Rigorous derivation of the nonlinear Schrödinger equation and Davey-Stewartson systems from quadratic hyperbolic systems.

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Abstract: In this paper, we derive rigorously the nonlinear Schrödinger equation and Davey-Stewartson systems from quadratic hyperbolic systems using nonlinear diffractive geometric optics. We construct approximate solutions and prove the convergence of the asymptotic expansion. The keys of this work is to consider only systems of a particular form introduced in [20], which includes Maxwell-Bloch system, and which satisfy a "transparency" hypothesis.

Key words: Geometric optics, nonlinear Schrödinger equation, Davey-Stewartson systems.
AMS subject classification: 35L05, 35Q55.

1 Introduction

1.1 Motivations.

The aim of this paper is to give some rigorous results concerning the derivation of the nonlinear Schrödinger equation and the Davey-Stewartson systems. These kind of systems are universal models in physics and mechanics, see for example [7], [8], [4] for water waves theory, [13] for internal gravity waves, [24, 25] for ferromagnetism, [27] for nonlinear optics and [26] for plasma theory. All the examples given in these references have a commun point: one start from nonlinear hyperbolic systems with quadratic nonlinearities and one finds cubic Davey-Stewartson limit systems that sometimes degenerate into nonlinear Schrödinger equations. In this paper, we try to explain why a generic asymptotic expansion leads to Davey-Stewartson-type system and we prove a convergence result for a special class of systems taken from [20].
The general framework used in this paper in that used in Joly, Métivier and Rauch's works of the last decade. We are concerned by hyperbolic systems under the form:

$$\partial_t u + \sum_{j=1}^{n} A_j \partial_{x_j} u + \frac{Eu}{\varepsilon} = f(u), \tag{1.1}$$

where the matrix A_j are real symmetric, E is skew-symmetric and f is homogeneous of order p. The initial data is oscillatory and reads:

$$u(0,x) = \varepsilon^{\alpha}(a(x)e^{i\frac{k\cdot x}{\varepsilon}} + \bar{a}(x)e^{-i\frac{k\cdot x}{\varepsilon}})$$

General theorems give that for such initial data, there exists a solution to system (1.1) defined on a time interval of size $O(1/\varepsilon^{(p-1)\alpha})$. Moreover, some asymptotic expansions are available. For geometric optics ($\alpha = 0$), the results are very complete [15, 17, 16, 18]. More recently, the cubic case $p = 3$, $\alpha = 1/2$ has been studied in [9], [10]. The existence time is then $O(1/\varepsilon)$ and diffractive effects occur. The asymptotics is driven by a cubic nonlinear Schrödinger equation, or by a system of nonlinear Schrödinger equations. For even nonlinearities, an oscillatory initial data can lead to the creation of a low frequency wave. This is called rectification [19], [22]. The expansion in this situation leads to coupled systems of nonlinear Schrödinger equations. In all these cases, for the simplest even nonlinearity $p = 2$, the solution is of size $O(\varepsilon)$ and the existence time is of size $O(1/\varepsilon)$. When one tries to apply these expansions to physical problems, one often find **linear** enveloppe equations. An algebraic property called "transparency" in [20] is responsible for that. In order to obtain nonlinear results, one then tries to obtain larger solutions in the expansion. In this case, the general theorems do not give existence results on suitable time interval. It is however possible to conclude in some cases. The geometric optics case is done in [20]. In this paper, we are concerned with the diffractive case.

1.2 Description of the results

We restrict ourselves to the case where $f(u)$ is quadratic and we denote from now on the nonlinearity as being a bilinear mapping $f(u, u)$. We are looking for solutions to (1.1) with initial data

$$u(0, x) = a(x)e^{i\frac{k\cdot x}{\varepsilon}} + \bar{a}(x)e^{-i\frac{k\cdot x}{\varepsilon}}$$

defined on $[0, O(\frac{1}{\varepsilon})]$. We have organized this paper as follows: in the second section, we make a formal asymptotic expansion. We derive a set of necessary conditions in order to have an approximate solution (Theorem 2.1). One of these conditions is the Davey-Stewartson-type system (2.22)-(2.26). This system in a coupling between a Schrödinger type equation with dispersion $\lambda''_{l_0}(k)$ where ($k \mapsto \lambda_{l_0}(k)$ denotes the dispersion relation of the system) and a static equation concerning the low frequency part of the solution. This system reads as follows:

$$\begin{aligned} i\partial_\tau A + \frac{1}{2}\lambda''_{l_0}(k)(\partial_X, \partial_X)A &= \alpha f(A, \varphi) + \mathcal{C}(A, A, \bar{A}) \\ H_1(\partial_X)\varphi &= H_2(\partial_X)f(A, \bar{A}) \end{aligned} \tag{1.1}$$

where α, H_1 and H_2 are matrices and $\mathcal{C}$ is a trilinear mapping. A is the amplitude of the oscillatory part of the solution and φ its low frequency part. We adapt the classification of [11] in section 2.3. If $det(H_1(\xi))$ does not vanish on the unit sphere, the system is called elliptic and the Cauchy problem for regular initial data for (1.1) is quite straightforward. If $det(H_1(\xi))$ vanishes at order one, the system is called hyperbolic and no general result for the Cauchy problem is available. If this quantity vanishes at higher order, the asymptotic is degenerate and one has to change the ansatz see [5]. In the hyperbolic and elliptic case, the system can reduce to a system of nonlinear Schrödinger type equations if $H_1 = H_2$. These equation are "usual" Schrödinger equations provided that the matrix $\lambda''_{l_0}(k)(\partial_X, \partial_X)$ is proportionnal to the Laplace operator. Section 3 is devoted to the explicitation of the system in some physical examples. The last section is devoted to the proof of the convergence of the expansion. As said above, general theorems do not give the result. Our results (theorems 4.1 and 4.2) rely on the work of [20] and is valid in the context of Maxwell-Bloch systems. We use as in [20] a strong transparency condition that has two consequences. The first one is that the Davey-Stewartson system is elliptic or degenerate into the nonlinear Schrödinger equation. The second consequence is the existence of a nonlinear change of unknown that transform the system into a cubic one for which it is easy to obtain the right existence part. Finally, see [30, 31] for justifications in 1-D cases.

2 Formal asymptotic expansion.

The aim of this section is to construct approximate solutions of the following system:

$$\partial_t u + \sum_{j=1}^{n} A_j \partial_{x_j} u + \frac{Eu}{\varepsilon} = f(u, u), \tag{2.1}$$

by the WKB proceedure.
Here $u(t, x) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$. Matrices A_j are real symmetric and E is real skew-symmetric. The mapping f is bilinear symmetric from $\mathbb{R}^n \times \mathbb{R}^n$ with values in $\mathbb{R}^d$.
For $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n$, we denote by $\mathcal{L}(\tau, \xi)$ the mapping

$$\mathcal{L}(\tau, \xi) = \tau I_d + \sum_{j=1}^{n} A_j \xi_j + E$$

and by $\mathcal{L}_1(\tau, \xi)$ and $A(\xi)$ the matrices:

$$\mathcal{L}_1(\tau, \xi) = \tau I_d + \sum_{j=1}^{n} A_j \xi_j \text{ and } A(\xi) = \sum_{j=1}^{n} A_j \xi_j.$$

For $\xi \in \mathbb{R}^n$, we denote by $(i\lambda_l(\xi))_{l=1\cdots p}$ the eigenvalues of $iA(\xi) + E$ and by $\Pi_l(\xi)$ the orthogonal projector on the null space of $-i\lambda_l(\xi) + iA(\xi) + E = \mathcal{L}(-i\lambda_l(\xi), i\xi)$.
From now one, we take $k \in \mathbb{R}^n$ and choose $\omega = \lambda_{l_0}(k)$ such that:

$$\text{the multiplicity of the eigenvalue } \lambda_{l_0}(\xi) \text{ is constant in a neighborhood of } \xi = k. \tag{2.2}$$

We search for a solution to (2.1) of the form

$$u^\varepsilon(t,x) = \sum_{j=0}^{2} \varepsilon^j u_j(\theta,T,X,\tau)\Big|_{\theta=\frac{k\cdot x-\omega t}{\varepsilon},T=t,X=x,\tau=\varepsilon t} \tag{2.3}$$

where the functions u_j are periodic with respect to θ.
The aim of this section is to find conditions satisfied by u_0, u_1, u_2 so that u^ε defined by (2.3) satisfies

$$\mathcal{L}(\partial_t,\partial_x)u^\varepsilon - f(u^\varepsilon,u^\varepsilon) = O(\varepsilon^2)$$

on $[0,\dfrac{T}{\varepsilon}]\times \mathbb{R}^n$ for a suitable norm.
We proceed as follows: let us consider the following problem:

$$\Big(\varepsilon\partial_\tau + \mathcal{L}_1(\partial_T,\partial_X) + \frac{1}{\varepsilon}\mathcal{L}(-\omega\partial_\theta,k\partial_\theta)\Big)\mathcal{U} = f(\mathcal{U},\mathcal{U}) \tag{2.4}$$

