Littlewood-Paley-Stein functions and R-boundedness

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Outline

Framework

- (M, g) is a complete and non-compact Riemannian manifold,
- μ and ρ are the Riemannian measure and the Riemannian distance induced by g on M,
- $L = \Delta + V$ is a Schrödinger operator with a potential $0 \leq V \in L^1_{loc}$ and Δ is the Laplace-Beltrami operator,
- $e^{-tL}f$ is the solution of the heat equation

$$\frac{\partial}{\partial_t} u(t,x) + Lu(t,x) = 0$$

with initial data $f \in L^p(M)$,
- $C_0^\infty(M)$ is the set of smooth and compactly supported functions on M.

Framework

Vertical LPS function for $L = \Delta + V$

$$H(f)(x) = \left(\int_0^\infty |\nabla e^{-tL} f(x)|^2 + V(x)|e^{-tL} f(x)|^2 dt\right)^{1/2}.$$

One may consider

$$H^\Gamma(f)(x) = \left(\int_0^\infty |\Gamma e^{-tL} f(x)|^2 dt\right)^{1/2}.$$

with $\Gamma = \nabla$ or $\Gamma = V^{1/2}$.

Problem of L^p boundedness

Is there a constant $C_p > 0$ such that for all $f \in C_0^\infty$,

$$\|H^\Gamma(f)\|_p \leq C_p \|f\|_p?$$

Vertical LPS functions for Δ

Here $V = 0$, so $L = \Delta$ is the Laplace-Beltrami operator.

1. If $M = \mathbb{R}^n$, H is bounded on $L^p(M)$ for all $p \in (1, +\infty)$ and of weak-type (1,1).[1]
2. On an arbitrary manifold, H is bounded on $L^p(M)$ for all $p \in (1, 2]$.[1]
3. On compact Lie groups, H is bounded $L^p(M)$ for all $p \in (1, +\infty)$.[1]
4. If there exists $C, \delta > 0$ such that

$$|\nabla e^{-t\Delta} f| \leq C e^{-t\delta\Delta} |\nabla f|, \tag{*}$$

   then H is bounded $L^p(M)$ for all $p \in (1, +\infty)$[2]. This is in particular true if the Ricci curvature is non-negative.

Remark: (*) is difficult to obtain on arbitrary manifold, or replacing Δ by $L = \Delta + V$ with $V \neq 0$.

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[1] Stein

[2] Coulhon-Duong

Motivation 1: the Riesz transform

It is a major question in harmonic analysis to determine for which values of $p \in (1, +\infty)$, the Riesz transform $\nabla \Delta^{-1/2}$ is bounded on $L^p(M)$, that is there exists $C_p > 0$ such that for all $f \in C_0^\infty(M)$,

$$\|\nabla f\|_p \leq C_p \|\Delta^{1/2} f\|_p. \tag{R_p}$$

The values of p such that (R_p) is satisfied form an interval containing 2. Assume that H is bounded on $L^p(M)$ and there exists $C > 0$ such that for all $\omega \in C_0^\infty(\Lambda^1 T^* M)$,

$$\left\| \left(\int_0^\infty \left| \frac{\partial}{\partial t} e^{-t\vec{\Delta}} \omega(x) \right|_x^2 \frac{dt}{t} \right)^{1/2} \right\|_{p'} \leq C \left\| \omega \right\|_{p'} .$$

Then the Riesz transform $\nabla \Delta^{-1/2}$ is bounded on $L^p(M)$.

Motivation 2: regularity

The boundedness of H on L^p gives the regularity estimate

$$\|\nabla e^{-t\Delta} f\|_p \leq \frac{C}{\sqrt{t}} \|f\|_p.$$

. $\Rightarrow \{\sqrt{t}\nabla e^{-t\Delta}, t \geq 0\}$ is **uniformly bounded** on $L^p(M)$.

