A numerical model for the Raman amplification for laser-plasma interaction

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Abstract : In this paper, we continue the study of the Raman amplification initiated in [3]. We use a dispersive, quasilinear system. The quasilinear part is not hyperbolic and this difficulty is overcomed using the dispersion. We give an asymptotic result on a reduced system. We then introduce a simple, robust and efficient numerical scheme on the whole system that takes into account the non-hyperbolicity of the quasilinear part as well as the nonlinear saturation of the Raman growth. The scheme is validated thanks to the asymptotic result. Finally, we present 1-D and 2-D simulations.

1 Introduction and statement of the result.

1.1 Position of the problem.

The aim of this paper is to provide and vaildate a robust numerical method for the study of simulated Raman scattering in a plasma. The starting point is the model introduced for example in [7] that we have modified in [3]. This model describes the coupling effects between the incident laser field, the backscattered Raman component, the electronic-plasma wave and the ionic accoustic wave. For practical reason, this system is written using the vector potential A_C of the incident laser field, the vector potential A_R of the Raman component, the electric field E corresponding to the electronic-plasma wave and p the modulation of density of ions. The 3-D system, can be written in a dimensionless form (the unit of time is $\frac{1}{\omega_0}$ and of space is $\frac{1}{k_0}$ where (k_0, ω_0) are respectively the wave

AMS classification scheme number: 35Q55, 35Q60, 78A60, 74S20 number and the frequency of the incident laser field)

$$\left(i(\partial_t + v_C\partial_y) + \alpha_1\partial_y^2 + \alpha_2\Delta_\perp\right) A_C = \frac{b^2}{2} pA_C - (\nabla \cdot E)\, A_R e^{-i\theta}, \qquad (1.1)$$

$$\left(i(\partial_t + v_R\partial_y) + \beta_1\partial_y^2 + \beta_2\Delta_\perp\right) A_R = \frac{bc}{2} pA_R - (\nabla \cdot E^*)\, A_C e^{i\theta}, \qquad (1.2)$$

$$(i\partial_t + \gamma\nabla\nabla \cdot -\delta\nabla \times \nabla\times)\, E = \frac{b}{2} pE + \nabla(A_R^* \cdot A_C e^{i\theta}), \qquad (1.3)$$

$$(\partial_t^2 - v_s^2\Delta)p = a\Delta\left(|E|^2 + b|A_C|^2 + c|A_R|^2\right). \qquad (1.4)$$

The vectors A_0, A_R and E are such that

$$A_0, A_R, E \;:\; \mathbb{R}^3 \longrightarrow \mathbb{C}^3.$$

See below for precise values of the constants.
If A and B are two vectors of $\mathbb{R}^3$, the inner product in $L^2(\mathbb{R}^3)$ is denoted by $A \cdot B$.

The direction of propagation of the laser is y. The tranverse directions are x and z. We denote $\Delta_\perp = \partial_x^2 + \partial_z^2$. The main frequency of the laser is ω_0 and k_0 is the corresponding wave number. They satisfy the dispersion relation for electromagnetic waves in a plasma

$$\omega_0^2 = \omega_{pe}^2 + k_0^2 c_0^2, \qquad (1.5)$$

where c_0 is the velocity of light in vaccum and ω_{pe} the electronic-plasma frequency. The main frequency of the Raman component will be denoted by ω_R and k_R will be the wave number. They satisfy the same condition as (1.5)

$$\omega_R^2 = \omega_{pe}^2 + k_R^2 c_0^2. \qquad (1.6)$$

Furthermore,

$$\theta = \frac{k_1}{k_0} y - \frac{\omega_1}{\omega_0} t,$$

and k_1, ω_1 satisfy

$$k_0 = k_R + k_1, \qquad (1.7)$$

$$\omega_0 = \omega_R + \omega_{pe} + \omega_1. \qquad (1.8)$$

This system describes therefore a three-waves interaction. The resonance condition is that the third wave $(\omega_{pe} + \omega_1, k_1)$ satisfies the dispersion relation of the electronic-plasma wave namely

$$(\omega_{pe} + \omega_1)^2 = \omega_{pe}^2 + v_{th}^2 k_1^2, \qquad (1.9)$$

where v_{th} is the thermal velocity of the electrons. The complete electronic field is recovered by

$$i\frac{\omega_0}{c_0}\tilde{A}_C e^{i(k_0 y - \omega_0 t)} + i\frac{\omega_R}{c_0}\tilde{A}_R e^{i(k_R y - \omega_R t)} + \tilde{E}_0 e^{-i\omega_{pe} t} + c.c.$$

where $\tilde{A}_C$, $\tilde{A}_R$ and $\tilde{E}$ are the dimensional value of the fields given by $\tilde{A}_i = \dfrac{\alpha c}{\omega_{pe}\sqrt{\omega_i}} A_i$ with $i = C, R$, $\tilde{E} = \sqrt{\alpha}\sqrt{\omega_{pe}} E$ and parameter α is given by $\alpha = \dfrac{2m_e\omega_0\sqrt{\omega_0\omega_R\omega_{pe}}}{ek_0}$.

With these notations, the constants in system $(1.1)-(1.4)$ are given below. The dispersion coefficients are

$$\alpha_1 = \frac{c_0^2 k_0^2 \omega_{pe}^2}{2\omega_0^4}, \quad \alpha_2 = \frac{c_0^2 k_0^2}{2\omega_0^2}, \quad \beta_1 = \frac{c_0^2 k_0^2 \omega_{pe}^2}{2\omega_R^3 \omega_0}, \quad \beta_2 = \frac{c_0^2 k_0^2}{2\omega_R \omega_0},$$

$$\gamma = \frac{v_{th}^2 k_0^2}{2\omega_{pe}\omega_0}, \quad \delta = \frac{c_0^2 k_0^2}{2\omega_{pe}\omega_0}, \quad v_s = \frac{c_s k_0}{\omega_0},$$

where c_s is the sound velocity in the plasma. The group velocity v_C and v_R are given by

$$v_C = \frac{k_0^2 c_0^2}{\omega_0^2}, \quad v_R = \frac{k_R k_0 c_0^2}{\omega_R \omega_0}.$$

The coefficients of the nonlinearities are

$$a = 4\frac{m_e}{m_i}\frac{\omega_0\omega_R}{\omega_{pe}^2}, \quad b = \frac{\omega_{pe}}{\omega_0}, \quad c = \frac{\omega_{pe}}{\omega_R},$$

where m_e and m_i are respectively the mass of electrons and ions.

Note that the resonance condition (1.9) can be written

$$2\omega_{pe}\omega_1 + \omega_1^2 = v_{th}^2 k_1^2.$$

Since $\omega_1 \ll \omega_{pe}$, one gets

$$\omega_1 \approx \frac{v_{th}^2 k_1^2}{2\omega_{pe}},$$

that is

$$\frac{\omega_1}{\omega_0} = \frac{v_{th}^2 k_0^2}{2\omega_{pe}\omega_0}\left(\frac{k_1}{\omega_0}\right)^2 = \gamma\left(\frac{k_1}{\omega_0}\right)^2.$$

It means that in this case, the oscillation $e^{i\theta}$ is resonant for the Schrödinger operator $i\partial_t + \gamma\nabla\nabla\cdot$.

Notations : As usual, we denote by $L^p(\mathbb{R}^d)$ the Lebesgue space

$$L^p(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \ / \ ||u||_p < +\infty \right\}$$

where

$$||u||_p = \left(\int_{\mathbb{R}^d} |u(x)|^p dx\right)^{\frac{1}{p}} \text{ if } 1 \leq p < +\infty$$

and

$$||u||_\infty = \text{ess.sup}\left\{|u(x)|; x \in \mathbb{R}^d\right\}.$$

We define the Sobolev space $H^s(\mathbb{R}^d)$ as follows

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d) \ / \ ||u||^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < +\infty \right\}$$

where $\widehat{u}(\xi)$ is the Fourier transform of u. Let $C(I,E)$ be the space of continuous functions from an interval I of $\mathbb{R}$ to a Banach space E. For $1 \leq j \leq d$, we set $\partial_{x_j} u = \dfrac{\partial u}{\partial x_j}$. Different positive constants might be denoted by the same letter C. We also denote by $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ the real part and the imaginary part of u.

1.2 The main result.

The Cauchy problem for system $(1.1)-(1.4)$ has been solved in [3]. The difficulty is that the quasilinear part

$$\begin{aligned} i\partial_t A_C &= -\nabla \cdot E A_R, \\ i\partial_t A_R &= -\nabla \cdot E^* A_C, \\ i\partial_t E &= -\nabla(A_R^* \cdot A_C), \end{aligned}$$

is not hyperbolic. This difficulty is overcomed using the full dispersion. We apply this method here to study a simplified problem in the semi-classical scaling, namely

$$\left\{ \begin{array}{l} i(\partial_t + v_C \partial_y) A_C + \varepsilon(\alpha_1 \partial_y^2 + \alpha_2 \Delta_\perp) A_C = -\varepsilon \left(\nabla \cdot E\right) A_R e^{-i\frac{(k_1 y - \omega_1 t)}{\varepsilon}}, \\ \\ i(\partial_t + v_R \partial_y) A_R + \varepsilon(\beta_1 \partial_y^2 + \beta_2 \Delta_\perp) A_R = -\varepsilon \left(\nabla \cdot E^*\right) A_C e^{i\frac{(k_1 y - \omega_1 t)}{\varepsilon}}, \\ \\ (i\partial_t + \varepsilon\gamma\Delta - \delta \nabla \times \nabla \times) E = \varepsilon \nabla(A_R^* \cdot A_C e^{i\frac{(k_1 y - \omega_1 t)}{\varepsilon}}), \end{array} \right. \quad (1.10)$$

where ε is a small parameter that will tend to 0. See section 3 to see how (1.10) is linked to $(1.1) - (1.4)$. We will show how to modify the proof of [3] in order to construct a solution of (1.10) on a time interval independent of ε and bounded independently of ε. Our main result reads as follows. The asymptotic behaviour of the solution is different wether the resonance condition is satisfied or not. Here this condition reads $\omega_1 = \gamma k_1^2$. Denoting

$$E = \mathcal{E}^\varepsilon e^{i\frac{(k_1 y - \omega_1 t)}{\varepsilon}},$$

in the resonant case we introduce the following limit system

$$\left\{ \begin{array}{l} (\partial_t + v_C \partial_y) A_C = -(k \cdot \mathcal{E}) A_R, \\ (\partial_t + v_R \partial_y) A_R = (k \cdot \mathcal{E}^*) A_C, \\ \partial_t \mathcal{E} + 2\gamma k \cdot \nabla \mathcal{E} = (A_R^* \cdot A_C) k, \end{array} \right. \quad (1.11)$$

where $k = (0, k_1, 0)$.