For $\tau\in[0,\underline{T}]$, $T\in\mathbb{R}$, $X\in\mathbb{R}^n$ and $\theta\in[0,2\pi]$. This equation (called the singular equation) has been introduced by Joly-Métivier-Rauch [16] and is of crucial importance in this type of problem. The basic remark is the following one:

Proposition 2.1 *If $\mathcal{U}\in\mathcal{C}_\tau([0,\underline{T}],H^s_{\theta,T,X}([0,2\pi]\times\mathbb{R}^{n+1}))$ is a solution to (2.4) $(s>(n+2)/2)$ then*

$$u(t,x) := \mathcal{U}(\frac{k\cdot x-\omega t}{\varepsilon},t,x,\varepsilon t)$$

is a solution to (2.1) defined on $[0,\dfrac{\underline{T}}{\varepsilon}]\times\mathbb{R}^n$ and

$$|u|_{L^\infty([0,\frac{\underline{T}}{\varepsilon}]\times\mathbb{R}^n)} \le C|\mathcal{U}|_{\mathcal{C}_\tau([0,\underline{T}],H^s_{\theta,T,X}([0,2\pi]\times\mathbb{R}^{n+1}))}.$$

The proof is straightforward.
We look for approximate solutions to (2.4) under the form:

$$\mathcal{U}(\theta,T,X,\tau) = \sum_{j=0}^{2}\varepsilon^j u_j(\theta,T,X,\tau).$$

2.1 The WKB expansion

Plugging this ansatz for $\mathcal{U}$ in (2.4) yields

$$\Big(\varepsilon\partial_\tau + \mathcal{L}_1(\partial_T,\partial_X) + \frac{1}{\varepsilon}\mathcal{L}(-\omega\partial_\theta,k\partial_\theta)\Big)\mathcal{U} - f(\mathcal{U},\mathcal{U}) = \sum_{j=-1}^{4}\varepsilon^j r_j$$

with

$$r_{-1} = \mathcal{L}(-\omega\partial_\theta,k\partial_\theta)u_0$$

$$r_0 = \mathcal{L}(-\omega\partial_\theta, k\partial_\theta)u_1 + \mathcal{L}_1(\partial_T, \partial_X)u_0 - f(u_0, u_0)$$
$$r_1 = \mathcal{L}(-\omega\partial_\theta, k\partial_\theta)u_2 + \mathcal{L}_1(\partial_T, \partial_X)u_1 + \partial_\tau u_0 - 2f(u_0, u_1)$$
$$r_2 = \mathcal{L}_1(\partial_T, \partial_X)u_2 + \partial_\tau u_1 - 2f(u_0, u_2) - f(u_1, u_1)$$
$$r_3 = \partial_\tau u_2 - 2f(u_1, u_2)$$
$$r_4 = -f(u_2, u_2)$$

We now solve the equations $r_{-1} = 0$, $r_0 = 0$ and $r_1 = 0$.

i) $r_{-1} = 0$**:** the main ingredient that gives the Davey-Stewartson systems (or the nonlinear Schrödinger equation) is that one takes for u_0 a purely oscillatory wave that is:

$$u_0 = \left(u_{01}(T, X, \tau)e^{i\theta} + \bar{u}_{01}(T, X, \tau)e^{-i\theta}\right).$$

The equation $r_{-1} = 0$ then reads

$$\mathcal{L}(-i\omega, ik)u_{01} = 0.$$

Since $det(\mathcal{L}(-i\omega, ik)) = 0$ by definition of ω, this equation as a nontrivial solution $u_{01}(T, X, \tau)$ which moreover satisfies

$$\Pi_{l_0}(k)u_{01} = u_{01}.$$

ii) $r_0 = 0$**:** One has $f(u_0, u_0) = 2Ref(u_{01}, \bar{u}_{01}) + (f(u_{01}, u_{01})e^{2i\theta} + c.c.)$ where $c.c.$ denotes the complex conjugate. We therefore look for a solution u_1 to $r_0 = 0$ under the form

$$u_1 = u_{10} + (u_{11}e^{i\theta} + u_{12}e^{2i\theta} + c.c.).$$

Let us investigate each Fourier mode:

$$(e^{i\theta})^0 : \quad Eu_{10} = 2Ref(u_{01}, \bar{u}_{01}) \tag{2.5}$$

$$(e^{i\theta})^1 : \quad \mathcal{L}(-i\omega, ik)u_{11} + \mathcal{L}_1(\partial_T, \partial_X)u_{01} = 0 \tag{2.6}$$

$$(e^{i\theta})^2 : \quad \mathcal{L}(-2i\omega, 2ik)u_{12} = f(u_{01}, u_{01}) \tag{2.7}$$

• Solution for (2.5):
The right-hand-side of (2.5) has to be in the range of E. Therefore, if one denotes by Π_0 the orthogonal projector on the null space of E, necessary, one has:

$$\Pi_0 f(u_{01}, \bar{u}_{01}) = 0.$$

This implies that the following transparency condition has to be satisfied:

$$\forall V, W \in \mathbf{C}^d, \ \forall \xi \in \mathbb{R}^n \ \Pi_0 Ref(\Pi_l(\xi)V, \overline{\Pi_l(\xi)}W) = 0 \tag{2.8}$$

Note that if E is one-to-one, (2.8) is allways satisfied since $\Pi_0 = 0$. This condition is satisfied by a lot of physical systems (see the next section for Maxwell Bloch equations), in particular, all systems from which Davey-Stewartson systems can be derived ([4], [7], [8], [13]). This kind of condition is also introduced

in [20].
Once (2.8) is satisfied, one gets:

$$(1-\Pi_0)u_{10} = 2E^{(-1)}Ref(u_{01}, \bar{u}_{01}) \tag{2.9}$$

where $E^{(-1)}$ denotes the partial inverse of E such that $EE^{(-1)}V = E^{(-1)}EV = (1-\Pi_0)V\ \forall V \in \mathbf{C}^d$.
• Solution for (2.6):
Since $\mathcal{L}(-i\omega, ik)$ is not one-to-one, a necessary and sufficient condition for (2.6) to have a solution reads

$$\Pi_{l_0}(k)\mathcal{L}_1(\partial_T, \partial_X)u_{01} = 0$$

As in [10] this last equality reads thanks to hypothesis (2.2)

$$\left(\partial_T + \lambda'_{l_0}(k)\cdot\partial_X\right)u_{01} = 0. \tag{2.10}$$

We therefore obtain:

$$(1-\Pi_{l_0}(k))u_{11} = -\mathcal{L}^{(-1)}\left(\mathcal{L}(\partial_T, \partial_X)\right)u_{01} \tag{2.11}$$

where $\mathcal{L}^{(-1)}$ is the partial inverse of $\mathcal{L}(-i\omega, ik)$.
• Solution for (2.7):
we suppose that

$$\mathcal{L}(-2i\omega, 2ik) \text{ is one-to-one} \tag{2.12}$$

i.e. $(2\omega, 2k)$ is not in the caracteristic variety of $\mathcal{L}$ and one gets:

$$u_{12} = \left(\mathcal{L}(-2i\omega, 2ik)\right)^{-1} f(u_{01}, u_{01}). \tag{2.13}$$

iii) $r_1 = 0$**:** we compute

$$f(u_0, u_1) = 2Ref(u_{01}, \bar{u}_{11}) + \Big\{e^{i\theta}\left(f(u_{01}, u_{10}) + f(\bar{u}_{01}, u_{12})\right) + e^{2i\theta}f(u_{01}, u_{11})$$
$$+e^{3i\theta}f(u_{01}, u_{12}) + c.c.\Big\}$$

Writting

$$r_1 = r_{10} + \Big\{r_{11}e^{i\theta} + r_{12}e^{2i\theta} + r_{13}e^{3i\theta} + c.c.\Big\}$$

leads to 4 equations to solve:
• Solution of $r_{10} = 0$:
This equation reads

$$Eu_{20} + \mathcal{L}_1(\partial_T, \partial_X)u_{10} = 4Ref(u_{01}, \bar{u}_{11})$$

As before this equation can be solved with respect to u_{20} if and only if

$$\Pi_0\mathcal{L}_1(\partial_T, \partial_X)u_{10} = 4\Pi_0 Ref(u_{01}, \bar{u}_{11}) \tag{2.14}$$

and in this case one gets

$$(1-\Pi_0)u_{20} = E^{(-1)}\left[4Ref(u_{01},\bar{u}_{11}) - \mathcal{L}_1(\partial_T,\partial_X)u_{10}\right] \tag{2.15}$$

