Well-posedness of the stratified Euler equations

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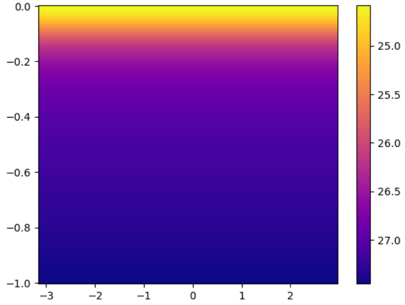
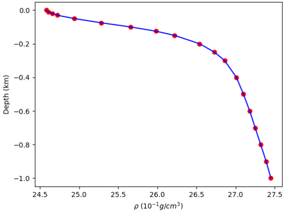
May 14th 2025

Setting





Mean stratification profile in the Atlantic ocean [**?**]

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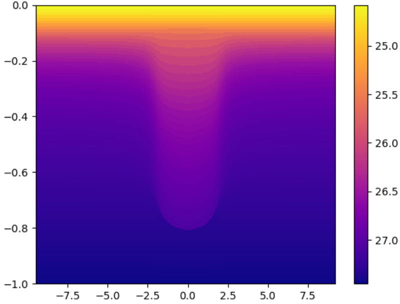

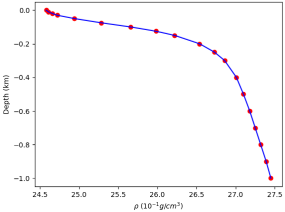

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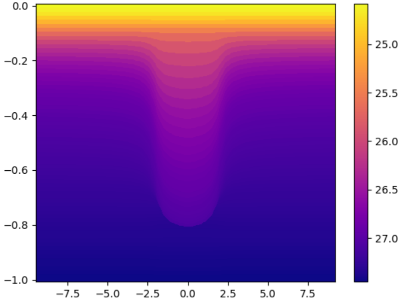

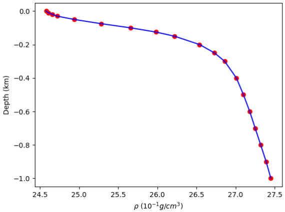

Mean stratification profile in the Atlantic ocean [?]

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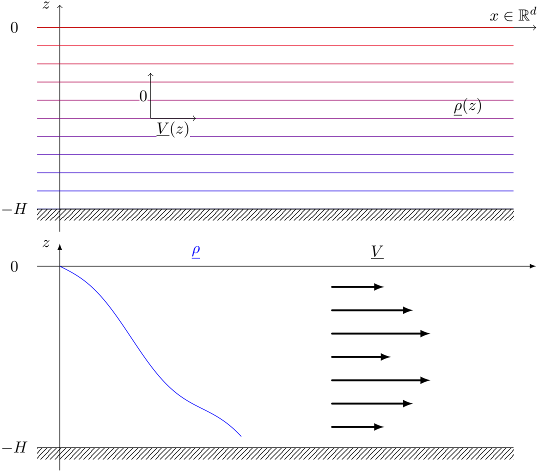
Setting at equilibrium



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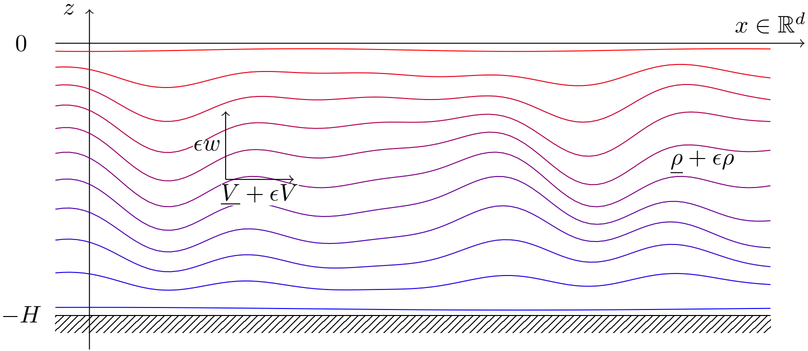
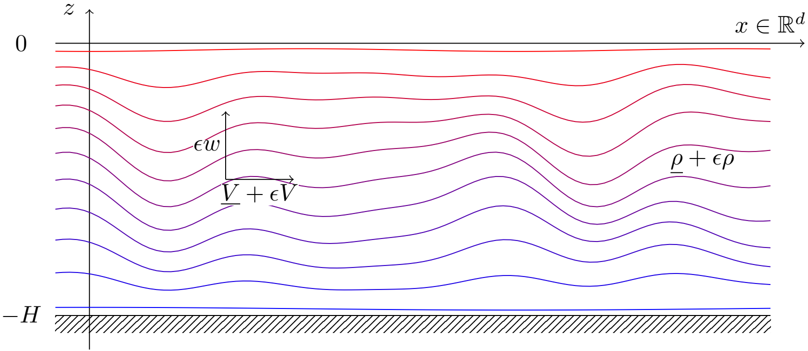