In the non-resonant case, the limit system reads, introducing

$$\mathcal{E}^\varepsilon = F^\varepsilon e^{i(\omega_1-\gamma k_1^2)\frac{t}{\varepsilon}},$$

$$\begin{cases} (\partial_t + v_C\partial_y)A_C = 0, \\ (\partial_t + v_R\partial_y)A_R = 0, \\ (\partial_t + 2\gamma k\cdot\nabla)F = 0. \end{cases} \tag{1.12}$$

Theorem 1.1. *Take A_C^0, A_R^0, $\mathcal{E}^0$ in $H^s(\mathbb{R}^d)$ for s large enough. There exists a time T independent of ε and a unique solution $(A_C^\varepsilon, A_R^\varepsilon, \mathcal{E}^\varepsilon)$ of (1.10) such that*

$$A_C^\varepsilon(0) = A_C^0, \;\; A_R^\varepsilon(0) = A_R^0, \;\; \mathcal{E}^\varepsilon(0) = \mathcal{E}^0 e^{i\frac{k_1 y}{\varepsilon}}.$$

i) Suppose $\omega_1 = \gamma k_1^2$ and let $(A_C, A_R, \mathcal{E})$ be the solution to (1.11) such that

$$A_C(0) = A_C^0, \;\; A_R(0) = A_R^0, \;\; \mathcal{E}(0) = \mathcal{E}^0,$$

one has

$$\left(A_C^\varepsilon - A_C, A_R^\varepsilon - A_R, Ee^{-i\frac{(k_1 y-\omega_1 t)}{\varepsilon}} - \mathcal{E}\right) \longrightarrow 0$$

as ε goes to 0 in $\left[L^\infty(0,T;H^\sigma(\mathbb{R}^d))\right]^{3d}$ for $\sigma > \frac{d}{2}$.
ii) If $\omega_1 \neq \gamma k_1^2$, let (A_C, A_R, F) be the solution to (1.12) such that

$$A_C(0) = A_C^0, \;\; A_R(0) = A_R^0, \;\; F(0) = \mathcal{E}^0,$$

one has

$$\left(A_C^\varepsilon - A_C, A_R^\varepsilon - A_R, Ee^{-i\frac{(k_1 y-\omega_1 t)}{\varepsilon}} - Fe^{i(\omega_1-\gamma k_1^2)\frac{t}{\varepsilon}}\right) \longrightarrow 0$$

as ε goes to 0 in $\left[L^\infty(0,T;H^\sigma(\mathbb{R}^d))\right]^{3d}$ for $\sigma > \frac{d}{2}$.

Remark In the non-resonant case (ii), the limit system is linear. Basically, one starts with $A_R(t=0)$ and $E(t=0)$ nearly zero. Therefore A_R and F do not grow in time and the Raman effect does not hold.
In the resonant case (i), the limit system is non linear and it is easy to see that for

$$\begin{aligned} \partial_t A_c &= -(k\cdot\mathcal{E})A_R, \\ \partial_t A_R &= -(k\cdot\mathcal{E}^*)A_C, \\ \partial_t \mathcal{E} &= (A_R^*\cdot A_C)k, \end{aligned}$$

the stationnary point $(\alpha, 0, 0)$ is unstable. Therefore, the Raman component and the electronic plasma wave will grow in time.

1.3 The numerical scheme.

In view of the preceding result, and because of the ill-posedness of the quasilinear part, we want to construct a numerical scheme that couples the nonlinearity with the dispersion. It will be done using a Crank-Nicolson-type scheme. In order to avoid expensive nonlinear steps, we use a relaxation method inspired of [1]. For the accoustic part, we use Glassey's scheme (see [5]).

Another important feature is the conservation of an L^2-invariant by system (1.1) – (1.4). Our scheme has the same property. It corresponds to a nonlinear saturation of the Raman amplification. We validate our scheme thanks to the properties described with the asymptotic expansion presented above. We perform 1-D and 2-D computations. The outline of the paper is the following one. Section 2 is devoted to the proof of Theorem (1.1). In section 3, we introduce the numerical scheme in 1-D and validate our results with Theorem (1.1). In section 4, we extend our scheme to the 2-D case and present some results of computations.

Acknowledgments: This work was partially supported by the European network HYKE, funded by the EC as contract HPRN-CT-2002-00282 and GDR EAPQ 2103 CNRS.

2 Asymptotic analysis of the Raman amplification.

2.1 Presentation of the problem.

The aim of this section is to provide a simple explanation of the Raman amplification observed in physics as well as in the numerical experiment of Section 3 and 4. We start with the system considered in the introduction

$$\left(i(\partial_t + v_C\partial_y) + \alpha_1\partial_y^2 + \alpha_2\Delta_\perp\right) A_C = \frac{b^2}{2} p A_C - (\nabla\cdot E)\, A_R e^{-i\theta}, \tag{2.1}$$

$$\left(i(\partial_t + v_R\partial_y) + \beta_1\partial_y^2 + \beta_2\Delta_\perp\right) A_R = \frac{bc}{2} p A_R - (\nabla\cdot E^*)\, A_C e^{i\theta}, \tag{2.2}$$

$$(i\partial_t + \gamma\nabla\nabla\cdot - \delta\nabla\times\nabla\times)\, E = \frac{b}{2} p E + \nabla(A_R^*\cdot A_C e^{i\theta}), \tag{2.3}$$

$$(\partial_t^2 - v_s^2\Delta) p = a\Delta\left(|E|^2 + b|A_C|^2 + c|A_R|^2\right), \tag{2.4}$$

with $\theta = k_1 y - \omega_1 t$. The precise values of all parameters are given in the introduction. We will consider here the adiabatic limit which consists in taking $\partial_t^2 p = 0$ in (2.4). This implies

$$p = -\frac{a}{v_s^2}\left(|E|^2 + b|A_C|^2 + c|A_R|^2\right). \tag{2.5}$$

Since we will consider very regular solutions, the cubic terms won't play any roles in the analysis and we will omit them in the sequel. Finally, we will consider a semi-classical regime, which is the best one in order to study the instability:

$$i(\partial_t + v_C\partial_y)A_C + \varepsilon(\alpha_1\partial_y^2 + \alpha_2\Delta_\perp)A_C = -\varepsilon\,(\nabla\cdot E)\,A_R e^{-i\frac{(k_1 y-\omega_1 t)}{\varepsilon}}, \tag{2.6}$$

$$i(\partial_t + v_R\partial_y)A_R + \varepsilon(\beta_1\partial_y^2 + \beta_2\Delta_\perp)A_R = -\varepsilon\,(\nabla\cdot E^*)\,A_C e^{i\frac{(k_1 y-\omega_1 t)}{\varepsilon}}, \tag{2.7}$$

$$i\partial_t E + \varepsilon(\gamma\nabla\nabla\cdot - \delta\nabla\times\nabla\times)E = \varepsilon\nabla(A_R^*\cdot A_C e^{i\frac{(k_1 y-\omega_1 t)}{\varepsilon}}), \tag{2.8}$$

We will consider this system when $\varepsilon\to 0$ and we will show that the behaviour of the solution strongly depends on the value of $\omega_1-\gamma k_1^2$ is satisfied or not. First, one can decouple the transverse and longitudinal part of the electronic-plasma wave E as follows. Write $E = E_\perp + E_{||}$ with

$$E_\perp = -\nabla\times(\Delta^{-1}\nabla\times E), \quad E_{||} = \nabla\Delta^{-1}\mathrm{div}(E).$$

Then (2.6) – (2.8) becomes (using $\nabla\cdot E_\perp = 0$ and $\nabla\times E_{||}=0$)

$$i(\partial_t + v_C\partial_y)A_C + \varepsilon(\alpha_1\partial_y^2 + \alpha_2\Delta_\perp)A_C = -\varepsilon\left(\nabla\cdot E_{||}\right)A_R e^{-i\frac{(k_1 y-\omega_1 t)}{\varepsilon}}, \tag{2.9}$$

$$i(\partial_t + v_R\partial_y)A_R + \varepsilon(\beta_1\partial_y^2 + \beta_2\Delta_\perp)A_R = -\varepsilon\left(\nabla\cdot E_{||}^*\right)A_C e^{i\frac{(k_1 y-\omega_1 t)}{\varepsilon}}, \tag{2.10}$$

$$(i\partial_t + \varepsilon\gamma\Delta)E_{||} = \varepsilon\nabla(A_R^*\cdot A_C e^{i\frac{(k_1 y-\omega_1 t)}{\varepsilon}}), \tag{2.11}$$

$$(i\partial_t + \varepsilon\delta\Delta)E_\perp = 0. \tag{2.12}$$

From now on, we take $E_\perp = 0$ and $E = E_{||}$. Denoting $k=(0,k_1,0)$ and

$$E = \mathcal{E}e^{i\frac{(k_1 y-\omega_1 t)}{\varepsilon}},$$

one gets after a straightforward computation:

$$i(\partial_t + v_C\partial_y)A_C + \varepsilon(\alpha_1\partial_y^2 + \alpha_2\Delta_\perp)A_C = -i(k\cdot\mathcal{E})A_R - \varepsilon\,(\nabla\cdot\mathcal{E})\,A_R, \tag{2.13}$$

$$i(\partial_t + v_R\partial_y)A_R + \varepsilon(\beta_1\partial_y^2 + \beta_2\Delta_\perp)A_R = -i(k\cdot\mathcal{E}^*)A_C - \varepsilon\,(\nabla\cdot\mathcal{E}^*)\,A_C, \tag{2.14}$$

$$(\,i\partial_t + \frac{\omega_1-\gamma k_1^2}{\varepsilon} + 2i\gamma k\cdot\nabla + \varepsilon\gamma\Delta)\mathcal{E} = i(A_R^*\cdot A_C)k + \varepsilon\nabla(A_R^*\cdot A_C), \tag{2.15}$$

This is the system that we are going to study in the remaining of this section.

