On a quasilinear Zakharov system describing laser-plasma interactions

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Abstract : In this paper, starting from the bi-fluid Euler-Maxwell system, we derive a complete set of Zakharov's equations type describing laser-plasma interactions. This system involves a quasilinear part which is not hyperbolic and exhibits some elliptic zones. This difficulty is overcome by making a change of unknowns that are strongly related to the dispersive part. This change of variable is a symmetrization of the quasilinear part and is the key of this paper. This shows that the Cauchy problem is locally well-posed.

1 Introduction.

1.1 Physical context.

The construction of always more powerful lasers allows new experiments where hot plasma are created in which laser beams can propagate. The main goal is to simulate nuclear fusion by inertial confinement in a laboratory. Of course, one wants also to simulate numerically these experiences (it is much cheaper than true experiences !). We therefore need some precise and reliable models of laser-plasma interactions. The kinetic-type models are the more precise ones but the cost in term of computations is exorbitant and no physically relevant situation for nuclear fusion can be simulated in this context. Therefore, we use a bi-fluid model for the plasma : we couple two compressible Euler systems with Maxwell equations. Even under this form, it is not possible to perform direct computations because of high frequency motions and the small wavelength

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involved in the problem. At the beginning of the 70's, Zakharov and its collaborators introduced the so-called Zakharov's equation [20] in order to describe the electronic plasma waves. Basically, the slowly varying envelope of the electric field $E = \nabla\psi$ is coupled to the low-frequency variation of the density of the ions δn by the following equations written in a dimensionless form :

$$\begin{cases} i\partial_t \nabla\psi + \Delta(\nabla\psi) = \nabla\Delta^{-1}\mathrm{div}(\delta n \nabla\psi), \\ \\ \partial_t^2 \delta n - \Delta\delta n = \Delta(|\nabla\psi|^2). \end{cases} \tag{1.1}$$

Of course, variations of this systems exist (see [17] for example). For laser propagation, one uses the paraxial approximation and the Zakharov system reads

$$\begin{cases} i(\partial_t + \partial_y)E + \Delta_\perp E = nE, \\ \\ (\partial_t^2 - \Delta_\perp)n = \Delta_\perp(|E|^2), \end{cases} \tag{1.2}$$

where $\Delta_\perp = \partial_x^2 + \partial_z^2$. (See [14] or [17] for a symmetric use of this kind of model for numerical simulation). It is however not clear if system (1.2) is well-posed or not and no results are available at the moment.

Physically, the complete situation is the following one : the laser enters the plasma. Part of it is backscattered through a Raman-type process and a Brillouin-type process. The Raman and Brillouin parts and the laser combine together to create an electron-plasma wave. These four waves interact in order to create a low-frequency variation of density of the ions which has itself an influence on the four preceding waves. This situation can be described by two systems of the form (1.2) and one of the form (1.1) coupled by quasilinear terms. Such a system is used in [15] for numerical simulations.

Notations : As usual, we denote by $L^p(\mathbb{R}^d)$ the Lebesgue space

$$L^p(\mathbb{R}^d) = \left\{ u \in \mathcal{S}^{'}(\mathbb{R}^d) \ \ / \ \ ||u||_p < +\infty \right\}$$

where

$$||u||_p = \left(\int_{\mathbb{R}^d} |u(x)|^p dx \right)^{\frac{1}{p}} \ \text{ if } 1 \leq p < +\infty$$

and

$$||u||_\infty = \text{ess.sup} \left\{ |u(x)|; x \in \mathbb{R}^d \right\}.$$

We define the Sobolev space $H^s(\mathbb{R}^d)$ as follows

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \ \ / \ \ ||u||^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < +\infty \right\}$$

where $\widehat{u}(\xi)$ is the Fourier transform of u. Let $C(I, E)$ be the space of continuous functions from an interval I of $\mathbb{R}$ to a Banach space E. For $1 \leq j \leq d$, we set

$\partial_{x_j} u = \dfrac{\partial u}{\partial x_j}$. For $k \in \mathbb{N}^d$, $k = (k_1, ..., k_d)$, we denote $\partial^k u = \partial_{x_1}^{k_1} ... \partial_{x_d}^{k_d} u$. Different positive constants might be denoted by the same letter C. We also denote by $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ the real part and the imaginary part of u.

1.2 Presentation of the problem.

The system used in [15] is

$$
\begin{aligned}
&(\partial_t + v_0 \partial_y - \frac{ic^2}{2\omega_0}\Delta_\perp)A_0 = -\frac{2i\pi e^2}{m\omega_0} n A_0 + \frac{ie}{4m\omega_0}\left(\nabla \cdot E\right) A_R e^{-i\theta},\\
&(\partial_t + v_R \partial_y - \frac{ic^2}{2\omega_R}\Delta_\perp)A_R = -\frac{2i\pi e^2}{m\omega_R} n A_R + \frac{ie}{4m\omega_R}\left(\nabla \cdot E^*\right) A_0 e^{i\theta},\\
&(2i\omega_{pe}\partial_t + 3v_{th}^2 \Delta)E = \frac{4\pi e^2}{m}\nabla \Delta^{-1}\mathrm{div}(nE) + \frac{e}{2mc^2}\omega_{pe}^2 \nabla(A_0 A_R^* e^{i\theta}),\\
&(\partial_t^2 - c_s^2 \Delta)n = \frac{Z}{16\pi m_i}\Delta\left(\frac{\omega_{pe}^2}{c^2}(|A_0|^2 + |A_R|^2) + |E|^2\right),
\end{aligned}
$$

where $\theta = k_1 y - \omega_1 t$, $t \in \mathbb{R}$, $y \in \mathbb{R}$ and $\Delta_\perp = \partial_x^2 + \partial_z^2$. For simplicity, if A and B are two vectors of $\mathbb{R}^3$, the inner product $A \cdot B$ is denoted by AB. In this system, A_0 denotes the envelope of the incident laser field, A_R is the scattered light, E is the electronic-plasma wave and n the variation of density of ions. Furthermore, the vector fields A_0, A_R and E are such that

$$A_0, A_R, E \ : \ \mathbb{R}^3 \longrightarrow \mathbb{C}^3.$$

At this point, we do not give the orders of the different constants. We will use them carefully in the next section.

The authors observe some numerical instability explained by the fact that the paraxial approximation is done on A_0 and A_R and not on E. This appears through the presence of the operators

$$\partial_t + v_0 \partial_y - \frac{ic^2}{2\omega_0}\Delta_\perp \quad \text{and} \quad \partial_t + v_R \partial_y - \frac{ic^2}{2\omega_R}\Delta_\perp$$

on A_0 and A_R while the evolution operator on E is the complete dispersive one $(2i\omega_{pe}\partial_t + 3v_{th}^2\Delta)$. As mentioned before, no existence results are known for (1.2). It is probably ill-posed. See however [19] for variations around this kind of system. In order to obtain a system having better properties, we will derive a new set of equations starting from the bi-fluid Euler-Maxwell system. This will be done in section 2. Our system involves four Schrödinger equations coupled by quasilinear terms and a wave equation. It reads :

$$
\begin{aligned}
i\left(\partial_t + \frac{k_0 c^2}{\omega_0}\partial_y\right) A_0 + \frac{c^2}{2\omega_0}\Delta A_0 - \frac{k_0^2 c^4}{2\omega_0^3}\partial_y^2 A_0 = \frac{\omega_{pe}^2}{2\omega_0} < \delta n > (A_0 + e^{-2ik_0 y} A_B)\\
- \frac{e}{2m_e\omega_0}\left(\nabla \cdot E_0\right) A_R e^{-i(k_1 y - \omega_1 t)}, \qquad (1.3)
\end{aligned}
$$

$$i\left(\partial_t - \frac{k_0c^2}{\omega_0}\partial_y\right)A_B + \frac{c^2}{2\omega_0}\Delta A_B - \frac{k_0^2c^4}{2\omega_0^3}\partial_y^2 A_B = \frac{\omega_{pe}^2}{2\omega_0} < \delta n > (A_0 + e^{-2ik_0y}A_B)$$
$$- \frac{e}{2m_e\omega_0}\left(\nabla\cdot E_0\right)A_R e^{i((2k_0-k_1)y-\omega_1 t)}, \qquad (1.4)$$

$$i\left(\partial_t + \frac{k_Rc^2}{\omega_R}\partial_y\right)A_R + \frac{c^2}{2\omega_R}\Delta A_R - \frac{k_R^2c^4}{2\omega_R^3}\partial_y^2 A_R = \frac{\omega_{pe}^2}{2\omega_R} < \delta n > A_R$$
$$- \frac{e}{2m_e\omega_R}\left(\nabla\cdot E_0^*\right)(A_0 + e^{-2ik_0y}A_B)e^{i(k_1y-\omega_1 t)}, \qquad (1.5)$$

$$i\partial_t E_0 + \frac{v_{th}^2}{2\omega_{pe}}\Delta E_0 = \frac{\omega_{pe}}{2}\nabla\Delta^{-1}\mathrm{div}\,(< \delta n > E_0)$$
$$+ \frac{e\omega_{pe}}{2c^2m_e}\nabla\left(A_R^*(A_0 + e^{-2ik_0y}A_B)e^{i(k_1y-\omega_1 t)}\right), \qquad (1.6)$$

$$\left(\partial_t^2 - c_s^2\Delta\right) < \delta n >= \frac{1}{4\pi n_0 m_i}\Delta\left(|E_0|^2 + \frac{\omega_{pe}^2}{c^2}\left(|A_0 + e^{-2ik_0y}A_B|^2\right.\right.$$
$$\left.\left. + |A_R|^2\right)\right). \qquad (1.7)$$

Here A_0 is the incident laser field, A_B is the Brillouin component, A_R is the Raman field, E_0 the electronic-plasma field and $< \delta n >$ the low-frequency variation of the density of the ions. The coefficients are described in section 2. The complete electric field is recovered by

$$E = \frac{i\omega_0}{c}A_0(t,X)e^{i(k_0y-\omega_0t)} + \frac{i\omega_R}{c}A_R(t,X)e^{i(k_Ry-\omega_Rt)}$$
$$+ \frac{i\omega_0}{c}A_B(t,X)e^{-i(k_0y+\omega_0t)} + E_0(t,X)e^{-i\omega_{pe}t} + c.c.,$$

where (k_1,ω_1) are defined by

$$k_0 = k_R + k_1, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1. \qquad (1.8)$$

We are namely in a situation where we want to describe a three-waves interaction. It is easy to see that k_R has to be non positive if k_0 is non negative. Therefore, the velocity of A_R is non positive. System (1.3) $-$ (1.7) differs from that used in [15] by the presence of A_B and the complete dispersion in (1.3), (1.4), (1.5). This will play a fundamental role in the existence theory. However, A_B can be eliminated as shown in section 3.2 and the final system is using $A_C = A_0 + e^{-2ik_1y}A_B$:

$$i\left(\partial_t A_C + \frac{k_0c^2}{\omega_0}\partial_y A_C\right) + \frac{c^2}{2\omega_0}\Delta A_C - \frac{k_0^2c^4}{2\omega_0^3}\partial_y^2 A_C = \frac{\omega_{pe}^2}{2\omega_0} < \delta n > A_C$$
$$- \frac{e}{2m_e\omega_0}\left(\nabla\cdot E_0\right)A_R e^{-i(k_1y-\omega_1 t)}, \qquad (1.9)$$

$$i\left(\partial_t A_R + \frac{k_R c^2}{\omega_R}\partial_y A_R\right) + \frac{c^2}{2\omega_R}\Delta A_R - \frac{k_R^2 c^4}{2\omega_R^3}\partial_y^2 A_R = \frac{\omega_{pe}^2}{2\omega_R} < \delta n > A_R - \frac{e}{2m_e\omega_R}\left(\nabla\cdot E_0^*\right) A_C e^{i(k_1 y - \omega_1 t)}, \tag{1.10}$$

$$i\partial_t E_0 + \frac{v_{th}^2}{2\omega_{pe}}\Delta E_0 = \frac{\omega_{pe}}{2}\nabla\Delta^{-1}\mathrm{div}\left(< \delta n > E_0\right) + \frac{e\omega_{pe}}{2c^2 m_e}\nabla\left(A_R^* A_C e^{i(k_1 y - \omega_1 t)}\right), \tag{1.11}$$

$$\left(\partial_t^2 - c_s^2\Delta\right) < \delta n > = \frac{1}{4\pi n_0 m_i}\Delta\left(|E_0|^2 + \frac{\omega_{pe}^2}{c^2}\left(|A_C|^2 + |A_R|^2\right)\right). \tag{1.12}$$

1.3 Statement of the results.

The local in time Cauchy problem for the usual Zakharov equations (1.1) is now well understood in the context of regular solutions (see [1], [12], [16] [18] for local models, see [3] for the non-local case (1.1)). For weak solutions, one can see [7]. For finite-time blow-up see [8, 9]. Here, we are interested in proving local existence in time for strong solutions. We want to apply Ozawa-Tsutsumi's method (see [12]). The main problem is to deal with the quasilinear part of the system :

$$i\partial_t A_C = -\left(\nabla\cdot E_0\right) A_R, \tag{1.13}$$
$$i\partial_t A_R = -\left(\nabla\cdot E_0^*\right) A_C, \tag{1.14}$$
$$i\partial_t E_0 = \nabla(A_R^* A_C). \tag{1.15}$$

We omit here the exponentials which play no role regarding the local existence method. Even if any regular solution of $(1.13)-(1.15)$ satisfies the conservation law

$$\frac{d}{dt}\int(2|A_C|^2 + |A_R|^2 + |E_0|^2) = 0,$$

the system is **not** symmetric, neither hyperbolic ! It is therefore ill-posed (see section 3.3).
However, when one adds the dispersion part, one can show that

$$i\partial_t A_C + \alpha\Delta A_C = -\left(\nabla\cdot E_0\right) A_R, \tag{1.16}$$
$$i\partial_t A_R + \beta\Delta A_R = -\left(\nabla\cdot E_0^*\right) A_C, \tag{1.17}$$
$$i\partial_t E_0 + \gamma\Delta E_0 = \nabla(A_R^* A_C), \tag{1.18}$$

is locally well-posed. The dispersion seems thus to play a fundamental role (see section 4). The Cauchy problem for quasilinear Schrödinger equations has been studied by many authors [4], [10], [11]. We want here a proof that can be adapted to Zakharov-type models and that involves usual energy estimates. In fact, we show that the dispersion plays the same role than dissipation for non hyperbolic diphasic model (see [13]). The way we proceed is the following one : we add new unknowns to the system and we use the dispersive terms to symmetrize the quasilinear part. This is the key of this paper.

