section, it is quite reasonable to consider that $E \sim \underline{E} \neq 0$ at infinity. Our result has therefore a practical application and means that the paraxial approximation is not a good model in this case: one should add the longitudinal dispersion, that is replace $\Delta_x E$ in the first equation by $(\alpha \partial_z^2 + \Delta_x)E$. A natural mathematical question will then be: how does the instability grow when one let

the longitudinal dispersion parameter α tend to zero? We will address this question in a future work.

We look for solutions $U = (E, n)$ of (3), which are periodic in x , with period 2π in x and periodic in z with period $2\pi Z$, where Z is arbitrary. We denote by $\mathbb{T}$ the corresponding torus $\mathbb{R}/2\pi Z \times (\mathbb{R}/2\pi)^2$.

We consider the constant solution

$$\underline{U} = (\underline{E}, 0), \quad \underline{E} \neq 0, \quad (5)$$

which of course does not belong to the spaces used in [LPS], and we prove that this solution is strongly unstable.

Theorem 1.1 *For all s , there are families of solutions $U_k = \underline{U} + (e_k, n_k)$, in $C^1([0, T_k]; H^s(\mathbb{T}))$ such that*

$$\|e_k(0), n_k(0), \partial_t n_k(0)\|_{H^s(\mathbb{T})} \rightarrow 0, \quad (6)$$

$$T_k \rightarrow 0, \quad (7)$$

$$\|e_k(T_k), n_k(T_k)\|_{L^2(\mathbb{T})} \rightarrow \infty. \quad (8)$$

This nonlinear instability result is pretty strong: not only the *amplification* $\|u(T)\|_0/\|u(0)\|_s$ is arbitrarily large, in arbitrarily small time T , with arbitrary loss of derivatives s , but there is an effective *blow up* of the L^2 norm.

2 The instability mechanism

Our construction is based on the analysis of the *dispersion relation* for the Zakharov system. Consider the linearized equations around $(\underline{E}, 0)$:

$$\begin{cases} i(\partial_t + \partial_z)e + \Delta_x e - \underline{E} n & = f, \\ (\partial_t^2 - \Delta_x)n - \Delta_x(2\operatorname{Re} \underline{E} \bar{e}) & = g \end{cases} \quad (9)$$

With $(e, \bar{e}, n)$ as unknowns the system reads:

$$\begin{cases} -i(\partial_t + \partial_z)e - \Delta_x e + \underline{E} n & = -f, \\ i(\partial_t + \partial_z)\bar{e} - \Delta \bar{e} + \underline{E} n & = -\bar{f}, \\ (\partial_t^2 - \Delta_x)n - \underline{E} \Delta_x \bar{e} - \bar{\underline{E}} \Delta_x e & = g. \end{cases} \quad (10)$$

Denoting by (τ, ζ, ξ) the frequency variables dual to (t, z, x) , its symbol is

$$\begin{pmatrix} (\tau + \zeta) + |\xi|^2 & 0 & E \\ 0 & -(\tau + \zeta) + |\xi|^2 & \overline{E} \\ |\xi|^2 \overline{E} & |\xi|^2 E & |\xi|^2 - \tau^2 \end{pmatrix} \quad (11)$$

and the relation dispersion is $P = 0$, where P is the determinant of the system, that is

$$P = (|\xi|^2 - \tau^2)(|\xi|^4 - (\tau + \zeta)^2) - 2|\underline{E}|^2|\xi|^4 = P_0 - 2|\underline{E}|^2|\xi|^4. \quad (12)$$

The remark is that for (ζ, ξ) real, P_0 has 4 real roots in τ

$$-|\xi|, \quad +|\xi|, \quad -\zeta - |\xi|^2, \quad -\zeta + |\xi|^2, \quad (13)$$

with an intermediate double root when $0 < |\xi| = -\zeta - |\xi|^2$. Note that P_0 is of degree 6 in ξ while the perturbation $-2|\underline{E}|^2|\xi|^4$ is of degree 4 and negative. Therefore, for ξ large and $\zeta = -|\xi| - |\xi|^2$, the double root of P_0 is perturbed in two conjugated *complex roots*. More precisely, for

$$|\xi| \gg 0, \quad \zeta = -|\xi| - |\xi|^2 \quad \text{and} \quad \tau = |\xi|(1 + \sigma), \quad (14)$$

the determinant P is

$$P = -|\xi|^5 \left(\sigma^2(2 - \sigma/|\xi|)(2 + \sigma) + 2|\underline{E}|^2/|\xi| \right). \quad (15)$$

The implicit function theorem shows that there are two non-real roots

$$\tau = \xi \pm i \frac{|\underline{E}|}{\sqrt{2}} |\xi|^{\frac{1}{2}} + 0(1). \quad (16)$$

This means that waves at frequency (ζ, ξ) with $\zeta = -|\xi| - |\xi|^2$ are amplified by the exponential factor

$$e^{\gamma t |\xi|^{\frac{1}{2}}}, \quad \gamma = \frac{|\underline{E}|}{\sqrt{2}} > 0. \quad (17)$$

This implies that the Cauchy problem for the linearized equations (9) is ill-posed in H^∞ : there are Cauchy data in H^∞ such that the homogeneous problem with $f = g = 0$ has no solution in $C^0([0, T]; H^{-\infty})$.

The goal of this paper is to translate this *spectral instability* into a non-linear instability result for the Zakharov system (3).

Remark 2.1 How is it that this spectral instability does not intervene in the analysis of [LPS]? The first answer is that the condition $\underline{E} \neq 0$ is crucial for γ to be positive. In their case, where solutions vanish at infinity, linearizing the equation around non-vanishing constants has no real significance. However, the symbolic calculus above also makes sense in the case of variable coefficients and one expects that the dispersion relation $P = 0$, with $\underline{E}$ replaced by $E(t, z, x)$, which still has non-real roots, should play an important role in the analysis. For instance, the symbolic analysis appears when one replaces the plane wave analysis used for constant coefficients, by *geometric optics* expansions associated to localized wave packets. In this case, for a wave packet with mean frequency $(-|\xi| - |\xi|^2, \xi)$ an exponential amplification similar to (17) is expected. But the group velocity in x of this packet is of order 2ξ ; therefore if E is confined (think of it as compactly supported) the time of amplification is short (typically $O(|\xi|^{-1})$) so that the overall effect of the amplification is bounded. Of course, this is just a very rough explanation, but it is rather intuitive. The detailed balance between amplification and localization is indeed given by the dispersive estimates proved in [LPS].