Remark: The representation formula for $\Delta^{-1/2}$ gives

$$\nabla \Delta^{-1/2} f = \int_0^\infty \left[\sqrt{t}\nabla e^{-t\Delta} f\right] \frac{dt}{t}.$$

The case $p = 2$

By integration by parts we obtain

$$\begin{aligned}
\|H(f)\|_2^2 &= \int_M \int_0^\infty |\nabla e^{-tL} f|^2 + V|e^{-tL} f|^2 dt\, dx \\
&= \int_M \int_0^\infty \Delta e^{-tL} f . e^{-tL} f + V|e^{-tL} f|^2 dt\, dx \\
&= \int_M \int_0^\infty L e^{-tL} f . e^{-tL} f dt\, dx \\
&= -\frac{1}{2} \int_M \int_0^\infty \frac{\partial}{\partial t} |e^{-tL} f|^2 dt\, dx \\
&= \frac{1}{2} \int_M |f|^2 dx = \frac{1}{2} \|f\|_2^2.
\end{aligned}$$

Then H is always bounded on $L^2(M)$.

Vertical Littlewood-Paley-Stein functions

Questions:

1. Is H^Γ bounded on L^p for $p \neq 2$?
2. Can we replace e^{-tL} by a more general function of tL ?
3. Can we obtain better than $\|\sqrt{t}\Gamma e^{-tL}\|_p \leq C$ from the boundedness of H^Γ ? What can we deduce about the Riesz transform $\Gamma L^{-1/2}$.

Some results about H ...

- H is bounded[1] on $L^p(M)$ for $p \in (1,2]$ for arbitrary manifolds,
- Assume $V \neq 0$. H is unbounded[2] on $L^p(M)$ for $p > N$ if the manifold satisfies the Sobolev inequality

$$|f(x) - f(y)| \leq C\rho(x,y)^{1-\frac{N}{p}} \|\nabla f\|_p$$

and if there exists $0 < \phi \in L^\infty(M)$ such that $e^{-tL}\phi = \phi$.
- Positive results for $p < 2$ in the case of potentials with non trivial negative part[3],

[1] Ouhabaz
[2] Chen-Magniez-Ouhabaz
[3] C.

... and the Riesz transform

For $p \in (1,2]$, the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^p(M)$ if we assume two assumptions[1].

1. Doubling property : there exists $C > 0$ such that for all $x \in M, r > 0$,

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \tag{D}$$

2. Gaussian upper estimate for the heat kernel : there exist $C, c > 0$ such that for all $x, y \in M, t > 0$,

$$p_t(x,y) \leq C \frac{exp(-c\frac{\rho(x,y)^2}{t})}{\mu(B(x,t^{1/2}))} \tag{G}$$

No proof of this result without (D) or (G) whereas we know examples where (G) is false[2].

[1] Coulhon-Duong
[2] Chen-Coulhon-Feneuil-Russ

Outline

Riesz $\Rightarrow$ LPS

Vector-valued extension

If $\Gamma L^{-1/2}$ is bounded $L^p(M)$, then it is bounded on $L^p(M, L^2(\mathbb{R}_+))$.

Let f be in $L^p(M)$. One has

$$\left\| \left(\int_0^\infty |\Gamma e^{-tL} f(x)|^2 dt \right)^{1/2} \right\|_p \leq C \left\| \left(\int_0^\infty |L^{1/2} e^{-tL} f(x)|^2 dt \right)^{1/2} \right\|_p \leq C \|f\|_p.$$

Question: Can we have better than the boundedness of H from the boundeness of $\Gamma L^{-1/2}$?

R-boundedness

Rademacher variables

A random variable is a Rademacher variable if

$$\mathbb{P}(\tau = 1) = \mathbb{P}(\tau = -1) = 1/2.$$

Definition

A family of operators $(T_t)_{t \in I}$ on L^p is R-bounded if there exists $C > 0$ such that for all $n \in \mathbb{N}$, $t_1, ..., t_n \in I$ and $f_1, ...f_n \in L^p$,

$$\mathbb{E} \left\| \sum_{i=1}^{n} \tau_i T_{t_i} f_i \right\|_p \leq C \mathbb{E} \left\| \sum_{i=1}^{n} \tau_i f_i \right\|_p ,$$

where (τ_i) is a sequence of Rademacher variable.