Our main result reads (see Theorem 5.1) :

Theorem. *Let* $(a_C, a_R, e) \in (H^{s+2}(\mathbb{R}^d))^{3d}$*,* $n_0 \in H^{s+1}(\mathbb{R}^d)$ *and* $n_1 \in H^s(\mathbb{R}^d)$ *with* $s > \frac{d}{2}+3$*. There exists* $T^* > 0$ *and a unique maximal solution* $(A_C, A_R, E, \delta n)$ *to* $(1.9) - (1.12)$ *such that*

$$(A_C, A_R, E) \in \mathcal{C}([0, T^*[; H^{s+2})^{3d}, \qquad \delta n \in \mathcal{C}([0, T^*[; H^{s+1}) \cap \mathcal{C}^1([0, T^*[; H^s)$$

satisfying

$$(A_C, A_R, E, \delta n)(0) = (a_C, a_R, e, n_0), \quad \partial_t \delta n(0) = n_1.$$

It is clear that the same theorem can be proved for system $(1.3) - (1.7)$

The paper is organized as follows. In section 2, we carefully derive our new model. In section 3, we discuss the algebraic structure of the system : we introduce a dimensionless form, we eliminate the Brillouin component and study the lack of hyperbolicity of the quasilinear part. In section 4, we solve the Cauchy problem for $(1.16) - (1.18)$ using the dispersive part. In section 5, we apply the same method to $(1.9) - (1.12)$. In section 6, we give some complements and open problems.

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2 The nonlinear coupled model.

In this part, we derive the nonlinear coupled model describing the interaction of a laser beam in a plasma with the Raman and Brillouin backscattered fields. In section 2.1, we introduce the starting point : the bi-fluid Euler-Maxwell system. We recall the polarization conditions for the transverse and longitudinal unknowns. In section 2.2, we obtain the amplitude equations for the different electric fields. In section 2.3, we couple these amplitude equations with the linearized hydrodynamical equations for the ions.

2.1 The bi-fluid Euler-Maxwell system.

The Euler-Maxwell system describes the evolution of the electromagnetic field in a plasma. The plasma is modelized by two fluids (one for the ions, one for the electrons). Therefore, we couple Maxwell system with two Euler systems. This coupling reads

$$\partial_t B + c\nabla \times E = 0, \tag{2.1}$$
$$\partial_t E - c\nabla \times B = 4\pi e\left((n_0+n_e)v_e - (n_0+n_i)v_i\right), \tag{2.2}$$
$$(n_0+n_e)\left(\partial_t v_e + v_e\cdot\nabla v_e\right) = -\frac{\gamma_e T_e}{m_e}\nabla n_e - \frac{e(n_0+n_e)}{m_e}(E+\frac{1}{c}v_e\times B), \tag{2.3}$$

$$(n_0+n_i)\left(\partial_t v_i + v_i\cdot\nabla v_i\right) = -\frac{\gamma_i T_i}{m_i}\nabla n_i + \frac{e(n_0+n_i)}{m_i}(E+\frac{1}{c}v_i\times B), \tag{2.4}$$
$$\partial_t n_e + \nabla\cdot((n_0+n_e)v_e) = 0, \tag{2.5}$$
$$\partial_t n_i + \nabla\cdot((n_0+n_i)v_i) = 0. \tag{2.6}$$

The unknowns are :
- E and B are respectively the electric and magnetic field.
- v_e and v_i denote respectively the electron's and ion's velocity.
- n_0 is the mean density of the plasma.
- n_e and n_i are the variation of density respectively of electrons and ions with respect to the mean density n_0.

The constants are :
- c is the velocity of light in the vacuum; e is the elementary electric charge.
- m_e and m_i are respectively the electron's and ion's mass.
- T_e and T_i are respectively the electronic and ionic temperature and γ_e and γ_i the thermodynamic coefficient.

For a precise description of this kind of model, see classical textbooks [5]. One of the main points is that the mass of the electrons is very small compared to the mass of the ions : $m_e \ll m_i$. Since the Lorentz force is the same for the ions and the electrons, the velocity of the ions will be neglectable with respect to the velocity of the electrons. The consequence is that we neglect the contribution of the ions in equation (2.2).

Now, we recall the polarizations for the longitudinal and transverse components of the fields in the linear regime. We begin with the linear theory of system $(2.1)-(2.6)$. The linearized version around $(E,B,v_e,v_i,n_e,n_i) = (0,0,0,0,0,0)$ is (neglecting the contribution of the ions in (2.2)) :

$$\partial_t B + c\nabla\times E = 0, \tag{2.7}$$
$$\partial_t E - c\nabla\times B = 4\pi e n_e v_e, \tag{2.8}$$
$$n_0\partial_t v_e = -\frac{\gamma_e T_e}{m_e}\nabla n_e - \frac{e n_0}{m_e}E, \tag{2.9}$$
$$\partial_t n_e + n_0\nabla\cdot v_e = 0. \tag{2.10}$$

Now writing $B = B_\parallel + B_\perp$ with $\nabla \times B_\parallel = 0$ and $\nabla \cdot B_\perp = 0$ and using similar notations for E and v_e, we get

$$\partial_t B_\parallel = 0, \tag{2.11}$$
$$\partial_t E_\parallel = 4\pi e n_0 v_{e\parallel}, \tag{2.12}$$
$$\partial_t v_{e\parallel} = -\frac{\gamma_e T_e}{m_e n_0}\nabla n_e - \frac{e}{m_e}E_\parallel, \tag{2.13}$$
$$\partial_t n_e + n_0 \nabla \cdot v_{e\parallel} = 0, \tag{2.14}$$

and

$$\partial_t B_\perp + c\nabla \times E_\perp = 0, \tag{2.15}$$
$$\partial_t E_\perp - c\nabla \times B_\perp = 4\pi e n_0 v_{e\perp}, \tag{2.16}$$
$$\partial_t v_{e\perp} = -\frac{e}{m_e}E_\perp. \tag{2.17}$$

(2.11) – (2.14) and (2.15) – (2.17) are two decoupled systems that can be treated separately. We begin with the longitudinal part (2.11) – (2.14). Computing ∂_t(2.13) and using (2.12) and (2.14) leads to

$$\partial_t^2 v_{e\parallel} = \frac{\gamma_e T_e}{m_e}\nabla\nabla \cdot v_{e\parallel} - \frac{4\pi e^2 n_0}{m_e}v_{e\parallel}.$$

That is

$$[\partial_t^2 - v_{th}^2\Delta + \omega_{pe}^2]v_{e\parallel} = 0, \tag{2.18}$$

since $\nabla \times v_{e\parallel} = 0$ and where ω_{pe} is the electronic plasma pulsation

$$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}},$$

and v_{th} is the thermal velocity of electrons

$$v_{th} = \sqrt{\frac{\gamma_e T_e}{m_e}}.$$

The dispersion relation associated to (2.18) is therefore

$$\omega^2 = \omega_{pe}^2 + k^2 v_{th}^2. \tag{2.19}$$

For the transverse part, we compute ∂_t(2.16) and we use (2.15) and (2.17) to obtain

$$\partial_t^2 E_\perp - c^2\Delta E_\perp + \omega_{pe}^2 E_\perp = 0, \tag{2.20}$$

since $\nabla \cdot E_\perp = 0$. The dispersion relation associated to (2.20) is

$$\omega^2 = \omega_{pe}^2 + k^2 c^2. \tag{2.21}$$

For practical applications, the thermal velocity v_{th} is always one order of magnitude (at least) smaller than c. Therefore, the status of equations (2.18) and (2.20) are quite different. One has to think to the solution of (2.20) under the form $e^{i(kx-\omega t)}E(t,x)$ with $\partial_t E \ll \omega E$ and $\partial_x E \ll kE$ while the one of (2.18) will be $e^{-i\omega_{pe}t}v_e$ with $\partial_t v_e \ll \omega_{pe} v_e$.

We are then able to write the polarization conditions on the fields. As usual, we introduce the variation of density $\delta n_e = \dfrac{n_e}{n_0}$.

• Longitudinal part : the linear equations for the longitudinal part are (2.11) – (2.14). Using the time envelope approximation, one sets

$$(B_\parallel, E_\parallel, v_{e\parallel}, \delta n_e) = e^{-i\omega_{pe}t}(\widetilde{B}_\parallel, \widetilde{E}_\parallel, \widetilde{v}_{e\parallel}, \widetilde{\delta n}_e)$$

with

$$\partial_t(\widetilde{B}_\parallel, \widetilde{E}_\parallel, \widetilde{v}_{e\parallel}, \widetilde{\delta n}_e) \ll \omega_{pe}(\widetilde{B}_\parallel, \widetilde{E}_\parallel, \widetilde{v}_{e\parallel}, \widetilde{\delta n}_e).$$

One obtains :

$$\widetilde{B}_\parallel = 0, \tag{2.22}$$

$$\widetilde{v}_{e\parallel} = -\frac{i\omega_{pe}}{4\pi e n_0}\widetilde{E}_\parallel, \tag{2.23}$$

$$\widetilde{\delta n}_e = -\frac{1}{4\pi e n_0}\nabla\cdot\widetilde{E}_\parallel. \tag{2.24}$$

• Transverse part : in this case, in the physical literature, it is usual to work with the vector potential A such that $B = \nabla \times A$ and the scalar potential ψ in the Lorentz jauge $\partial_t \psi = c\nabla \cdot A$. The linearized equations become :

$$\begin{aligned}
&\partial_t \psi = c\nabla\cdot A,\\
&\partial_t A + cE = c\nabla\psi,\\
&\partial_t E - c\nabla\times\nabla\times A = 4\pi e n_0 v_e,\\
&\partial_t v_e = -\frac{\gamma_e T_e}{m_e}\nabla\delta n_e - \frac{e}{m_e}E,\\
&\partial_t \delta n_e + \nabla\cdot v_e = 0.
\end{aligned}$$

The transverse part is :

$$\partial_t A_\perp + cE_\perp = 0, \tag{2.25}$$

$$\partial_t E_\perp - c\nabla\times\nabla\times A_\perp = 4\pi e n_0 v_{e\perp}, \tag{2.26}$$

$$\partial_t v_{e\perp} = -\frac{e}{m_e}E_\perp. \tag{2.27}$$

Looking for the solution in the form $(A_\perp, E_\perp, v_{e\perp}) = e^{i(kx-\omega t)}(\widetilde{A}_\perp, \widetilde{E}_\perp, \widetilde{v}_{e\perp})$ with

$$\partial_t(\widetilde{A}_\perp, \widetilde{E}_\perp, \widetilde{v}_{e\perp}) \ll \omega(\widetilde{A}_\perp, \widetilde{E}_\perp, \widetilde{v}_{e\perp}),$$

$$\partial_x(\widetilde{A}_\perp, \widetilde{E}_\perp, \widetilde{v}_{e\perp}) \ll k(\widetilde{A}_\perp, \widetilde{E}_\perp, \widetilde{v}_{e\perp}),$$

one obtains

$$\widetilde{E}_\perp = \frac{i\omega}{c}\widetilde{A}_\perp, \tag{2.28}$$

$$\widetilde{v}_{e\perp} = \frac{e}{cm_e}\widetilde{A}_\perp. \tag{2.29}$$

2.2 Amplitude equations for the electric fields.

We now present a weakly nonlinear theory of the laser-plasma interaction. We will generalized the equations used in [15]. The electromagnetic transverse part is decomposed into three parts :

i) the incident wave with vector potential $A_0 e^{i(k_0 x - \omega_0 t)} + c.c.$

ii) a Raman backscattered wave with vector potential $A_R e^{i(k_R x - \omega_R t)} + c.c.$

iii) a Brillouin backscattered wave with vector potential $A_B e^{i(-k_0 x - \omega_0 t)} + c.c.$