• Solution of $r_{11}=0$:
one has to solve

$$\partial_\tau u_{01} + \mathcal{L}(-i\omega, ik)u_{21} + \mathcal{L}_1(\partial_T,\partial_X)u_{11} = 2\left(f(u_{01},u_{10}) + f(\bar{u}_{01},u_{12})\right)$$

As before, it has a solution if and only if

$$\partial_\tau u_{01} + \Pi_{l_0}(k)\mathcal{L}_1(\partial_T,\partial_X)u_{11} = 2\Pi_{l_0}(k)\left(f(u_{01},u_{10}) + f(\bar{u}_{01},u_{12})\right) \tag{2.16}$$

and then

$$(1-\Pi_{l_0}(k))u_{21} = -\mathcal{L}^{(-1)}\mathcal{L}_1(\partial_T,\partial_X)u_{11} + 2\mathcal{L}^{(-1)}\left(f(u_{01},u_{10}) + f(\bar{u}_{01},u_{12})\right) \tag{2.17}$$

• Solution of $r_{12}=0$:
this equation becomes

$$\mathcal{L}(-2i\omega, 2ik)u_{22} + \mathcal{L}(\partial_T,\partial_X)u_{12} = f(u_{01},u_{11})$$

which gives (since $\mathcal{L}(-2i\omega,2ik)$ is one-to-one)

$$u_{22} = \left(\mathcal{L}(-2i\omega,2ik)\right)^{-1}\left(f(u_{01},u_{11}) - \mathcal{L}(\partial_T,\partial_X)u_{12}\right) \tag{2.18}$$

• Solution of $r_{13}=0$:
We assume that

$$\mathcal{L}(-3i\omega, 3ik) \text{ is one-to-one} \tag{2.19}$$

and obtain

$$u_{23} = \left(\mathcal{L}(-3i\omega,3ik)\right)^{-1} f(u_{01},u_{12}) \tag{2.20}$$

We now have to show that equations (2.9), (2.10), (2.11), (2.13), (2.14), (2.15), (2.16), (2.17), (2.18), (2.20) give enough information in order to find an approximate solution.

2.2 Study of the profile equations.

The aim is to find a system of two equations for u_{01} and $\Pi_0 u_{10}$ and to show that one can take $\Pi_{l_0}(k)u_{11}=0$ and $\Pi_{l_0}(k)u_{21}=0$. If we are able to solve this system (and therefore, if we obtain u_{01} and $\Pi_0 u_{10}$), equation (2.11) will give $(1-\Pi_{l_0}(k))u_{11}$, (2.13) gives u_{12}, (2.15) gives $(1-\Pi_0)u_{20}$, (2.17) gives $(1-\Pi_{l_0}(k))u_{21}$ and (2.18) and (2.20) give respectively u_{22} and u_{23}. This gives the complete approximate solution. In order to achieve our goal, we use the average operators introduced by D. Lannes [22]. For $w(T,X)\in\mathcal{C}^0(\mathbb{R}^{n+1})$, let

$$(G^h w)(T,X) = \frac{1}{h}\int_0^h w(T+s, X+s\lambda'_{l_0}(k))ds$$

and

$$< w >= \lim_{h \to +\infty} G^h w$$

when it exists. We summarize here the properties taken from [22] that we need concerning G^h and $< \cdot >$:

Lemma 2.1 *If $w \in L^\infty(\mathbb{R}; H^s(\mathbb{R}^n))$ and*

$$(\partial_T + \lambda'_{l_0}(k) \cdot \partial_X) w = 0$$

then $< w >$ exists and $< w >= w$.

Lemma 2.2 • *If $w \in \mathcal{C}^0(\mathbb{R}; H^s(\mathbb{R}^n))$ and w satisfies a sublinear growth condition*

$$\frac{1}{t} ||w||_{H^s}(t) \to 0 \text{ as } t \to +\infty$$

then

$$< (\partial_T + \lambda'_{l_0}(k) \cdot \partial_X) w >= 0.$$

• *If $w \in L^\infty(\mathbb{R}; H^s(\mathbb{R}^n))$ (resp. $L^\infty(\mathbb{R}^{n+1})$), then up to a subsequence, $G^h w$ converges in $L^\infty(\mathbb{R}; H^s(\mathbb{R}^n))$ $w*$ (resp. in $L^\infty(\mathbb{R}^{n+1})$ $w*$). We still denote $< w >$ a weak limit and*

$$\partial_T < w > + \lambda'_{l_0}(k) \cdot \partial_X < w >= 0$$

Lemma 2.3 *If w satisfies $(\partial_T + \lambda'_{l_0}(k) \cdot \partial_X) w = 0$ then for all function v*

$$G^h(vw) = w G^h(v)$$

We begin with equation (2.16): writting

$$u_{11} = (1 - \Pi_{l_0}(k)) u_{11} + \Pi_{l_0}(k) u_{11}$$

and

$$u_{10} = (1 - \Pi_0) u_{11} + \Pi_0 u_{10}$$

and using (2.11) and (2.9) give

$$\begin{aligned}
&\partial_\tau u_{01} - \Pi_{l_0}(k) \mathcal{L}_1(\partial_T, \partial_X) \mathcal{L}^{(-1)} \mathcal{L}_1(\partial_T, \partial_X) \Pi_{l_0}(k) u_{01} \\
&+ \Pi_{l_0}(k) \mathcal{L}_1(\partial_T, \partial_X) \Pi_{l_0}(k) u_{11} = 2 \Pi_{l_0}(k) \left(f(u_{01}, \Pi_0 u_{10}) \right. \\
&\left. + 2 f(u_{01}, E^{(-1)} f(u_{01}, \bar{u}_{01}) + f(\bar{u}_{01}, \mathcal{L}(-2i\omega, 2ik)^{-1} f(u_{01}, u_{01})) \right)
\end{aligned} \quad (2.21)$$

One has

Lemma 2.4

$$-\Pi_{l_0}(k) \mathcal{L}_1(\partial_T, \partial_X) \mathcal{L}^{(-1)} \mathcal{L}_1(\partial_T, \partial_X) \Pi_{l_0}(k) = -\frac{i}{2} \lambda''_{l_0}(k)(\partial_X, \partial_X)$$

See [10]. Recall moreover that

$$\Pi_{l_0}(k)\mathcal{L}_1(\partial_T,\partial_X)\Pi_{l_0}(k)=(\partial_T+\lambda'_{l_0}(k)\cdot\partial_X)\Pi_{l_0}(k)$$

and that εu_{11} is a corrector for the oscillatory part of the approximate solution. Therefore, one has $\varepsilon u_{11}(t,x,\varepsilon t)\to 0$ as $\varepsilon\to 0$ for $t\in[0,T_0/\varepsilon]$, $x\in\mathbb{R}^n$ for some $T_0>0$. As in [10], this is implied by the following stronger condition

$$\frac{1}{T}u_{11}(T,X,\tau)\to 0 \text{ as } T\to+\infty$$

and therefore, it satisfies the sublinear growth condition of lemma 2 and

$$<(\partial_T+\lambda'_{l_0}(k)\cdot\partial_X)\Pi_{l_0}(k)u_{11}>=0.$$

On the other hand, $\Pi_0 u_{10}$ is the first term of the asymptotic expansion of the mean value of u, therefore it has to be bounded with respect to T and we impose (at least)

$$\Pi_0 u_{10}\in L^\infty(\mathbb{R}^{n+1}).$$

Since $(\partial_T+\lambda'_{l_0}(k)\cdot\partial_X)u_{01}=0$, thanks to lemma (2.1), (2.2) and (2.3), applying the averaging operator on (2.21) leads to

$$\partial_\tau u_{01}-\frac{i}{2}\lambda''_{l_0}(k)(\partial_X,\partial_X)u_{01}=\mathcal{C}(u_{01},u_{01},\bar{u}_{01})+2\Pi_{l_0}(k)f(u_{01},<\Pi_0 u_{01}>) \tag{2.22}$$

where $\mathcal{C}$ denotes the cubic terms in the right-hand-side of (2.21).
We now study equation (2.14):
using (2.11) and (2.9), one gets:

$$\begin{aligned}&\Pi_0\mathcal{L}_1(\partial_T,\partial_X)\Pi_0 u_{10}+\Pi_0\mathcal{L}_1(\partial_T,\partial_X)2E^{(-1)}Ref(u_{01},\bar{u}_{01})\\ &=4\Pi_0 Ref(u_{01},\overline{\Pi_{l_0}(k)u_{11}})-4\Pi_0 Ref(u_{01},\overline{\mathcal{L}^{(-1)}\mathcal{L}_1(\partial_T,\partial_X)u_{01}})\end{aligned}$$