Remark 2.2 The system can be reduced to first order in t , introducing $(\partial_x e, \partial_t n, \partial_x n)$ as unknowns, but it is not first order in x , because of the Schrödinger part of the system. However, there is a good analogy with the analysis of weakly hyperbolic system. Indeed, the analysis of the symbol (11) shows that when for $\zeta = -|\xi|^2 - |\xi|$, there is a double eigenvalue with a 2×2 Jordan block. The existence of non-real eigenvalues (16), simply means that the natural analogue of the Levi condition for first order system is not satisfied. Pursuing the analogy, the exponential growth (51) indicates that the Cauchy problem should be well posed in Gevrey classes G^s for $s \leq 2$.

3 Scheme of the proof

It is certainly sufficient to prove the theorem with functions of $x = (x_1, x_2)$ independent of x_2 . To simplify notations, we assume from now on that x is one real variable. Consider spatially periodic solutions of (3), with period 2π in x and $2\pi Z$ in z . Moreover, we look for solutions n and E of the form

$$\begin{aligned} n &= n(kx - mz, t) \\ E &= \underline{E} + e(kx - mz, t) \end{aligned} \tag{18}$$

with new functions $n(\theta, t)$ and $e(\theta, t)$ 2π periodic in θ . For the functions to be 2π periodic in x and $2\pi/Z$ periodic in z , it is sufficient that

$$k \in \mathbb{N}, \quad mZ \in \mathbb{N}. \quad (19)$$

To be close to the unstable frequencies, we require that $|m - k - k^2| \ll \sqrt{k}$ and therefore we choose $m \in \mathbb{N}/Z$ such that

$$(k^2 + k) - 1/Z < m \leq (k^2 + k). \quad (20)$$

The new equations read

$$\begin{cases} i(\partial_t - m\partial_\theta)e + k^2\partial_\theta^2 e - \underline{E}n = ne, \\ (\partial_t^2 - k^2\partial_\theta^2)n - k^2\partial_\theta^2(\underline{E}e + \underline{E}\bar{e}) = k^2\partial_\theta^2|e|^2. \end{cases} \quad (21)$$

written in short

$$L_k(\partial_t, \partial_\theta)U = N_k(U) \quad (22)$$

where $U = (e, n)$, L_k is the linear operator defined in the left hand side of (21), and $N_k(u)$ the quadratic term in the right hand side.

The first step concerns the homogeneous equation

$$L_k U = 0, \quad (23)$$

which is studied using Fourier series expansions in θ . The choice (20) together with the spectral analysis of Section 2 and the choice (20) imply that for k large, the harmonic 1 is unstable :

Proposition 3.1 *There is k_0 such that for $k \geq k_0$, there are solutions $U^a = (e^a, n^a)$ of (23) such that*

$$\begin{cases} e^a &= \hat{e}_1^a(t)e^{i\theta} + \hat{e}_{-1}^a(t)e^{-i\theta} \\ n^a &= \sinh(t\sigma) \cos(t\operatorname{Re} \lambda + \theta) \end{cases} \quad (24)$$

with

$$\hat{e}_{\pm 1}^a(t) = (e_{\pm 1, +}^a e^{t\gamma} + e_{\pm 1, -}^a e^{-t\gamma})e^{it\lambda}, \quad (25)$$

where the parameters $\lambda, \sigma, e_{\pm 1, \pm}^a$ depend on k , λ and σ being real positive and satisfy as $k \rightarrow +\infty$:

$$e_{+1, +}^a \sim -i\underline{E}/4\sigma, \quad e_{+1, -}^a \sim -i\underline{E}/4\sigma, \quad e_{-1, \pm}^a = O(k^{-2}). \quad (26)$$

$$\lambda \sim k, \quad \sigma \sim |\underline{E}|\sqrt{k/2}. \quad (27)$$

The proof is given in Section 4.

Next, we consider δU^a as a first approximation of the solution of (22) to construct, with δ a small parameter to be chosen. More precisely look for solutions of (22) as

$$U = \delta(U^a + u), \quad u = (e, n) \quad (28)$$

with the same initial data as δU^a . Because the nonlinearity is exactly quadratic, the equation for u reads

$$L_k(\partial_t, \partial_\theta)u = \delta N_k(U^a + u), \quad e|_{t=0} = n|_{t=0} = \partial_t n|_{t=0} = 0. \quad (29)$$

This equation is solved by Picard's iteration; therefore the main step is to solve the linear equation

$$L_k U = F, \quad e|_{t=0} = n|_{t=0} = \partial_t n|_{t=0} = 0. \quad (30)$$

in Banach spaces which are also well adapted to the nonlinearity. The choice of these spaces, more precisely of their norm, is technical and dictated by the computations detailed in the next sections. We just give here their definition.

For a periodic function v of θ , we denote by $\hat{v}_p$ its Fourier coefficients so that

$$v = \sum_{p \in \mathbb{Z}} \hat{v}_p e^{ip\theta}. \quad (31)$$

The first Fourier coefficient $\hat{e}_1$ plays a special role and we use the notations

$$e(t, \theta) = \hat{e}_1(t) e^{i\theta} + e'(t, \theta). \quad (32)$$

For $s \geq 1$ and $T > 0$, we denote by $\mathbb{E}^1(T)$ the space of $u = (e, n)$ with n real valued, such that

$$e \in C^0([0, T]; H^{s+2}) \cap C^1([0, T]; H^s), \quad n \in C^1([0, T]; H^s) \quad (33)$$

equipped with the norm

$$\begin{aligned} \|u\|_{\mathbb{E}^1(T)} = \sup_{t \in [0, T]} e^{-\sigma t} \Big\{ & k^{\frac{1}{2}} |\hat{e}_1(t)| + k^{-\frac{1}{2}} |\partial_t \hat{e}_1(t)| \\ & + k^{\frac{3}{4}} \|e'(t)\|_{H^{s+2}} + k^{-\frac{1}{2}} \|\partial_t e'(t)\|_{H^s} \\ & + \|n(t)\|_{H^s} + k^{-1} \|\partial_t n(t)\|_{H^s} \Big\} \end{aligned} \quad (34)$$

where σ is defined at Proposition 3.1. The norm depends on $k \geq 1$ and s , but, to lighten the text, we do not mention this dependence explicitly in the notations.