These three waves create a longitudinal electronic plasma wave at the pulsation ω_{pe}. The corresponding electric field is $E_0 e^{-i\omega_{pe}t} + c.c.$ These four electric fields combine in order to create a low-frequency density modulation $< \delta n >$. Using the polarization condition (2.28) for the transverse part, we look for the electric field in the form

$$\begin{aligned} E =& \frac{i\omega_0}{c} A_0(t,X) e^{i(K_0 \cdot X - \omega_0 t)} + \frac{i\omega_R}{c} A_R(t,X) e^{i(K_R \cdot X - \omega_R t)} \\ &+ \frac{i\omega_0}{c} A_B(t,X) e^{-i(K_0 \cdot X + \omega_0 t)} + E_0(t,X) e^{-i\omega_{pe} t} + c.c., \end{aligned} \tag{2.30}$$

with $X = (x, y, z)$. The aim of this section is to find the envelope equations satisfied by A_0, A_R, A_B and E_0. We introduce (K_1, ω_1) satisfying

$$K_0 = K_R + K_1, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1. \tag{2.31}$$

Note that (K_R, ω_R) and (K_0, ω_0) satisfy the same dispersion relation (2.21). In order to match the notations used in [15], we take $K_0 = k_0(0,1,0)$ and $K_R = k_R(0,1,0)$. (2.30) becomes

$$\begin{aligned} E =& \frac{i\omega_0}{c} A_0(t,X) e^{i(k_0 y - \omega_0 t)} + \frac{i\omega_R}{c} A_R(t,X) e^{i(k_R y - \omega_R t)} \\ &+ \frac{i\omega_0}{c} A_B(t,X) e^{-i(k_0 y + \omega_0 t)} + E_0(t,X) e^{-i\omega_{pe} t} + c.c., \end{aligned} \tag{2.32}$$

and (2.31)

$$k_0 = k_R + k_1, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1.$$

We begin with the nonlinear Maxwell equations with the vector potential :

$$\partial_t \psi = c\nabla \cdot A,$$
$$\partial_t A + cE = c\nabla\psi,$$
$$\partial_t E - c\nabla \times \nabla \times A = 4\pi e((n_0 + n_e)v_e - (n_0 + n_i)v_i).$$

Combining together these equations and neglecting the effects of the ions, we obtain

$$\partial_t^2 A - c^2 \Delta A = -4\pi e n_0 (1 + \delta n_e) v_e. \tag{2.33}$$

Using the form (2.30), the relations (2.31), the dispersion relation (2.21), the polarization condition (2.24) and (2.29) and collecting the factor of $e^{i(k_0 y - \omega_0 t)}$ in the right-hand-side of (2.33), we get

$$\begin{aligned}\partial_t^2 A_0 - 2i\omega_0 \partial_t A_0 - c^2 \Delta A_0 - 2ik_0 c^2 \partial_y A_0 = &- \omega_{pe}^2 < \delta n > (A_0 + e^{-2ik_0 y} A_B) \\ &+ \frac{e}{m_e} (\nabla \cdot E_0)\, A_R e^{-i(k_1 y - \omega_1 t)}.\end{aligned}$$

We now relax the usual time envelope approximation : at the first order, one has $-2i\omega_0 \partial_t A_0 - 2ik_0 c^2 \partial_y A_0 = 0$, that is

$$\partial_t A_0 = -\frac{k_0 c^2}{\omega_0} \partial_y A_0$$

and

$$\partial_t^2 A_0 = \frac{k_0^2 c^4}{\omega_0^2} \partial_y^2 A_0.$$

We therefore get

$$\begin{aligned}i(\partial_t A_0 + \frac{k_0 c^2}{\omega_0} \partial_y A_0) + \frac{c^2}{2\omega_0} \Delta A_0 - \frac{k_0^2 c^4}{2\omega_0^3} \partial_y^2 A_0 = &\frac{\omega_{pe}^2}{2\omega_0} < \delta n > (A_0 + e^{-2ik_0 y} A_B) \\ &- \frac{e}{2m_e \omega_0} (\nabla \cdot E_0)\, A_R e^{-i(k_1 y - \omega_1 t)}.\end{aligned} \tag{2.34}$$

We make the same computation for the harmonic $e^{i(k_R y - \omega_R t)}$ to obtain

$$\begin{aligned}&i(\partial_t A_R + \frac{k_R c^2}{\omega_R} \partial_y A_R) + \frac{c^2}{2\omega_R} \Delta A_R - \frac{k_R^2 c^4}{2\omega_R^3} \partial_y^2 A_R = \frac{\omega_{pe}^2}{2\omega_R} < \delta n > A_R \\ &- \frac{e}{2m_e \omega_R} (\nabla \cdot E_0^*)\, (A_0 + e^{-2ik_0 y} A_B) e^{i(k_1 y - \omega_1 t)}.\end{aligned} \tag{2.35}$$

The equation for the Brillouin component $e^{-i(k_0 y + \omega_0 t)}$ is similar

$$\begin{aligned}i(\partial_t A_B - \frac{k_0 c^2}{\omega_0} \partial_y A_B) + \frac{c^2}{2\omega_0} \Delta A_B - \frac{k_0^2 c^4}{2\omega_0^3} \partial_y^2 A_B = &\frac{\omega_{pe}^2}{2\omega_0} < \delta n > (A_B + e^{2ik_0 y} A_0) \\ &- \frac{e}{2m_e \omega_0} (\nabla \cdot E_0)\, A_R e^{i((2k_0 - k_1) y - \omega_1 t)}.\end{aligned} \tag{2.36}$$

For the electronic-plasma part, we go back to the Maxwell equations in (B, E) $(2.1) - (2.2)$ and we get

$$\partial_t^2 E + c^2 \nabla \times \nabla \times E = 4\pi e \partial_t ((n_0 + n_e) v_e). \tag{2.37}$$

We use the Euler equations (2.3) and (2.5) in order to determine the right-hand-side of (2.37) keeping only at most quadratic terms :

$$\begin{aligned}\partial_t^2 E + c^2 \nabla \times \nabla \times E = \quad & 4\pi e(-n_0 v_e \nabla v_e - \tfrac{\gamma_e T_e}{m_e} n_e \nabla \delta n_e - \tfrac{e(n_0+n_e)}{m_e} E \\ & -\frac{e n_0}{c m_e} v_e \times B - n_e v_e \nabla \cdot v_e).\end{aligned}$$

We write $E = E_0 e^{-i\omega_{pe} t}$ and we replace δn_e by $-\frac{1}{4\pi e n_0} \nabla \cdot E_0$ thanks to the polarization condition (2.24) and get

$$\begin{aligned}&\partial_t^2 E_0 - 2i\omega_{pe} \partial_t E_0 + c^2 \nabla \times \nabla \times E - v_{th}^2 \nabla \nabla \cdot E_0 \\ &= -2e\omega_{pe} n_e \int_0^{\frac{2\pi}{\omega_{pe}}} (v_e \cdot \nabla v_e + \frac{e}{m_e} \delta n_e E + \frac{e}{c m_e} v_e \times B + v_e \nabla \cdot v_e) e^{i\omega_{pe} t} dt.\end{aligned}$$

The right-hand-side is computed using the polarization relations $(2.28) - (2.29)$. One gets

$$\begin{aligned}v_e \cdot \nabla v_e + \frac{e}{c m_e} v_e \times B + v_e \nabla \cdot v_e &= \frac{e^2}{c^2 m_e^2}(A \cdot \nabla A + A \nabla \cdot A) + \frac{e^2}{c^2 m_e^2} A \times (\nabla \times A) \\ &= \frac{e^2}{c^2 m_e^2}(\nabla \frac{|A|^2}{2} + A \nabla \cdot A),\end{aligned}$$

since for all v

$$v \cdot \nabla v = \nabla \frac{|v|^2}{2} + (\nabla \times v) \times v.$$

Moreover, at first order $\nabla \cdot A = 0$. One finally gets using the time envelope approximation

$$\begin{aligned}i\partial_t E_0 + \frac{v_{th}^2}{2\omega_{pe}} \nabla \nabla \cdot E_0 - \frac{c^2}{2\omega_{pe}} \nabla \times \nabla \times E_0 &= \frac{\omega_{pe}}{2} < \delta n > E_0 \\ &+ \frac{e\omega_{pe}}{2c^2 m_e} \nabla (A_R^* (A_0 + e^{-2ik_0 y} A_B) e^{i(k_1 y - \omega_1 t)}).\end{aligned} \tag{2.38}$$

Now since $v_{th} \ll c$, we perform the limit $c \to +\infty$ in this expression. We find $\nabla \times E_0 = 0$ and therefore E_0 is a gradient. We then eliminate the term $\nabla \times \nabla \times E_0$ in (2.38) by applying the divergence on it :

$$\begin{aligned}i\partial_t \nabla \cdot E_0 + \frac{v_{th}^2}{2\omega_{pe}} \Delta \nabla \cdot E_0 &= \frac{\omega_{pe}}{2} \nabla \cdot (< \delta n > E_0) \\ &+ \frac{e\omega_{pe}}{2c^2 m_e} \Delta (A_R^* (A_0 + e^{-2ik_0 y} A_B) e^{i(k_1 y - \omega_1 t)}).\end{aligned}$$

Recall now that $E_0 = \nabla\psi$ and apply $\nabla\Delta^{-1}$ on this equation yields

$$\begin{aligned} i\partial_t E_0 + \frac{v_{th}^2}{2\omega_{pe}}\Delta E_0 =& \frac{\omega_{pe}}{2}\nabla\Delta^{-1}\mathrm{div}(< \delta n > E_0) \\ &+ \frac{e\omega_{pe}}{2c^2 m_e}\nabla\left(A_R^*(A_0 + e^{-2ik_0 y}A_B)e^{i(k_1 y - \omega_1 t)}\right). \end{aligned} \tag{2.39}$$

See [6] for the justification of such an approximation in the context of nonlinear Schrödinger equations. We still have to obtain an equation for $< \delta n >$.

2.3 Acoustic approximation for the low-frequency part.

Since $\nabla\cdot E = -e(n_e - n_i)$ and since the electric field has a high-frequency at the first order, one gets that electron's low-frequency part $< \delta n >$ equals ion's low-frequency part $< \delta n_i >$. We look for an equation on $< \delta n_i >$. The linearization of (2.4) – (2.6) yields :

$$\begin{aligned} &\partial_t v_i = -\frac{\gamma_i T_i}{m_i}\nabla < \delta n_i > + \frac{e}{m_i} < E >, \\ &\partial_t < \delta n_i > + \nabla\cdot v_i = 0, \end{aligned}$$

where $< E >$ is the low-frequency part of the electric field. One gets

$$\partial_t^2 < \delta n_i > - \frac{\gamma_i T_i}{m_i}\Delta < \delta n_i > = -\frac{e}{m_i}\nabla\cdot < E > . \tag{2.40}$$

In order to express $< E >$, we divide (2.3) by $n_0 + n_e$ and we take the mean value component

$$< v_e\cdot\nabla v_e > = -v_{th}^2\nabla < \delta n_e > - \frac{e}{m_e} < E > - \frac{e}{m_e c} < v_e \times B >,$$

that is

$$\frac{e}{m_e} < E > = -v_{th}^2\nabla < \delta n_i > - < v_e\cdot\nabla v_e > - \frac{e}{m_e c} < v_e \times B > . \tag{2.41}$$

The contribution of the longitudinal part to the right-hand-side of (2.41) is only in the term $< v_e\cdot\nabla v_e >$ which gives

$$< \nabla\frac{|v_e|^2}{2} > = \frac{\omega_{pe}^2}{16\pi^2 e^2 n_0^2}\nabla|E_0|^2,$$

thanks to the polarization condition (2.23) and since $\nabla\times E_0 = 0$ at the first order.
The contribution of the transverse part is

$$\begin{aligned} < -v_e\cdot\nabla v_e - \frac{e}{m_e c}v_e\times B > &= -\frac{e^2}{c^2 m_e^2} < A\cdot\nabla A + A\times(\nabla\times A) >, \\ &= -\frac{e^2}{c^2 m_e^2} < \nabla|A|^2 >, \end{aligned}$$

thanks to the polarization condition (2.29) and $B = \nabla \times A$. We obtain

$$< v_e \cdot \nabla v_e + \frac{e}{m_e c} v_e \times B >= -\frac{e^2}{c^2 m_e^2} \nabla(|A_0 + e^{-2ik_0 y} A_B|^2 + |A_R|^2).$$

Finally, the low-frequency component of E is

$$\begin{aligned}\frac{e}{m_e} < E > &= -v_{th}^2 \nabla < \delta n_i > - \frac{\omega_{pe}^2}{16\pi^2 e^2 m_e^2} \nabla |E_0|^2 \\ &- \frac{e^2}{c^2 m_e^2} \nabla(|A_0 + e^{-2ik_0 y} A_B|^2 + |A_R|^2).\end{aligned}$$

Plugging this expression in (2.40) gives

$$\left(\partial_t^2 - c_s^2 \Delta\right) < \delta n >= \frac{1}{4\pi n_0 m_i} \Delta \left(|E_0|^2 + \frac{\omega_{pe}^2}{c^2} (|A_0 + e^{-2ik_0 y} A_B|^2 + |A_R|^2) \right) \quad (2.42)$$

where $c_s^2 = \frac{\gamma_i T_i}{m_i} + \frac{\gamma_e T_e}{m_e}$ is the acoustic velocity.