2.2 Local existence for fixed ε.

In this section, we recall how the proof of [3] can be adapted to (2.13) – (2.15) in order to obtain local existence for fixed ε.

Proposition 2.1. *Let $s > \frac{d}{2}+3$ and $(A_C^0, A_R^0, \mathcal{E}^0) \in \left[H^s(\mathbb{R}^d)\right]^{3d}$. There exist $T(\varepsilon)>0$ and a unique $(A_C, A_R, \mathcal{E}) \in \left[L^\infty(0,T(\varepsilon); H^s(\mathbb{R}^d))\right]^{3d}$ solution to (2.13) – (2.15) such that $(A_C,A_R,\mathcal{E})(0) = (A_C^0, A_R^0, \mathcal{E}^0)$.*

Proof: In order to simplify the proof, we omit the transport terms $v_C \partial_y A_C$, $v_R \partial_y A_R$, $2i\gamma k \cdot \nabla \mathcal{E}$, the semilinear terms as well as $\frac{\omega_1 - \gamma k_1^2}{\varepsilon}$ in (2.15). We also set $\varepsilon = 1$ and $\gamma = 1$ and we consider:

$$i\partial_t A_C + L_C A_C = -(\nabla \cdot \mathcal{E}) A_R, \tag{2.16}$$

$$i\partial_t A_R + L_C A_R = -(\nabla \cdot \mathcal{E}^*) A_C, \tag{2.17}$$

$$i\partial_t \mathcal{E} + \Delta \mathcal{E} = \nabla (A_R^* \cdot A_C), \tag{2.18}$$

where

$$L_C = \alpha_1 \partial_y^2 + \alpha_2 \Delta_\perp, \quad L_R = \beta_1 \partial_y^2 + \beta_2 \Delta_\perp.$$

As noted in [3], the quasilinear part of (2.16) – (2.18) is not hyperbolic. It is easy to see in 1-D:

$$\begin{aligned} i\partial_t A_C &= -(\partial_x \mathcal{E}) A_R, \\ i\partial_t A_R &= -(\partial_x \mathcal{E}^*) A_C, \\ i\partial_t \mathcal{E} &= (\partial_x A_R^*) A_C + A_R^* \partial_x A_C. \end{aligned}$$

The part

$$\begin{aligned} i\partial_t A_C &= -(\partial_x \mathcal{E}) A_R, \\ i\partial_t \mathcal{E} &= A_R^* \partial_x A_C, \end{aligned}$$

is symmetric whereas the part

$$\begin{aligned} i\partial_t A_R &= -(\partial_x \mathcal{E}^*) A_C, \\ i\partial_t \mathcal{E} &= A_C \partial_x A_R^*, \end{aligned}$$

which can be rewritten using $e = \mathcal{E}^*$ as

$$\begin{aligned} \partial_t A_R &= i A_C \partial_x e, \\ \partial_t e &= i A_C^* \partial_x A_R, \end{aligned}$$

is elliptic.

In order to overcome this difficulty, we introduce the same kind of unknowns as in [3]. Let

$$\begin{aligned} &B_C = L_C A_C, \ \ B_R = L_R A_R, \ \ C_C = \partial_t A_C, \\ &C_R = \partial_t A_R, \ \ F = \Delta \mathcal{E}, \ \ G = \partial_t \mathcal{E}. \end{aligned} \tag{2.19}$$

Applying successively ∂_t and L_C on (2.16), ∂_t and L_R on (2.17), ∂_t and Δ on (2.18) and keeping the leading order terms leads to:

$$i\partial_t B_C + L_C B_C = -\nabla \cdot (L_C \mathcal{E}) A_R, \tag{2.20}$$

$$i\partial_t C_C + L_C C_C = -(\nabla \cdot G) A_R, \tag{2.21}$$

$$i\partial_t B_R + L_R B_R = -\nabla \cdot (L_R \mathcal{E}^*) A_C, \tag{2.22}$$

$$i\partial_t C_R + L_R C_R = -(\nabla \cdot G^*) A_C, \tag{2.23}$$

$$i\partial_t F + \Delta F = \nabla (\Delta A_R^* \cdot A_C + A_R^* \cdot \Delta A_C) \tag{2.24}$$

$$i\partial_t G + \Delta G = \nabla (C_R^* \cdot A_C + A_R^* \cdot C_C). \tag{2.25}$$

Note that the terms that have been omitted in this system are semilinear in the unknown $(B_C, C_C, B_R, C_R, F, G)$.

In order to express $L_C\mathcal{E}$ and $L_R\mathcal{E}$ in terms of $\Delta\mathcal{E}$, ΔA_R in terms of $L_R A_R$ and ΔA_C in terms of $L_C A_C$ one introduces

$$P_C = \sqrt{L_C\Delta^{-1}} \text{ and } P_R = \sqrt{L_R\Delta^{-1}}. \tag{2.26}$$

These operators are one-to-one on L^2 and homogeneous of order 0. Equations (2.20), (2.22) and (2.24) become

$$i\partial_t B_C + L_C B_C = -\nabla\cdot(P_C^2 F)A_R, \tag{2.27}$$
$$i\partial_t B_R + L_R B_R = -\nabla\cdot(P_R^2 F^*)A_C, \tag{2.28}$$
$$i\partial_t F + \Delta F = \nabla\left[(P_R^{-2}B_R^*)\cdot A_C + A_R^*\cdot(P_C^{-2}C_C)\right]. \tag{2.29}$$

Following [3], using equations (2.16) – (2.18), we write

$$\begin{aligned} &iC_C + B_C = \text{semilinear terms,}\\ &iC_R + B_R = \text{semilinear terms,}\\ &iG + F = \text{semilinear terms,}\end{aligned} \tag{2.30}$$

where the semilinear terms are semilinear in the new unknowns C_C, B_C, C_R, B_R, G, F.

Using (2.30) in the terms responsible for the elliptic part of the quasilinear terms gives the following system:

$$\begin{aligned}
&i\partial_t B_C + L_C B_C = -\nabla\cdot(P_C^2 F)A_R,\\
&i\partial_t C_C + L_C C_C = -(\nabla\cdot G)A_R,\\
&i\partial_t B_R + L_R B_R = -i\nabla\cdot(P_R^2 G^*)A_C,\\
&i\partial_t C_R + L_R C_R = i(\nabla\cdot F^*)A_C,\\
&i\partial_t F + \Delta F = i\nabla(P_R^{-2}C_R^*\cdot A_C) + \nabla(A_R^*\cdot P_C^{-2}B_C)\\
&i\partial_t G + \Delta G =_i \nabla(B_R^*\cdot A_C) + \nabla(A_R^*\cdot C_C).
\end{aligned}$$

If P_R and P_C would be equal to identity, this system would be symmetric, just like in [3]. We here have to make another change of unknowns:

$$\widetilde{B}_R = P_R^{-1}B_R,\quad \widetilde{B}_C = P_C^{-2}B_C,\quad \widetilde{C}_R = P_R^{-1}C_R,$$

to obtain

$$i\partial_t\widetilde{B}_C + L_C\widetilde{B}_C = -P_C^{-2}\left[\nabla\cdot(P_C^2 F)A_R\right], \tag{2.31}$$
$$i\partial_t\widetilde{C}_C + L_C\widetilde{C}_C = -(\nabla\cdot G)A_R, \tag{2.32}$$
$$i\partial_t\widetilde{B}_R + L_R\widetilde{B}_R = -iP_R^{-1}\left[\nabla\cdot(P_R^2 G^*)A_C\right], \tag{2.33}$$
$$i\partial_t\widetilde{C}_R + L_R\widetilde{C}_R = iP_R^{-1}\left[(\nabla\cdot F^*)A_C\right], \tag{2.34}$$
$$i\partial_t F + \Delta F = i\nabla(P_R^{-1}\widetilde{C}_R^*\cdot A_C) + \nabla(A_R^*\cdot\widetilde{B}_C) \tag{2.35}$$
$$i\partial_t G + \Delta G = -i\nabla(P_R\widetilde{B}_R^*\cdot A_C) + \nabla(A_R^*\cdot C_C). \tag{2.36}$$

In order to conclude we need:

Proposition 2.2. *For $A \in \mathbb{C}^n$ fixed, the adjoint of $-P_C^{-1}(A\nabla\cdot)$ in L^2 is $-m_{A^*}(P_C^{-1}\cdot)$ where $m_A(B) = A \cdot \nabla B + A \times (\nabla \times B)$.*

Proof: It is a simple consequence of Proposition 4.1 in [3].