Now since $\Pi_{l_0}(k)u_{01}=u_{01}$, thanks to (2.8), one obtains

$$\begin{aligned}&\Pi_0\mathcal{L}_1(\partial_T,\partial_X)\Pi_0 u_{10}+\Pi_0\mathcal{L}_1(\partial_T,\partial_X)2E^{(-1)}Ref(u_{01},\bar{u}_{01})\\ &=-4\Pi_0 Ref(u_{01},\overline{\mathcal{L}^{(-1)}\mathcal{L}_1(\partial_T,\partial_X)u_{01}}).\end{aligned} \tag{2.23}$$

At this point, we need the following lemma:

Lemma 2.5

$$(1-\Pi_{l_0}(k))\,u_{11}=-i\sum_{j=1}^{n}\partial_{k_j}\Pi_{l_0}(k)\partial_{X_j}u_{01}.$$

Proof : One as $\Pi_{l_0}(\xi)^2=\Pi_{l_0}(\xi)$ since $\Pi_{l_0}(\xi)$ is a projector. Differentiating with respect to ξ leads to

$$\Pi_{l_0}(\xi)\partial_{\xi_j}\Pi_{l_0}(\xi)+\partial_{\xi_j}\Pi_{l_0}(\xi)\Pi_{l_0}(\xi)=\partial_{\xi_j}\Pi_{l_0}(\xi),$$

so that

$$\Pi_{l_0}(\xi)\partial_{\xi_j}\Pi_{l_0}(\xi)\Pi_{l_0}(\xi) = 0$$

for ξ in a neighborhood of k. Therefore the range of $\partial_{\xi_j}\Pi_{l_0}(\xi)\Pi_{l_0}(k)_{l_0}(\xi)$ is included in that of $(1 - \Pi_{l_0}(\xi))$.
On the other hand,

$$(-i\lambda_{l_0}(\xi) + iA(\xi) + E)\,\Pi_{l_0}(\xi) = 0.$$

Differentiating with respect to ξ leads to

$$\left(-i\partial_{\xi_j}\lambda_{l_0}(\xi_j) + iA_j\right)\Pi_{l_0}(\xi) + (-i\lambda_{l_0}(\xi) + iA(\xi) + E)\,\partial_{\xi_j}\Pi_{l_0}(\xi) = 0.$$

We apply $\mathcal{L}^{(-1)}$ on this last inequality and we get

$$\partial_{\xi_j}\Pi_{l_0}(\xi) = -i\mathcal{L}^{(-1)}\left(-\partial_{\xi_j}\lambda_{l_0}(\xi_j) + A_j\right)$$

which leads to the result thanks to (2.10) and (2.11). ∎

One has also

Lemma 2.6

$$2Re\Pi_0\left(f(u_{01}, \overline{-i\sum_{j=1}^{n}\partial_{k_j}\Pi_{l_0}(k)\partial_{X_j}u_{01}})\right) = i\Pi_0\sum_{j=1}^{n}\partial_{X_j}f(\partial_{k_j}\Pi_{l_0}(k)u_{01}, \overline{u}_{01}).$$

Proof : Thanks to (2.8), one has that for any u_1, u_2 in $\mathbf{C}^n$

$$\Pi_0 f(\Pi_{l_0}(\eta)u_1, \Pi_{l_0}(-\eta)u_2) = 0.$$

We differentiate this identity with respect to η_j and get:

$$\Pi_0 f(\partial_{\eta_j}\Pi_{l_0}(\eta)u_1, \Pi_{l_0}(-\eta)u_2) = g(\Pi_{l_0}(\eta)u_1, \partial_{\eta_j}\Pi_{l_0}(-\eta)u_2). \tag{2.24}$$

The result follows from (2.24) since f is symmetric and $\partial_{\eta_j}\overline{\Pi}_{l_0}(\eta) = -\partial_{\eta_j}\Pi_{l_0}(-\eta)$. ∎

It follows from this lemma that (2.23) reads

$$\begin{aligned}
&\Pi_0\mathcal{L}_1(\partial_T, \partial_X)\Pi_0 u_{10} \\
&= 2i\Pi_0\sum_{j=1}^{n}\partial_{X_j}f(\partial_{k_j}\Pi_{l_0}(k)u_{01}, \bar{u}_{01}) \qquad (2.25)\\
&-\Pi_0\mathcal{L}_1(-\lambda'_{l_0}(k)\cdot\partial_X, \partial_X)2E^{(-1)}Ref(u_{01}, \bar{u}_{01})
\end{aligned}$$

We now apply the averaging projector on (2.25). Recall that since u_{01} is the first term of the expansion of the mean value of the solution, it has to be bounded in (T,X). We therefore impose $u_{10} \in L^\infty_{T,X}(\mathbb{R}^{n+1})$. Denote by $< \Pi_0 u_{10} >$ a weak limit of $G^h \Pi_0 u_{10}$, it satisfies

$$(-\lambda'_{l_0}(k)\cdot\partial_X + \Pi_0 A(\partial_X)\Pi_0) < \Pi_0 u_{10} >$$

$$= 2i\Pi_0 \sum_{j=1}^{n} \partial_{X_j} f(\partial_{k_j}\Pi_{l_0}(k)u_{01}, \bar{u}_{01}) \tag{2.26}$$

$$-\Pi_0\mathcal{L}_1(-\lambda'_{l_0}(k)\cdot\partial_X, \partial_X)2E^{(-1)}Ref(u_{01}, \bar{u}_{01})$$

Concerning the operator $\Pi_0 A(\partial_X)\Pi_0$, one has the following result proved by D. Lannes [22]:

Proposition 2.2 *If $(0,0)$ is an isolated singular point of the caracteristic variety of $\mathcal{L}$ then the caracteristic variety of $\Pi_0\partial_T + \Pi_0 A(\partial_X)\Pi_0$ is the tangent cone of the caracteristic variety of $\mathcal{L}$ at $(0,0)$.*

That is, the velocities corresponding to $\Pi_0\partial_T + \Pi_0 A(\partial_X)\Pi_0$ are those of the long waves.
Equations (2.22) and (2.26) are the generalized Davey-Stewartson system.
We therefore have proved:

Theorem 2.1 *Let $\mathcal{U}^\varepsilon(\theta,T,X,\tau) = \sum_{j=0}^{2}\varepsilon^j u_j(\theta,T,X,\tau)$ with $u_0 = u_{01}e^{i\theta} + c.c.$ and $u_1 = u_{10} + (u_{11}e^{i\theta} + u_{12}e^{2i\theta} + c.c.)$. Suppose that u_{01} and u_{10} are bounded in τ, T, X and that u_{11} is a corrector that is $\sup_{\tau,X} \frac{1}{T}||u_{11}||(T) \to 0$ as $T \to +\infty$. Suppose moreover that*

$$\mathcal{L}\mathcal{U}^\varepsilon - f(\mathcal{U}^\varepsilon, \mathcal{U}^\varepsilon) = O(\varepsilon^2)$$

in $L^\infty_\tau([0,T_0]; L^\infty(\mathbb{R}^{n+1}_{T,X} \times [0,2\pi]_\theta)$.
Then necessarily:
i) Transparency:

$$\forall V, W \in \mathbf{C}^d \ \forall \xi \in \mathbb{R}^n,\ \ Re\Pi_0 f(\Pi_{l_0}(k)V, \overline{\Pi_{l_0}(k)}W) = 0.$$

ii) Polarization:

$$\Pi_{l_0}(k)u_{01} = u_{01}$$

iii) Transport at the group velocity:

$$(\partial_T + \lambda'_{l_0}(k)\cdot\partial_X)u_{01} = 0$$

iv) The average value of $\Pi_0 u_{10}$ (denoted by $< \Pi_0 u_{10} >$) and u_{01} satisfy the Davey-Stewartson system (2.22)-(2.26).