We denote by $\mathbb{E}^2(T)$ the same space (33), equipped with the norm

$$\begin{aligned} \|u\|_{\mathbb{E}^2(T)} = \sup_{t \in [0, T]} e^{-2\sigma t} \Big\{ & k|\hat{e}_1(t)| + |\partial_t \hat{e}_1(t)| \\ & + k\|e'(t)\|_{H^{s+2}} + k^{-\frac{1}{4}}\|\partial_t e'(t)\|_{H^s} \\ & + k^{\frac{1}{2}}\|n(t)\|_{H^s} + k^{-\frac{1}{2}}\|\partial_t n(t)\|_{H^s} \Big\}. \end{aligned} \quad (35)$$

There are two differences between (34) and (35) : first the weight $e^{-\sigma t}$ is replaced by $e^{-2\sigma t}$ and second all the powers of k in the coefficients are increased, at least by a factor $\frac{1}{4}$. In particular,

$$\|u\|_{\mathbb{E}^1(T)} \leq k^{-\frac{1}{4}} e^{\sigma T} \|u\|_{\mathbb{E}^2(T)} \quad (36)$$

For the right hand sides, we denote by $\mathbb{F}^2(T)$ the space of $F = (f, g)$ with g real valued such that

$$f \in C^1([0, T]; H^s), \quad g \in C^0([0, T]; H^s) \quad \text{with } \hat{g}_0 = 0, \quad (37)$$

equipped with the norm

$$\begin{aligned} \|F\|_{\mathbb{E}^1(T)} = \sup_{t \in [0, T]} e^{-2\sigma t} \Big\{ & k^{\frac{1}{2}}\|f(t)\|_{H^s} + k^{-\frac{1}{2}}\|\partial_t f(t)\|_{H^s} \\ & + k^{-\frac{3}{4}}\|g(t)\|_{H^s} \Big\} \end{aligned} \quad (38)$$

The next three results justify the choices of these norms. We assume that the parameter $s \geq 1$ is fixed.

The first estimate is an immediate consequence of Proposition 3.1 and (26) (27).

Lemma 3.2 *There is a constant K^a such that for all $k \geq k_0$ and all $T \leq 1$, the approximate solution U^a of Proposition 3.1 satisfies*

$$\|U^a\|_{\mathbb{E}^1(T)} \leq K^a. \quad (39)$$

The next two propositions are proved in Section 6.

Proposition 3.3 *There is $C_1 > 0$, such that for all $k \geq k_0$, all $T \leq 1$ and all $F \in \mathbb{F}^2(T)$, the Cauchy problem (30) has a unique solution $U \in \mathbb{E}^2(T)$ and*

$$\|U\|_{\mathbb{E}^2(T)} \leq C_1 \|F\|_{\mathbb{F}^2(T)}. \quad (40)$$

The nonlinearity $N_k(U)$ occurring in (22) is quadratic. Denote by $\mathcal{N}_k(U, V)$ the associated bilinear form such that $N_k(U) = \mathcal{N}_k(U, U)$.

Proposition 3.4 *There is $C_2 > 0$, such that for all $k \geq k_0$, all $T \leq 1$ and all U and V in $\mathbb{E}^1(T)$, there holds $\mathcal{N}_k(U, V) \in \mathbb{F}^2(T)$ and*

$$\|\mathcal{N}_k(U, V)\|_{\mathbb{F}^2(T)} \leq C_2 \|U\|_{\mathbb{E}^1(T)} \|V\|_{\mathbb{E}^1(T)}. \quad (41)$$

These estimates easily imply the following:

Corollary 3.5 *There are $c_0 > 0$, C and k_0 , such that for all $k \geq k_0$ and all $\delta \in]0, 1]$, the problem (29) has a unique solution $u = (e, n)$ in the unit ball of $\mathbb{E}^1(T)$, provided that*

$$\delta k^{-\frac{1}{4}} e^{\sigma T} \leq c_0. \quad (42)$$

Moreover, the solution satisfies

$$\|n(t)\|_{H^s} \leq C k^{-\frac{1}{4}} e^{\sigma t} \quad (43)$$

Denote by $L_k^{-1}F$ the solution of (30), and consider the mapping

$$u \mapsto \mathcal{T}u := \delta L_k^{-1} N_k(u^a + u)$$

which, by the lemma and propositions above, is well defined from $\mathbb{E}^1(T)$ to $\mathbb{E}^1(T)$. Moreover,

$$\|\mathcal{T}u\|_{\mathbb{E}^1(T)} \leq C_1 C_2 \delta k^{-\frac{1}{4}} e^{\sigma T} (K^a + \|u\|_{\mathbb{E}(T)})^2.$$

Thus it maps the unit ball to of $\mathbb{E}^1(T)$ to itself, if (42) holds with c_0 small enough. Similarly, decreasing c_0 if necessary, one shows that this mapping is contractive on the unit ball, implying the existence and uniqueness of the solution of $u = \mathcal{T}u$ in the unit ball.

The equation $u = \mathcal{T}u$ and the estimates also imply that

$$\begin{aligned} \|n(t)\|_{H^s} &\leq k^{-\frac{1}{2}} e^{2\sigma t} \|u\|_{\mathbb{E}^2(T)} \leq C_1 C_2 \delta k^{-\frac{1}{2}} e^{2\sigma t} (K^a + 1)^2 \\ &\leq C_1 C_2 c_0 k^{-\frac{1}{4}} e^{\sigma t} (K^a + 1)^2 \end{aligned}$$

finishing the proof of the Corollary.