Proposition 2.3. *The adjoint of $A\nabla\cdot$ in L^2 is $m_{A^*}(\cdot)$.*

Writting

$$P_C^{-2}\nabla \cdot (P_C^2 F \cdot A_R) = \nabla \cdot (F \cdot A_R) + R_1,$$

with

$$|R_1|_{H^s(\mathbb{R}^d)} \leq C|F|_{H^s(\mathbb{R}^d)}|A_R|_{H^s(\mathbb{R}^d)},$$

(see [2] for example), this shows that $(2.31) - (2.36)$ has a symmetric structure which ends the proof of Proposition 2.1. □

2.3 Estimates independent of ε.

Proposition 2.4. *Let $s > \frac{d}{2} + 3$ and $(A_C^0, A_R^0, \mathcal{E}^0) \in \left[H^s(\mathbb{R}^d)\right]^{3d}$. There exists $T > 0$ independent of ε and a unique $(A_C^\varepsilon, A_R^\varepsilon, \mathcal{E}^\varepsilon) \in \left[L^\infty(0, T; (H^s(\mathbb{R}^d))\right]^{3d}$ solution to $(2.13) - (2.15)$ such that $(A_C^\varepsilon, A_R^\varepsilon, \mathcal{E}^\varepsilon)(0) = (A_C^0, A_R^0, \mathcal{E}^0)$. Moreover, there exists a positive C such that*

$$|(A_C^\varepsilon, A_R^\varepsilon, \mathcal{E}^\varepsilon)|_{[L^\infty(0,T;(H^s(\mathbb{R}^d))]^{3d}} \leq C.$$

This means that the existence time does not shrink to 0 as ε goes to 0 and that the solution is bounded.

Proof: We go back to $(2.13) - (2.15)$ and in order to simplify, we take $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 1$ (the general case can be handled by adapting the techniques of the previous section).

The idea is then to perform the semi-classical version of the change of unknowns (2.19) by letting

$$\begin{array}{l} B_R = \varepsilon^2 \Delta A_R, \;\; B_C = \varepsilon^2 \Delta A_C, \;\; F = \varepsilon^2 \Delta \mathcal{E}, \\ C_R = \varepsilon \partial_t A_R, \;\; C_C = \varepsilon \partial_t A_C, \;\; G = \varepsilon \partial_t E. \end{array} \tag{2.37}$$

We have to investigate the relationships between B_R and C_R and so on. Let us multiply $(2.13) - (2.15)$ by ε and obtain:

$$i\varepsilon \partial_t A_C + i\varepsilon v_C \partial_y A_C + \varepsilon^2 \Delta A_C = -i\varepsilon (k \cdot \mathcal{E}) A_R + \varepsilon^2 (\nabla \cdot \mathcal{E}) A_R, \tag{2.38}$$

$$i\varepsilon \partial_t A_R + i\varepsilon v_R \partial_y A_R + \varepsilon^2 \Delta A_R = i\varepsilon (k \cdot \mathcal{E}^*) A_C - \varepsilon^2 (\nabla \cdot \mathcal{E}^*) A_C, \tag{2.39}$$

$$i\varepsilon (\partial_t + 2\gamma k \cdot \nabla)\mathcal{E} + (\omega_1 - \gamma k_1^2)\mathcal{E} + \varepsilon^2 \gamma \Delta \mathcal{E} = i\varepsilon (A_R^* \cdot A_C) k + \varepsilon^2 \nabla (A_R^* \cdot A_C), \tag{2.40}$$

Remark 2.1. *The singular term in (2.40) $(\omega_1 - \gamma k_1^2)$ is now of order zero with respect to ε.*

Remark 2.2. *The quasilinear terms are semilinear with respect to the new unknowns given in* (2.37) *uniformly with respect to* ε *(since* ε^2 *is in front of it).*

It follows that

$$\begin{aligned} iC_C + i\varepsilon v_C \partial_y A_C + B_C &= \text{semilinear terms}, \\ iC_R + i\varepsilon v_R \partial_y A_R + B_R &= \text{semilinear terms}, \\ iG + 2i\gamma\varepsilon k \cdot \nabla\mathcal{E} + \gamma F &= \text{semilinear terms}. \end{aligned}$$

The control of the order one terms is obtained by using, for all $f \in H^s(\mathbb{R}^d)$:

$$|\varepsilon\partial_y f|_{H^s(\mathbb{R}^d)} \le |f|_{H^s(\mathbb{R}^d)} + |\varepsilon^2 \Delta f|_{H^s(\mathbb{R}^d)}.$$

Therefore,

$$\begin{aligned} iC_C + B_C &= \text{semilinear terms}, \\ iC_R + B_R &= \text{semilinear terms}, \\ iG + \gamma F &= \text{semilinear terms}, \end{aligned}$$

where the controls of the semilinear terms are uniform with respect to ε. The remaining of the proof is the same than in the previous section. $\square$

2.4 Asymptotic description.

In this section, we investigate the limit $\varepsilon \to 0$ in (2.13) – (2.15). The behaviour depends on the value of ω_1.

Proposition 2.5. *Resonant case. Suppose* $\omega_1 = \gamma k_1^2$. *Then the solution of* (2.13) – (2.15) *given by Proposition* 2.4 *satisfies*

$$(A_C^\varepsilon, A_R^\varepsilon, \mathcal{E}^\varepsilon) \longrightarrow (A_C, A_R, \mathcal{E})$$

as ε *goes to 0 in* $\left[L^\infty(0,T,H^{s-2}(\mathbb{R}^d))\right]^{3d}$ *where* $(A_C, A_R, \mathcal{E})$ *satisfy*

$$\begin{aligned} \partial_t A_C + v_C \partial_y A_C &= -(k \cdot \mathcal{E}) A_R, \\ \partial_t A_R + v_R \partial_y A_R &= (k \cdot \mathcal{E}^*) A_C, \\ \partial_t \mathcal{E} + 2\gamma k \cdot \nabla\mathcal{E} &= (A_R^* \cdot A_C) k. \end{aligned}$$

Proof: It is a straightforward consequence of Proposition 2.4 since the solutions are bounded in $L^\infty(0,T,H^s(\mathbb{R}^d))$. $\square$

Remark 2.3. *Suppose that* A_C *is fixed to a constant, the last two equations read using* $e = \mathcal{E}^*$ *as new unknown*

$$\begin{aligned} \partial_t A_R + v_R \partial_y A_R &= (k \cdot e) A_C, \\ \partial_t \mathcal{E} + 2\gamma k \cdot \nabla e &= (A_R \cdot A_C^*) k, \end{aligned}$$

and the amplification rate is $|k \cdot e|$.

Proposition 2.6. *Non resonant case. Suppose $\omega_1 \neq \gamma k_1^2$. For $\sigma > \frac{d}{2}$, there exists $s > 0$ large enough such that the solution to* (2.13) − (2.15) *with initial data $(A_C^0, A_R^0, \mathcal{E}^0) \in \left[H^s(\mathbb{R}^d)\right]^{3d}$ satisfies*

$$(A_C^\varepsilon, A_R^\varepsilon, \mathcal{E}^\varepsilon) - (A_C, A_R, F e^{i(\omega_1 - \gamma k_1^2)\frac{t}{\varepsilon}}) \longrightarrow 0$$

as ε goes to 0 in $\left[L^\infty(0,T,H^\sigma(\mathbb{R}^d))\right]^{3d}$ where

$$\begin{aligned}
&\partial_t A_C + v_C \partial_y A_C = 0,\\
&\partial_t A_R + v_R \partial_y A_R = 0,\\
&\partial_t F + 2\gamma k \cdot \nabla F = 0.
\end{aligned}$$

Proof: It is a geometrical optics type result. See [6] for example. □

Remark 2.4. *In this case, of course, no amplification occurs.*

We therefore have proved Theorem 1.1. □

3 Numerical scheme in 1-D.

The aim of this section is to present an efficient numerical scheme in 1-D for our problem. In 1-D, the system restricts to

$$i(\partial_t + v_C \partial_y) A_C + \alpha \partial_y^2 A_C = \frac{b^2}{2} p A_C - \partial_y E A_R e^{-i\theta}, \tag{3.1}$$

$$i(\partial_t + v_R \partial_y) A_R + \beta \partial_y^2 A_R = \frac{bc}{2} p A_R - \partial_y E^* A_C e^{i\theta}, \tag{3.2}$$

$$i\partial_t E + \gamma \partial_y^2 E = \frac{b}{2} p E + \partial_y (A_R^* A_C e^{i\theta}) \tag{3.3}$$

$$\partial_t^2 p - v_s^2 \partial_y^2 p = a \partial_y^2 (|E|^2 + b|A_C|^2 + c|A_R|^2), \tag{3.4}$$

with the coefficients given in Section 1. In order to introduce our scheme, we make three remarks

Remark 3.1. *The quasilinear part is not hyperbolic. Therefore, any splitting technique with one step of resolution of this part will be unstable.*

Remark 3.2. *As seen in the previous section, in the resonant case, one can expect an exponential growth on A_R and E, we will therefore look for an implicit scheme.*

Remark 3.3. *The invariant $\int_{\mathbb{R}} 2|A_C|^2 + |A_R|^2 + |E|^2 = constant$ corresponds to a nonlinear saturation process (which stops the exponential growth in finite time). Therefore, we want to find a scheme that, at least asymptotically, also preserves this quantity.*

