Well-posedness of the stratified Euler equations

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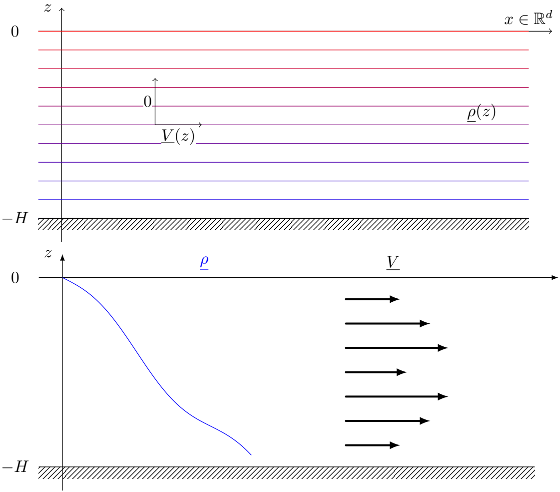
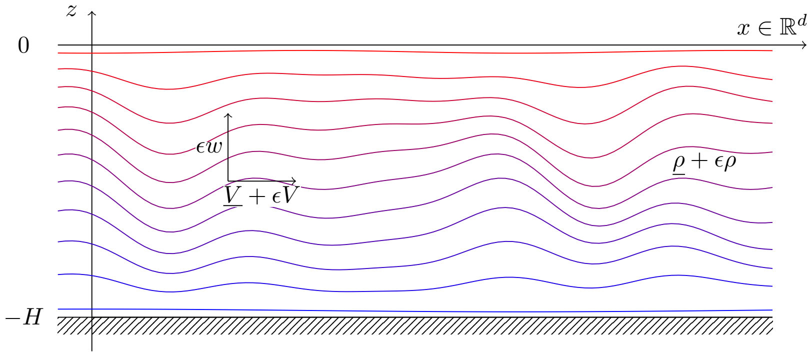
Setting at equilibrium



A **stably stratified shear flow** is an equilibrium for the Euler equations.

Setting

We study the **evolution in time** of a **perturbation** of the equilibrium.



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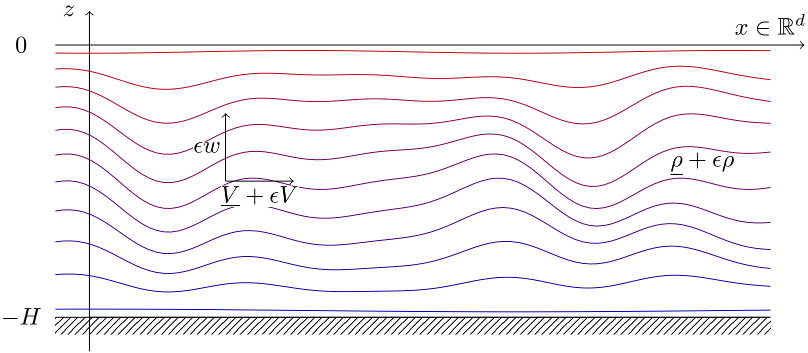

Assumption

The stratification is **stable**, i.e. $-\partial_z(\underline{\rho} + \epsilon\rho) \geq c_* > 0$.

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The equations

The Euler equations linearized around the equilibrium $(\underline{V}, 0, \underline{\rho}, \underline{P})$ read

$$\begin{cases} \partial_t V + \underline{V} \cdot \nabla_x V + w\underline{V}' + \dfrac{1}{\underline{\rho}}\nabla_x P = O(\epsilon), \\ \mu(\partial_t w + \underline{V} \cdot \nabla_x w) + \dfrac{1}{\underline{\rho}}\partial_z P + g\dfrac{\rho}{\underline{\rho}} = O(\epsilon), \\ \partial_t \rho + \underline{V} \cdot \nabla_x \rho + w\underline{\rho}' = O(\epsilon), \\ \nabla_x \cdot V + \partial_z w = 0. \end{cases}$$

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<u>Boundary conditions</u>

$$\begin{cases} w_{|z=-1} = w_{|z=0} = 0, \\ \rho_{|z=-1} = \rho_{|z=0} = 0. \end{cases}$$

<u>Initial conditions</u>

$$\begin{cases} V_{|t=0} = V_{\mathrm{in}}, \quad w_{|t=0} = w_{\mathrm{in}}, \\ \rho_{|t=0} = \rho_{\mathrm{in}}. \end{cases}$$

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Previous results

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Theorem (Desjardins, Lannes, Saut [DLS20])

With no shear flow, if $N^2 := -g\frac{\underline{\rho}'}{\underline{\rho}}$ *is independent of* z *and under additional assumptions, there* **exists a unique solution** *to the stratified Euler equations on* $[0, \frac{1}{\epsilon/\sqrt{\mu}}\tilde{T}]$.

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Theorem (Bianchini, Duchêne [BD24])

With an additional diffusion term, there **exists a unique solution** *to the stratified Euler equations on $[0,T]$.*

Main result

Theorem (F. [Fra24])

There **exists a unique solution** *to the stratified Euler equations on the time interval*

$$\left[0, \frac{\tilde{T}}{1 + |\underline{V}'|_{L^\infty}/\sqrt{\mu} + \epsilon/\sqrt{\mu}}\right].$$

+ $\underline{V} \neq 0$
+ No additional assumptions.
+ No diffusion.

- Additional $|\underline{V}'|_{L^\infty}/\sqrt{\mu}$
- Additional 1: short time well-posedness.

An energy estimate at the linear level

$$\mathcal{E}_0 := \int_S \underline{\rho} V^2 + \int_S \underline{\rho}(\sqrt{\mu} w)^2 + \int_S \frac{g}{-\underline{\rho}'} \rho^2.$$

$$\begin{cases} \partial_t V + \underline{V} \cdot \nabla_x V + w \underline{V}' + \frac{1}{\underline{\rho}} \nabla_x P = 0 \\ \mu \left(\partial_t w + \underline{V} \cdot \nabla_x w \right) + \frac{1}{\underline{\rho}} \partial_z P + \frac{g \rho}{\underline{\rho}} = 0 \\ \partial_t \rho + \underline{V} \cdot \nabla_x \rho + \underline{\rho}' w = 0 \end{cases}$$

$$\nabla_x \cdot V + \partial_z w = 0$$

$$\boxed{\frac{d}{dt} \mathcal{E}_0 \lesssim \frac{|\underline{V}'|_{L^\infty}}{\sqrt{\mu}} \mathcal{E}_0}$$

Extension to the non-linear case

- Define the higher order energies

$$\mathcal{E}_s \approx \|V\|^2_{H^s(S)} + \mu\|w\|^2_{H^s(S)} + \|\rho\|^2_{H^s(S)}, \qquad s \geq \frac{d+1}{2} + 2.$$

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- Characteristic Initial Boundary Value Problem : use of semi-Lagrangian coordinates.
- Loss of derivatives due to the semi-Lagrangian coordinates : use of Alinhac's good unknown.

Conclusion

Recap:

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- Remove the smallness assumptions.
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Thank you for your attention !

Bibliography I

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