R-boundedness

Khintchine-Kahane inequality

Let p be in $(0,+\infty)$. There exists $A, B > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{C}^d$,

$$A\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \leq \left(\mathbb{E}\left|\sum_{i=1}^{n} \tau_i x_i\right|^p\right)^{1/p} \leq B\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

Theorem (Weis)

Let $I \subset \mathbb{R}$ be an interval. If $(T_t)_{t\in I}$ is R-bounded on $L^p(M)$, there exists $C > 0$ such that

$$\forall f \in L^p(M, L^2(I)), \left\|\left(\int_I |T_t f(t,.)|^2 dt\right)^{1/2}\right\|_p \leq C\left\|\left(\int_I |f(t,.)|^2 dt\right)^{1/2}\right\|_p.$$

Link Riesz - R-boundedness

Theorem

Given a $p \in (1, \infty)$ and suppose that the Riesz transform $\Gamma L^{-1/2}$ is bounded on $L^p(M)$. Then the set $\{\sqrt{t}\,\Gamma e^{-tL}, \quad t > 0\}$ is R-bounded on $L^p(M)$.

Let $T_k := \sqrt{t_k}\Gamma e^{-t_k L}$ for $t_k > 0$ and $f_k \in L^p(M)$ for $k = 1, ..., n$. We have

$$\begin{aligned} \mathbb{E}\left\|\sum_{k=1}^{n} \tau_k T_k f_k\right\|_p &= \mathbb{E}\left\|\Gamma L^{-1/2} \sum_{k=1}^{n} \tau_k (t_k L)^{1/2} e^{-t_k L} f_k\right\|_p \\ &\leq C\,\mathbb{E}\left\|\sum_{k=1}^{n} \tau_k (t_k L)^{1/2} e^{-t_k L} f_k\right\|_p . \end{aligned}$$

The R-boundedness of $\{(tL)^{1/2}e^{-tL}, t > 0\}$ completes the proof.

Link LPS - R-boundedness

Theorem (C.-Ouhabaz 2020)

Let $L = \Delta + V$ be a Schrödinger operator with $0 \leq V \in L^1_{loc}$ and $\Gamma = \nabla$ or $\Gamma = V^{1/2}$. The following properties are equivalent

1. There exists $C > 0$ such that for all $f \in L^p$,

$$\left\| \left(\int_0^\infty |\Gamma e^{-tL} f|^2 dt \right)^{1/2} \right\|_p \leq C \|f\|_p,$$

2. The operator family $\{\sqrt{t}\Gamma e^{-tL}, t \geq 0\}$ is R-bounded on L^p,

- The set $\{\sqrt{t}\nabla e^{-tL}, t \geq 0\}$ and $\{\sqrt{t}V^{1/2}e^{-tL}, t \geq 0\}$ are always R-bounded on L^p for $p \in (1, 2]$.
- (2) $\Rightarrow$ (1): we prove a more general result in the next section.

$(1) \Rightarrow (2)$: idea of proof

Set $I := \mathbb{E}\left|\sum_k \tau_k \sqrt{t_k}\, \Gamma\, e^{-t_k L} f_k\right|^2$. Using the independance of the Rademacher variables we obtain

$$
\begin{aligned}
I &= -\int_0^\infty \frac{d}{dt} \mathbb{E}|\Gamma e^{-tL} \sum_k \tau_k \sqrt{t_k} e^{-t_k L} f_k|^2 \, dt \\
&= 2\int_0^\infty \mathbb{E}\left[(\Gamma e^{-tL} \sum_k \tau_k \sqrt{t_k} e^{-t_k L} f_k) \cdot (\Gamma e^{-tL} \sum_k \tau_k \sqrt{t_k} L e^{-t_k L} f_k) \right] dt \\
&= 2\int_0^\infty \mathbb{E} \sum_k \Gamma e^{-tL} \tau_k \sqrt{t_k} e^{-t_k L} f_k \cdot \Gamma e^{-tL} \tau_k \sqrt{t_k} L e^{-t_k L} f_k \, dt \\
&= 2\int_0^\infty \mathbb{E} \sum_k \Gamma e^{-tL} \tau_k e^{-t_k L} f_k \cdot \Gamma e^{-tL} \tau_k (t_k L) e^{-t_k L} f_k \, dt \\
&= 2\int_0^\infty \mathbb{E}\left[(\Gamma e^{-tL} \sum_k \tau_k e^{-t_k L} f_k) \cdot (\Gamma e^{-tL} \sum_k \tau_k (t_k L) e^{-t_k L} f_k) \right] dt.
\end{aligned}
$$