3.1 Semi-discretization in time.

We propose here a fractional-step, Crank-Nicolson-type scheme with relaxation directly inspired by that of C. Besse for NLS or Davey-Stewartson system (see [1]). For the accoustic part, we use the scheme introduced by Glassey (see [5]). The scheme reads:

$$i\frac{A_C^{n+1}-A_C^n}{\delta t}+(iv_C\partial_y+\alpha\partial_y^2)\left(\frac{A_C^{n+1}+A_C^n}{2}\right)=\frac{b^2}{2}\left(\frac{p^{n+1}+p^n}{2}\right)\left(\frac{A_C^{n+1}+A_C^n}{2}\right)$$
$$-\frac{1}{2}\phi^{n+\frac{1}{2}}\left(\frac{A_R^{n+1}+A_R^n}{2}\right)e^{-i\theta^{n+\frac{1}{2}}}-\frac{1}{2}\psi^{n+\frac{1}{2}}\left(\frac{\partial_yE^{n+1}+\partial_yE^n}{2}\right)e^{-i\theta^{n+\frac{1}{2}}}, \tag{3.5}$$

$$i\frac{A_R^{n+1}-A_R^n}{\delta t}+(iv_R\partial_y+\beta\partial_y^2)\left(\frac{A_R^{n+1}+A_R^n}{2}\right)=\frac{bc}{2}\left(\frac{p^{n+1}+p^n}{2}\right)\left(\frac{A_R^{n+1}+A_R^n}{2}\right)$$
$$-(\phi^{n+\frac{1}{2}})^*\left(\frac{A_C^{n+1}+A_C^n}{2}\right)e^{i\theta^{n+\frac{1}{2}}}, \tag{3.6}$$

$$i\frac{E^{n+1}-E^n}{\delta t}+\gamma\partial_y^2\left(\frac{E^{n+1}+E^n}{2}\right)=\frac{b}{2}\left(\frac{p^{n+1}+p^n}{2}\right)\left(\frac{E^{n+1}+E^n}{2}\right)$$
$$+\partial_y\left[(\psi^{n+\frac{1}{2}})^*\left(\frac{A_C^{n+1}+A_C^n}{2}\right)e^{i\theta^{n+\frac{1}{2}}}\right], \tag{3.7}$$

$$\frac{p^{n+1}-2p^n+p^{n-1}}{\delta t^2}-v_s^2\partial_y^2\left(\frac{p^{n+1}+p^{n-1}}{2}\right)=a\partial_y^2\left(|E^n|^2+b|A_C^n|^2+c|A_R^n|^2\right), \tag{3.8}$$

where the auxiliary functions ϕ and ψ are given by

$$\frac{\phi^{n+\frac{1}{2}}+\phi^{n-\frac{1}{2}}}{2}=\partial_yE^n,\quad \frac{\psi^{n+\frac{1}{2}}+\psi^{n-\frac{1}{2}}}{2}=A_R^n. \tag{3.9}$$

Note that

$$\theta^{n+\frac{1}{2}}=k_1y-\omega_1(n+\frac{1}{2})\delta t.$$

$\phi^{n+\frac{1}{2}}$ and $\psi^{n+\frac{1}{2}}$ are prediction respectively of ∂_yE and A_R at time $(n+\frac{1}{2})\delta t$. Therefore the value $\phi^{-\frac{1}{2}}$ and $\psi^{-\frac{1}{2}}$ are obtained by explicit integration of the system on one half-time step backwards. This scheme conserves the L^2-invariant, namely:

Proposition 3.1. *Any regular solution of* (3.5) − (3.9) *satisfies:*

$$\int_{\mathbb{R}}2|A_C^n|^2+|A_R^n|^2+|E^n|^2=\int_{\mathbb{R}}2|A_C^0|^2+|A_R^0|^2+|E^0|^2.$$

Proof: We compute

$$\int_{\mathbb{R}} (3.5) \left(\frac{A_C^{n+1}+A_C^n}{2}\right)^* + \int_{\mathbb{R}} (3.6) \left(\frac{A_R^{n+1}+A_R^n}{2}\right)^* + \int_{\mathbb{R}} (3.7) \left(\frac{E^{n+1}+E^n}{2}\right)^*,$$

and take the imaginary part:

$$\begin{aligned}
&\frac{1}{2\delta t}\int_{\mathbb{R}} \left(2|A_C^{n+1}|^2+|A_R^{n+1}|^2+|E^{n+1}|^2\right) - \frac{1}{2\delta t}\int_{\mathbb{R}} \left(2|A_C^{n}|^2+|A_R^{n}|^2+|E^{n}|^2\right) \\
&= -\mathrm{Im}\int_{\mathbb{R}} \phi^{n+\frac{1}{2}} \left(\frac{A_R^{n+1}+A_R^n}{2}\right) e^{-i\theta^{n+\frac{1}{2}}} \left(\frac{A_C^{n+1}+A_C^n}{2}\right)^* \\
&- \mathrm{Im}\int_{\mathbb{R}} \psi^{n+\frac{1}{2}} \left(\frac{\partial_y E^{n+1}+\partial_y E^n}{2}\right) e^{-i\theta^{n+\frac{1}{2}}} \left(\frac{A_C^{n+1}+A_C^n}{2}\right)^* \\
&- \mathrm{Im}\int_{\mathbb{R}} (\phi^{n+\frac{1}{2}})^* \left(\frac{A_C^{n+1}+A_C^n}{2}\right) e^{i\theta^{n+\frac{1}{2}}} \left(\frac{A_R^{n+1}+A_R^n}{2}\right)^* \\
&+ \mathrm{Im}\int_{\mathbb{R}} \partial_y \left((\psi^{n+\frac{1}{2}})^* \left(\frac{A_C^{n+1}+A_C^n}{2}\right) e^{i\theta^{n+\frac{1}{2}}}\right) \left(\frac{E^{n+1}+E^n}{2}\right)^* \\
&= -\mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

It is clear that $\mathrm{I} = -\mathrm{III}$. Moreover

$$\mathrm{IV} = -\mathrm{Im}\int_{\mathbb{R}} (\psi^{n+\frac{1}{2}})^* \left(\frac{A_C^{n+1}+A_C^n}{2}\right) e^{i\theta^{n+\frac{1}{2}}}\, \partial_y \left(\frac{E^{n+1}+E^n}{2}\right)^* = \mathrm{II}.$$

Hence the result. □

3.2 Space discretization.

We will consider a regular mesh in space. The fields are approximated by $(E_i)_{i=0,..,N_y}$. We consider here periodic boundary conditions $E_{N_y} = E_0$ and $E_{N_y+1} = E_1$. We consider centered discretization for each differential operator. Such discretization are known to be dispersive and bad for solutions involving chocs. It is not the case here since the equations that we deal with are already dispersive.

Therefore, ∂_y is approximated by the centered finite difference operator D_0:

$$(D_0 E)_i = \frac{E_{i+1} - E_{i-1}}{2\delta y},$$

and ∂_y^2 by $D_+ D_-$:

$$(D_+ D_- E)_i = \frac{E_{i+1} - 2E_i + E_{i-1}}{\delta y^2}.$$

The scheme reads:

$$
i\frac{A_C^{n+1}-A_C^n}{\delta t}+(iv_C D_0+\alpha D_+D_-)\left(\frac{A_C^{n+1}+A_C^n}{2}\right)=\frac{b^2}{2}\left(\frac{p^{n+1}+p^n}{2}\right)\left(\frac{A_C^{n+1}+A_C^n}{2}\right)
$$
$$
-\frac{1}{2}\phi^{n+\frac{1}{2}}\left(\frac{A_R^{n+1}+A_R^n}{2}\right)e^{-i\theta^{n+\frac{1}{2}}}-\frac{1}{2}\psi^{n+\frac{1}{2}}\left(\frac{D_0E^{n+1}+D_0E^n}{2}\right)e^{-i\theta^{n+\frac{1}{2}}}, \tag{3.10}
$$

$$
i\frac{A_R^{n+1}-A_R^n}{\delta t}+(iv_R D_0+\beta D_+D_-)\left(\frac{A_R^{n+1}+A_R^n}{2}\right)=\frac{bc}{2}\left(\frac{p^{n+1}+p^n}{2}\right)\left(\frac{A_R^{n+1}+A_R^n}{2}\right)
$$
$$
-(\phi^{n+\frac{1}{2}})^*\left(\frac{A_C^{n+1}+A_C^n}{2}\right)e^{i\theta^{n+\frac{1}{2}}}, \tag{3.11}
$$

$$
i\frac{E^{n+1}-E^n}{\delta t}+\gamma D_+D_-\left(\frac{E^{n+1}+E^n}{2}\right)=\frac{b}{2}\left(\frac{p^{n+1}+p^n}{2}\right)\left(\frac{E^{n+1}+E^n}{2}\right)
$$
$$
+D_0\left[(\psi^{n+\frac{1}{2}})^*\left(\frac{A_C^{n+1}+A_C^n}{2}\right)e^{i\theta^{n+\frac{1}{2}}}\right], \tag{3.12}
$$

$$
\frac{p^{n+1}-2p^n+p^{n-1}}{\delta t^2}-v_s^2D_+D_-\left(\frac{p^{n+1}+p^{n-1}}{2}\right)=aD_+D_-\left(|E^n|^2+b|A_C^n|^2+c|A_R^n|^2\right), \tag{3.13}
$$

$$
\frac{\phi^{n+\frac{1}{2}}+\phi^{n-\frac{1}{2}}}{2}=D_0E^n,\quad \frac{\psi^{n+\frac{1}{2}}+\psi^{n-\frac{1}{2}}}{2}=A_R^n. \tag{3.14}
$$

In this context, one has

Proposition 3.2. *Any solution of* (3.10) – (3.14) *satisfies*

$$
2|A_C^n|_2^2+|A_R^n|_2^2+|E^n|_2^2=2|A_C^0|_2^2+|A_R^0|_2^2+|E^0|_2^2,
$$

where

$$
|f|_2^2=\sum_{j=1}^{N_y}|f_j|^2.
$$

Proof: It is the same that in the semi-discrete case. □

Proposition 3.3. *For all* $(A_C^n, A_R^n, E^n, p^{n-1}, p^n, \phi^{n-\frac{1}{2}}, \psi^{n-\frac{1}{2}})_i$, $i=0,...,N_y$, *there exists a unique* $(A_C^{n+1}, A_R^{n+1}, E^{n+1}, p^{n+1}, \phi^{n+\frac{1}{2}}, \psi^{n+\frac{1}{2}})_i$, $i=0,...,N_y$, *solution to* (3.10) – (3.14).

That means that the matrix that one has to inverse at each time step is never singular.