z
0
$x \in \mathbb{R}^d$
ϵw
$\underline{V} + \epsilon V$
$\underline{\rho} + \epsilon \rho$
$-H$

Setting



Assumption

The stratification is stable, i.e.

$$-\partial_z(\underline{\rho} + \epsilon\rho) \geq c_* > 0$$

The equations

The Euler equations linearized around the equilibrium $(\underline{V}, 0, \underline{\rho}, \underline{P})$ read

$$\begin{cases} \partial_t V + \underline{V}\cdot\nabla_x V + w\underline{V}' + \frac{1}{\underline{\rho}}\nabla_x P = O(\epsilon), \\ \mu(\partial_t w + \underline{V}\cdot\nabla_x w) + \frac{1}{\underline{\rho}}\partial_z P + g\frac{\rho}{\underline{\rho}} = O(\epsilon), \\ \partial_t \rho + \underline{V}\cdot\nabla_x \rho + w\underline{\rho}' = O(\epsilon), \\ \nabla_x \cdot V + \partial_z w = 0. \end{cases}$$

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Boundary conditions

$$\begin{cases} w_{|z=-1} = w_{|z=0} = 0, \\ \rho_{|z=-1} = \rho_{|z=0} = 0. \end{cases}$$

Initial conditions

$$\begin{cases} V_{|t=0} = V_{\rm in}, \quad w_{|t=0} = w_{\rm in}, \\ \rho_{|t=0} = \rho_{\rm in}. \end{cases}$$

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Previous results

Goal : Construct regular solutions *(in Sobolev spaces)* on $[0, T] \times \mathbb{R}^d \times [-1, 0]$, with T independent of μ.

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Theorem (Desjardins, Lannes, Saut [DLS20])

With no shear flow, if $N^2 := -g\frac{\underline{\rho}'}{\underline{\rho}}$ *is independent of* z *and under additional assumptions, there* **exists a unique solution** *to the stratified Euler equations on* $[0, \frac{1}{\epsilon/\sqrt{\mu}} T] \times \mathbb{R}^d \times [-1, 0]$.

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Theorem (Desjardins, Lannes, Saut [DLS20])

With no shear flow, if $N^2 := -g\frac{\underline{\rho}'}{\underline{\rho}}$ is independent of z and under additional assumptions, there **exists a unique solution** *to the stratified Euler equations on* $[0, \frac{1}{\epsilon/\sqrt{\mu}}T] \times \mathbb{R}^d \times [-1, 0]$.

Theorem (Bianchini, Duchêne [BD24])

With an additional diffusion term, there **exists a unique solution** *to the stratified Euler equations on* $[0, T] \times \mathbb{R}^d \times [-1, 0]$.

Main result

Theorem (F. [Fra24])

There **exists a unique solution** *to the stratified Euler equations on the time interval*

$$\left[0, \frac{T}{1 + |\underline{V}'|_{L^\infty}/\sqrt{\mu} + \epsilon/\sqrt{\mu}}\right].$$

Main result

Theorem (F. [Fra24])

There **exists a unique solution** *to the stratified Euler equations on the time interval*

$$\left[0, \frac{T}{1 + |\underline{V}'|_{L^\infty}/\sqrt{\mu} + \epsilon/\sqrt{\mu}}\right].$$

+ $\underline{V} \neq 0$
+ No additional assumptions.
+ No diffusion.

- Additional $|\underline{V}'|_{L^\infty}/\sqrt{\mu}$
- Additional 1: short time well-posedness.