2.3 Solvability of the Davey-Stewartson system

The main difficulty for constructing solutions to (2.22) and (2.26) is the second equation (2.26). This last equation can be rewritten as

$$\left(-\lambda'_{l_0}(k)\cdot\partial_X+\Pi_0A(\partial_X)\Pi_0\right)\Pi_0u_{10}=P(\partial_X)g(u_{01},\bar{u}_{01}) \tag{2.27}$$

where g is bilinear and $P(\partial_X)$ is an homogeneous differential operator of first order. We use a spectralization of $\Pi_0A(\partial_X)\Pi_0$ under the form

$$\Pi_0A(\partial_X)\Pi_0=\sum_{j=1}^{p}i\mu_j(\partial_X)E_j(\partial_X)$$

where $E_j(\xi)$ are orthogonal projectors that are homogeneous of degree 0 in ξ and regular outside 0. The eigenvalue $i\mu_j(\xi)$ are imaginary and homogeneous of degree 1. Equation (2.27) then reads

$$\begin{gathered}\left(-\lambda'_{l_0}(k)\cdot\partial_X+i\mu_j(\partial_X)\right)E_j(\partial_X)\Pi_0u_{10}=E_j(\partial_X)P(\partial_X)g(u_{01},\bar{u}_{01})\\ \forall j=1\text{ to }p\end{gathered} \tag{2.28}$$

Three cases can occur:
Case i): For all $\xi\in\mathbf{S}^{n-1}$ and all $j=1$ to p, on has:

$$-\lambda'_{l_0}(k)\cdot\xi+\mu_j(\xi)\neq 0$$

Equation (2.28) is then called elliptic and

$$E_j(\partial_X)\Pi_0u_{10}=\left(-\lambda'_{l_0}(k)\cdot\partial_X+i\mu_j(\partial_X))E_j(\partial_X)\right)^{-1}E_j(\partial_X)P(\partial_X)g(u_{01},\bar{u}_{01})$$

where

$$\left(-\lambda'_{l_0}(k)\cdot\partial_X+i\mu_j(\partial_X)\right)^{-1}E_j(\partial_X)P(\partial_X)$$

is a Fourier multiplier that acts continuously on $H^s(\mathbb{R}^n)$ for all s. It follows that is $s>n/2$, equation (2.28) can be solved and

$$|E_j(\partial_X)\Pi_0u_{10}|_{H^s}\leq C|u_{01}|^2_{H^s}.$$

Using this estimates, it is easy to construct local in time solutions to (2.22) and (2.26) for regular initial data:

Proposition 2.3 *Let $s>n/2$ and $a(X)\in H^s(\mathbb{R}^n)$. Then there exists $T_0>0$ and a unique maximal solution $(u_{01},<\Pi_0u_{10}>)$ to the generalized Davey-Stewartson system (2.22) and (2.26) in $\mathcal{C}([0,T_0[,H^s)$.*

Case ii): For some j, the surfaces $\tau=\lambda'_{l_0}(k)\cdot\xi$ and $\tau=\mu_j(\xi)$ intersect transversally in $\mathbb{R}^{n+1}$ that is

$$\forall\xi_0\neq 0,\ \text{if }\ -\lambda'_{l_0}(k)\cdot\xi_0+\mu_j(\xi_0)=0\text{ then }\ -\lambda'_{l_0}(k)+\mu'_j(\xi_0)\neq 0 \tag{2.29}$$

In this case, equation (2.28) is called hyperbolic. In this case, on has:

Proposition 2.4 *Suppose that (2.29) is satisfied. Then for all* $u \in H^s(\mathbb{R}^n)$ *such that* $\sqrt{1+|x|^2}u \in H^s(\mathbb{R}^n)$*, there exists* $\psi \in L^\infty(\mathbb{R}^n)$ *solution to*

$$\left(-\lambda'_{l_0}(k) \cdot \partial_X + i\mu_j(\partial_X)\right) E_j(\partial_X)\psi = E_j(\partial_X)P(\partial_X)g(u,\bar{u})$$

Remark 2.1 *The result of proposition 2.4 is not sufficient in order to construct local solutions to the Davey-Stewartson systems (2.22) and (2.26). In fact, no general theory is available concerning this sytsems. See however section 3 for accurate results concerning the usual DS systems in the context of water-waves.*

Proof of proposition 2.4:
Let us consider $F(\xi) = -\lambda'_{l_0}(k) \cdot \xi + \mu_j(\xi)$ and

$$G: \begin{array}{ccc} \mathbf{S}^{n-1} & \to & \mathbb{R} \\ \xi & \mapsto & F(\xi) \end{array}$$

Let $\xi_0 \neq 0$ such that $G(\xi_0) = 0$. Since F is homogeneous of degree 1, one has $F'(\xi_0)\cdot\xi_0 = F(\xi_0) = 0$. Therefore, the radial part of $F'(\xi_0)$ is equal to zero and since $F'(\xi_0) \neq 0$ the tangential part is nonzero so that $G'(\xi_0) \neq 0$. Therefore,

$$Z = \{\xi \in \mathbf{S}^{n-1} \text{ such that } G(\xi) = 0\}$$

is a compact submanifold of $\mathbf{S}^{n-1}$. One can find K open sets U_k of $\mathbf{S}^{n-1}$ such that for all $k = 1$ to K, there exists a function ν_j satisfying $\xi \in Z \cap U_j \leftrightarrow \xi_{p_k} = \nu_k(\check{\xi})$, where $\check{\xi} = (\xi_1, \cdots, \xi_{p_k-1}, \xi_{p_k+1}, \cdots, \xi_n)$. Let U_0 be an open set such that $(U_0, U_1, \cdots, U_K)$ is a covering of $\mathbf{S}^{n-1}$ and $\bar{U}_0 \cap Z = \emptyset$ and let $(\varphi_0, \cdots, \varphi_K)$ be a partition of 1 attached to this covering:

$$1 = \sum_{m=1}^{K} \varphi_m(\xi) + \varphi_0(\xi).$$

In order to solve

$$\left(-\lambda'_{l_0}(k) \cdot \partial_X + i\mu_j(\partial_X)\right) E_j(\partial_X)\psi = E_j(\partial_X)P(\partial_X)g(u,\bar{u})$$

one then solves the $K+1$ equations:

$$\left(-\lambda'_{l_0}(k) \cdot \partial_X + i\mu_j(\partial_X)\right) E_j(\partial_X)\mathcal{F}^{-1}(\varphi_m)(\partial_X)\psi = \mathcal{F}^{-1}(\varphi_m)(\partial_X)E_j(\partial_X)P(\partial_X)g(u,\bar{u}) \tag{2.30}$$

That is taking

$$\psi_m = E_j(\partial_X)\mathcal{F}^{-1}(\varphi_m)(\partial_X)\psi$$

on has to solve

$$\left(-\lambda'_{l_0}(k) \cdot \partial_X + i\mu_j(\partial_X)\right) \psi_m = \mathcal{F}^{-1}(\varphi_m)(\partial_X)E_j(\partial_X)P(\partial_X)g(u,\bar{u}) \tag{2.31}$$

for $m = 0$ to K.
For $m = 0$: the operator

$$\left(-\lambda'_{l_0}(k)\cdot\partial_X + i\mu_j(\partial_X)\right)^{-1}\mathcal{F}^{-1}(\varphi_m)(\partial_X)E_j(\partial_X)P(\partial_X)$$

is a bounded operator on H^s since the symbol of $\left(-\lambda'_{l_0}(k)\cdot\partial_X + i\mu_j(\partial_X)\right)$ does not vanish on the support of $\varphi_)$. It follows that $|\psi_0|_{H^s} \le C|u|^2_{H^s}$ for $s > n/2$ so that $|\psi_0|_{L^\infty} \le C|u|^2_{H^s}$.
For $m \neq 0$: taking the Fourier transform of (2.31) yields

$$\left(-i\lambda'_{l_0}(k)\cdot\xi + i\mu_j(\xi)\right)\hat{\psi}_m = \varphi_m(\xi)P(\xi)E_j(\xi)\hat{g} \tag{2.32}$$

Moreover, since $G'(\xi) \neq 0$ when $G(\xi) = 0$, the function

$$\chi(\xi) = \frac{\xi_{p_m} - \nu_m(\check{\xi})}{-\lambda'_{l_0}(k)\cdot\xi + \mu_j(\xi)}$$

is bounded and (2.32) can be written

$$\left(\xi_{p_m} - \nu_m(\check{\xi})\right)\hat{\psi_m} = \chi(\xi)\varphi_m(\xi)E_j(\xi)P(\xi)\hat{g} \tag{2.33}$$

Note that $\chi E_j P$ is homogeneous of degree 1. Therefore, if u and $\sqrt{1+|x|^2}u$ are in H^s, then $h(x) := \mathcal{F}^{-1}(\chi(\xi)E_j(\xi)P(x)g\widehat{(u,\bar{u})})$ satisfies $h \in H^{s-1}$ and $\sqrt{1+|x|^2}h \in H^{s-1}$. Taking the inverse Fourier transform of (2.33) yields

$$(\partial_{x_{p_m}} - i\nu_m(\partial_{\check{x}}))\psi_m = h$$

which is solved as an evolution equation in the x_{p_m} variable. Now remark that $h\ inL^1_{x_{p_m}}(\mathbb{R}; H^s_{\check{x}}(\mathbb{R}^{n-1}))$ so that any solution ψ_m of this equation lies in $L^\infty(\mathbb{R}^n)$ and proposition 2.4 is proved. ∎

Remark 2.2 *The solution such constructed is not unique, and there are no natural way to choose a particular one. A choice is usually done in the case of water-waves, see next section.*

Case iii): The surfaces $\tau = \lambda'_{l_0}(k)\cdot\xi$ and $\tau = \mu_j(\xi)$ intersect but are tangent in some point of intersection. The system is degenerate and one has to modify the ansatz in order to describe this phenomena [5].