We end this section by proving that the main Theorem 1.1 is a consequence of this analysis.

Proof of Theorem 1.1

We fix an integer s . With

$$\delta = k^{-(2s+2)}, \quad (44)$$

Corollary 3.5 provides us with solutions of (22), $U_k = \underline{U} + \delta(U^a + u_k)$, with u_k in the unit ball of $\mathbb{E}^1(T_k)$ and $T_k = \frac{1}{\sigma} \ln(k^{2s+2+\frac{1}{4}}/c_0)$ satisfies

$$\delta k^{-1/4} e^{\sigma T_k} = c_0. \quad (45)$$

Since σ is of order $k^{\frac{1}{2}}$ by (88), T_k tends to 0 as k tends to infinity, at the rate

$$T_k \approx \frac{\ln k}{\sqrt{k}}. \quad (46)$$

Going back to the (z, x) variables, according to the change of variables (18), we obtain solutions, denoted by $\tilde{U}_k = \underline{U} + \tilde{u}_k$, of the original Zakharov system (3). Set $\tilde{u}_k = (\tilde{e}_k, \tilde{n}_k)$; these functions are deduced from $\delta(U^a + u_k)$ by the change of variables (18). Since $m \leq k^2 + k$, we can evaluate the H^s norm (in the variables (z, x)) of the Cauchy data

$$\begin{aligned} \|(\tilde{e}_k|_{t=0}, \tilde{n}_k|_{t=0}, \partial_t \tilde{n}_k|_{t=0})\|_{H^s(\mathbb{T})} &\leq C \delta k^{2s+1} \|U^a + u_k\|_{\mathbb{E}^1(T)} \\ &\leq C \delta k^{2s+1} (K^a + 1). \end{aligned}$$

Note that there is no Jacobian factor because the L^2 norms are taken for $(z, x) \in \mathbb{T}$ in the left hand side and for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ in the right hand side so that

$$\int_{\mathbb{T}} v(kx - mz) dz dx = \frac{\text{meas}\mathbb{T}}{2\pi} \int_0^{2\pi} v(\theta) d\theta. \quad (47)$$

Therefore, with our choice of δ , the left hand side tends to zero as k tends to infinity.

Finally we compute the L^2 norm of $\tilde{n}_k$ at time T_k . Using (47) and (24) we see that

$$\|\tilde{n}_k(T_k)\|_{L^2(\mathbb{T})} \geq c_1 \delta \sinh(T_k \sigma) - \delta \|n_k(T_k)\|_{L^2}$$

with $c_1 > 0$ independent of k . Therefore, (43) (45) imply that

$$\begin{aligned} \|\tilde{n}_k(T_k)\|_{L^2(\mathbb{T})} &\geq \frac{1}{2} c_1 \delta e^{\sigma T_k} - C \delta k^{-\frac{1}{4}} e^{\sigma T_k} - O(\delta e^{-\sigma T_k}) \\ &\geq \frac{1}{2} c_1 c_0 k^{\frac{1}{4}} - C c_0 - o(1). \end{aligned}$$

Therefore this L^2 norm tends to $+\infty$ and the proof of the theorem is complete.

4 The linear instability

We study the linear equation for $U = (e, n)$ and $F = (f, g)$

$$L_k U = F \quad (48)$$

using Fourier series expansions in θ :

$$e(\theta, t) = \sum \hat{e}_p(t) e^{ip\theta}, \quad n(\theta, t) = \sum \hat{n}_p(t) e^{ip\theta}. \quad (49)$$

Since n and g are real,

$$\hat{n}_{-p} = \overline{\hat{n}_p}, \quad \hat{g}_{-p} = \overline{\hat{g}_p}, \quad (50)$$

and (48) reduces to

$$\tilde{L}_k(\partial_t, 0)U_0 := (\partial_t)_t \hat{e}_0 - E_0 \hat{n}_0 \partial_t^2 \hat{n}_0 = F_0 := (\hat{f}_0) \hat{g}_0 \quad (51)$$

and for $p \geq 1$

$$\begin{cases} (i\partial_t + mp - k^2 p^2) \hat{e}_p - \underline{E} \hat{n}_p = \hat{f}_p, \\ (i\partial_t + mp + k^2 p^2) \tilde{e}_p + \overline{\underline{E}} \hat{n}_p = \tilde{f}_p, \\ (\partial_t^2 + k^2 p^2) \hat{n}_p + k^2 p^2 (\overline{\underline{E}} \hat{e}_p + \underline{E} \tilde{e}_p) = \hat{g}_p, \end{cases} \quad (52)$$

with

$$\tilde{e}_p = \overline{\tilde{e}_{-p}}, \quad \tilde{f}_p = -\overline{\tilde{f}_{-p}} \quad (53)$$

are the Fourier coefficients of $\tilde{e}$ and $-\tilde{f}$ respectively. For $p > 0$, we denote by $\tilde{L}_k(\partial_t, p)$ the linear operator in the left hand side of (52).