$(1) \Rightarrow (2)$: idea of proof

Cauchy-Schwarz inequality gives

$$\begin{aligned} I &\leq 2\int_0^\infty \left(\mathbb{E}|\Gamma e^{-tL}\sum_k \tau_k e^{-t_k L} f_k|^2 \cdot \mathbb{E}|\Gamma e^{-tL}\sum_k \tau_k(t_k L) e^{-t_k L} f_k|^2\right)^{1/2} dt \\ &\leq \int_0^\infty \mathbb{E}|\Gamma e^{-tL}\sum_k \tau_k e^{-t_k L} f_k|^2\, dt + \int_0^\infty \mathbb{E}|\Gamma e^{-tL}\sum_k \tau_k(t_k L) e^{-t_k L} f_k|^2\, dt \\ &\leq \mathbb{E}\left[\left(H^\Gamma(\sum_k \tau_k e^{-t_k L} f_k)\right)^2\right] + \mathbb{E}\left[\left(H^\Gamma(\sum_k \tau_k(t_k L) e^{-t_k L} f_k)\right)^2\right]. \end{aligned}$$

Therefore, by Khintchine inequality,

$$c_p\sqrt{I} \leq \left|\mathbb{E}\left[\left(H^\Gamma(\sum_k \tau_k e^{-t_k L} f_k)\right)^p\right]\right|^{1/p} + \left|\mathbb{E}\left[\left(H^\Gamma(\sum_k \tau_k(t_k L) e^{-t_k L} f_k)\right)^p\right]\right|$$

$(1) \Rightarrow (2)$: idea of proof

The boundedness of H^Γ implies

$$\left\|\sqrt{I}\right\|_p \leq C\left(\left|\mathbb{E}\left\|\sum_k \tau_k e^{-t_k L} f_k\right\|_p^p\right|^{1/p} + \left|\mathbb{E}\left\|\sum_k \tau_k (t_k L) e^{-t_k L} f_k\right\|_p^p\right|^{1/p}\right)$$

$$\leq C'\left(\mathbb{E}\left\|\sum_k \tau_k e^{-t_k L} f_k\right\|_p + \mathbb{E}\left\|\sum_k \tau_k (t_k L) e^{-t_k L} f_k\right\|_p\right).$$

Then (2) comes from the R-boundedness of the sets $\{e^{-tL}, t \geq 0\}$ and $\{tLe^{-tL}, t \geq 0\}$.

Link LPS - Riesz

$(a)-$ The Riesz transform $\Gamma L^{-1/2}$ is bounded on L^p,

$$\Downarrow$$

$(b)-$ $\{\sqrt{t}\,\Gamma\, e^{-tL},\ t \geq 0\}$ is R-bounded on L^p,

$$\Updownarrow$$

$(c)-$ the LPS functional H^Γ is bounded on L^p,

$$\Downarrow$$

$(d)-$ $\{\sqrt{t}\,\Gamma\, e^{-tL},\ t \geq 0\}$ is uniformly bounded on L^p.

Remark : Under some assumptions, $(c) \Rightarrow (a)$, $(d) \Rightarrow (a)$.