Sketch of proof at the linear level

- Define the energy

$$\mathcal{E}_0 := \int_{\mathbb{R}^d\times[-1,0]} \underline{\rho} V^2 + \mu \int_{\mathbb{R}^d\times[-1,0]} \underline{\rho} w^2 + \int_{\mathbb{R}^d\times[-1,0]} \frac{g}{-\underline{\rho}'} \rho^2.$$

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- Prove the estimate

$$\frac{d}{dt}\mathcal{E}_0(t) \lesssim \frac{|\underline{V}'|_{L^\infty}}{\sqrt{\mu}} \mathcal{E}_0(t).$$

- Conclude by constructing a sequence of approximate solutions.

An energy estimate

$$\mathcal{E}_0 := \int_S \underline{\rho} V^2 + \int_S \underline{\rho}(\sqrt{\mu} w)^2 + \int_S \frac{g}{-\underline{\rho}'} \rho^2.$$

$$\begin{cases} \partial_t V + \underline{V} \cdot \nabla_x V + w \underline{V}' + \frac{1}{\underline{\rho}} \nabla_x P = 0 \\ \mu \left(\partial_t w + \underline{V} \cdot \nabla_x w \right) + \frac{1}{\underline{\rho}} \partial_z P + \frac{g \rho}{\underline{\rho}} = 0 \\ \partial_t \rho + \underline{V} \cdot \nabla_x \rho + \underline{\rho}' w = 0 \end{cases}$$

$$\nabla_x \cdot V + \partial_z w = 0$$

$$\boxed{\frac{d}{dt} \mathcal{E}_0 \lesssim \frac{|\underline{V}'|_{L^\infty}}{\sqrt{\mu}} \mathcal{E}_0}$$

Extension to the non-linear case

- Define the higher order energies

$$\mathcal{E}_s \approx \|V\|^2_{H^s(S)} + \mu\|w\|^2_{H^s(S)} + \|\rho\|^2_{H^s(S)}, \qquad s \geq \frac{d+1}{2} + 2.$$

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Main difficulties:

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Main difficulties:

- Characteristic Initial Boundary Value Problem : use of semi-Lagrangian coordinates.
- Loss of derivatives due to the semi-Lagrangian coordinates : use of Alinhac's good unknown.

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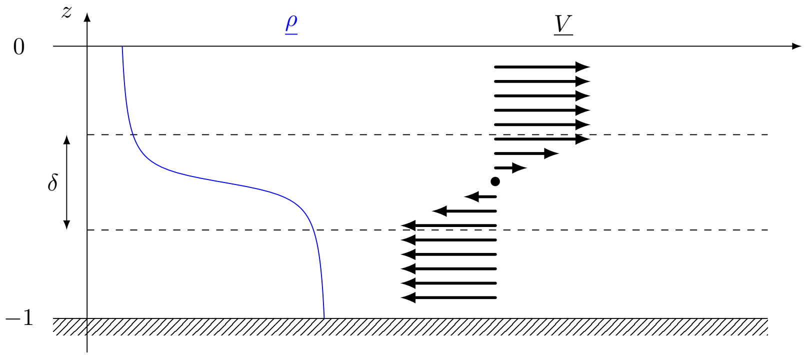
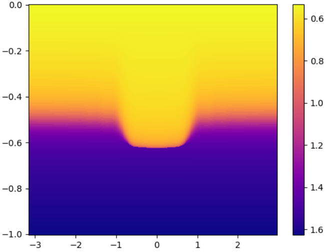
The sharp stratification



$\delta > 0$ is the thickness of the pycnocline.

The sharp stratification



Sharp stratification, $\underline{V} = 0$.

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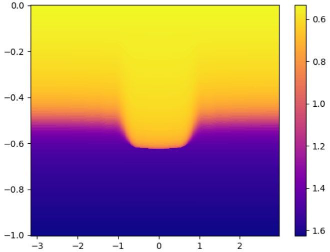

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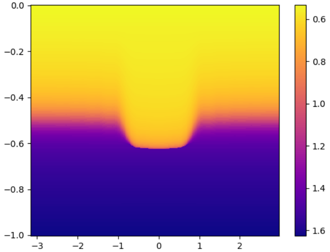