In the remaining part of this section we concentrate on the case $p = 1$ and prove Proposition 3.1. We reduce (52) for $p = 1$ to a first order system by introducing $v_1 = -ik^{-1} \partial_t \hat{n}_1$. The equation reads

$$i\partial_t V_1 + AV_1 = F_1, \quad (54)$$

with

$$V_1 = (\hat{e}_1, \tilde{e}_1, \hat{n}_1, v_1), \quad F_1 = (\hat{f}_1, \tilde{f}_1, 0, k^{-1} \hat{g}_1) \quad (55)$$

and

$$A = \begin{pmatrix} m - k^2 & 0 & -\underline{E} & 0 \\ 0 & m + k^2 & \overline{\underline{E}} & 0 \\ 0 & 0 & 0 & k \\ k\overline{\underline{E}} & k\underline{E} & k & 0 \end{pmatrix} \quad (56)$$

Lemma 4.1 *If $\underline{E} \neq 0$ and k is large enough, A has four distinct eigenvalues; two, called λ_1 and λ_2 are real and the other two, λ_3 and λ_4 , are non-real and complex conjugated. There holds*

$$\lambda_1 \sim 2k^2, \quad \lambda_2 \sim -k, \quad \operatorname{Re} \lambda_3 \sim k, \quad \sigma := \operatorname{Im} \lambda_3 \sim |\underline{E}| \sqrt{k/2}. \quad (57)$$

This follows from the analysis of the determinant equation in Section 2. The eigenvalue equation is

$$P = (\lambda^2 - k^2)((\lambda - m)^2 - k^4) - 2|\underline{E}|^2 k^4 = 0 \quad (58)$$

Following (20), we write $m = k^2 + k + m'$, and the equation reads

$$(\lambda^2 - k^2)(\lambda - k + m')(\lambda - 2k^2 - k + m') = 2|\underline{E}|^2 k^4$$

Because $m' = O(1)$, the lemma easily follows by perturbation analysis of the roots of

$$(\lambda^2 - k^2)(\lambda - k + m')(\lambda - 2k^2 - k + m') = 0.$$

Next, to evaluate e^{itA} , we need to analyze the eigenprojectors of A . Denote by r_j [resp. l_j] right [resp. left] eigenvectors of A associated to the eigenvalue λ_j . Then

$$e^{itA} \Phi = \sum_{j=1}^4 e^{it\lambda_j} \frac{(l_j \cdot \Phi)}{(l_j \cdot r_j)} r_j. \quad (59)$$

A detailed inspection of the eigenvector equations implies the following

$$r_1 = \begin{bmatrix} O(k^{-4}) \\ 1 \\ O(k^{-2}) \\ O(k^{-1}) \end{bmatrix}, \quad l_1 = [O(k^{-4}), \quad 1, \quad O(k^{-2}), \quad O(k^{-3})], \quad (60)$$

$$r_2 \sim \begin{bmatrix} O(k^{-1}) \\ O(k^{-2}) \\ 1 \\ -1 \end{bmatrix}, \quad l_2 \sim [O(1), \quad O(k^{-1}), \quad 1, \quad -1], \quad (61)$$

where, for vectors a and b , $a \sim b$ means that all the components satisfy $a_k \sim b_k$. Moreover,

$$r_3 \sim \begin{bmatrix} i\underline{E}/\sigma \\ O(k^{-2}) \\ 1 \\ 1 \end{bmatrix}, \quad l_3 \sim [k\overline{\underline{E}}/i\sigma, \quad O(k^{-1}), \quad 1, \quad 1], \quad (62)$$

$$r_4 = \bar{r}_3, \quad l_4 = \bar{l}_3 \quad (63)$$

where $\sigma^2 = k|\underline{E}|^2/2 \approx k$. Note that $r_{3,4} = 0(1)$ and $r_3 - r_4 = O(|\underline{E}|/\sqrt{k})$ and $l_{3,4} = O(|E|\sqrt{k})$ while $r_{3,4} \cdot l_{3,4} \sim 4$. This reflects that for $\underline{E} = 0$, the corresponding matrix has a Jordan block.

Proof of Proposition 3.1.

With notations as above,

$$V_1^a = (\hat{e}_1^a) \tilde{e}_1^a \hat{n}_1^a v_1^a := \frac{1}{4}(e^{it\lambda_4} r_4 - e^{it\lambda_3} r_3) \quad (64)$$

is a solution of (54) with $F_1 = 0$. It corresponds to a solution $(\hat{e}_1^a, \tilde{e}_1^a, n_1^a)$ of $\tilde{L}_1 \tilde{U}_1^a = 0$ and therefore to a solution

$$e^a = \hat{e}_1^a e^{i\theta} + \overline{\tilde{e}_1^a} e^{-i\theta}, \quad n^a = \hat{n}_1^a e^{i\theta} + \overline{n_1^a} e^{-i\theta} \quad (65)$$

of $L_1 U^a = 0$.

Choosing, as we may, r_3 and r_4 such that the third component is exactly equal to one, we obtain that

$$n^a(t, \theta) = \sinh(t\sigma) \cos(t\operatorname{Re} \lambda_3 + \theta)$$

and the estimate (26) follows from the estimates of the eigenvectors above. Moreover, (27) follows from Lemma 4.1.

Next we turn to the analysis of (54). The solution with vanishing initial data is

$$V_1(t) = \sum_{j=1}^4 \int_0^t e^{i(t-s)\lambda_j} \frac{(l_j \cdot F_1(s))}{(l_j \cdot r_j)} r_j ds. \quad (66)$$

Introduce $\Phi_j = l_j \cdot F_1$. With f denoting $(\hat{f}_1, \tilde{f}_1)$ and $g = \hat{g}_1$ there holds

$$\begin{aligned} \Phi_1 &= *f + *k^{-4}g, \\ \Phi_2 &= *f + *k^{-1}g, \\ \Phi_{3,4} &= *\sqrt{k}f + *k^{-1}g. \end{aligned} \quad (67)$$

where $*$ denotes constants coefficients that are uniformly bounded in k . Let

$$\Psi_j(t) = \int_0^t e^{i\lambda_j(t-s)} \Phi_j(s) ds. \quad (68)$$

The properties of the r_j 's and (66) imply that the components $(\hat{e}_1, \tilde{e}_1, \hat{n}_1, v_1)$ of V_1 satisfy:

$$\begin{aligned}\hat{e}_1 &= *k^{-4}\Psi_1 + *k^{-1}\Psi_2 + *k^{-1/2}\Psi_{3,4}, \\ \tilde{e}_1 &= *\Psi_1 + *k^{-2}\Psi_2 + k^{-2}\Psi_{3,4}, \\ \hat{n}_1 &= *k^{-2}\Psi_1 + *\Psi_2 + *\Psi_{3,4}, \\ v_1 &= *k^{-1}\Psi_1 + *\Psi_2 + *\Psi_{3,4}.\end{aligned}\tag{69}$$