H^∞ functionnal calculus

For $\omega > 0$, set

$$\Sigma(\omega) = \{z \neq 0, |arg(z)| < \omega\}.$$

Let $H^\infty(\Sigma(\omega))$ be the set of bounded holomorphic functions on $\Sigma(\omega)$.

$$H_0^\infty(\Sigma(\omega)) = \{f \in H^\infty(\Sigma(\omega)) : \exists \gamma, C > 0\, \forall z \in \Sigma(\omega), |F(z)| \leq C \frac{|z|^\gamma}{1 + |z|^{2\gamma}}\}.$$

Let $F \in H_0^\infty(\Sigma(\omega))$. For an appropriate contour γ one defines

$$F(L) = \frac{1}{2i\pi} \int_\gamma f(z)(zI - L)^{-1} dz.$$

One says that L has an bounded holomorphic functional calculus with angle ω if for some constant C_ω and for all $F \in H_0^\infty(\Sigma(\omega))$,

$$\|F(L)\|_{\mathcal{L}(L^p)} \leq C_\omega \sup_{z \in \Sigma(\omega)} |F(z)|. \tag{CF}$$

Square functions estimates

In this case, $F(L)$ is well-defined for all $F \in H^\infty(\Sigma(\omega))$ and satisfies (CF).
$-L$ is the generator of a sub-Markovian semigroup, then L has a bounded holomorphic functional calculus on $L^p(M)$ for all $p \in (1, +\infty)$.
$\to$ Most recent result[1] : $\omega_p = \arcsin |\frac{2}{p} - 1| + \epsilon$.

Square functions estimates[2]

Let L be a Schrödinger operator. Let $p \in (1, +\infty)$. If $\phi \in H_0(\Sigma(\omega_p))$ then

$$\|f\|_p \simeq \left\| \left(\int_0^\infty |\phi(tL)f|^2 \frac{dt}{t} \right)^{1/2} \right\|_p .$$

[1] Carbonaro-Dragičević
[2] Cowling-Doust-McIntosh-Yagi

Theorem

Theorem (C.-Ouhabaz, 2020)

Let $L = \Delta + V$ be a Schrödinger operator with $V \geq 0$. Let $\Gamma = \nabla$ or $\Gamma = V^{1/2}$. Let $F \in H^\infty(\Sigma(\omega_p))$. Assume there exists $\epsilon > 0$ and $\delta > 1/2$ such that $|F(z)| \leq \frac{C}{|z|^\delta}$ when $|z| \to +\infty$ and $|F'(z)| \leq C|z|^{\epsilon-1}$ when $z \to 0$.

If $\{\sqrt{t}\Gamma e^{-tL}, t \geq 0\}$ is R-bounded on $L^p(M)$, then there exists $C > 0$ such that for all $f \in L^p(M)$,

$$\left\| \left(\int_0^\infty |\Gamma F(tL) f|^2 dt \right)^{1/2} \right\|_p \leq C \left\| f \right\|_p .$$

<u>Remark</u> : Always true for $p \in (1, 2]$.

Proof

1. Set $I(x) = \left(\int_0^\infty |\Gamma F(tL)f(x)|^2 dt\right)^{1/2}$. An integration by parts gives

$$\begin{aligned} I^2 &= \lim_{t\to+\infty} t|\Gamma F(tL)f|^2 - 2\int_0^\infty t\Gamma LF'(tL)f \cdot \Gamma F(tL)f dt \\ &= -2\int_0^\infty t\Gamma LF'(tL)f \cdot \Gamma F(tL)f dt \\ &= 2\left(\int_0^\infty |\Gamma tLF'(tL)f|^2 dt\right)^{1/2} I. \end{aligned}$$

Then it is sufficient to bound $\| \left(\int_0^\infty |\Gamma G(tL)f|^2 dt\right)^{1/2} \|_p$ where $G(z) = zF'(z)$. Note that $G(z) \leq |z|^\epsilon$ when $z \to 0$.