3 Some examples.

The aim of this section is to present some physical contextes in which the Davey-Stewartson systems occurs and to show that each of the three cases founded above can happen.

3.1 Water-waves problem.

Davey-Stewartson systems were derived first in [7] from Euler equation for incompressible, irrotationnal, perfect fluid with free surface with no surface tension, then in [8] when one adds surface tension effects. See also [4] in the case of non irrotationnal flows. These systems read

$$\begin{cases} i\partial_\tau + \alpha\partial_x^2 u + \beta\partial_y^2 u & = \quad \gamma|u|^2 u + \delta u\varphi \\ \\ (gh - c_g^2)\partial_x^2\varphi + gh\partial_y^2\varphi & = -a \quad \partial_x^2|u|^2 \end{cases}$$

where α, β, γ, δ and a are real parameters that can be expressed in terms of the physical quantities of this problem: the acceleration of gravity g, the mean depth of the water h the wave number k, the surface tension coefficient T. Here the dispersion relation reads

$$\omega^2 = k(g + Tk^2)\tanh(kh)$$

and $c_g = \frac{d\omega}{dk}$. Note that in this case $\sqrt{gh}$ denotes the velocity of long waves. This system is included in the class of the form (2.22)-(2.26) with $u = u_{01}$. The second equation can be seen as comming from a system of two first order equations and φ is one component of $\Pi_0 u_{10}$. Now each of the three cases of the discussion of the preceeding section occurs:
If $c_g^2 < gh$ we are in case i) and according to the classification of [11], the system is either elliptic-elliptic or hyperbolic-elliptic. In both cases, the local in time Cauchy problem is well understood see [11].
If $c_g^2 > gh$, we are in case ii) and the local Cauchy problem is much more difficult see [14] for recent results. Note that in this case, after a change of variable, the second equation writtes

$$(\partial_x^2 - \partial_y^2)\varphi = \partial_x^2|u|^2$$

and if one gives $u \in H^s$, it is easy to construct bounded solutions to this equation. Usually, one impose that $\varphi(x, y) \to 0$ as $x + y \to -\infty$ and $x - y \to -\infty$. Obviously, one could choose non homogeneous boundary conditions, or other boundary conditions at infinity, which change the solution to the system!
If $c_g^2 = gh$, we are in case iii) called the long-wave short-wave resonnance. One has to change the ansatz. See [8] for the one dimensional case and [32] or [23] for the 2-d case. See [5] for a general study of this phenomena.

3.2 Internal gravity waves.

For internal gravity waves, Grimshaw [13] derives the following system:

$$\begin{aligned} i\partial_t u + \delta\partial_x^2 u + \partial_y^2 u &= u(S_1 + S_2) + \chi|u|^2 u, \\ \partial_x^2 S_1 + \partial_y^2 S_1 &= \partial_x^2|u|^2, \\ \partial_x^2 S_2 + m\partial_y^2 S_2 &= \partial_y^2|u|^2. \end{aligned}$$

The parameter m can be of both sign. Again this system is included in the class of the form (2.22)-(2.26). The equations on S_1 and S_2 can be seen as a system of four first order equations.

3.3 Maxwell-Bloch system.

In the case of nonlinear optics, the equation discribing the interaction of a laser beam with a gaz are known as the Maxwell-Bloch system and read in a non dimensionnal form:

$$
\begin{aligned}
&\partial_t B + \nabla \times E = 0, \\
&\partial_t E - \nabla \times B + \frac{1}{\varepsilon} Q, \\
&\partial_t P - \frac{1}{\varepsilon} Q = 0, \\
&\partial_t Q + \frac{1}{\varepsilon} P - \frac{1}{\varepsilon} E = NE, \\
&\partial_t N = -Q \cdot E
\end{aligned}
\tag{3.1}
$$

see for example [29], [27] or [9]. This system enters directly in the theory developped above, one obtain that the profile equation reduces in fact to the cubic nonlinear schrödinger equation see [9] for example. The reason is simple: in this case, one has $\Pi_0 A \Pi_0 = 0$ so that the second equation of the Davey-Stewartson system can be integrated and one can express explicitely u_{10} with u_{01}.

4 Convergence in the case of Maxwell-Bloch type systems.

In this section, we impose two restrictions:

H1: The system is of Maxwell-Bloch type, *i.e.* it can be written:

$$
\left\{
\begin{array}{rcl}
\left(\partial_t + A(\partial_X) + \dfrac{E}{\varepsilon}\right) u &=& f(u, m) \\
\left(\partial_t + B(\partial_X) + \dfrac{F}{\varepsilon}\right) m &=& g(u, u)
\end{array}
\right.
\tag{4.1}
$$

This operator has a block-diagonal structure wich is very particular. This is one which is used in [20]. Its caracteristic variety is the union of that of

$$\mathcal{L}(\partial_t, \partial_X) = \partial_t + A(\partial_X) + E$$

and that of

$$\mathcal{G}(\partial_t, \partial_X) = \partial_t + B(\partial_X) + F$$

Let us call $i\lambda_l(\xi)$ and $\Pi_l^1(\xi)$ the eigenvalue and orthogonal projector of $iA(\xi)+E$ and $i\mu_l(\xi)$ and $\Pi_l^2(\xi)$ those of $iB(\xi)+F$. Finally, let us call Π_0^2 the orthogonal projector on $KerF$. We choose k and $\omega = \lambda_0(k)$ and we make the following assumption:

H2: $(k, \lambda_{l_0}(k))$ is a smooth point of the caracteristic variety of $\mathcal{L}$. Moreover $(k, \lambda_{l_0}(k))$ and $(2k, 2\lambda_{l_0}(k))$ are not in the caracteristic variety of $\mathcal{G}$.

We now construct the approximate solution as in the previous sections. One obtains:

$$\mathcal{U}_a^\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$$
$$\mathcal{M}_a^\varepsilon = \varepsilon m_1 + \varepsilon^2 m_2$$

with $u_0 = u_{01}e^{i\theta} + c.c.$, $u_1 = (1 - \Pi_{l_0}^1(k))u_{11}e^{i\theta} + c.c.$ with

$$(1 - \Pi_{l_0}^1(k))u_{11} = -i\sum_{j=1}^{n} \partial_{k_j}\Pi_{l_0}^1(k)\partial_{X_j}u_{01}$$

The third term $u_2 = ((1 - \Pi_{l_0}^1(k))u_{21}e^{i\theta} + u_{23}e^{3i\theta} + c.c.)$ is given by

$$(1-\Pi_{l_0}^1(k))u_{21} = \mathcal{L}^{(-1)}\left[-(\partial_T + A(\partial_X))(1 - \Pi_{l_0}^1(k))u_{11} + f(u_{01}, m_{10}) + f(\bar{u}_{01}, m_{12})\right]$$

and

$$\begin{aligned} m_1 = m_{10} + (m_{12}e^{2i\theta} + c.c.) &= (2F^{(-1)}g(u_{01}, \bar{u}_{01}) + \Pi_0^2 m_{10}) \\ &+ \Big((\mathcal{G}(-2i\omega, 2ik))^{-1}g(u_{01}, u_{01})e^{2i\theta} + c.c.\Big) \end{aligned}$$

Moreover

$$m_2 = (1 - \Pi_0^2)m_{20} + (m_{22}e^{2i\theta} + c.c.)$$

and

$$(1 - \Pi_0^2)m_{20} = F^{(-1)}(-\mathcal{G}(\partial_T, \partial_X)m_{10} + 4Reg(u_{01}, (1 - \overline{\Pi}_{l_0}^1(k))\bar{u}_{11}))$$

and

$$m_{22} = \mathcal{G}(-2i\omega, 2ik))^{-1}(-\mathcal{G}(\partial_T, \partial_X)m_{20} + g(u_{01}, (1 - \Pi_{l_0}^1(k))u_{11})$$