Proof: Indeed, equation (3.13) gives p^{n+1} while (3.14) gives $\phi^{n+\frac{1}{2}}$ and $\psi^{n+\frac{1}{2}}$. Let us consider the mapping

$$\mathcal{X}: \quad (X,Y,Z) \longrightarrow \mathcal{X}(X,Y,Z),$$

given by

$$\begin{aligned}\mathcal{X}(X,Y,Z) = \Bigg(& \frac{i}{\delta t}X + \frac{1}{2}(iv_C D_0 + \alpha D_+ D_-)X - \frac{b^2}{8}(p^{n+1}+p^n)X + \frac{1}{4}\phi^{n+\frac{1}{2}}e^{-i\theta^{n+\frac{1}{2}}}Y \\ & + \frac{1}{4}\psi^{n+\frac{1}{2}}D_0 Z e^{-i\theta^{n+\frac{1}{2}}}, \frac{i}{\delta t}Y + \frac{1}{2}(iv_R D_0 + \beta D_+ D_-)Y - \frac{bc}{8}(p^{n+1}+p^n)Y \\ & - \frac{1}{2}(\phi^{n+\frac{1}{2}})^* e^{i\theta^{n+\frac{1}{2}}}X, \frac{i}{\delta t}Z + \frac{1}{2}\gamma D_+ D_- Z - \frac{b}{8}(p^{n+1}+p^n)Z \\ & - \frac{1}{2}D_0(\psi^{n+\frac{1}{2}})e^{i\theta^{n+\frac{1}{2}}}\ X \Bigg).\end{aligned}$$

Then

$$\operatorname{Im}\left(\mathcal{X}(X,Y,Z),(X,Y,Z)\right)_2 = \frac{1}{\delta t}\left(|X|_2 + |Y|_2| + |Z|_2\right),$$

hence $\mathcal{X}$ is one-to-one.

3.3 Numerical results.

We have performed our simulations with the following set of values that are reasonable from the physical point of view. We give the velocity of light $c_0 = 3 * 10^8 m.s^{-1}$. The thermal velocity of the electrons is taken to be equal to $v_{th} = \frac{c_0}{10}$. The sound velocity of the electrons is $c_s = 0.005 * c_0$. The ratio of the mass of the electrons and ions is $\frac{m_e}{m_i} = 0.01$. The plasma frequency $\omega_{pe} = 3 * 10^{15} s^{-1}$ and the wave number of the laser is $k_0 = 2 * 10^7 m^{-1}$. We compute ω_0 by

$$\omega_0 = \sqrt{\omega_{pe}^2 + k_0^2 c_0^2},$$

and search for k_R, k_1, ω_R and ω_1 such that

$$k_0 = k_R - k_1, \quad \omega_0 = \omega_1 + \omega_R + \omega_{pe},$$

where

$$\omega_R = \sqrt{\omega_{pe}^2 + k_R^2 c_0^2}, \quad \omega_1 = \sqrt{\omega_{pe}^2 + k_1^2 v_{th}^2},$$

with a dichotomy process. One obtains

$$k_R = -6.65 * 10^6 m^{-1}, \;\; k_1 = 2.6 * 10^7 m^{-1}, \;\; \omega_R = 3.6 * 10^{15} s^{-1}, \;\; \frac{\omega_1}{\omega_0} = 0.01561.$$

We work on a system in dimensionless form. (The unit of length is $\frac{1}{k_0}$ and the unit of time is $\frac{1}{\omega_0}$). We compute on a space interval $[0, L]$ with $L = 200$ and on a time interval $[0, T]$ with $T = 100$.

We will consider gaussian initial data for A_C of the form

$$A_C(0) = \alpha e^{-\beta(x-\gamma)^2}.$$

Typical values of α, β and γ will be $\alpha = 0.3$, $\beta = 0.01$ and $\gamma = 40$. We will let them vary slightly.

Since we deal with simulated Raman effect, we have to begin with a small perturbation on A_R and we take $A_R(0) = 0.01A_C(0)$. Furthermore, E, p and $\partial_t p$ are taken equal to 0 at $t = 0$. Typical number of points discretization in space is $N_y = 500$ and in time $N_t = 200$.

- **Case 1:** The first test is with $A_C(0) = 0.3e^{-0.01(x-40)^2}$ and with ω_1 given by

$$\frac{\omega_1}{\omega_0} = 0.01561,$$

that is for the resonance case. This concerns figures 1 to 3. In figure 1, one can find snapshots of the modulus of the fields at nine different times ($t = \frac{n*100}{8}$, $n = 0$ to 8). The continuous line corresponds to A_C, the dashed line to A_R and the semi-doted one to E. Of course A_C and A_R travel in different directions. The growth is rapid. The interaction stops when the supports of A_C and A_R are disjoints.

Figure 2 reports the maximum of the fields with respect to the time with the same convention that before. It is quite clear that the growth stops when the supports become disjoint.

Figure 3 shows snapshots of p at the same time than figure 1. The disturbance of p is clearly localized on the support of E even if E is the smallest field. It can be explained by the fact that the caracteristic velocity of p is $\frac{c_s}{c_0} = 0.005$ which is near 0. Therefore the interaction is stronger with E than with the other fields.

- **Case 2:** For the second test, we took $T = 50$ and

$$A_C(0) = 0.3e^{-0.001(x-100)^2},$$

that is we took a wider gaussian in order to increase the interaction time. We are still in the resonnant case. Figure 4 shows the maximum of the fields with respect to time. The interaction between the different fields is more efficient and in this case, we have reached the nonlinear saturation regime.

- **Case 3:** For the third case, we decrease the interaction time and take

$$A_C(0) = 0.3e^{-0.02(x-40)^2},$$

and $T = 100$. Figure 5 shows the maximum of the fields and the Raman amplification is smaller since the supports becomes disjoint more rapidely. The process is less efficient than in the preceding case.

• **Case 4:** For the fourth case, we choose ω_1 far away from the exact resonance value with $\frac{\omega_1}{\omega_0} = 1$. Except that, the values are the same than in the first test corresponding to figure 1,2 and 3. The maximum of the fields are drawn in Figure 6 and the Raman process almost not exists.

• **Case 5:** The last case is intermediate between the first and the fourth. We take $\frac{\omega_1}{\omega_0} = 0.5$ and Figure 7 and 8 show respectively snapshots of the fields and maximum of the fields. The situation is really intermediate. The Raman effect exists but is weaker than in case 1.

Conclusion: Our scheme allows us to recover the main feature of the qualitative behaviour of the solution discribed in section 2. It is also possible to obtain some intermediate behaviours that are not described by the asymptotic analysis. In the next section, we extend it in 2-D.

4 The 2-D case.

In this section, we perform some numerical experiment in the 2-D case.

4.1 The 2-D system.

In this case A_C and A_R are scalar while E is a 2-D vector field. The system can be written:

$$
\begin{aligned}
&i(\partial_t + v_C\partial_y)A_C + (\alpha_1\partial_y^2 + \alpha_2\partial_x^2)A_C = \frac{b^2}{2}pA_C - (\nabla\cdot E)A_R e^{-i\theta},\\
&i(\partial_t + v_R\partial_y)A_R + (\beta_1\partial_y^2 + \beta_2\partial_x^2)A_R = \frac{bc}{2}pA_R - (\nabla\cdot E^*)A_C e^{i\theta},\\
&(i\partial_t + \gamma\nabla\nabla\cdot - \delta\nabla\times\nabla\times)E = \frac{b}{2}pE + \nabla(A_R^*\cdot A_C e^{i\theta}),\\
&(\partial_t^2 - v_s^2\Delta)p = a\Delta\left(|E|^2 + b|A_C|^2 + c|A_R|^2\right),
\end{aligned}
$$

with the coefficients given in Section 1. We now make some comments.

• Since the system has some transverse dispersion, one expects that the amplitude of the fields will decay more rapidly than in the 1-D case. Indeed, the solution of the linear Schrödinger equation

$$
\begin{cases} i\partial_t u + \Delta u = 0, \quad x\in\mathbb{R}^d,\\ u(0) = u_0, \end{cases}
$$

satisfies

$$
\operatorname*{Max}_{x\in\mathbb{R}^d} |u(t,x)| \le \frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} |u_0(x)|dx.
$$

The time-decay is therefore stronger in 2-D than in 1-D.

• Since $\delta \gg \gamma$, one expects E to be a gradient see [4] in which the author proves that the solution E^α to

$$
i\partial_t E^\alpha + (\nabla\nabla\cdot - \alpha^2\nabla\times\nabla\times)E^\alpha = |E^\alpha|^2E^\alpha
$$

behaves when $\alpha \to +\infty$ like

$$E^\alpha = E_1^\alpha + E_2 + o(1)$$

where E_1^α is the solution to

$$\begin{cases} i\partial_t E_1^\alpha - \alpha^2 \nabla \times \nabla \times E_1^\alpha = 0 \\ E_1^\alpha(0) = -\nabla \times (\Delta^{-1}) \nabla \times E^\alpha(0) \end{cases}$$

and $E_2 = \nabla\psi$ with

$$i\partial_t(\nabla\psi) + \Delta(\nabla\psi) = \nabla\Delta^{-1}\mathrm{div}(|\nabla\psi|^2\nabla\psi).$$

4.2 Numerical results.

Like in Section 3, we begin by the resonant case. The spatial domain is $y \in [0, 100]$ and $x \in [0, 60]$ and the time interval is $[0, 50]$. We took $N_y = 100$ points in the y direction, $N_x = 60$ in the x direction and $N_t = 96$ time steps. The physical parameters are the same than in 1-D. The initial data for A_C is

$$A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}.$$

In Figure 9, the neperian logarithm of the modulus of the fields are drawn. The Raman effect is less efficient than in 1-D as expected. The decrease of A_C in time is more rapid than in 1-D because of the 2-D dispersion. Since $\alpha_2 > \alpha_1$ ($\alpha_2 \approx 0.4$ while $\alpha_1 \approx 0.08$), one can see that the transverse dispersion is stronger that the longitudinal ones on A_C in figure 10. The same remark apply on A_R.

In Figure 10, 11, 12, 13, the level curves of respectively A_C, A_R, E_x, E_y are drawn at time $\frac{n*50}{8}$ for $n = 0$ to 8.