We use the following elementary estimates:

Lemma 4.2 *Let*

$$\psi(t) = \int_0^t e^{i\lambda(t-s)} \phi_j(s) ds.\tag{70}$$

There holds

$$\begin{aligned}|\psi(t)| &\leq \int_0^t e^{-\text{Im } \lambda(t-s)} |\phi(s)| ds, \\ |\partial_t \psi(t)| &\leq |\lambda_j| |\psi(t)| + |\phi(t)|, \\ |\partial_t \psi(t)| &\leq e^{-\text{Im } \lambda t} |\phi(0)| + \int_0^t e^{-\text{Im } \lambda(t-s)} |\partial_t \phi(s)| ds, \\ |\lambda| |\psi(t)| &\leq |\partial_t \psi(t)| + |\phi(t)|.\end{aligned}\tag{71}$$

To simplify notations, we note $A \lesssim B$ to mean that there is a constant C independent of k such that $A \leq CB$. We use the first and second estimate of Lemma 4.2 to bound the contributions of g to the integrals in (66), and we use the third and fourth estimate, when necessary, to bound the contributions of f . Therefore,

$$\begin{aligned}|\Psi_1(t)| &\lesssim \int_0^t |f(s), k^{-4}g(s)| ds, \\ |\partial_t \Psi_1(t)| &\lesssim |f(0)| + |k^{-4}g(t)| + \int_0^t |\partial_t f(s), k^{-2}g(s)| ds, \\ k^2 |\Psi_1(t)| &\lesssim |f(0)| + |f(t)| + \int_0^t |\partial_t f(s), k^{-2}g(s)| ds\end{aligned}\tag{72}$$

$$\begin{aligned}|\Psi_2(t)| &\lesssim \int_0^t |f(s), k^{-1}g(s)| ds, \\ |\partial_t \Psi_2(t)| &\lesssim |f(0)| + |k^{-1}g(t)| + \int_0^t |\partial_t f(s), g(s)| ds, \\ k |\Psi_2(t)| &\lesssim |f(0)| + |f(t)| + \int_0^t |\partial_t f(s), g(s)| ds,\end{aligned}\tag{73}$$

$$\begin{aligned}|\Psi_{3,4}(t)| &\lesssim \int_0^t e^{(t-s)\sigma} |\sqrt{k}f(s), k^{-1}g(s)| ds, \\ |\partial_t \Psi_{3,4}(t)| &\lesssim e^{t\sigma} |\sqrt{k}f(0)| + |k^{-1}g(t)| + \int_0^t e^{(t-s)\sigma} |\sqrt{k}\partial_t f(s), g(s)| ds, \\ k |\Psi_{3,4}(t)| &\lesssim e^{t\sigma} |\sqrt{k}f(0)| + |\sqrt{k}f(t)| + \int_0^t e^{(t-s)\sigma} |\sqrt{k}\partial_t f(s), g(s)| ds.\end{aligned}\tag{74}$$

Adding up the various estimates, we obtain:

Proposition 4.3 *For $p = 1$, the solution $(\hat{e}_1, \tilde{e}_1, \hat{n}_1)$ of (52) with vanishing initial data satisfies:*

$$\begin{aligned} |\hat{e}_1(t)| &\lesssim \int_0^t e^{\sigma(t-s)} |f_1(s), k^{-\frac{3}{2}} \hat{g}_1(s)| ds, \\ |\partial_t \hat{e}_1(t)| &\lesssim e^{\sigma t} |f_1(0)| + |k^{-\frac{3}{2}} \hat{g}_1(t)| \\ &\quad + \int_0^t e^{\sigma(t-s)} |\partial_t f_1(s), k^{-\frac{1}{2}} \hat{g}_1(s)| ds, \end{aligned} \quad (75)$$

$$\begin{aligned} k^2 |\tilde{e}_1(t)| + |\partial_t \tilde{e}_1(t)| &\lesssim e^{\sigma t} |f_1(0)| + |f_1(t)| + |k^{-3} \hat{g}_1(t)| \\ &\quad + \int_0^t e^{\sigma(t-s)} |\partial_t f_1(s), k^{-1} \hat{g}_1(s)| ds, \end{aligned} \quad (76)$$

$$\begin{aligned} k |\hat{n}_1(t)| + |\partial_t \hat{n}_1(t)| &\lesssim e^{\sigma t} k^{\frac{1}{2}} |f_1(0)| + |k^{\frac{1}{2}} f_1(t)| + |k^{-1} \hat{g}_1(t)| \\ &\quad + \int_0^t e^{\sigma(t-s)} |f_1(s), k^{\frac{1}{2}} \partial_t f_1(s), \hat{g}_1(s)| ds, \end{aligned} \quad (77)$$

where $f_1 = (\hat{f}_1, \tilde{f}_1)$.

Corollary 4.4 *There are k_0 and C such that for all $k \geq k_0$, $K, T > 0$, and all $f_1 = (\hat{f}_1, \tilde{f}_1)$, g_1 satisfying for $t \in [0, T]$*

$$k^{\frac{1}{2}} |f_1(t)| + k^{-\frac{1}{2}} |\partial_t f_1(t)| + k^{-\frac{3}{4}} |\hat{g}_1(t)| \leq K e^{2\sigma t},$$

then the solution of (52) for $p = 1$ with vanishing initial data satisfies

$$\begin{aligned} k |\hat{e}_1(t)| + |\partial_t \hat{e}_1(t)| &\leq C K e^{2\sigma t}, \\ k |\tilde{e}_{-1}(t)| + k^{-\frac{1}{4}} |\partial_t \tilde{e}_{-1}(t)| &\leq C K e^{2\sigma t}, \\ k^{\frac{1}{2}} |\hat{n}_1(t)| + k^{-\frac{1}{2}} |\partial_t \hat{n}_1(t)| &\leq C K e^{2\sigma t}. \end{aligned}$$

a) From Proposition 4.3 we deduce that

$$k |\hat{e}_1(t)| \leq C K_1 \sqrt{k} \int_0^t e^{\sigma(t-t')} e^{2\sigma t'} dt' \leq C K e^{2\sigma t}, \quad (78)$$

where we have used that $\sigma \approx \sqrt{k}$. Similarly,

$$|\partial_t \hat{e}_1(t)| \leq C K_1 \left(k^{-1/2} e^{\sigma t} + k^{-3/4} e^{2\sigma t} + \int_0^t \sqrt{k} e^{\sigma(t-t')} e^{2\sigma t'} dt' \right) \leq C K e^{2\sigma t}. \quad (79)$$