Proof

2. Multiply by $(I+tL)^{\delta'}(I+tL)^{-\delta'}$, with $\delta > \delta' > 1/2$, to obtain

$$\left(\int_0^\infty |\Gamma G(tL)f|^2 dt\right)^{1/2} = \left(\int_0^\infty |\sqrt{t}\Gamma(1+tL)^{-\delta'}(1+tL)^{\delta'}G(tL)f|^2 \frac{dt}{t}\right)^{1/2}.$$

3. The R-boundedness of $\{\sqrt{t}\Gamma(I+tL)^{-\delta'}\}$ (equivalent to the R-boundedness of $\sqrt{t}\Gamma e^{-tL}$) gives

$$\left\|\left(\int_0^\infty |\Gamma F(tL)f|^2 dt\right)^{1/2}\right\|_p \leq C\left\|\left(\int_0^\infty |G(tL)(I+tL)^{\delta'}f|^2 \frac{dt}{t}\right)^{1/2}\right\|_p = C\left\|\left(\int_0^\infty |\phi(tL)f|^2 \frac{dt}{t}\right)^{1/2}\right\|_p$$

with $\phi(z) = zF'(z)(1+z)^{\delta'}$.

Proof

$$\phi(z) = zF'(z)(1+z)^{\delta'}$$

.

4. The choice $\delta' < \delta$ gives that $\phi \in H_0^\infty(\Sigma(\omega_p'))$ for $\omega_p' \in (\arcsin|\frac{2}{p} - 1|, \omega_p)$:
   - Cauchy's integral formula for f' implies $|\phi(z)| \leq \frac{C}{|z|^{\delta-\delta'}}$ when $|z| \to +\infty$,
   - The assumption $|F'(z)| \leq C|z|^{\epsilon-1}$ implies $|\phi(z)| \leq |z|^\epsilon$ when $z \to 0$.

   Then
$$\left\| \left(\int_0^\infty |\phi(tL)f|^2 \frac{dt}{t} \right)^{1/2} \right\|_p \leq \|f\|_p$$

   and the theorem follows.

Generalized LPS functions

Using the R-boundedness of holomorphic functional calculus and Khintchine-Kahane inequality we obtain the more general result.

Theorem (C.-Ouhabaz, 2020)

Let $m_1, ..., m_n \in H^\infty(\Sigma(\omega_p))$.

Under the same assumptions on F, there exists $C > 0$ (independent of m_k) such that for all $f_1, ..., f_n \in L^p(M)$,

$$\left\| \left(\int_0^\infty \sum_{k=1}^n |\Gamma m_k(L) F(tL) f_k|^2 dt \right)^{1/2} \right\|_p \leq C \sup_k \|m_k\|_\infty \left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{1/2} \right\|_p .$$

A result for spectral multipliers

Here we assume (D) and (G). Then,

$$\forall x \in M, \forall \lambda, r > 0,\ \mu(B(x,\lambda r)) \leq C\lambda^N \mu(B(x,r)).$$

Theorem (C.-Ouhabaz, 2020)

Let $m_k : [0,+\infty) \to \mathbb{C}$ with support contained in $[\frac{1}{2}, 2]$ for every k. If $\{\sqrt{t}\Gamma e^{-tL}, t \geq 0\}$ is R-bounded then for some $C > 0$ independent of n,

$$\left\| \left(\int_0^\infty \sum_{k=1}^n |\Gamma m_k(tL) f_k|^2 dt \right)^{1/2} \right\|_p \leq C \sup_k \|m_k\|_{W^{\delta,2}} \left\| \left(\sum_{k=1}^n |f_k|^2 \right)^{1/2} \right\|_p .$$

- δ depends on N,
- This relies on the same proof and on results by Deléaval-Kriegler.