The source term in the equation for E is $\nabla(A_R^* \cdot A_C e^{i\theta})$ and $A_R^* \cdot A_C$ is mainly a gaussian. This source term is

$$\nabla\left(e^{-a_1x^2}e^{-b_1y^2}e^{i(k_1y-\omega_1t)}\right),$$

with $k_1 \gg \sqrt{a_1}$ and $k_1 \gg \sqrt{b_1}$. Therefore $E = \nabla\psi$ will have a similar form that means that E_y is one order of magnitude greater than E_x (see Figure 9). Moreover, $E_x = \partial_x\psi$ is basically the derivative of a gaussian (see Figure 12) while E_y is a gaussian (see Figure 13).

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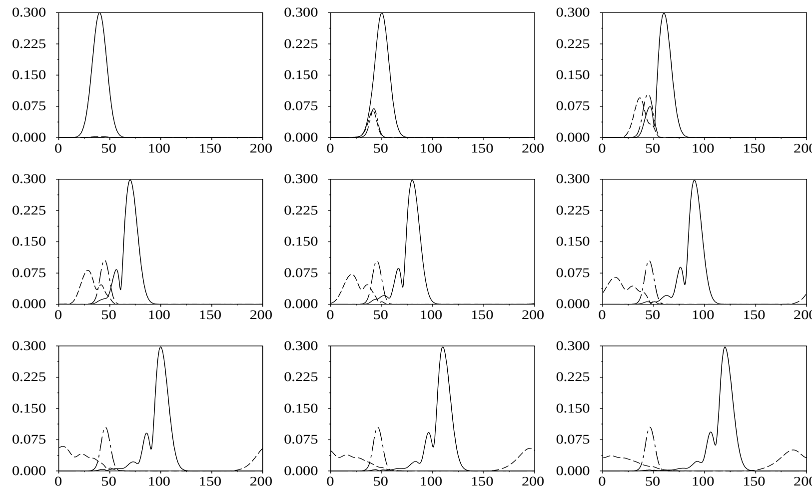

Figure 1: Case 1, 1-D geometry. Modulus of the fields at time $t = n\dfrac{100}{8}$ for $n = 0 \cdots 8$ with $A_C(0) = 0.3e^{-0.01(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. First line, from left to right, $n = 0, 1, 2$, second line, from left to right, $n = 3, 4, 5$, third line, from left to right, $n = 6, 7, 8$. The continuous line corresponds to A_C, the dashed line to A_R and the semi-doted line to E. The value of ω_1 corresponds to the resonant case.

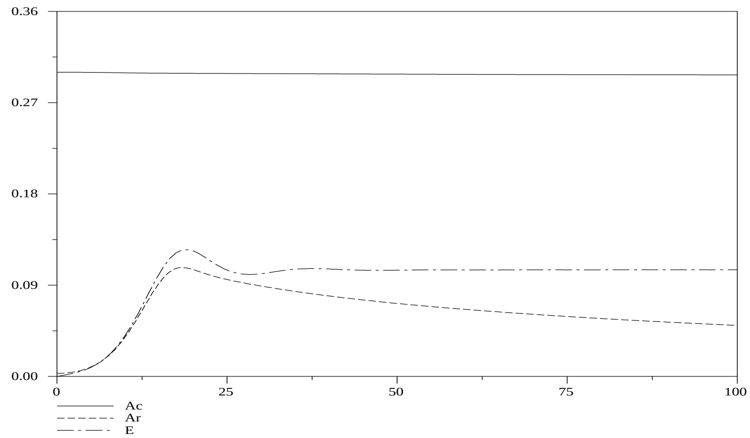

Figure 2: Case 1, 1-D geometry. Maximum of the fields with respect to time with $A_C(0) = 0.3e^{-0.01(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. This value of ω_1 corresponds to the resonant case.

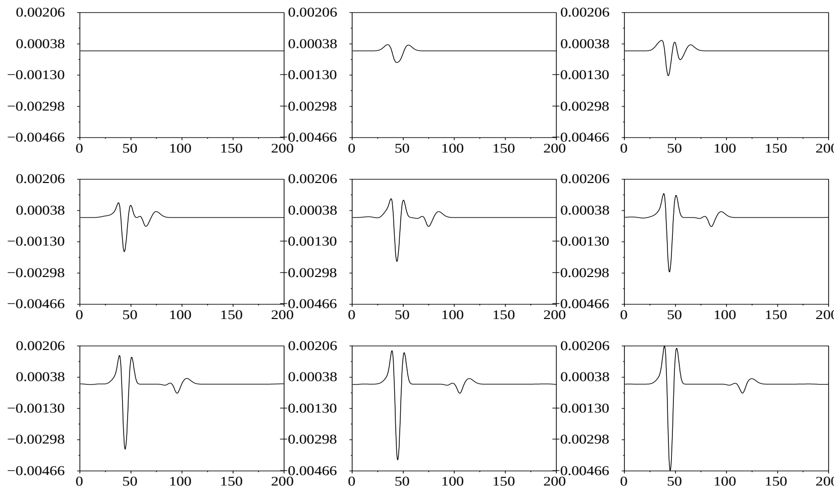

Figure 3: Case 1, 1-D geometry. p at time $t = n\dfrac{100}{8}$ for $n = 0 \cdots 8$ with $A_C(0) = 0.3e^{-0.01(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. First line, from left to right, $n = 0, 1, 2$, second line, from left to right, $n = 3, 4, 5$, third line, from left to right, $n = 6, 7, 8$. The value of ω_1 corresponds to the resonant case.

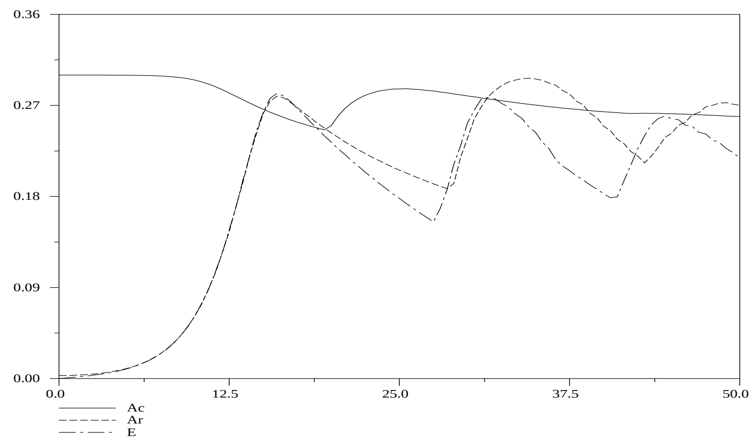

Figure 4: Case 2, 1-D geometry. Maximum of the fields with respect to time with $A_C(0) = 0.3e^{-0.001(x-100)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. This value of ω_1 corresponds to the resonant case. The interaction time is long: the Raman process is strong and the saturation regime is reached.

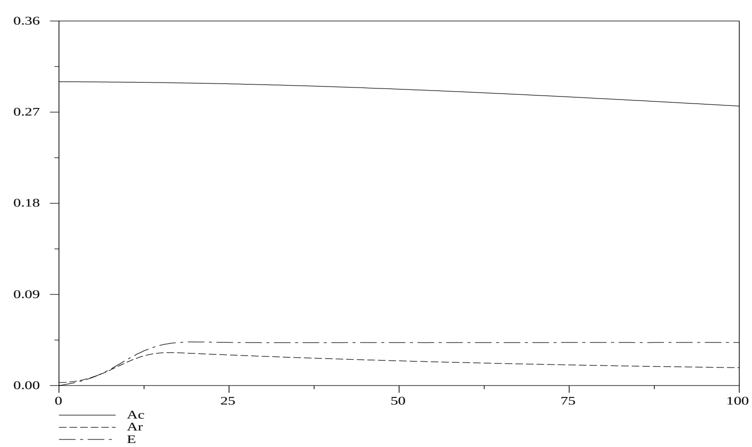

Figure 5: Case 3, 1-D geometry. Maximum of the fields with respect to time with $A_C(0) = 0.3e^{-0.02(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. This value of ω_1 corresponds to the resonant case. The interaction time is short: the Raman process is weak.

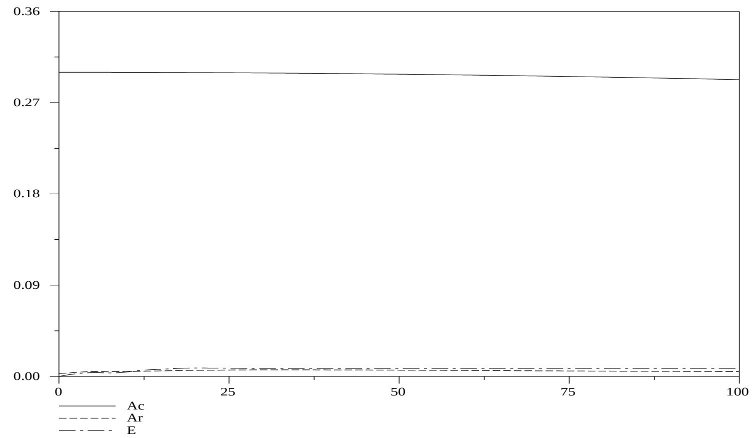

Figure 6: Case 4, 1-D geometry. Maximum of the fields with respect to time with $A_C(0) = 0.3e^{-0.01(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 1$. It is the same case than in figure 1 but far away from the resonance: the Raman process is quasi inexistant.

