Well-posedness of the stratified Euler equations

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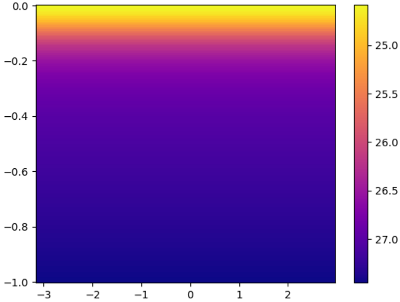
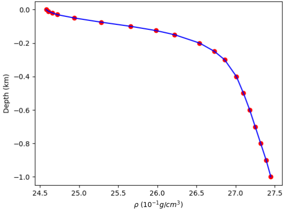
Setting





Mean stratification profile in the Atlantic ocean [Lev82]

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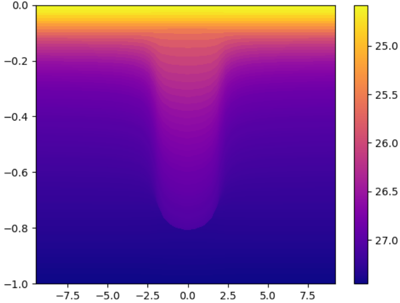

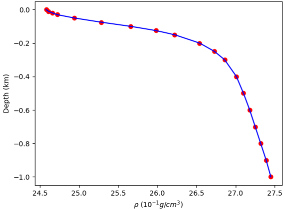

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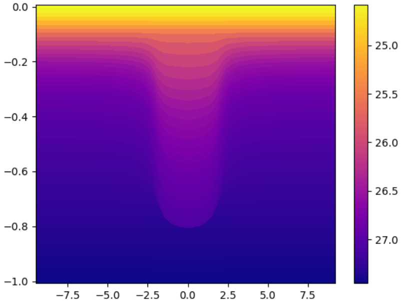

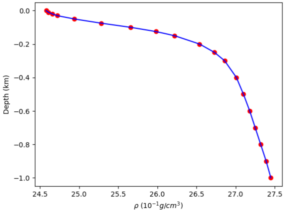

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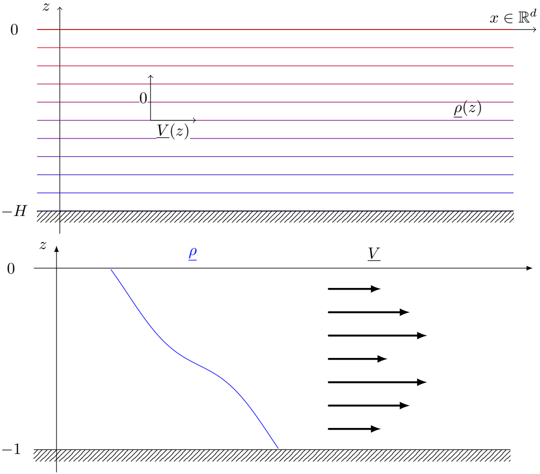
Setting at equilibrium



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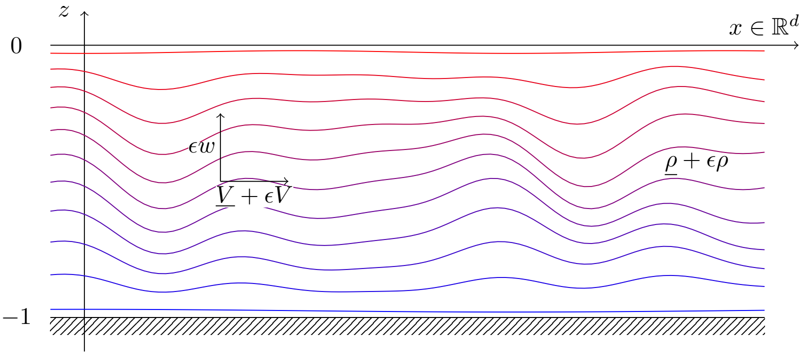
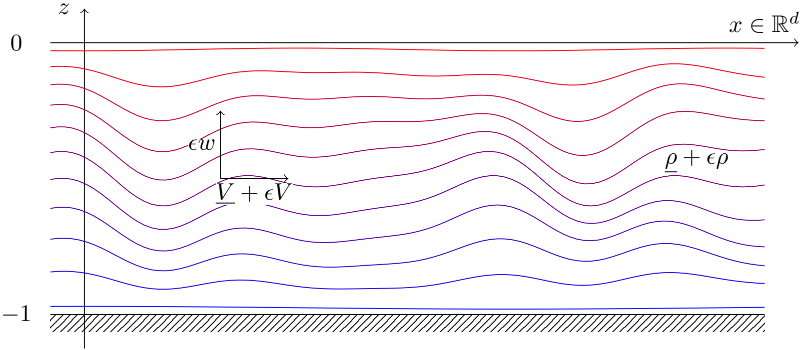