Outline

Other Riesz transforms

One can define a local Riesz transform and a Riesz transform "at infinity"[1].

$$R^{\Gamma}_{loc} = \Gamma(L+I)^{-1/2}, \qquad R^{\Gamma}_{\infty} = \Gamma e^{-L} L^{-1/2}.$$

Note that

$$\Gamma L^{-1/2} \text{ is bounded on } L^p \iff R^{\Gamma}_{loc} \text{ and } R^{\Gamma}_{\infty} \text{ are bounded on } L^p.$$

Proof :

$$\begin{aligned}
\|\Gamma L^{-1/2} f\|_p &\leq \|\Gamma L^{-1/2} e^{-L} f\|_p + \|\Gamma(L+I)^{-1/2} L^{-1/2}(L+I)^{1/2}(I-e^{-L}) f\|_p \\
&\leq C\left[\|f\|_p + \|L^{-1/2}(L+I)^{1/2}(I-e^{-L}) f\|_p\right] \\
&\leq C\|f\|_p.
\end{aligned}$$

[1] Chen

Other LPS functionals

The boundedness of these functionals implies the boundedness the following LPS functions.

$$H^{\Gamma}_{loc}(f)(x) = \left(\int_0^1 |\Gamma e^{-tL} f(x)|^2 dt \right)^{1/2},$$
$$H^{\Gamma}_{(\infty)}(f)(x) = \left(\int_1^\infty |\Gamma e^{-tL} f(x)|^2 dt \right)^{1/2}.$$

Theorem

1. If R^{loc} is bounded on L^p, then H^{Γ}_{loc} is bounded on L^p,
2. If R^{∞} is bounded on L^p, then $H^{\Gamma}_{(\infty)}$ is bounded on L^p,

$$H_\Gamma \text{ is bounded on } L^p \iff H^{\Gamma}_{loc} \text{ and } H^{\Gamma}_{(\infty)} \text{ are bounded on } L^p.$$

All these functionals are always bounded on L^p for $p \in (1, 2]$.

Other LPS functionals

Proposition

Let $p \in (2, +\infty)$,

1. If the set $\{\sqrt{t}\Gamma e^{-tL}, t \in (0,1]\}$ is R-bounded on L^p, then H^{Γ}_{loc} is bounded on L^p.
2. If the set $\{\sqrt{t-1}\Gamma e^{-tL}, t \geq 1\}$ is R-bounded on L^p, then $H^{\Gamma}_{(\infty)}$ is bounded on L^p.

Theorem (Bakry)

Let M be a complete Riemannian manifold with Ricci curvature bounded from below, then $\nabla(\Delta + I)^{-1/2}$ is bounded on L^p for all $p \in (1, +\infty)$.

Corollary

On these manifolds, the local LPS functional H_{loc} for $L = \Delta$ is bounded on L^p for all $p \in (1, +\infty)$.

Outline

Lowers bounds - a useful example

The boundedness of these generalized functionals implies lower bounds on the dual space.

Example: $Q(f) = |e^{-L}f| + H_{loc}(f)$.

Theorem (C.-Ouhabaz 2020)

Let $p \in (1, \infty)$ and suppose that H_{loc} is bounded on $L^p(M)$. Then there exists a constant $C > 0$ such that for all g in the dual space $L^q(M)$.

$$C\|g\|_q \leq \|Q(g)\|_q$$

Proof

Let $f \in L^p(M) \cap L^2(M)$ and $g \in L^q(M) \cap L^2(M)$. We have

$$\int_0^1 \int_M \nabla e^{-tL} f . \nabla e^{-tL} g + \sqrt{V} e^{-tL} f . \sqrt{V} e^{-tL} g \, dx \, dt$$

$$= \int_0^1 \int_M (L e^{-2tL} f) g \, dx \, dt$$

$$= -\frac{1}{2} \int_M \int_0^1 \frac{d}{dt} (e^{-2tL} f) g \, dx \, dt$$

$$= \frac{1}{2} \int_M fg \, dx - \frac{1}{2} \int_M (e^{-L} f)(e^{-L} g) \, dx.$$

Proof

Therefore,

$$\begin{aligned}\left|\int_M fg\,dx\right| &\leq \int_M |e^{-L}f||e^{-L}g|\,dx + 2\int_M H_{loc}(f)H_{loc}(g)\,dx \\ &\leq 2\int_M (|e^{-L}f| + H_{loc}(f))(|e^{-L}g| + H_{loc}(g))dx \\ &\leq 2\|Q(f)\|_p\|Q(g)\|_q \\ &\leq C\,\|f\|_p\|Q(g)\|_q.\end{aligned}$$

The latter inequality extends by density to all $f \in L^p(M)$ and the proposition follows.