Equations (2.34), (2.35), (2.36), (2.39) and (2.42) form a closed set of equations describing the laser-plasma interactions. In the next section, we investigate the algebraic properties of this system.

3 Structural properties of the system.

This system is strongly quasilinear and we investigate its structure. In order to make more comprehensible computations, we first introduce a dimensionless form (3.1) – (3.5). Then we show that the quasilinear part in the equations of A_0, A_R, A_B, E_0 is not symmetric. Then we prove that the Brillouin component A_B can be eliminated. The system becomes (3.6)–(3.9). We finish by discussing a resonance relation on (k_1, ω_1) and by some remarks.

3.1 Dimensionless form.

We use $\frac{1}{\omega_0}$ as time scale, $\frac{1}{k_0}$ as space scale and introduce

$$\begin{aligned}&\widetilde{A}_i = \sqrt{\omega_i} \frac{\omega_{pe}}{c} \frac{A_i}{\alpha} \qquad \text{with } i = 0, R \text{ or } B, \\ &\widetilde{E} = \frac{\sqrt{\omega_{pe}}}{\alpha} E \qquad \text{with } \alpha = \frac{2 m_e \omega_0 \sqrt{\omega_0 \omega_R \omega_{pe}}}{e k_0}, \\ &\widetilde{k}_1 = \frac{k_1}{k_0}, \quad \widetilde{\omega}_1 = \frac{\omega_1}{\omega_0},\end{aligned}$$

and dropping the tildes, we get the following system :

$$\left(i(\partial_t + \frac{k_0^2c^2}{\omega_0^2}\partial_y) + \frac{c^2k_0^2}{2\omega_0^2}\Delta - \frac{k_0^4c^4}{2\omega_0^4}\partial_y^2\right) A_0 = \frac{\omega_{pe}^2}{2\omega_0^2} < \delta n > (A_0 + e^{-2iy}A_B) - (\nabla \cdot E_0)\, A_R e^{-i(k_1y-\omega_1t)}, \quad (3.1)$$

$$\left(i(\partial_t + \frac{k_Rk_0c^2}{\omega_R\omega_0}\partial_y) + \frac{c^2k_0^2}{2\omega_R\omega_0}\Delta - \frac{k_0^2k_R^2c^4}{2\omega_R^3\omega_0}\partial_y^2\right) A_R = \frac{\omega_{pe}^2}{2\omega_R\omega_0} < \delta n > A_R - (\nabla \cdot E_0^*)\,(A_0 + e^{-2iy}A_B)e^{i(k_1y-\omega_1t)}, \quad (3.2)$$

$$\left(i(\partial_t - \frac{k_0^2c^2}{\omega_0^2}\partial_y) + \frac{c^2k_0^2}{2\omega_0^2}\Delta - \frac{k_0^4c^4}{2\omega_0^4}\partial_y^2\right) A_B = \frac{\omega_{pe}^2}{2\omega_0^2} < \delta n > (A_B + e^{2iy}A_0) - (\nabla \cdot E_0)\, A_R e^{i((2-k_1)y-\omega_1t)}, \quad (3.3)$$

$$i\partial_t E_0 + \frac{v_{th}^2k^2}{2\omega_{pe}\omega_0}\Delta E_0 = \frac{\omega_{pe}}{2\omega_0}\nabla\Delta^{-1}\mathrm{div}(< \delta n > E_0) + \nabla\left(A_R^*(A_0 + e^{-2iy}A_B)e^{i(k_1y-\omega_1t)}\right), \quad (3.4)$$

$$\partial_t^2 < \delta n > - \frac{c_s^2k_0^2}{\omega_0^2}\Delta < \delta n > = \frac{4m_e}{m_i}\frac{\omega_0\omega_R}{\omega_{pe}^2}\Delta\left(\frac{\omega_{pe}}{\omega_0}|A_0 + e^{-2iy}A_B|^2 + |E_0|^2 + \frac{\omega_{pe}}{\omega_R}|A_R|^2\right). \quad (3.5)$$

Note that $\frac{m_e}{m_i}$ is small, as well as $\frac{c_s^2k_0^2}{\omega_0^2}$. All other coefficients are of order 1 except $\frac{v_{th}^2k_2}{2\omega_{pe}\omega_0}$ which is small.

3.2 Elimination of the Brillouin component.

In this section, we prove that we can in fact decrease the number of unknowns. We introduce

$$A_C = A_0 + e^{-2iy}A_B.$$

Let us compute

$$\left(i(\partial_t + \frac{k_0^2c^2}{\omega_0^2}\partial_y) + \frac{c^2k_0^2}{2\omega_0^2}\Delta - \frac{k_0^4c^4}{2\omega_0^4}\partial_y^2\right) A_C,$$

one obtains after a straightforward computation :

$$\left(i(\partial_t + \frac{k_0^2 c^2}{\omega_0^2}\partial_y) + \frac{c^2 k_0^2}{2\omega_0^2}\Delta - \frac{k_0^4 c^4}{2\omega_0^4}\partial_y^2\right) A_C = \frac{\omega_{pe}^2}{2\omega_0^2} < \delta n > A_C \\ - (\nabla \cdot E_0)\, A_R e^{-i(k_1 y - \omega_1 t)}. \qquad (3.6)$$

Of course (3.2), (3.4) and (3.5) become

$$\left(i(\partial_t + \frac{k_R k_0 c^2}{\omega_R \omega_0}\partial_y) + \frac{c^2 k_0^2}{2\omega_R \omega_0}\Delta - \frac{k_0^2 k_R^2 c^4}{2\omega_R^3 \omega_0}\right) A_R = \frac{\omega_{pe}^2}{2\omega_R \omega_0} < \delta n > A_R \\ - (\nabla \cdot E_0^*)\, A_C e^{i(k_1 y - \omega_1 t)}, \qquad (3.7)$$

$$i\partial_t E_0 + \frac{v_{th}^2 k^2}{2\omega_{pe}\omega_0}\Delta E_0 = \frac{\omega_{pe}}{2\omega_0}\nabla \Delta^{-1} \mathrm{div}(< \delta n > E_0) \\ + \nabla \left(A_R^* A_C e^{i(k_1 y - \omega_1 t)}\right), \qquad (3.8)$$

and

$$\left(\partial_t^2 - \frac{c_s^2 k_0^2}{\omega_0^2}\Delta\right) < \delta n >= \frac{4m_e}{m_i}\frac{\omega_0 \omega_R}{\omega_{pe}^2}\Delta \left(\frac{\omega_{pe}}{\omega_0}|A_C|^2 + |E_0|^2 + \frac{\omega_{pe}}{\omega_R}|A_R|^2\right). \qquad (3.9)$$

(3.6) – (3.9) form a closed set of equations that we solve in the sequel.

3.3 Study of the quasilinear part.

In this section, we consider the quasilinear part of the system (3.6) – (3.8) and we omit the exponentials :

$$\partial_t A_C = i\,(\nabla \cdot E_0)\, A_R, \qquad (3.10)$$
$$\partial_t A_R = i\,(\nabla \cdot E_0^*)\, A_C, \qquad (3.11)$$
$$\partial_t E_0 = -i\nabla(A_R^* A_C). \qquad (3.12)$$

Unfortunately, this system is not obviously symmetrizable as we show below for the scalar 1-D model. Let us write $A_C = u_1 + iu_2$, $A_R = u_3 + iu_4$, $E_0 = u_5 + iu_6$. System (3.10) – (3.12) reads :

$$\begin{aligned}
\partial_t u_1 &= -\partial_x u_6 u_3 - \partial_x u_5 u_4, \\
\partial_t u_2 &= \partial_x u_5 u_3 - \partial_x u_6 u_4, \\
\partial_t u_3 &= \partial_x u_6 u_1 - \partial_x u_5 u_2, \\
\partial_t u_4 &= \partial_x u_5 u_1 + \partial_x u_6 u_2, \\
\partial_t u_5 &= \partial_x (u_2 u_3 - u_1 u_4), \\
\partial_t u_6 &= -\partial_x (u_1 u_3 + u_2 u_4),
\end{aligned}$$

and it can be rewritten in the form

$$\partial_t \mathcal{U} = M(\mathcal{U})\partial_x \mathcal{U},$$

with $\mathcal{U} = (u_1, u_2, u_3, u_4, u_5, u_6)^t$ and

$$M(\mathcal{U}) = \begin{pmatrix} 0 & 0 & 0 & 0 & -u_4 & -u_3 \\ 0 & 0 & 0 & 0 & u_3 & -u_4 \\ 0 & 0 & 0 & 0 & -u_2 & u_1 \\ 0 & 0 & 0 & 0 & u_1 & u_2 \\ -u_4 & u_3 & u_2 & -u_1 & 0 & 0 \\ -u_3 & -u_4 & -u_1 & -u_2 & 0 & 0 \end{pmatrix}.$$

This matrix is obviously not symmetric and the blocks

$$M_1 = \begin{pmatrix} -u_2 & u_1 \\ u_1 & u_2 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} u_2 & -u_1 \\ -u_1 & -u_2 \end{pmatrix}$$

are such that $M_1^t = -M_2$. Therefore the Cauchy problem for (3.10) – (3.12) is certainly ill-posed and the Cauchy problem for (3.6), (3.7), (3.8) and (3.9) can not be handled directly by coupling Ozawa-Tsutsumi's method with hyperbolic-type energy estimates for symmetric system. We will explain in section 4 how to overcome this difficulty by using the dispersive part.

3.4 The resonance condition (2.31).

Let us recall that the numbers k_0, k_R, k_1, ω_0, ω_R and ω_1 satisfy

$$k_0 = k_1 + k_R, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1.$$

We moreover impose (as in [15]) that $\omega_1 = \omega_{pe}(k_1\lambda_D)^2$. This corresponds to the fact that

$$\omega_{pe} + \omega_1 \simeq \sqrt{\omega_{pe}^2 + v_{th}^2 k_1^2}.$$

Therefore, condition (2.31) is a usual 3-waves interaction between two electromagnetic waves and a plasma-electronic wave :

$$k_0 = k_R + k_1, \quad \sqrt{\omega_{pe}^2 + c^2k_0^2} = \sqrt{\omega_{pe}^2 + c^2k_R^2} + \sqrt{\omega_{pe}^2 + v_{th}^2 k_1^2}.$$

We expect a Raman growth on the A_R component. It is therefore not surprising that the quasilinear part is not hyperbolic.

3.5 A hierarchy of model.

Note that since $|\omega_0| > |k_0 c|$ and $|\omega_R| > |k_R c|$, the second order operators in (3.6), (3.7) are elliptic. From now on, we will restrict ourself to the case where this operator is

$$\frac{c^2k_0^2}{2\omega_0^2}\Delta.$$

This is the result that one obtains when making the time envelope approximation. In order to decompose the difficulty, we will consider successively more and more complex model. The first one is the system of quasilinear Schrödinger equations obtained by setting $< \delta n >= 0$ in (3.6), (3.7) and (3.8). We will show how to handle the Cauchy problem in section 4. One could also use the complete equation on E_0 that is

$$i\partial_t E_0 + \frac{v_{th}^2 k^2}{2\omega_{pe}\omega_0}\nabla\nabla\cdot E_0 - \frac{c^2 k_0^2}{2\omega_{pe}\omega_0}\nabla\times\nabla\times E_0 = \frac{\omega_{pe}}{2\omega_0}(< \delta n > E_0) + \nabla(A_R^* A_C e^{-i(k_1 y - \omega_1 t)}). \tag{3.13}$$

We do not discuss (3.13) in this paper. It can certainly be handled by the same method but the proof may be more technical.