z
$x \in \mathbb{R}^d$
0
-1
ϵw
$\underline{V} + \epsilon V$
$\underline{\rho} + \epsilon\rho$

Setting



Assumption

The stratification is stable, i.e.

$$-\partial_z(\underline{\rho} + \epsilon\rho) \geq c_* > 0$$

The equations

The Euler equations linearized around the equilibrium $(\underline{V}, 0, \underline{\rho}, \underline{P})$ read

$$\begin{cases} \partial_t V + \underline{V} \cdot \nabla_x V + w\underline{V}' + \frac{1}{\underline{\rho}} \nabla_x P = O(\epsilon), \\ \mu(\partial_t w + \underline{V} \cdot \nabla_x w) + \frac{1}{\underline{\rho}} \partial_z P + g\frac{\rho}{\underline{\rho}} = O(\epsilon), \\ \partial_t \rho + \underline{V} \cdot \nabla_x \rho + w\underline{\rho}' = O(\epsilon), \\ \nabla_x \cdot V + \partial_z w = 0. \end{cases}$$

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<u>Boundary conditions</u>

$$\begin{cases} w_{|z=-1} = w_{|z=0} = 0, \\ \rho_{|z=-1} = \rho_{|z=0} = 0. \end{cases}$$

<u>Initial conditions</u>

$$\begin{cases} V_{|t=0} = V_{\rm in}, \quad w_{|t=0} = w_{\rm in}, \\ \rho_{|t=0} = \rho_{\rm in}. \end{cases}$$

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Previous results

Goal : Construct regular solutions *(in Sobolev spaces)* on $[0, T] \times \mathbb{R}^d \times [-1, 0]$, with T independent of μ.

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Theorem (Desjardins, Lannes, Saut [DLS20])

With no shear flow, if $N^2 := -g\frac{\underline{\rho}'}{\underline{\rho}}$ *is independent of* z *and under additional assumptions, there* **exists a unique solution** *to the stratified Euler equations on* $[0, \frac{1}{\epsilon/\sqrt{\mu}}\tilde{T}] \times \mathbb{R}^d \times [-1, 0]$.

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Theorem (Bianchini, Duchêne [BD24])

With an additional diffusion term, there **exists a unique solution** *to the stratified Euler equations on $[0, T] \times \mathbb{R}^d \times [-1, 0]$.*

Main result

Theorem (F. [Fra24])

There **exists a unique solution** *to the stratified Euler equations on the time interval*

$$\left[0, \frac{\tilde{T}}{1 + |\underline{V}'|_{L^\infty}/\sqrt{\mu} + \epsilon/\sqrt{\mu}}\right].$$

+ $\underline{V} \neq 0$
+ No additional assumptions.
+ No diffusion.

- Additional $|\underline{V}'|_{L^\infty}/\sqrt{\mu}$
- Additional 1: short time well-posedness.

Sketch of proof at the linear level

- Define the energy

$$\mathcal{E}_0 := \int_{\mathbb{R}^d\times[-1,0]} \underline{\rho} V^2 + \mu \int_{\mathbb{R}^d\times[-1,0]} \underline{\rho} w^2 + \int_{\mathbb{R}^d\times[-1,0]} \frac{g}{-\underline{\rho}'} \rho^2.$$

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- Prove the estimate

$$\frac{d}{dt}\mathcal{E}_0(t) \lesssim \frac{|\underline{V}'|_{L^\infty}}{\sqrt{\mu}} \mathcal{E}_0(t).$$

- Conclude by constructing a sequence of approximate solutions.