Outline

Application 1: Δ on $\mathbb{R}^n \# \mathbb{R}^n$

Let $M_n = \mathbb{R}^n \# \mathbb{R}^n$ be the connected sum of two copies of $\mathbb{R}^n \setminus B(0,1)$ glued among the unit spheres. On M_n,

- The Riesz transform is bounded on L^p for $p \in (1,n)$ and this is sharp[1],
- M_n has Ricci curvature bounded from below.

Consequently,

1. The LPS function associated with Δ is bounded on L^p for all $p \in (1,n)$,
2. The local Riesz transform and LPS function are bounded on L^p for all $p \in (1,+\infty)$,
3. The Riesz transform at infinity is unbounded for $p > n$.

[1] Coulhon-Duong, Carron-Coulhon-Hassell

Application 1: Δ on $\mathbb{R}^n \# \mathbb{R}^n$

- The heat kernel on M_n satisfies (G),
- The balls on M_n have polynomial growth,
- M_n satisfies a global Sobolev inequality.

$$\Rightarrow p_t(x,y) > C \frac{exp(-c\frac{\rho(x,y)^2}{t})}{\mu(B(x,t^{1/2}))},$$

$\Rightarrow$ H is unbounded on L^p for $p > n$. As H_{loc} is bounded, $H_{(\infty)}$ is necessarily unbounded.

Application 2: Elliptic operators on domains

All our results remain valid for elliptic operators with real bounded measurable coefficients a_{kl}. Let Ω be an open subset of $\mathbb{R}^N$ and $L = -div(A(x)\nabla \cdot)$ be a self-adjoint elliptic operator with Dirichlet conditions on Ω.

Theorem (C.-Ouhabaz 2020)

Let $L = -div(A(x)\nabla \cdot)$ be as previously. For $q \in [2, +\infty)$,

$$C\|f\|_q \leq \|e^{-L}f\|_q + \left\| \left(\int_0^1 |\nabla e^{-tL} f|^2 dt \right)^{1/2} \right\|_q$$

and

$$C\|f\|_q \leq \left\| \left(\int_0^\infty |\nabla e^{-tL} f|^2 dt \right)^{1/2} \right\|_q .$$

No assumption of regularity on A or Ω !

Application 2: Elliptic operators on domains

Theorem (C.-Ouhabaz 2020)

Let $L = -div(A(x)\nabla\cdot)$ be a self-adjoint elliptic operator with real bounded measurable coefficients a_{kl}. Then for all $q \in (1, \infty)$

$$C\,\|f\|_{L^q(\mathbb{R}^N)} \leq \left\| \left(\int_0^\infty |\nabla e^{-tL} f|^2 dt \right)^{1/2} \right\|_{L^q(\mathbb{R}^N)},$$

and

$$C\,\|f\|_{L^q(\mathbb{R}^N)} \leq \|e^{-L} f\|_{L^q(\mathbb{R}^N)} + \left\| \left(\int_0^1 |\nabla e^{-tL} f|^2 dt \right)^{1/2} \right\|_{L^q(\mathbb{R}^N)}.$$

Proof

We consider the case $q \in (1,2]$ only. There exists a Calderòn-Zygmund operator U such that $L^{1/2}f = U\nabla f$[1]. Therefore, square functions estimates for L give

$$\begin{aligned} \|f\|_q &\leq C\left\|\left(\int_0^\infty |L^{1/2}e^{-tL}f|^2 dt\right)^{1/2}\right\|_q \\ &= C\left\|\left(\int_0^\infty |U\nabla e^{-tL}f|^2 dt\right)^{1/2}\right\|_q \\ &\leq C\left\|\left(\int_0^\infty |\nabla e^{-tL}f|^2 dt\right)^{1/2}\right\|_q \leq C\|f\|_q. \end{aligned}$$

[1] Auscher-Tchamitchian

Merci pour votre attention !