4 The Cauchy problem for the nonlinear Schrödinger system.

Let us consider the system obtained from (3.6), (3.7), (3.8) by setting $< \delta n >= 0$. It can be rewritten in the form (omitting the exponentials that have no role in this context)

$$(i(\partial_t + v_C\partial_y) + \alpha\Delta)\, A_C = -\,(\nabla\cdot E_0)\, A_R, \tag{4.1}$$

$$(i(\partial_t + v_R\partial_y) + \beta\Delta)\, A_R = -\,(\nabla\cdot E_0^*)\, A_C, \tag{4.2}$$

$$(i\partial_t + \gamma\Delta)\, E_0 = \nabla(A_R^* A_C), \tag{4.3}$$

where

$$v_C = \frac{k_0^2 c^2}{\omega_0^2}, \quad v_R = \frac{k_R k_0 c^2}{\omega_R\omega_0},$$

$$\alpha = \frac{k_0^2 c^2}{2\omega_0^2}, \quad \beta = \frac{k_0^2 c^2}{2\omega_R\omega_0}, \quad \gamma = \frac{v_{th}^2 k^2}{2\omega_{pe}\omega_0}.$$

Theorem 4.1. *Let $(a_C, a_R, e_0) \in H^s(\mathbb{R}^d)$ with $s > \frac{d}{2} + 3$. There exists $T^* > 0$ and a unique solution $(A_C, A_R, E_0) \in (\mathcal{C}([0, T^*[; H^s(\mathbb{R}^d)))^{3d}$ of (4.1), (4.2), (4.3) satisfying $(A_C, A_R, E_0)(0, X) = (a_C, a_R, e_0)(X)$.*

Proof : A lot of work has already been done on quasilinear Schrödinger equations. In [11], Kenig, Ponce and Vega use local smoothing effects in order to obtain local existence. In [4] or [10], the authors use nonlinear changes of unknowns in order to obtain local existence. Here we want a generalization of such methods that can be adapted to Zakharov-type equations. In order to prove Theorem 4.1, we transform system $(4.1) - (4.2)$ into a dispersive perturbation of a symmetric quasilinear system as follows. Let us first explain how the method works. Consider a system of PDE's of the form (for simplicity, we consider the 1-D case) :

$$\partial_t\mathcal{U} + B(\mathcal{U})\partial_x\mathcal{U} + K\partial_x^2\mathcal{U} = 0, \tag{4.4}$$

where

$$\mathcal{U} \ : \ [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}^d,$$

K is skew-adjoint and one-to-one, and

$$B(\mathcal{U}) = B_1(\mathcal{U}) + B_2(\mathcal{U}),$$

where $B_1(\mathcal{U})$ is symmetric and $B_2(\mathcal{U})$ is skew-symmetric. Except if $B_2 = 0$, one cannot handle directly (4.4) by usual energy methods. One introduces $\mathcal{V} = \partial_t \mathcal{U}$, $\mathcal{X} = \partial_x \mathcal{U}$ and $\mathcal{W} = \partial_x^2 \mathcal{U}$. The equations on $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{W}$ read

$$\partial_t \mathcal{V} + B(\mathcal{U})\partial_x \mathcal{V} + K\partial_x^2 \mathcal{V} = \text{ semilinear terms,} \tag{4.5}$$

$$\partial_t \mathcal{X} + B(\mathcal{U})\partial_x \mathcal{X} + K\partial_x^2 \mathcal{X} = \text{ semilinear terms,} \tag{4.6}$$

$$\partial_t \mathcal{W} + B(\mathcal{U})\partial_x \mathcal{W} + K\partial_x^2 \mathcal{W} = \text{ semilinear terms,} \tag{4.7}$$

$$\partial_t \mathcal{U} = \text{ semilinear terms,} \tag{4.8}$$

and

$$\mathcal{V} + K\mathcal{W} = \text{ semilinear terms,} \tag{4.9}$$

where the semilinear terms depend on $\mathcal{U}$, $\mathcal{V}$, $\mathcal{X}$ and $\mathcal{W}$. Now in (4.5), we replace $\mathcal{V}$ by $-K\mathcal{W}$ + semilinear terms (depending only on $\mathcal{U}$ and $\mathcal{X}$) in $B_2(\mathcal{U})\partial_x\mathcal{V}$ and in (4.7) we replace $\mathcal{W}$ by $-K^{-1}\mathcal{V}$ + semilinear terms (depending only on $\mathcal{U}$ and $\mathcal{X}$) in $B_2(\mathcal{U})\partial_x\mathcal{W}$. The new system on $\mathcal{V}$ and $\mathcal{W}$ reads

$$\partial_t \mathcal{V} + B_1(\mathcal{U})\partial_x \mathcal{V} - B_2(\mathcal{U})K\partial_x\mathcal{W} + K\partial_x^2 \mathcal{V} = \text{ semilinear terms,} \tag{4.10}$$

$$\partial_t \mathcal{W} + B_1(\mathcal{U})\partial_x \mathcal{W} - B_2(\mathcal{U})K^{-1}\partial_x\mathcal{V} + K\partial_x^2 \mathcal{W} = \text{ semilinear terms.} \tag{4.11}$$

Now system $(4.10)-(4.11)$ may be symmetrizable, depending on the properties of B_2 and K. In our case, it will possible and we explain now why.
Let us introduce the following notations (for simplicity, we denote $E = E_0$):

$$\begin{aligned}
&B_R^j = \partial_{x_j} A_R, \ \ B_C^j = \partial_{x_j} A_C, \ \ F^j = \partial_{x_j} E,\\
&C_R = \Delta A_R, \ \ C_C = \Delta A_C, \ \ G = \Delta E,\\
&D_R = \partial_t A_R, \ \ D_C = \partial_t A_C, \ \ H = \partial_t E.
\end{aligned}$$

We write the equations satisfied by these new unknowns. We set

$$\mathcal{U} := (A_C, A_R, E, B_C, B_R, F, C_C, C_R, G, D_C, D_R, H).$$

We also introduce the following linear operator

$$L = \begin{pmatrix} i(\partial_t + v_C\partial_y) + \alpha\Delta \\ i(\partial_t + v_R\partial_y) + \beta\Delta \\ i\partial_t + \gamma\Delta \end{pmatrix}.$$

The system becomes

$$L \begin{pmatrix} A_C \\ A_R \\ E \end{pmatrix} = f_1(\widetilde{\mathcal{U}}), \tag{4.12}$$

$$L \begin{pmatrix} B_C^j \\ B_R^j \\ F^j \end{pmatrix} = f_2^j(\mathcal{U}), \tag{4.13}$$

where $\widetilde{\mathcal{U}} = (A_C, A_R, E, B_R, B_C, F)$. Here f_1 is a smooth function and the f_2^j are non-local functions satisfying $|f_2^j(\mathcal{U})| \leq K|\mathcal{U}|^\sigma_{H^s}$ for some σ, K and $s > \frac{n}{2}$. Indeed, the $f_2^j(\mathcal{U})$ are polynomial in A_C, A_R, E, B_R^j, B_C^j, F^j and $\partial^2_{x_i x_j}(A_C, A_R, E)$ that can be expressed in terms of C_C, C_R, G by

$$\partial^2_{x_i x_j}(A_C, A_R, E) = \partial^2_{x_i x_j}\Delta^{-1}(C_C, C_R, G). \tag{4.14}$$

In order to find the equations satisfied by C_C, C_R and G, we need to express the nonlinear term $\nabla(A_R^* A_C)$ in (4.3) more precisely using

$$\nabla(AB) = A \cdot \nabla B + B \cdot \nabla A + A \times (\nabla \times B) + B \times (\nabla \times A). \tag{4.15}$$

We moreover set

$$m_A(B) := A \cdot \nabla B + A \times (\nabla \times B). \tag{4.16}$$

The system satisfied by (C_C, C_R, G) is therefore

$$L \begin{pmatrix} C_C \\ C_R \\ G \end{pmatrix} = \begin{pmatrix} -A_R \nabla \cdot G \\ -A_C \nabla \cdot G^* \\ m_{A_R^*}(C_C) + m_{A_C}(C_R^*) \end{pmatrix} + f_3(\mathcal{U}), \tag{4.17}$$

where $f_3(\mathcal{U})$ is a non-local function as $f_2^j(\mathcal{U})$. In the same way, the system for (D_C, D_R, H) is

$$L \begin{pmatrix} D_C \\ D_R \\ H \end{pmatrix} = \begin{pmatrix} -A_R \nabla \cdot H \\ -A_C \nabla \cdot H^* \\ m_{A_R^*}(D_C) + m_{A_C}(D_R^*) \end{pmatrix} + f_4(\mathcal{U}), \tag{4.18}$$

where $f_4(\mathcal{U})$ is a non-local function as $f_2^j(\mathcal{U})$. As mentionned before, the quasi-linear part of (4.17) and (4.18) is not symmetric and therefore usual energy estimates are inoperative. We now prove the following proposition.

Proposition 4.1. *Let $b \in \mathbb{C}^3$ be a constant. The adjoint in L^2 of $-b\nabla\cdot$ is $m_{b^*}(\cdot)$.*

Proof : Let V and W two $\mathcal{C}^\infty$ vector fields in $\mathbb{C}^3$. Let us compute

$$\begin{aligned}\int m_{b^*}(V)\cdot W^* &= \int (b^*\cdot\nabla V + b^*\times(\nabla\times V))\cdot W^* \quad \text{by (4.16)},\\ &= \int ((b^*\cdot\nabla)V\cdot W^* + (W^*\times b^*)\cdot\nabla\times V)\\ &= \int (b^*\cdot\nabla)V\cdot W^* + \int \nabla\times(W^*\times B^*)\cdot V\end{aligned}$$

since the operator $\nabla\times$ is self-adjoint. Now using the relation

$$\nabla\times(A\times B) = A\nabla\cdot B - B\nabla\cdot A + B\cdot\nabla A - A\cdot\nabla B$$

and the fact that b is constant, one gets

$$\nabla\times(W^*\times b^*) = -b^*\nabla\cdot W^* + b^*\cdot\nabla W^*.$$

Moreover

$$\int (b^*\cdot\nabla)V\cdot W^* = -\int V\cdot(b^*\cdot\nabla)W^*.$$

It follows that

$$\begin{aligned}\int m_{b^*}(V)\cdot W^* &= -\int V\cdot(b^*\cdot\nabla)W^* - \int b^*\nabla\cdot W^*\cdot V + \int b^*\cdot\nabla W^*\cdot V\\ &= -\int b^*\nabla\cdot W^*\cdot V = -\overline{\int b\nabla\cdot W\cdot V^*}.\end{aligned}$$

This ends the proof of Proposition 4.1. □

In the quasilinear part of (4.17) or (4.18) the terms $-A_R\nabla\cdot H$ in the first equation and $m_{A_R^*}(D_C)$ in the third one can be rewritten as

$$\begin{pmatrix} 0 & 0 & -A_R\nabla\cdot\\ 0 & 0 & 0\\ m_{A_R^*}(\cdot) & 0 & 0\end{pmatrix}\begin{pmatrix} D_C\\ D_R\\ H\end{pmatrix}$$

where the operator

$$\begin{pmatrix} 0 & 0 & -A_R\nabla\cdot\\ 0 & 0 & 0\\ m_{A_R^*}(\cdot) & 0 & 0\end{pmatrix}$$

is skew-adjoint. Therefore usual energy estimates will apply. Unfortunately, this is not the case for the other part. For the terms $-A_C\nabla\cdot G^*$, $m_{A_C}(C_R^*)$, $-A_C\nabla\cdot H^*$ and $m_{A_C}(D_R^*)$ we use equations (4.2) – (4.3)

$$iD_R + \beta C_R = g_1(\widetilde{\mathcal{U}}), \tag{4.19}$$

and

$$i\partial_{x_j}D_R + \beta\partial_{x_j}C_R = g_2^j(\mathcal{U}), \tag{4.20}$$

where g_1 is a smooth function and the g_2^j are functions satisfying the same properties as the f_2^j.
In the same way, one has

$$iH + \gamma G = g_3(\widetilde{\mathcal{U}}), \tag{4.21}$$

and

$$i\partial_{x_j} H + \gamma \partial_{x_j} G = g_4^j(\mathcal{U}). \tag{4.22}$$

Plugging these relationships into the bad terms of equations (4.17) and (4.18), we obtain :

$$L \begin{pmatrix} C_C \\ C_R \\ G \end{pmatrix} = \begin{pmatrix} -A_R \nabla \cdot G \\ -\frac{iA_C}{\gamma} \nabla \cdot H^* \\ m_{A_R^*}(C_C) + \frac{i}{\beta} m_{A_C}(D_R^*) \end{pmatrix} + f_5(\mathcal{U}), \tag{4.23}$$

and

$$L \begin{pmatrix} D_C \\ D_R \\ H \end{pmatrix} = \begin{pmatrix} -A_R \nabla \cdot H \\ i\gamma A_C \nabla \cdot G^* \\ m_{A_R^*}(D_C) - i\beta m_{A_C}(C_R^*) \end{pmatrix} + f_6(\mathcal{U}). \tag{4.24}$$

The complete systems reads :

$$\begin{pmatrix} L \\ L \\ L \\ L \end{pmatrix} \mathcal{U} = \begin{pmatrix} 0 & 0 \\ & \\ & \\ 0 & M(A_C, A_R, \partial) \end{pmatrix} \begin{pmatrix} \widetilde{\mathcal{U}} \\ \\ \\ \mathcal{V} \end{pmatrix} + F_1(\mathcal{U}), \tag{4.25}$$

where

$$\widetilde{\mathcal{U}} = (A_C, A_R, E, B_C, B_R, F),$$
$$\mathcal{V} = (C_C, C_R, G, D_C, D_R, H),$$

and M is a 6x6 matrix of operators which value is

$$M(V, W, \partial) = \begin{pmatrix} 0 & 0 & -W\nabla\cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{iV}{\gamma}\nabla\cdot\sigma \\ m_{W^*}(\cdot) & 0 & 0 & 0 & \frac{i}{\beta} m_V(\sigma\cdot) & 0 \\ 0 & 0 & 0 & 0 & 0 & -W\nabla\cdot \\ 0 & 0 & i\gamma V\nabla\cdot\sigma & 0 & 0 & 0 \\ 0 & -i\beta m_V(\sigma\cdot) & 0 & m_{W^*}(\cdot) & 0 & 0 \end{pmatrix} \tag{4.26}$$

where σ is the complex conjugation (for a vector field A in $\mathbb{R}^3$, $\sigma A = A^*$). A straightforward change of variable enables us to restrict to the case $\gamma = \beta = 1$. In this case, the operator $M(V, W, \partial)$ with V and W fixed constants is symmetric. (4.25) then reads

$$(\partial_t + C(\nabla) + iD\Delta)\mathcal{U} = \begin{pmatrix} 0 & 0 \\ 0 & iM(A_C, A_R, \partial) \end{pmatrix} \mathcal{U} + iF_1(\mathcal{U}), \tag{4.27}$$

where

$$C(\nabla) = \sum_{j=1}^{d} C_j \partial x_j,$$

corresponds to the order one terms in (4.25) and D is the diagonal dispersion matrix. Usual energy estimates for the system (4.27) yield Theorem 4.1.