Functions $\Pi_{l_0}^1(k)u_{01}(X, T, \tau) = A(X - \lambda'_{l_0}(k)T, \tau)$ and $\Pi_0^2 m_{10}(X, T, \tau) = \varphi(X - \lambda'_{l_0}(k)T, \tau)$ satisfy the Davey-Stewartson system:

$$\begin{cases} i\partial_\tau A + \dfrac{1}{2}\lambda''_{l_0}(k)(\partial_X, \partial_X)A = \Pi_{l_0}^1(k)f\left(A, 2\Pi_0^2 g(A, \bar{A})\right) \\ +\Pi_{l_0}^1(k)f(A, \varphi) + \Pi_{l_0}^1(k)f\left(\bar{A}, \mathcal{G}(-2i\omega, 2ik)^{-1}g(A, A)\right), \\ \Pi_0^2\mathcal{G}_1(-\lambda'_{l_0}(k)\cdot\partial_X, \partial_X)\Pi_0^2\varphi - 2\Pi_0^2\mathcal{G}_1(-\lambda'_{l_0}(k)\cdot\partial_X, \partial_X)F^{(-1)}g(A, \bar{A}) \\ = +\dfrac{2i}{\lambda'_{l_0}(k)^2}\lambda'_{l_0}(k)\cdot\partial_X g(\lambda'_{l_0}(k)\cdot\nabla_k\Pi_{l_0}(k)A, \bar{A}). \\ A(0, X) = A_0(X) \end{cases} \quad (4.2)$$

4.1 The transparency hypothesis.

The transparency hypothesis which is necessary to obtain an approximate solution and that reads here:

$$\Pi_0^2 g(\Pi_{l_0}^1(k)U, \bar{\Pi}_{l_0}^1(k)V) = 0, \ \forall U, V \in \mathbf{C}^n$$

is not strong enough to perform the convergence proof. We need a more restrive one as in [20]:

H3: There exists a constant $C > 0$ such that $\forall \eta_1, \ \eta_2, \ \eta_3 \in \mathbb{R}^n$, for any eigenvalues $\lambda_j(\eta_1), \ \lambda_{j'}(\eta_2), \ \mu_{j''}(\eta_3)$ respectivelly of $iA(\eta_1) + E, \ iA(\eta_2) + E$, $iB(\eta_3) + F$ and all $u, v \in \mathbf{C}^d$, one has

$$\left| \frac{1}{\lambda_j(\eta_1) + \lambda_{j'}(\eta_2) - \mu_{j''}(\eta_3)} \Pi_{j''}^2(\eta_3) g\left(\Pi_j^1(\eta_1)u, \Pi_{j'}^1(-\eta_2)v\right) \right| \leq C|u||v|. \quad (4.3)$$

As in [5], this hypothesis has the following important consequence:

Proposition 4.1 *Under the transparency hypothesis* **H3***, then either the system (2.22) and (2.26) reduces to the nonlinear Schrödinger equation, or for all* $j = 1$ *to* p *equation (2.28) is elliptic, that means that* $det(-\lambda_{l_0}'(k) \cdot \xi + \Pi_0 A(\xi)\Pi_0)$ *does not vanish on the unit sphere of* $\mathbf{R}^n$.

Proof: The proof basically follows that of [5]. Take α a unitary vector and t in a neighborhood of 0 in $\mathbb{R}$ and η any vector. Take λ_j and $\lambda_{j'}$ two eigenvalues such that for all $\xi \in \mathbb{R}^n$ $\lambda_j(\xi) = -\lambda_{j'}(-\xi)$. Choose finally $\eta_3 = \alpha$, $\eta_1 = \eta + \frac{t}{2}\alpha$ and $\eta_2 = -\eta + \frac{t}{2}\alpha$. We now compute the denominator and the numerator of the left-hand-side of inequality (4.3) with μ denoting any of the μ_l. On the one hand, one has:

$$\lambda_j(\eta_1) + \lambda_{j'}(\eta_2) - \mu_{j''}(\eta_3) = \lambda_j(\eta + \frac{t}{2}\alpha) + \lambda_{j'}(\eta + \frac{t}{2}\alpha) - \mu_{j''}(t\alpha)$$

$$= \lambda_j(\eta) + \lambda_j'(\eta)\frac{t}{2}\alpha + \lambda_{j'}(-\eta) + \lambda_{j'}'(-\eta)\frac{t}{2}\alpha - \mu(t\alpha) + O(t^2)$$

It follows that

$$\lambda_j(\eta_1) + \lambda_{j'}(\eta_2) - \mu_{j''}(\eta_3) = \lambda_j'(\eta)t\alpha - \mu_{j''}(t\alpha) + O(t^2) \quad (4.4)$$

On the other hand:

$$\Pi_{j''}^2(t\alpha)g(\Pi_j^2(\eta + \frac{t\alpha}{2}U, \Pi_{j'}^1(-\eta + \frac{t\alpha}{2})V)$$

$$= \Pi_{j''}^2(0)g(\Pi_j^1(\eta)U, \Pi_{j'}(-\eta)V) + t\Pi_{j''}^2(0)\alpha g(\Pi_j^1(\eta)U, \Pi_{j'}^1(-\eta))V$$

$$+t\Pi_{j''}^2(0)g(\Pi_j^1(\eta)U, \frac{1}{2}\Pi_{j''}^{1'}(-\eta)\alpha V) + t\Pi_{j''}^2(0)g(\frac{1}{2}\Pi_j^{1'}(\eta)\alpha U, \Pi_{j''}^1(-\eta)V) + O(t^2)$$

It follows from this last expression and from (4.4) that

$$\begin{aligned}&\frac{1}{\lambda_j'(\eta)\alpha-\frac{1}{t}\mu_{j''}(t\alpha)}\Big(\Pi^2_{j''}(0)\alpha g(\Pi^1_j(\eta)U,\Pi^1_{j'}(-\eta)V)\\&+\Pi^2_{j''}(0)g(\Pi^1_j(\eta)U,\frac{1}{2}\Pi^{1'}_{j''}(-\eta)\alpha V)+\Pi^2_{j''}(0)g(\frac{1}{2}\Pi^{1'}_j(\eta)\alpha U,\Pi^1_{j''}(-\eta)V)\Big)\end{aligned}\tag{4.5}$$

has to be bounded by $C|U||V|$. We now have to make this terms explicit. Recall that for all α, on has:

$$\Pi^2_{j''}(t\alpha)(-i\mu_{j''}(t\alpha)+iB(t\alpha)+F)=0$$

We take the derivative of this equality (with respect to t) and let $t\to 0$, one gets

$$\partial_\alpha\Pi^2_{j''}(0)F+\Pi^2_{j''}(0)(-i\partial_\alpha\mu_{j''}(0)+iB(\alpha))=0.$$

Applying the partial inverse $F^{(-1)}$ of F to the right of this equality gives:

$$\Pi^{2'}_{j''}(0)\alpha(1-\Pi^2_{j''}(0))+\Pi^2_{j''}(\alpha)(-i\partial_\alpha\mu_{j''}(0)+iB(\alpha)F^{(-1)}=0$$

Now lemma 2.5 with the boundedness of expression (4.5), this last inequality combined with the second equation of (4.2) leads to the result. ■

4.2 The convergence proof.

As quoted in the introduction, the main difficulty in proving the stability of the above expansion is the existence time of the solution. We adopt here the point of view of [20]. Let (u,m) satisfy

$$\begin{cases}\left(\partial_t+A(\partial_x)+\dfrac{E}{\varepsilon}\right)u=f(u,m)\\[2ex]\left(\partial_t+B(\partial_x)+\dfrac{F}{\varepsilon}\right)m=g(u,u)\end{cases}$$

Changing m to εm in order to deal with $O(1)$ unknown gives:

$$\begin{cases}\left(\partial_t+A(\partial_x)+\dfrac{E}{\varepsilon}\right)u=\varepsilon f(u,m)\\[2ex]\left(\partial_t+B(\partial_x)+\dfrac{F}{\varepsilon}\right)m=\dfrac{1}{\varepsilon}g(u,u)\end{cases}\tag{4.6}$$

We recall here how to build a nonlinear change of unknow imported from [20] in order to have the correct existence time:
Take

$$N=m-\varepsilon h(u,u)$$

where h has to be determined. System (4.6) reads in Fourier variables:

$$\begin{cases} \left(\partial_t + \frac{i}{\varepsilon}\lambda_l(\varepsilon\xi)\right)\Pi_l^1(\varepsilon\xi)\hat{u}(\xi) = \Pi_l^1(\varepsilon\xi)\widehat{f(u,m)} \\ \left(\partial_t + \frac{i}{\varepsilon}\mu_{l'}(\varepsilon\xi)\right)\Pi_{l'}^2(\varepsilon\xi)\hat{m}(\xi) = \frac{1}{\varepsilon}\Pi_{l'}^2(\varepsilon\xi)\widehat{g(u,u)} \end{cases}$$