This implies the first estimate.

b) Similarly, (76) implies that

$$\begin{aligned} k^2|\hat{e}_{-1}(t)| + |\partial_t \hat{e}_{-1}(t)| &\leq CK_1 \left(e^{\sigma t} + e^{2\sigma t} + \int_0^t \sqrt{k} e^{\sigma(t-t')} e^{2\sigma t'} dt' \right) \\ &\leq CK e^{2\sigma t}. \end{aligned} \quad (80)$$

c) The estimate (77) implies that

$$\begin{aligned} k|\hat{n}_1(t)| + |\partial_t \hat{n}_1(t)| &\leq CK_1 \left(e^{\sigma t} + e^{2\sigma t} + \int_0^t k e^{\sigma(t-t')} e^{2\sigma t'} dt' \right) \\ &\leq CK \sqrt{k} e^{2\sigma t} \end{aligned} \quad (81)$$

and the lemma is proved.

5 The linear equation

We continue the analysis of the linear equation (48). As seen in (52), when expanded in Fourier series, this equation couples the coefficients of indices p and $-p$. The case of indices $+1$ and -1 is studied in the previous section. Using the notations

$$v = \hat{v}_1 e^{i\theta} + \hat{v}_{-1} e^{-i\theta} + v'' \quad (82)$$

we consider the equation (48) for functions with vanishing Fourier coefficients of indices ± 1 :

$$L_k'' U'' = F'', \quad (83)$$

which reduces to the analysis of equations (52) for Fourier $p \neq 1$.

The symbol of $\tilde{L}_k(\partial_t, p)$ is

$$\tilde{L}_k(i\tau, p) = \begin{pmatrix} -\tau + mp - k^2 p^2 & 0 & -E_0 \\ 0 & -\tau + mp + k^2 p^2 & \bar{E}_0 \\ k^2 p^2 \bar{E}_0 & k^2 p^2 E_0 & k^2 p^2 - \tau^2 \end{pmatrix} \quad (84)$$

which is of course equal to the symbol (11) with $\xi = kp$, $\zeta = -mp$, up to a change of sign in the first line.

Assume first that $p > 1$. In this case, we consider $\tilde{L}_k(\partial_t, p)$ as a perturbation of

$$M_k(\partial_t, p) := \begin{pmatrix} i\partial_t + mp - k^2 p^2 & 0 & 0 \\ 0 & i\partial_t + mp + k^2 p^2 & 0 \\ 0 & 0 & \partial_t^2 + k^2 p^2 \end{pmatrix} \quad (85)$$

For the wave operator, we use the classical estimates:

Lemma 5.1 *There is $C > 0$, such that for all $k \geq 1$ and $p \geq 1$, the solution n of*

$$\partial_t^2 n + k^2 p^2 n = g, \quad n(0) = \partial_t n(0) = 0 \quad (86)$$

satisfies

$$kp|n(t)| + |\partial_t n(t)| \leq C\|g\|_{L^1([0,t])}. \quad (87)$$

For the Schrödinger equations, we use the following estimates.

Lemma 5.2 *There are $C > 0$ and $k_0 \geq 1$, such that for all $k \geq k_0$ and $p \geq 2$, the solutions of*

$$(i\partial_t + mp \pm k^2 p^2)e = f, \quad e(0) \quad (88)$$

satisfy

$$k^2 p^2 |e(t)| + |\partial_t e(t)| \leq C(\|f\|_{L^1([0,t])} + \|\partial_t f\|_{L^1([0,t])} + |f(0)|) \quad (89)$$

Standard energy estimates imply that

$$|e(t)| \leq C(|e(0)| + \|f\|_{L^1([0,t])}). \quad (90)$$

Differentiating in time the equation, we obtain

$$|\partial_t e(t)| \leq C(|\partial_t e(0)| + \|\partial_t f\|_{L^1([0,t])}). \quad (91)$$

The initial condition in (88) implies that $\partial_t e(0) = -if(0)$. Therefore,

$$|(k^2 p^2 \pm mp)e(t)| + |\partial_t e(t)| \leq C(\|f\|_{L^1([0,t])} + \|\partial_t f\|_{L^1([0,t])} + |f(0)| + |f(t)|) \quad (92)$$

Recall that m is linked to k through (20). Thus $mp \leq k^2 p + kp$ and $k^2 p^2 - mp \geq k^2(p^2 - p) - kp \geq ck^2 p^2$ for all $p \geq 2$ if k is large enough.

Proposition 5.3 *Consider the equation (52) with initial data*

$$\hat{e}_p(0) = \tilde{e}_p(0) = \hat{n}_p(0) = \partial_t \hat{n}_p(0) = 0 \quad (93)$$

Then, for $p \geq 2$, $k \geq k_0$, there holds for $t \in [0, 1]$:

$$\begin{aligned} k^2 p^2 |\hat{e}_p(t), \tilde{e}_p(t)| + |\partial_t \hat{e}_p(t), \partial_t \tilde{e}_p(t)| + kp|\hat{n}_p(t)| + |\partial_t \hat{n}_p(t)| \\ \leq C(\|\hat{f}_p, \tilde{f}_p\|_{L^1([0,t])} + \|\partial_t \hat{f}_p, \partial_t \tilde{f}_p\|_{L^1([0,t])} \\ + |\hat{f}_p(0), \tilde{f}_p(0)| + |\hat{f}_p(t), \tilde{f}_p(t)| + \|\hat{g}_p\|_{L^1([0,t])}). \end{aligned} \quad (94)$$

The lemmas above imply that the left hand side is estimated by the right hand side plus

$$C\left(|\hat{n}_p(t)| + \|\hat{n}_p, \partial_t \hat{n}_p, k^2 p^2 \hat{e}_p(t), k^2 p^2 \tilde{e}_p\|_{L^1([0,t])}\right) \quad (95)$$

The first term is absorbed in the left hand side by $kp|\hat{n}_p(t)|$ for k large enough. With Gronwall's lemma, this implies (94) for $t \in [0, 1]$, with a larger constant C .