5 The Cauchy problem for the nonlinear Zakharov system.

In this section, we consider the following non-dimensional form of system (3.6) – (3.9). We have omitted the exponentials which have no role in this context.

$$(i(\partial_t + v_C \partial_y) + \alpha\Delta)\, A_C = \frac{b^2}{2} n A_C - (\nabla \cdot E)\, A_R, \tag{5.1}$$

$$(i(\partial_t + v_R \partial_y) + \beta\Delta)\, A_R = \frac{bc}{2} n A_R - (\nabla \cdot E^*)\, A_C, \tag{5.2}$$

$$(i\partial_t + \gamma\Delta)\, E = \frac{b}{2} n E + \nabla(A_R^* A_C), \tag{5.3}$$

$$(\partial_t^2 - v_s^2 \Delta) n = a\Delta \left(|E|^2 + b|A_C|^2 + c|A_R|^2\right), \tag{5.4}$$

where

$$a = 4\frac{m_e}{m_i}\frac{\omega_0\omega_R}{\omega_{pe}^2}, \quad b = \frac{\omega_{pe}}{\omega_0}, \quad c = \frac{\omega_{pe}}{\omega_R}, \quad v_s \frac{c_s k_0}{\omega_0}.$$

Here, $< \delta n >$ is denoted by n for simplicity.

Theorem 5.1. *Let $(a_C, a_R, e) \in (H^{s+2}(\mathbb{R}^d))^{3d}$, $n_0 \in H^{s+1}(\mathbb{R}^d)$ and $n_1 \in H^s(\mathbb{R}^d)$ with $s > \frac{d}{2} + 3$. There exists $T^* > 0$ and a unique maximal solution (A_C, A_R, E, n) to (5.1) – (5.4) such that*

$$(A_C, A_R, E) \in \mathcal{C}([0, T^*[; H^{s+2})^{3d}, \quad n \in \mathcal{C}([0, T^*[; H^{s+1}) \cap \mathcal{C}^1([0, T^*[; H^s)$$

with the initial value

$$(A_C, A_R, E, n)(0) = (a_C, a_R, e, n_0), \quad \partial_t n(0) = n_1.$$

The proof relies on the transformation described in section 4. In addition to the variables of section 4, we have to use Δn and $\partial_t n$. We therefore write the equations satisfied by n and $\partial_t n$. Take

$$p = \nabla n, \quad q = \Delta n, \quad r = \partial_t n, \quad \lambda = \nabla \Delta^{-1} \partial_t n, \quad \mu = \nabla\lambda, \quad \nu = \Delta\lambda, \quad \tau = \partial_t \lambda,$$

and set

$$\mathcal{N} := (n, p, q, r, \lambda, \mu, \nu, \tau),$$

$$\widetilde{\mathcal{U}} := (A_C, A_R, E, B_C, B_R, F, C_C, C_R, G, D_C, D_R, H).$$

The equations for $\mathcal{N}$ are

$$\begin{aligned}
&\partial_t n = \nabla \cdot \lambda,\\
&\partial_t \lambda = \nabla n + h_1(\widetilde{\mathcal{U}}),\\
&\partial_t p_j = \nabla \cdot \mu_j,\\
&\partial_t \mu_j = \nabla p_j + h_{2j}(\widetilde{\mathcal{U}}),\\
&\partial_t q = \nabla \cdot \nu,\\
&\partial_t \nu = \nabla q + \nabla(h_3(\widetilde{\mathcal{U}},\widetilde{\mathcal{U}})),\\
&\partial_t r = \nabla \cdot \tau,\\
&\partial_t \tau = \nabla r + \nabla(h_4(\widetilde{\mathcal{U}},\widetilde{\mathcal{U}})),
\end{aligned}$$

where h_1, h_{2j}, h_3 and h_4 are smooth functions. $h_3(\cdot,\cdot)$ and $h_4(\cdot,\cdot)$ are bilinear. The system is :

$$\partial \mathcal{N} + \mathcal{A}(\nabla)\mathcal{N} + F_2(\widetilde{\mathcal{U}}) + \nabla(F_3(\widetilde{\mathcal{U}},\widetilde{\mathcal{U}})) = 0, \tag{5.5}$$

where

$$\mathcal{A}(\nabla) = \sum_{j=1}^{d} \mathcal{A}_j \partial_{x_j},$$

the matrices $\mathcal{A}_j$ are symmetric, F_2 is a smooth function and $F_3(\cdot,\cdot)$ a smooth bilinear function. The equations satisfied by $\widetilde{\mathcal{U}}$ are

$$\left(\partial_t + \widetilde{C}(\nabla) + i\widetilde{D}\Delta\right)\widetilde{U} = \begin{pmatrix} 0 & 0 \\ 0 & iM(A_C, A_R, \partial) \end{pmatrix}\widetilde{\mathcal{U}} + i\widetilde{F}_1(\widetilde{\mathcal{U}}) + \widetilde{B}(\mathcal{N},\widetilde{\mathcal{U}}), \tag{5.6}$$

where $\widetilde{B}(\cdot,\cdot)$ is a bilinear mapping, $M(A_C, A_R, \partial)$ is the symmetric operator defined by (4.26) and

$$\widetilde{C} = \sum_{j=1}^{d} \widetilde{C}_j \partial_{x_j},$$

corresponds to the order one terms in (5.1) – (5.2).
Now using the real form of $\widetilde{\mathcal{U}}$, one obtains the following system for $\mathcal{U} = \left(\mathrm{Re}(\widetilde{\mathcal{U}}), \mathrm{Im}(\widetilde{\mathcal{U}})\right)$

$$\partial_t \mathcal{U} + C(\nabla)\mathcal{U} + D\Delta\mathcal{U} = \sum_{j=1}^{d} A_j(\mathcal{U})\partial_{x_j}\mathcal{U} + F_1(\mathcal{U}) + B(\mathcal{N},\mathcal{U}), \tag{5.7}$$

where

$$C(\nabla) = \sum_{j=1}^{d} C_j \partial_{x_j},$$

with C_j being constant symmetric matrices. D is a skew-symmetric non-singular matrix, $B(\cdot,\cdot)$ is a (possibly non-local of order 0) bilinear mapping, for all j, $A_j(\mathcal{U})$ is symmetric and

$$\mathcal{U} \longmapsto A_j(\mathcal{U})$$

is linear. Moreover, system (5.5) becomes

$$\partial_t \mathcal{N} + E(\nabla)\mathcal{N} + F_2(\mathcal{U}) + \nabla(F_3(\mathcal{U},\mathcal{U})) = 0, \tag{5.8}$$

where F_2 and F_3 are smooth functions. We will now apply Ozawa-Tsutsumi's method to (5.7). Set $\mathcal{A} = \partial_t \mathcal{U}$. The system satisfied by $\mathcal{A}$ is

$$\begin{aligned} \partial_t \mathcal{A} + C(\nabla)\mathcal{A} + D\Delta\mathcal{A} =& B(\partial_t \mathcal{N},\mathcal{U}) + B(\mathcal{N},\mathcal{A}) + F_1^{'}(\mathcal{U})\mathcal{A} \\ &+ \sum_{j=1}^{d} A_j(\mathcal{U})\partial_{x_j}\mathcal{A} + \sum_{j=1}^{d} A_j(\mathcal{A})\partial_{x_j}\mathcal{U}, \end{aligned} \tag{5.9}$$

and

$$\mathcal{A} + C(\nabla)\mathcal{U} + D\Delta\mathcal{U} = \sum_{j=1}^{d} A_j(\mathcal{U})\partial_{x_j}\mathcal{U} + F_1(\mathcal{U}) + B(\mathcal{N},\mathcal{U}). \tag{5.10}$$

Let us consider an initial datum $\mathcal{U}_0 \in H^{s+2}$, $\mathcal{A}_0 \in H^s$ given by (5.10) and $\mathcal{N}_0 \in H^{s+1}$. Just as in [12], we first transform (5.10) by adding $-\alpha D\mathcal{U}$ on both sides of (5.10) :

$$\mathcal{A} + C(\nabla)\mathcal{U} + D(-\alpha + \Delta)\mathcal{U} = \sum_{j=1}^{d} A_j(\mathcal{U})\partial_{x_j}\mathcal{U} + F_1(\mathcal{U}) + B(\mathcal{N},\mathcal{U}) - \alpha D\mathcal{U}. \tag{5.11}$$

We now consider a linearized version of (5.9) – (5.11) and (5.8) as follows. Take $\mathcal{A} \in \mathcal{C}([0,T];H^s)$, $\mathcal{N} \in \mathcal{C}([0,T];H^{s+1}) \cap \mathcal{C}^1([0,T];H^s)$ and construct new functions $\mathcal{B}$ and $\mathcal{P}$ by the following procedure. First construct $\mathcal{U} \in \mathcal{C}([0,T];H^s)$ by

$$\mathcal{U} = \mathcal{U}_0 + \int_0^t \mathcal{A}(s)ds, \tag{5.12}$$

and obtain $\mathcal{V}$ by solving

$$\mathcal{A} + C(\nabla)\mathcal{V} + D(-\alpha + \Delta)\mathcal{V} = \sum_{j=1}^{d} A_j(\mathcal{U})\partial_{x_j}\mathcal{V} + F_1(\mathcal{U}) + B(\mathcal{N},\mathcal{U}) - \alpha D\mathcal{U}. \tag{5.13}$$

We will obtain a solution to (5.13) for α large enough thanks to the fact that $\mathcal{A}_j(\mathcal{U})$ are symmetric. Now take $(\mathcal{B},\mathcal{P})$ as being the solutions to

$$\begin{aligned} \partial_t \mathcal{B} + C(\nabla)\mathcal{B} + D\Delta\mathcal{B} =& B(\partial_t \mathcal{N},\mathcal{U}) + B(\mathcal{N},\mathcal{B}) + F_1^{'}(\mathcal{U})\mathcal{B} \\ &+ \sum_{j=1}^{d} A_j(\mathcal{B})\partial_{x_j}\mathcal{V} + \sum_{j=1}^{d} A_j(\mathcal{A})\partial_{x_j}\mathcal{B}, \end{aligned} \tag{5.14}$$

$$\partial_t \mathcal{P} + E(\nabla)\mathcal{P} + F_2(\mathcal{V}) + \nabla(F_3(\mathcal{V},\mathcal{V})) = 0. \tag{5.15}$$

Note that the symmetry of the matrices $\mathcal{A}_j$ will play a crucial role in the resolution of (5.14).
Let

$$\mathcal{T} \;\; : \;\; (\mathcal{A},\mathcal{N}) \longmapsto (\mathcal{B},\mathcal{P}).$$

The proof consists in showing that $\mathcal{T}$ is a contraction on a suitable ball of $\mathcal{C}([0,T];H^s)$. We split the proof in 5 steps. Take

$$R = 2\left(|\mathcal{U}_0|_{H^s} + |\mathcal{N}_0|_{H^s}\right),$$

and consider $B(R)$ the ball of radius R in $\mathcal{C}([0,T];H^s(\mathbb{R}^d)\times H^{s+1}(\mathbb{R}^d))$.

• step 1 : Solving the elliptic equation (5.13). Assume that $(\mathcal{A},\mathcal{N}) \in B(R)$. Then

$$|\mathcal{U}|_{H^s} \le |\mathcal{U}_0|_{H^s} + TR \le R,$$

provided that T is small enough. Applying D^{-1} on (5.13) yields

$$D^{-1}C(\nabla)\mathcal{V} + (-\alpha+\Delta)\mathcal{V} - \sum_{j=1}^{d} D^{-1}A_j(\mathcal{U})\partial_{x_j}\mathcal{V} = D^{-1}F,$$

where

$$F = -A + F_1(\mathcal{U}) + B(\mathcal{N},\mathcal{U}) - \alpha D\mathcal{U}.$$

Lemma 5.1. *The bilinear form*

$$a(\mathcal{V}_1,\mathcal{V}_2) = \int (D^{-1}C(\nabla)\mathcal{V}_1\cdot\mathcal{V}_2 - \alpha\mathcal{V}_1\cdot\mathcal{V}_2 - \nabla\mathcal{V}_1\cdot\nabla\mathcal{V}_2 - \sum_{j=1}^{d} D^{-1}\mathcal{A}_j(\mathcal{U})\partial_{x_j}\mathcal{V}_1\cdot\mathcal{V}_2)$$

is elliptic for α large enough.