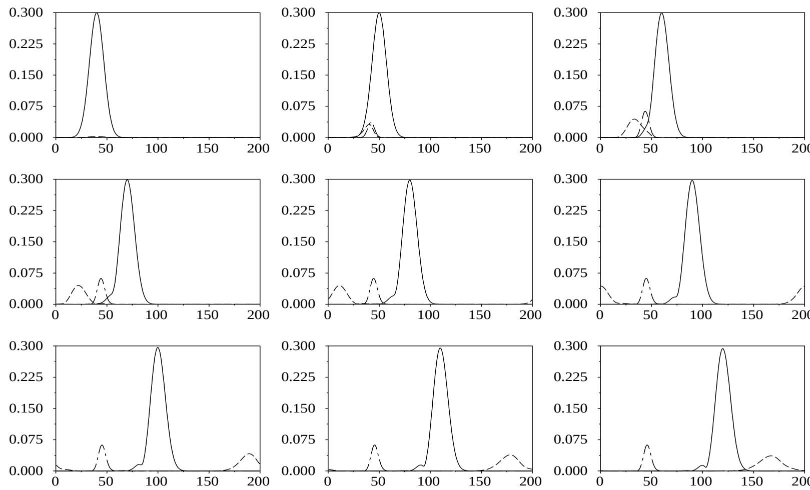

Figure 7: Case 5, 1-D geometry. Modulus of the fields at time $t = n\dfrac{100}{8}$ for $n = 0 \cdots 8$ with $A_C(0) = 0.3e^{-0.01(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 0.5$. The continuous line corresponds to A_C, the dashed line to A_R and the semi-doted line to E. First line, from left to right, $n = 0, 1, 2$, second line, from left to right, $n = 3, 4, 5$, third line, from left to right, $n = 6, 7, 8$. The value of ω_1 is near the resonance but not exactly at the resonance. The efficiency of the process is intermediate between case 1 (fig. 1,2,3) and case 4 (fig.6).

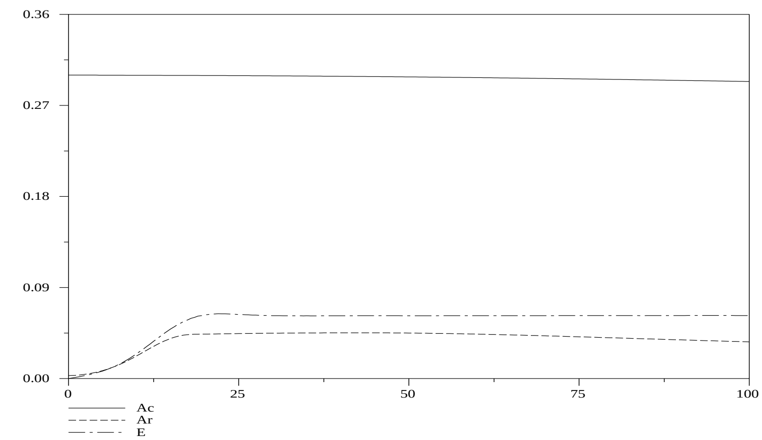

Figure 8: Case 5, 1-D geometry. Maximum of the fields with respect to time with $A_C(0) = 0.3e^{-0.01(x-40)^2}$, $\frac{\omega_1}{\omega_0} = 0.5$. This value of ω_1 is near the resonance. See comments of figure 7.

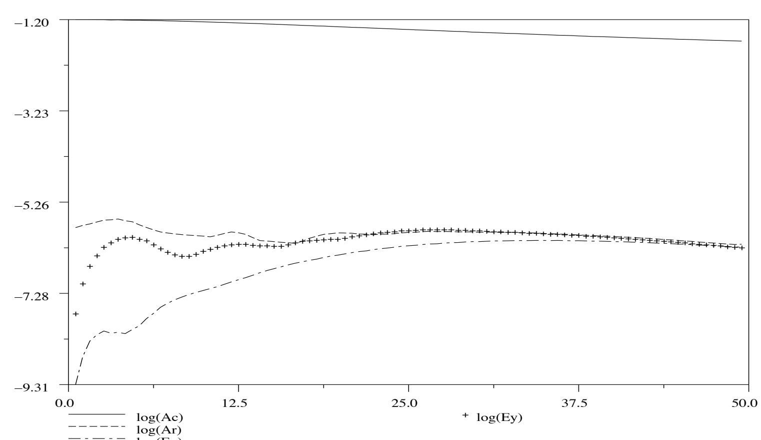

Figure 9: 2-D geometry. Neperian logarithm of the fiels with $A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$

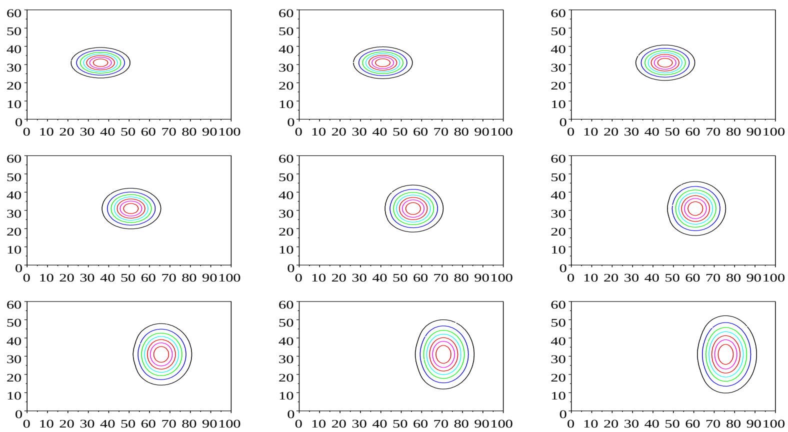

Figure 10: 2-D geometry. Level curves of A_C at time $t = n\dfrac{100}{8}$ for $n = 0 \cdots 8$ with $A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. First line, from left to right, $n = 0, 1, 2$, second line, from left to right, $n = 3, 4, 5$, third line, from left to right, $n = 6, 7, 8$.

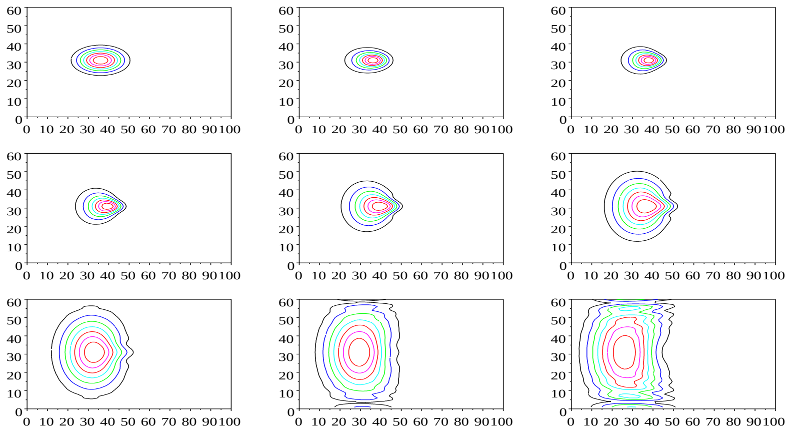

Figure 11: 2-D geometry. Level curves of A_R at time $t = n\dfrac{100}{8}$ for $n = 0 \cdots 8$ with $A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. First line, from left to right, $n = 0, 1, 2$, second line, from left to right, $n = 3, 4, 5$, third line, from left to right, $n = 6, 7, 8$.

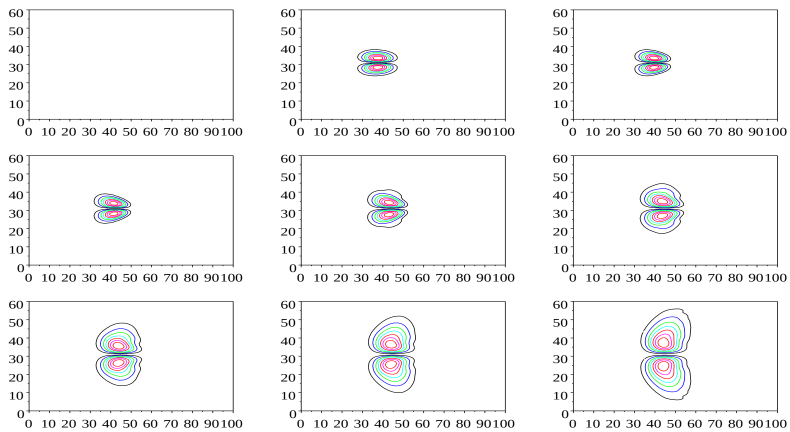

Figure 12: 2-D geometry. Level curves of E_x at time $t = n\dfrac{100}{8}$ for $n = 0 \cdots 8$ with $A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. First line, from left to right, $n = 0, 1, 2$, second line, from left to right, $n = 3, 4, 5$, third line, from left to right, $n = 6, 7, 8$.

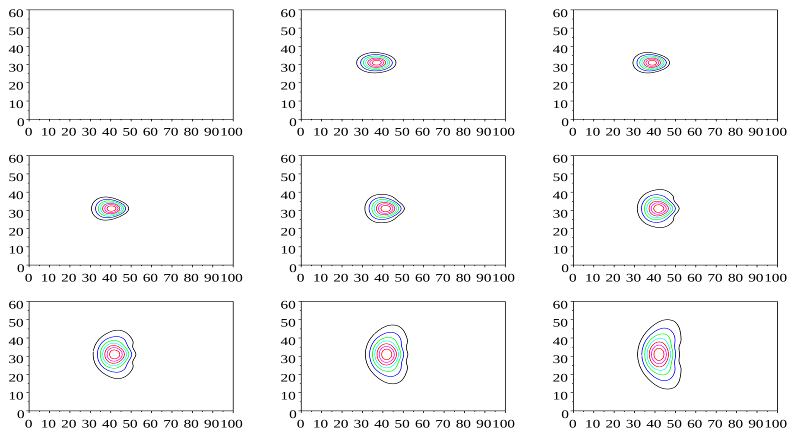

Figure 13: 2-D geometry. Level curves of E_y at time $t = n\dfrac{100}{8}$ for $n = 0 \cdots 8$ with $A_C(0) = 0.3e^{-0.01(y-35)^2}e^{-0.03(x-30)^2}$, $\frac{\omega_1}{\omega_0} = 0.01561$. First line, from left to right, $n = 0, 1, 2$, second line, from left to right, $n = 3, 4, 5$, third line, from left to right, $n = 6, 7, 8$.