When $p = 0$, there holds:

Lemma 5.4 *When $\hat{g}_0 = 0$, the solution of (51) with vanishing initial data is*

$$\hat{n}_0 = 0, \quad \hat{e}_0(t) = \int_0^t \hat{f}_0(t') dt'. \quad (96)$$

With the estimates (94), one deduces the following result

Corollary 5.5 *There are k_0 and C such that for all $k \geq k_0$, $K, T > 0$, and all (f'', g'') with $\hat{g}_0 = 0$, satisfying for $t \in [0, T]$*

$$\begin{aligned} k^{\frac{1}{2}} \|f''(t)\|_{H^s} + k^{-\frac{1}{2}} \|\partial_t f''(t)\|_{H^s} &\leq K e^{2\sigma t}, \\ \|g''(t)\|_{H^s} &\leq K k^{3/4} e^{2\sigma t}. \end{aligned}$$

the solution of (83) with vanishing initial data satisfies

$$\begin{aligned} k \|e''(t)\|_{H^{s+2}} + k^{-\frac{1}{4}} \|\partial_t e''(t)\|_{H^s} &\leq CK e^{2\sigma t}, \\ k^{\frac{1}{2}} \|n''(t)\|_{H^s} + k^{-\frac{1}{2}} \|\partial_t n''(t)\|_{H^s} &\leq CK e^{2\sigma t}. \end{aligned}$$

By Lemma 5.4, there holds

$$k|\hat{e}_0(t)| + |\partial_t \hat{e}_0(t)| \leq CK_1 \left(e^{2\sigma t} + \int_0^t \sqrt{k} e^{\sigma(t-t')} e^{2\sigma t'} dt' \right) \leq CK e^{2\sigma t}. \quad (97)$$

Next, Proposition 5.3 implies that e'' satisfies

$$\begin{aligned} k^2 \|\partial_\theta^2 e''(t)\|_{H^s} + \|\partial_t e''(t)\|_{H^s} \\ \leq CK \left((1 + e^{2\sigma t}) + \int_0^t k^{3/4} e^{\sigma(t-t')} e^{2\sigma t'} dt' \right) \leq k^{1/4} CK e^{2\sigma t}. \end{aligned} \quad (98)$$

Together with (97) this implies the first estimate.

Moreover, Proposition 5.3 implies that n'' satisfies

$$\begin{aligned} k \|n''(t)\|_{H^s} + \|\partial_t n''(t)\|_{H^s} \\ \leq CK \left((1 + e^{2\sigma t}) + \int_0^t k^{3/4} e^{\sigma(t-t')} e^{2\sigma t'} dt' \right) \leq k^{1/4} CK e^{2\sigma t}. \end{aligned} \quad (99)$$

Since $\hat{n}_0 = 0$, this implies the second estimate.

6 End of proofs

First, we note that Proposition 3.3 is an immediate consequence of Corollaries 4.4 and 5.5.

It remains to prove Proposition 3.4. With $U = (e, n)$ and $U^* = (e^*, n^*)$, there holds

$$\mathcal{N}_k(U, U^*) = (f, g) \text{ with} \quad (100)$$

$$f = ne^* + n^*e, \quad (101)$$

$$g = k^2 \partial_\theta^2 \{\text{Re}(\bar{e}e^*)\}. \quad (102)$$

Proposition 3.4 follows from the next estimates.

Lemma 6.1 *There is a constant C , independent of k , such that*

$$\sqrt{k} \|f(t)\|_{H^s} + \frac{1}{\sqrt{k}} \|\partial_t f(t)\|_{H^s} \leq C e^{2\sigma t} \|U\|_{\mathbb{E}^1(T)} \|U^*\|_{\mathbb{E}^1(T)}, \quad (103)$$

$$\|g(t)\|_{H^s} \leq C k^{3/4} e^{2\sigma t} \|U\|_{\mathbb{E}^1(T)} \|U^*\|_{\mathbb{E}^1(T)}. \quad (104)$$

Moreover, the mean value $\hat{g}_0$ of g vanishes.

The first estimate follows directly from the definitions and the inequality

$$\|ab\|_{H^s} \leq C \|a\|_{H^s} \|b\|_{H^s}. \quad (105)$$

Next, we note that for $e = \hat{e}_1 e^{i\theta} + e'$ and $e^* = \hat{e}_1^* e^{i\theta} + e^{*'}$

$$\partial_\theta^2(\bar{e}e^*) = \partial_\theta^2(\bar{e}'e^{*'}) + \bar{\hat{e}}_1 \partial_\theta^2(e^{*'}e^{-i\theta}) + \hat{e}_1^* \partial_\theta^2(\bar{e}'e^{i\theta}). \quad (106)$$

Hence, in H^s norms, there holds

$$\begin{aligned} \|\partial_\theta^2(\bar{e}e^*)\|_{H^s} &\lesssim \|\partial_\theta^2 e'\|_{H^s} (\|e^{*'}\| + \|\partial_\theta e^{*'}\|^2) + \|\partial_\theta^2 e^{*'}\|_{H^s} (\|e'\| + \|\partial_\theta e'\|^2) \\ &+ |\hat{e}_1| (\|\partial_\theta^2 e^{*'}\| + \|e^{*'}\|) + |\hat{e}_1^*| (\|\partial_\theta^2 e'\| + \|e'\|) \end{aligned} \quad (107)$$

and (104) follows.

In addition, the θ -mean value $\hat{g}_0$ vanishes since g is a θ -derivative.

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