Proof : Compute

$$\begin{aligned} I &= \left|\int (D^{-1}C(\nabla)\mathcal{V}\cdot\mathcal{V} - \sum_{j=1}^{d} D^{-1}\mathcal{A}_j(\mathcal{U})\partial_{x_j}\mathcal{V}\cdot\mathcal{V})\right| \\ &\le K\left(|\mathcal{V}|_{L^2}|\nabla\mathcal{V}|_{L^2} + |\mathcal{U}|_\infty|\mathcal{V}|_{L^2}|\nabla\mathcal{V}|_{L^2}\right). \end{aligned}$$

Recall that

$$|\mathcal{U}|_\infty \le K^{'}|\mathcal{U}|_{H^s} \le K^{'}R,$$

so that

$$I \le K^{''}(1+R^2)|\mathcal{V}|_{L^2}^2 + \frac{1}{2}|\nabla\mathcal{V}|_{L^2}^2.$$

Take $\alpha = 2K^{''}(1+R^2)$. This provides

$$a(\mathcal{V},\mathcal{V}) \ge \frac{\alpha}{2}|\mathcal{V}|_{L^2}^2 + \frac{1}{2}|\nabla\mathcal{V}|_{L^2}^2. \qquad \square$$

Then using usual arguments for elliptic equations, one obtains that there exists a unique $\mathcal{V} \in H^{s+2}$ solution to (5.13) such that

$$|\mathcal{V}|_{H^{s+2}} \leq C_1(R)|F|_{H^s},$$

and

$$|\mathcal{V}|_{L^\infty(0,T;H^{s+2})} \leq C_1(R)\big(|\mathcal{A}|_{L^\infty(0,T;H^s)} + C(|\mathcal{U}|_{L^\infty(0,T;H^s)}) \\ +|\mathcal{N}|_{L^\infty(0,T;H^s)}|\mathcal{U}|_{L^\infty(0,T;H^s)} + |\mathcal{U}|_{L^\infty(0,T;H^s)}\big),$$

since

$$|F(\mathcal{U})|_{H^s} \leq C(|\mathcal{U}|_{H^s}),$$

where $x \longmapsto C(x)$ is a continuous function. It follows that there exists $C_2(R)$ such that

$$|\mathcal{V}|_{L^\infty(0,T;H^{s+2})} \leq C_2(R). \tag{5.16}$$

• step 2 : Resolution of the dispersive equation (5.14). We first use a long-wave type regularization of (5.14) : take $\mathcal{B}^\varepsilon$ solution to

$$\partial_t(1-\varepsilon\Delta)\mathcal{B}^\varepsilon + C(\nabla)\mathcal{B}^\varepsilon + D\Delta\mathcal{B}^\varepsilon = B(\partial_t\mathcal{N},\mathcal{U}) + B(\mathcal{N},\mathcal{B}^\varepsilon) + F_1^{'}(\mathcal{U})\mathcal{B}^\varepsilon \\ + \sum_{j=1}^d A_j(\mathcal{B}^\varepsilon)\partial_{x_j}\mathcal{V} + \sum_{j=1}^d A_j(\mathcal{A})\partial_{x_j}\mathcal{B}^\varepsilon, \tag{5.17}$$

with $\mathcal{B}^\varepsilon(0) = \mathcal{B}_0$.
There exists a unique solution

$$\mathcal{B}^\varepsilon \in \mathcal{C}([0,T];H^s)$$

to (5.17) since (5.17) is a linear, zero order (in space) equation. We will now perform energy estimates on (5.17) and we will use several times the following inequality :

$$\forall\ (u,v) \in L^\infty \cap H^s,\ \forall\ (\alpha,\beta) \text{ such that } |\alpha| + |\beta| = s,$$

$$|\partial^\alpha u \partial^\beta v|_{L^2} \leq C\left(|u|_{L^\infty}|v|_{H^s} + |u|_{H^s}|v|_{L^\infty}\right). \tag{5.18}$$

See [2]. Note that ∂^α denotes here any spatial derivative of order α.

Proposition 5.1. *There exists $C_5(R) > 0$ such that*

$$|\mathcal{B}^\varepsilon|_{L^\infty(0,T;H^s)} \leq |\mathcal{B}_0|_{H^s} + C_5(R)T.$$

Proof : L^2 estimate : multiply (5.17) by $\mathcal{B}^\varepsilon$ and get

$$\begin{aligned}\frac{1}{2}\partial_t \int (|\mathcal{B}^\varepsilon|^2 + \varepsilon|\nabla \mathcal{B}^\varepsilon|^2) \leq & C \big(|\partial_t \mathcal{N}|_{L^\infty} |\mathcal{U}|_{L^2} |\mathcal{B}^\varepsilon|_{L^2} + |\mathcal{N}|_{L^\infty} |\mathcal{B}^\varepsilon|^2_{L^2} \\ & + \sum_{j=1}^d |\mathcal{B}^\varepsilon|^2_{L^2} |\partial_{x_j} \mathcal{V}|_{L^\infty} + C(|\mathcal{U}|_{L^\infty}) |\mathcal{B}^\varepsilon|^2_{L^2} \; \big) \\ & + \sum_{j=1}^d \int A_j(\mathcal{A}) \partial_{x_j} \mathcal{B}^\varepsilon \cdot \mathcal{B}^\varepsilon.\end{aligned}$$

Now, since $A_j(\mathcal{A})$ is symmetric, one obtains

$$\sum_{j=1}^d \int A_j(\mathcal{A}) \partial_{x_j} \mathcal{B}^\varepsilon \cdot \mathcal{B}^\varepsilon = -\frac{1}{2} \sum_{j=1}^d \int A_j(\partial_{x_j} \mathcal{A}) \mathcal{B}^\varepsilon \cdot \mathcal{B}^\varepsilon,$$

and this last quantity is controlled by

$$|\partial_{x_j} \mathcal{A}|_{L^\infty} |\mathcal{B}^\varepsilon|^2_{L^2}.$$

Therefore, on has

$$\int |\mathcal{B}^\varepsilon|^2 \leq |\mathcal{B}_0^\varepsilon|^2_{L^2} + TC_3(R). \tag{5.19}$$

H^s estimate : one applies ∂^s on (5.17) and takes the L^2 inner product with $\partial^s \mathcal{B}^\varepsilon$. The only non straightforward term is the quasilinear one

$$\int A_j(\mathcal{A}) \partial^s \partial_{x_j} \mathcal{B}^\varepsilon \cdot \partial^s \mathcal{B}^\varepsilon.$$

Again the symmetry of A_j gives

$$\int A_j(\mathcal{A}) \partial^s \partial_{x_j} \mathcal{B}^\varepsilon \cdot \partial^s \mathcal{B}^\varepsilon = -\frac{1}{2} \int A_j(\partial_{x_j} \mathcal{A}) \partial^s \mathcal{B}^\varepsilon \cdot \partial^s \mathcal{B}^\varepsilon,$$

which is controlled by

$$|\partial_{x_j} \mathcal{A}|_{L^\infty} |\partial^s \mathcal{B}^\varepsilon|^2_{L^2} \leq C |\mathcal{A}|_{H^s} |\partial^s \mathcal{B}^\varepsilon|^2_{L^2},$$

since $s > \frac{d}{2} + 1$. We then obtain

$$\frac{1}{2}\partial_t \left(|\mathcal{B}^\varepsilon|^2_{H^s} + \varepsilon |\mathcal{B}^\varepsilon|^2_{H^{s+1}} \right) \leq C_3(R) |\mathcal{B}^\varepsilon|^2_{H^s} + C_4(R) |\mathcal{B}^\varepsilon|^2_{H^{s+1}},$$

from which Proposition (5.1) follows. □
Letting $\varepsilon \to 0$ gives a solution

$$\mathcal{B} \in L^\infty(0,T; H^s) \cap \mathcal{C}([0,T]; H^{s-\eta}) \quad \forall\, \eta > 0,$$

to (5.14) satisfying

$$|\mathcal{B}|_{L^\infty(0,T;H^s)} \leq |\mathcal{B}|_{H^s} + C_5(R)T. \tag{5.20}$$

• step 3 : Solving the "wave equation" (5.15). Note that

$$|\nabla F_3(\mathcal{V},\mathcal{V})|_{L^\infty(0,T;H^{s+1})} \leq C|\mathcal{V}|_{L^\infty(0,T;H^{s+1})},$$

and

$$|\mathcal{V}|_{L^\infty(0,T;H^{s+2})} \leq C_2(R),$$

thanks to (5.16). It follows that

$$|\mathcal{P}|_{L^\infty(0,T;H^{s+1})} \leq |\mathcal{N}_0|_{H^s} + C_6(R)T. \tag{5.21}$$

Inequality (5.20) and (5.21) show that :

Proposition 5.2. *There exists $T_0 > 0$ such that if $T \leq T_0$, $\mathcal{T}$ maps $B(R)$ into itself.*

• step 4 : $\mathcal{T}$ is a contraction in $L^\infty(0,T;L^2)$ if T is small enough. It is obtained as usual. The crucial point is again the symmetry of $A_j(\mathcal{U})$.

Proposition 5.3. *There exists $T_1 \leq T_0$ such that if $T \leq T_1$, $\mathcal{T}$ is a contraction mapping on $B(R)$ in the $L^\infty(0,T;L^2)$-norm.*

It follows that there exists a unique $(\mathcal{A},\mathcal{N}) \in B(R)$ such that

$$\mathcal{T}(\mathcal{A},\mathcal{N}) = (\mathcal{A},\mathcal{N}),$$

that is a unique solution of

$$\begin{aligned}\partial_t\mathcal{A} + C(\nabla)\mathcal{A} + D\Delta\mathcal{A} =& B(\partial_t\mathcal{N},\mathcal{U}) + B(\mathcal{N},\mathcal{A}) + \sum_{j=1}^d A_j(\mathcal{A})\partial_{x_j}\mathcal{V} \\ &+ \sum_{j=1}^d A_j(\mathcal{A})\partial_{x_j}\mathcal{A} + F_1^{'}(\mathcal{U})\mathcal{A},\end{aligned}$$

$$\partial_t\mathcal{N} + E(\nabla)\mathcal{N} + F_2(\mathcal{V}) + \nabla(F_3(\mathcal{V},\mathcal{V})) = 0,$$

where $\mathcal{U}$ is given by

$$\mathcal{U} = \mathcal{U}_0 + \int_0^t \mathcal{A}(s)ds,$$

and $\mathcal{V}$ is the solution to the elliptic equation

$$\begin{aligned}\mathcal{A} + C(\nabla)\mathcal{V} + D(-\alpha + \Delta)\mathcal{V} =& \sum_{j=1}^d A_j(\mathcal{U})\partial_{x_j}\mathcal{V} + F_1(\mathcal{U}) \\ &+ B(\mathcal{N},\mathcal{U}) - \alpha D\mathcal{U}.\end{aligned} \tag{5.22}$$

• step 5 : Going back to the original problem. We still have to prove that $\mathcal{V} = \mathcal{U}$ which is done by making an estimate on the elliptic equation (5.22) just as in [12]. This ends the proof of Theorem (5.1).

6 Some complements.

6.1 Invariants.

System (3.6), (3.7), (3.8) and (3.9) has some invariants that we will make more precise now. We rewrite this system using the relations of section 4 :

$$(i(\partial_t + v_C\partial_y) + \alpha\Delta)\, A_C = \frac{b^2}{2} < \delta n > A_C - (\nabla\cdot E_0)\, A_R e^{-i\theta} \tag{6.1}$$

$$(i(\partial_t + v_R\partial_y) + \beta\Delta)\, A_R = \frac{bc}{2} < \delta n > A_R - (\nabla\cdot E_0^*)\, A_C e^{i\theta}, \tag{6.2}$$

$$i\partial_t E_0 + \gamma\Delta E_0 = \frac{b}{2}\nabla\Delta^{-1}\mathrm{div}(< \delta n > E_0) + \nabla\left(A_R^* A_C e^{i\theta}\right), \tag{6.3}$$

$$\partial_t^2 < \delta n > -v_s^2\Delta < \delta n >= a\Delta\left(|E_0|^2 + b|A_C|^2 + c|A_R|^2\right), \tag{6.4}$$

where $\theta = k_1 y - \omega_1 t$ and

$$v_C = \frac{k_0^2 c^2}{\omega_0^2},\quad v_R = \frac{k_R k_0 c^2}{\omega_R\omega_0},\quad \alpha = \frac{c^2 k_0^2}{2\omega_0^2},\quad \beta = \frac{c^2 k_0^2}{2\omega_R\omega_0},$$
$$\gamma = \frac{v_{th}^2 k^2}{2\omega_{pe}\omega_0},\quad a = \frac{4m_e}{m_i}\frac{\omega_0\omega_R}{\omega_{pe}^2},\quad b = \frac{\omega_{pe}}{\omega_0},\quad c = \frac{\omega_{pe}}{\omega_R},\quad v_s = \frac{c_s k_0}{\omega_0}.$$