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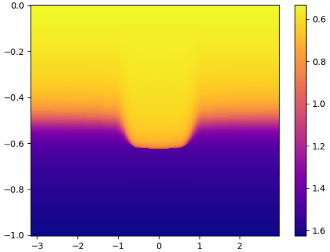
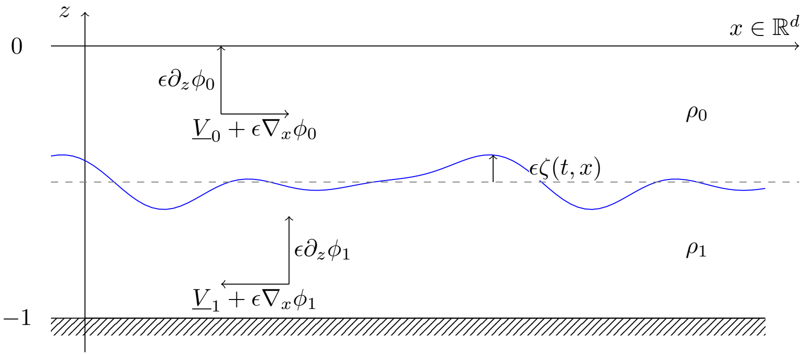


Sharp stratification, $\underline{V} = 0$.

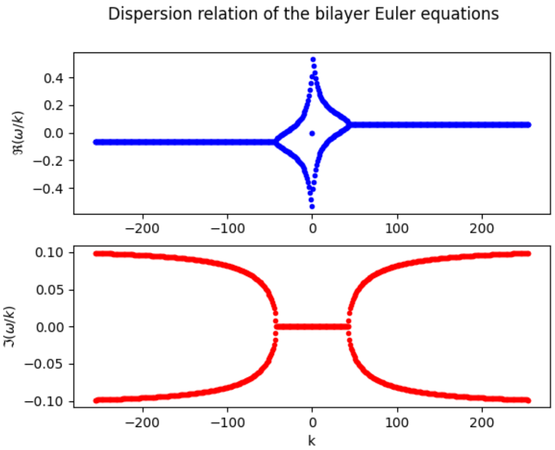
- Is the system described by the motion of the interface when $\delta << 1$?
- Solutions only defined on $[0, \delta T]$!

The expected limit ($\delta = 0$): the bilayer Euler equations



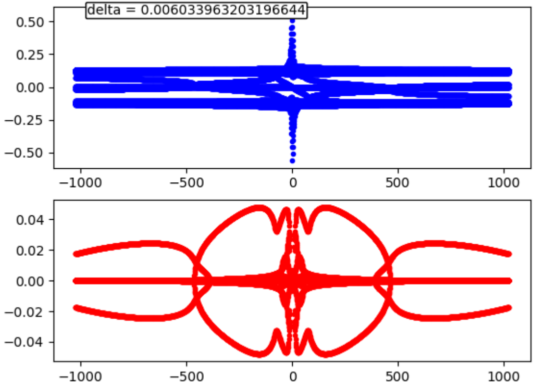
Two layers of constant density and irrotational velocity separated by an interface.

Dispersion relation of the bilayer Euler equations ($\delta = 0$)



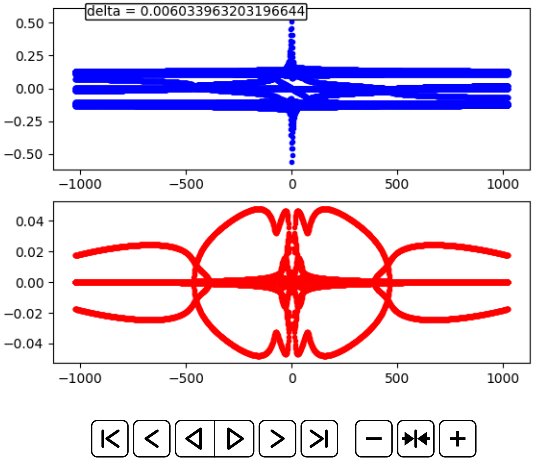
- Low frequencies are stable
- High frequencies are unstable (Kelvin-Helmholtz)

Dispersion relation of Stratified Euler



Dispersion relation of the stratified Euler equations as $\delta \to 0$.

Dispersion relation of Stratified Euler



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Conclusion

Recap:

- Well-posedness of the stratified Euler equations.
- The time interval is (*roughly*) independent of μ, but not δ.
- Numerical results for the spectral convergence towards the bilayer Euler equations.

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Open problems:

- Study the limit $\mu \to 0$ together with $\delta \to 0$.
- Spectral analysis on the stratified Euler equations in the limit $\delta \to 0$.

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Thank you for your attention !

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