Now computing $\partial_t N$ leads to

$$\partial_t \Pi_l^2(\varepsilon\xi)\hat{N} = -\frac{i}{\varepsilon}\mu_l(\varepsilon\xi)\Pi_l^2(\varepsilon\xi)\hat{m} + \Pi_l^2(\varepsilon\xi)\int g(\hat{u}(\xi-\eta),\hat{u}(\eta))d\eta$$
$$-\Pi_l^2(\varepsilon\xi)\int h(-i\sum_{l'}\lambda_{l'}(\varepsilon(\xi-\eta))\Pi_{l'}^1(\varepsilon(\xi-\eta))\hat{u}(\xi-\eta),\sum_{l''}\Pi_{l''}^1(\varepsilon\eta)\hat{u}(\eta))$$
$$-\Pi_l^2(\varepsilon\xi)\int h(\sum_{l'}\Pi_{l'}^1(\varepsilon(\xi-\eta))\hat{u}(\xi-\eta),\sum_{l''}-i\lambda_{l''}(\varepsilon\eta)\Pi_{l''}^1(\varepsilon\eta)\hat{u}(\eta))d\eta$$
$$-\varepsilon\Pi_l^2(\varepsilon\xi)\int h(\sum_{l'}\Pi_{l'}^1(\varepsilon(\xi-\eta))\widehat{f(u,m)}(\xi-\eta),\hat{u}(\eta))$$
$$-\varepsilon\Pi_l^2(\varepsilon\xi)\int h(\hat{u}(\xi-\eta),\sum_{l''}\Pi_{l''}^1(\varepsilon(\eta))\widehat{f(u,m)}(\eta))d\eta$$

We now choose h in order to cancel the quadratic terms in this expression, that is:

$$\Pi_l^2(\xi)h(\Pi_{l'}^1(\xi-\eta),\Pi_{l''}^1(\eta)) := \frac{1}{\lambda_{l'}(\xi-\eta)+\lambda_{l''}(\eta)-\mu_l(\xi)}\Pi_l^2(\xi)g(\Pi_{l'}(\xi-\eta),\Pi_{l''}^1(\eta))$$

h is well defined and bounded thanks to the transparency condition (4.3). The system satisfied by (u,N) is now:

$$\begin{cases} \left(\partial_t + A(\partial_x) + \frac{E}{\varepsilon}\right)u = f(u, N+\varepsilon h(u,u)) \\ \left(\partial_t + B(\partial_x) + \frac{F}{\varepsilon}\right)m = 2\varepsilon h(f(u,N+\varepsilon h(u,u)),u) \end{cases}$$

A solution of size $O(1)$ of this system obviously exist on a time interval of size $O(1/\varepsilon)$. Since we already have constructed an approximate solution of the system, the proof of the convergence theorem follows the usual way, see [20], [10], [22]. As usual, it is interesting to introduce the singular equation.

$$\begin{cases} \left(\partial_\tau + \frac{1}{\varepsilon^2}\left(-\omega\partial_\theta + A(k)\partial_\theta + E + \varepsilon(A(\partial_x) - \lambda_{l_0}'(k)\partial_x)\right)\right)U = \frac{1}{\varepsilon}f(U,M) \\ \left(\partial_\tau + \frac{1}{\varepsilon^2}\left(-\omega\partial_\theta + B(k)\partial_\theta + F + \varepsilon(B(\partial_x) - \lambda_{l_0}'(k)\partial_x)\right)\right)M = \frac{1}{\varepsilon}g(U,U) \end{cases} \tag{4.7}$$

The approximate solution to (4.7) is built as follows:

$$U_a(\tau, X, \theta) = A(\tau, X)e^{i\theta} + c.c.$$

$$\begin{aligned} M_a = \varepsilon \Big(&2F^{(-1)} g(A(\tau, X), \bar{A}(\tau, X)) + \varphi(\tau, X) \\ &+ \Big(\mathcal{G}(-2i\omega, 2ik)^{-1} g(A, A)E^{2i\theta} + c.c.\Big)\Big) \end{aligned} \tag{4.8}$$

where $A(\tau, X)$ and $\varphi(\tau, X)$ satisfy the Davey-Stewartson system (4.2).

Theorem 4.1 *Let $A_0(X) \in H^\sigma$ for σ large enough. There exists $T > 0$ and a unique solution $(A, \varphi) \in \mathcal{C}([0, T]; H^\sigma)^2$ to (4.2) such that $A(0, X) = A_0(X)$. Let $s > (n+1)/2$. There exists a unique solution $(U(\tau, X, \theta), M(\tau, X, \theta)) \in \mathcal{C}_\tau([0, T], H^s(\mathbb{R}^n_X \times [0, 2\pi]_\theta))^2$ to (4.7) such that*

$$(U(0, X, \theta), M(0, X, \theta)) = (U_a(0, X, \theta), M_a(0, X, \theta)).$$

Moreover the following estimate holds:

$$|U(\tau, X, \theta) - U_a(\tau, X, \theta)|_{L^\infty(0,T,H^s)} + \frac{1}{\varepsilon}\, |M(\tau, X, \theta) - M_a(\tau, X, \theta)|_{L^\infty(0,T,H^s)} \to 0 \tag{4.9}$$

as $\varepsilon \to 0$.

As usual, comming back to the original variables, one gets the following result:

Theorem 4.2 *Let $A_0(X) \in H^\sigma$ for σ large enough. There exists $T > 0$ and a unique solution $(A, \varphi) \in \mathcal{C}([0, T]; H^\sigma)^2$ to (4.2) such that $A(0, X) = A_0(X)$. Let $s > (n+1)/2$. There exists a unique solution*

$$(u(t, X), M(t, X)) \in \mathcal{C}([0, T/\varepsilon], H^s(\mathbb{R}^n_X))^2$$

to (4.1) such that $(U(0, X), M(0, X)) = (U_a(0, x, k \cdot x/\varepsilon), M_a(0, x, k \cdot x/\varepsilon))$. Moreover the following estimate holds:

$$\begin{aligned} &\left| u(t, x) - U_a(\varepsilon t, x - \lambda'_{l_0}(k)t, \frac{i(k \cdot x - \omega t}{\varepsilon}) \right|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^n)} \\ +\frac{1}{\varepsilon} &\left| m(t, x) - M_a(\varepsilon t, x - \lambda'_{l_0}(k)t, \frac{i(k \cdot x - \omega t}{\varepsilon}) \right|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^n)} \to 0 \end{aligned} \tag{4.10}$$

as $\varepsilon \to 0$.

The existence result for the Davey-Stewartson system follows from proposition 4.1: as in [20], the strong transparency hypothesis **H3** implies that the asymptotic system is well-posed. The convergence estimate of theorem 4.1 follows from standart energy estimates on $(U, M) - (U_a, M_a)$ in system (4.7).

■

This change of variable on the physical Maxwell-Bloch system (3.1) is made explicitely in [20] as follows: the fields B, E, P, Q are of order $O(1)$ while N is of order $O(\varepsilon)$. The new unknown is $\tilde{N} = N + \frac{\varepsilon}{2}(P^2 + Q^2)$. The equations on $(P, Q, \tilde{N})$ become

$$\partial_t P - \frac{1}{\varepsilon} Q = 0,$$

$$\partial_t Q + \frac{1}{\varepsilon} P - \frac{1}{\varepsilon} E = E[\tilde{N} - \frac{\varepsilon}{2}(P^2 + Q^2)],$$

$$\partial_t \tilde{N} = \varepsilon Q P(\tilde{N} - \frac{\varepsilon}{2}(P^2 + Q^2)).$$

This kind of change of variable is used in [6], [1] in order to compute numerically the solution of Maxwell-Bloch systems. In fact, in this particular case, the situation is even simpler. Recall that for this Bloch model with two levels, one has $P + iQ = c_1 \bar{c}_2$ and $N = 1 - |c_1|^2 + |c_2|^2$ where $|c_1|$ and $|c_2|$ denotes the probabilities of presence of the electrons at the energy level 1 or 2. Recall that $|c_1|^2 + |c_2|^2 = 1$. A simple computation [6] gives:

$$P^2 + Q^2 = |c_1|^2 |c_2|^2 = |c_1|^2 (1 - |c_1|^2)$$

On the other hand, one has

$$N/2 = 1 - |c_1|^2$$

so that

$$P^2 + Q^2 = (1 - N/2) N/2$$

and N is therefore the solution of a simple equation. Since we want $N = O(\varepsilon)$, one has

$$N = 1 - \sqrt{1 - 4(P^2 + Q^2)}.$$

It follows that fonction N is not necessary to solve the problem and the equation on P, Q read

$$\partial_t P - \frac{1}{\varepsilon} Q = 0,$$

$$\partial_t Q + \frac{1}{\varepsilon} P - \frac{1}{\varepsilon} E = E[1 - \sqrt{1 - 4(P^2 + Q^2)}].$$

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