Proposition 6.1. *For any regular solution of* (6.1) − (6.4)*, one has*

$$\frac{d}{dt}\int\left(2|A_C|^2 + |A_R|^2 + |E_0|^2\right) = 0.$$

Proof : Multiply (6.1) by $2A_C^*$, (6.2) by A_R^*, (6.3) by E_0^*, integrating over $\mathbb{R}^d$, summing the results and taking the imaginary part gives

$$\begin{aligned}
&\frac{1}{2}\partial_t\int\left(2|A_C|^2 + |A_R|^2 + |E_0|^2\right)\\
&= \mathrm{Im}\int\left(-2\nabla\cdot E_0 A_R A_C^* e^{-i\theta} - \nabla\cdot E_0^* A_R^* A_C e^{i\theta} + \nabla(A_R^* A_C e^{i\theta})E_0^*\right),\\
&= \mathrm{Im}\int\left(2\nabla\cdot E_0^* A_R^* A_C e^{i\theta} - \nabla\cdot E_0^* A_R^* A_C e^{i\theta} - A_R^* A_C\nabla\cdot E_0^* e^{i\theta}\right),\\
&= 0. \quad \square
\end{aligned}$$

Note that, this does not means that the quasilinear part is hyperbolic. In fact, as noticed before, it is not! For some particular values of the parameters, we have a Hamiltonian structure :

Proposition 6.2. *Assume that $\omega_1 = 0$ (that is $\theta = k_1 y$). Then, for any regular solution to* (6.1) − (6.4) *such that $\delta n \in \mathcal{C}([0,T]; H^{-1})$*

$$\frac{d}{dt}\Bigg(\int \frac{1}{2}\Bigg(\alpha|\nabla A_C|^2 + \beta|\nabla A_R|^2 + \gamma|\nabla E_0|^2 + \delta n(\frac{b^2}{2}|A_C|^2 + \frac{bc}{2}|A_R|^2$$
$$+ \frac{b}{2}|E_0|^2) + \frac{bv_s^2}{8a}(|\delta n|^2 + \frac{1}{v_s^2}\ |\nabla\Delta^{-1}\partial_t \delta n|^2)\Big) - Re\int \big(\nabla\cdot E_0^* A_R^* A_C e^{i\theta}\big)$$
$$+ \frac{1}{2} Im \int (v_C \partial_y A_C A_C^* + v_R \partial_y A_R A_R^*)\Bigg) = 0$$

Proof : Multiply (6.1) by $\partial_t A_C^*$, (6.2) by $\partial_t A_R^*$ and (6.3) by $\partial_t E_0^*$, summing the results, taking the real part and integrating over $\mathbb{R}^d$ leads to

$$\begin{aligned}
&-\operatorname{Im}\int (v_C \partial_y A_C \partial_t A_C^* + v_R \partial_y A_R \partial_t A_R^*)\\
&-\frac{1}{2}\partial_t \int (\alpha|\nabla A_C|^2 + \beta|\nabla A_R|^2 + \gamma|\nabla E_0|^2)\\
&= \frac{1}{2}\int \delta n\ (\frac{b^2}{2}\partial_t|A_C|^2 + \frac{bc}{2}\partial_t|A_R|^2 + \frac{b}{2}\partial_t|E_0|^2)\\
&+ \operatorname{Re}\int (-\nabla\cdot E_0 A_R \partial_t A_C^* e^{-i\theta} - \nabla\cdot E_0^* A_C \partial_t A_R^* e^{i\theta})\\
&+ \operatorname{Re}\int \nabla(A_R^* A_C e^{i\theta})\partial_t E_0^*.
\end{aligned} \tag{6.5}$$

We compute each terms separately.

$$\begin{aligned}
&\operatorname{Im}\int (v_C \partial_y A_C \partial_t A_C^* + v_R \partial_y A_R \partial_t A_R^*)\\
&= -\operatorname{Im}\int (v_C A_C \partial_t \partial_y A_C^* + v_R A_R \partial_t \partial_y A_R^*),\\
&= \operatorname{Im}\int (v_C A_C^* \partial_t \partial_y A_C + v_R A_R^* \partial_t \partial_y A_R).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\operatorname{Im}\int (v_C \partial_y A_C \partial_t A_C^* + v_R \partial_y A_R \partial_t A_R^*)\\
&= \frac{1}{2}\operatorname{Im}\ \partial_t \int (v_C \partial_y A_C A_C^* + v_R \partial_y A_R A_R^*.
\end{aligned} \tag{6.6}$$

Moreover

$$\begin{aligned}
&\int \delta n\ (\frac{b^2}{2}\partial_t|A_C|^2 + \frac{bc}{2}\partial_t|A_R|^2 + \frac{b}{2}\partial_t|E_0|^2)\\
&= \partial_t \int \delta n\ (\frac{b^2}{2}|A_C|^2 + \frac{bc}{2}|A_R|^2 + \frac{b}{2}|E_0|^2)\\
&- \int \partial_t \delta n\ (\frac{b^2}{2}|A_C|^2 + \frac{bc}{2}|A_R|^2 + \frac{b}{2}|E_0|^2).
\end{aligned} \tag{6.7}$$

The last term is

$$\mathrm{Re}\int(-\nabla\cdot E_0 A_R\partial_t A_C^* e^{-i\theta}-\nabla\cdot E_0^* A_C\partial_t A_R^* e^{i\theta}+\nabla(A_R^* A_C e^{i\theta})\partial_t E_0^*)$$
$$=-\mathrm{Re}\int\partial_t(\nabla\cdot E_0^* A_R^* A_C)e^{i\theta}. \tag{6.8}$$

Now (6.6), (6.7), (6.8) in (6.5) provides

$$\begin{aligned}
&\partial_t\Big(-\mathrm{Im}\frac{1}{2}\int(v_C\partial_y A_C A_C^*+v_R\partial_y A_R A_R^*)\\
&-\frac{1}{2}\int(\alpha|\nabla A_C|^2+\beta|\nabla A_R|^2+\gamma|\nabla E_0|^2)\\
&-\frac{1}{2}\int\delta n\;(\frac{b^2}{2}|A_C|^2+\frac{bc}{2}|A_R|^2+\frac{b}{2}|E_0|^2)\Big)\\
&=-\frac{1}{2}\int\partial_t\delta n\;(\frac{b^2}{2}|A_C|^2+\frac{bc}{2}|A_R|^2+\frac{b}{2}|E_0|^2)\\
&-\mathrm{Re}\int\partial_t(\nabla\cdot E_0^* A_R^* A_C)e^{i\theta}.
\end{aligned}\tag{6.9}$$

We still have to deal with the two terms in the right-hand-side of (6.9). For the second one, recalling that $\theta=k_1y-\omega_1t$, we take $\omega_1=0$ (which is not a physical case, see next section) and we get

$$\mathrm{Re}\int\partial_t(\nabla\cdot E_0^* A_R^* A_C)e^{i\theta}=\partial_t\int\mathrm{Re}(\nabla\cdot E_0^* A_R^* A_C e^{ik_1y}). \tag{6.10}$$

For the first term, we need to write the first order equation satisfied by δn. Introducing $V=\frac{1}{v_s}\nabla\Delta^{-1}\partial_t\delta n$ gives

$$\partial_t\delta n=v_s\nabla\cdot V, \tag{6.11}$$
$$\partial_t V=v_s\nabla\delta n+\frac{a}{v_s}\nabla(|E_0|^2+b|A_C|^2+c|A_R|^2). \tag{6.12}$$

Multiply (6.11) by δn, (6.12) by V provides

$$\begin{aligned}
\frac{1}{2}\partial_t\int(|\delta n|^2+V^2)&=-\frac{a}{v_s}\int(|E_0|^2+b|A_C|^2+c|A_R|^2)\nabla\cdot V,\\
&=-\frac{a}{v_s^2}\int(|E_0|^2+b|A_C|^2+c|A_R|^2)\partial_t\delta n.
\end{aligned}$$

Then

$$\begin{aligned}
\int\partial_t\delta n\;(\frac{b^2}{2}|A_C|^2+\frac{bc}{2}|A_R|^2+\frac{b}{2}|E_0|^2)=&\frac{b}{2}\int\partial_t\delta n\;(b|A_C|^2+c|A_R|^2+|E_0|^2)\\
&=-\frac{bv_s^2}{4a}\partial_t\int(|\delta n|^2+V^2).
\end{aligned}\tag{6.13}$$

Using (6.10) and (6.13) in (6.9) gives

$$\partial_t \Bigg(\int \frac{1}{2} \Bigg(\alpha|\nabla A_C|^2 + \beta|\nabla A_R|^2 + \gamma|\nabla E_0|^2 + \delta n(\frac{b^2}{2}|A_C|^2 + \frac{bc}{2}|A_R|^2$$
$$+ \frac{b}{2}|E_0|^2) + \frac{bv_s^2}{8a}(|\delta n|^2 + \frac{1}{v_s^2}\,|\nabla\Delta^{-1}\partial_t\delta n|^2)\Big) - \mathrm{Re}\int \big(\nabla\cdot E_0^* A_R^* A_C e^{i\theta}\big)$$
$$+ \frac{1}{2}\mathrm{Im}\int (v_C\partial_y A_C A_C^* + v_R\partial A_R A_R^*)\Bigg) = 0$$

which ends the proof of Proposition 6.2. □
At this point, we do not know how to use this conservation law.

6.2 Back to the resonance condition.

In order to obtain our model, we imposed the following relation (2.31) :

$$k_0 = k_R + k_1, \quad \omega_0 = \omega_R + \omega_{pe} + \omega_1,$$

where

$$\omega_0^2 = \omega_{pe}^2 + k_0^2 c^2, \quad \omega_R^2 = \omega_{pe}^2 + k_R^2 c^2.$$

In fact, in the physical literature, one finds

$$(\omega_{pe} + \omega_1)^2 = \omega_{pe}^2 + v_{th}^2 k_1^2.$$

This is the classical three waves resonance. At the first approximation, this gives (using $\omega_1 \ll \omega_{pe}$) :

$$2\omega_1\omega_{pe} \simeq v_{th}^2 k_1^2 \quad \Longrightarrow \quad \omega_1 \simeq \frac{v_{th}^2 k_1^2}{2\omega_{pe}}. \tag{6.14}$$

Now rewrite (2.38)

$$i\partial_t E_0 + \frac{v_{th}^2}{2\omega_{pe}}\Delta E_0 = \frac{\omega_{pe}}{2}\nabla\Delta^{-1}\mathrm{div}(< \delta n > E_0) + \frac{e\omega_{pe}}{2c^2 m_e}\nabla(A_R^* A_0 e^{i(k_1 y - \omega_1 t)}).$$

The dispersion relation of the linear part is

$$\omega = \frac{v_{th}^2 k^2}{2\omega_{pe}}.$$

Equation (6.14) means that (k_1, ω_1) satisfies the dispersion relation and therefore the term $A_R^* A_0 e^{i(k_1 y - \omega_1 t)}$ is resonant and the process will be efficient. More precisely, since $v_{th} = \lambda_D \omega_{pe}$ where λ_D is the Debye's length, the equation on E_0 can be written in 1-D (neglecting δn and considering A_R and A_0 as given)

$$i\partial_t E_0 + \frac{\lambda_D^2 \omega_{pe}}{2}\partial_y^2 E_0 = Ae^{i(k_1 y - \omega_1 t)}. \tag{6.15}$$

Let L be a characteristic scale in space, T in time. The dimensionless form of (6.15) reads :

$$i\partial_t E_0 + \frac{\lambda_D^2 \omega_{pe} T}{2L^2}\partial_y^2 E_0 = ATe^{i((k_1 L)y-(\omega_1 T)t)}.$$

Introduce $k_1 L = \frac{k}{\varepsilon}$, $\omega_1 T = \frac{\omega}{\varepsilon}$, and replace AT by A. Since $\omega_1 = \omega_{pe}\lambda_D^2 k_1^2$ and $\omega_1 T = \frac{\omega}{\varepsilon}$ (with $\omega, k = O(1)$), one gets

$$\omega_{pe}\frac{\lambda_D^2 T}{L^2} = \frac{\omega_1 T}{k_1^2 L^2} = \frac{\varepsilon^2 \omega}{k^2 \varepsilon} = \frac{\omega}{k^2}\varepsilon = O(\varepsilon).$$

The dimensionless form of (6.15) is therefore

$$i\partial_t E_0 + \frac{\varepsilon\omega}{k^2}\partial_y^2 E_0 = Ae^{i\frac{(ky-\omega t)}{\varepsilon}}. \tag{6.16}$$

A standard WKB expansion shows that

$$E_0 = Be^{i\frac{(ky-\omega t)}{\varepsilon}} + O(\varepsilon),$$

with

$$i(\partial_t B + \frac{\omega}{k}\partial_y B) = A.$$

This means a linear growth in time for B at the group velocity $\frac{\omega}{k}$. On the other hand, if the equation is

$$i\partial_t E_0 + \frac{\varepsilon\omega}{k^2}\partial_y^2 E_0 = Ae^{i\frac{(ky-\omega' t)}{\varepsilon}},$$

with $\omega' \neq \omega$, then E_0 will stay of size $O(\varepsilon)$. Of course, nonlinear versions of these results can be proved. We postpone the numerical study of this problem to a further work.

6.3 Open problems

The existence and stability of solitary waves for these equations is open. It can certainly be done for the coefficient corresponding to Proposition 6.2. It would be interesting also to have some existence proofs for system (6.1) – (6.4) with an existence time and bounds on the solutions which are independent of k_1 and ω_1 in order to investigate the limit $k_1 \to +\infty$ and $\omega_1 \to +\infty$ as in section 6.2. Our proof is clearly not uniform with respect to these parameters.

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