An energy estimate

$$\mathcal{E}_0 := \int_S \underline{\rho} V^2 + \int_S \underline{\rho}(\sqrt{\mu} w)^2 + \int_S \frac{g}{-\underline{\rho}'} \rho^2.$$

$$\begin{cases} \partial_t V + \underline{V} \cdot \nabla_x V + w \underline{V}' + \frac{1}{\underline{\rho}} \nabla_x P = 0 \\ \mu \left(\partial_t w + \underline{V} \cdot \nabla_x w \right) + \frac{1}{\underline{\rho}} \partial_z P + \frac{g \rho}{\underline{\rho}} = 0 \\ \partial_t \rho + \underline{V} \cdot \nabla_x \rho + \underline{\rho}' w = 0 \end{cases}$$

$$\nabla_x \cdot V + \partial_z w = 0$$

$$\boxed{\frac{d}{dt} \mathcal{E}_0 \lesssim \frac{|\underline{V}'|_{L^\infty}}{\sqrt{\mu}} \mathcal{E}_0}$$

Extension to the non-linear case

- Define the higher order energies

$$\mathcal{E}_N \approx \|V\|^2_{H^N(S)} + \mu\|w\|^2_{H^N(S)} + \|\rho\|^2_{H^N(S)}, \qquad N > \frac{d+1}{2} + 2.$$

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$$\frac{d}{dt}\mathcal{E}_N(t) \lesssim \left(1 + \frac{\epsilon}{\sqrt{\mu}} + \frac{|\underline{V}'|_{L^\infty}}{\sqrt{\mu}}\right)\mathcal{E}_N(t).$$

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Main difficulties:

- Characteristic Initial Boundary Value Problem : use of **isopycnal coordinates**.

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Main difficulties:

- Characteristic Initial Boundary Value Problem : use of **isopycnal coordinates**.
- Loss of derivatives due to the isopycnal coordinates : use of **Alinhac's good unknown**.

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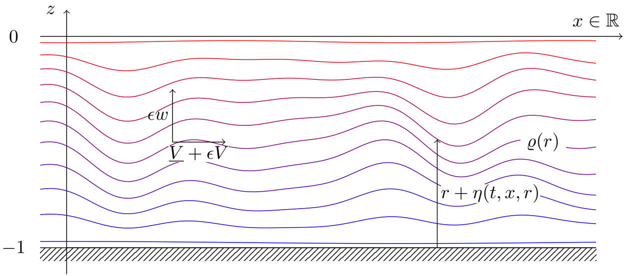
In this section, $\underline{V} = 0$ and $d = 1$ to simplify notations.

Change of coordinates (1)



- The density is constant along the isopycnals:

$$(\underline{\rho} + \epsilon \rho)(t, x, r + \epsilon \eta(t, x, r)) = \varrho(r).$$

- The stratification is admissible: $\partial_r (r + \epsilon \eta) \geq c_* > 0$.
- The stratification is stable: $-\partial_r \varrho \geq c_* > 0$.

Change of coordinates (2)

$$\varphi : \left| \begin{array}{r} [0,T) \times \mathbb{R} \times [-1,0] \to [0,T) \times \mathbb{R} \times [-1,0] \\ (t,x,r) \mapsto \varphi(t,x,r) := (t,x,r+\epsilon\eta(t,x,r)). \end{array} \right.$$

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Semi-Lagrangian property

$$\partial^{\eta}_t f^{\mathrm{iso}} + \epsilon V^{\mathrm{iso}} \partial^{\eta}_x f^{\mathrm{iso}} + \epsilon w^{\mathrm{iso}} \partial^{\eta}_r f^{\mathrm{iso}} = \partial_t f^{\mathrm{iso}} + \epsilon V^{\mathrm{iso}} \partial_x f^{\mathrm{iso}}$$

Euler equations in isopycnal coordinates

Recall $\nabla^\eta_{t,x,r} f = \nabla_{t,x,r} f - \epsilon \frac{\nabla_{t,x,r}\eta}{1+\epsilon\partial_r\eta}\partial_r f.$

The Euler equations in isopycnal coordinates read

$$\begin{cases} \partial_t V + \epsilon V \partial_x V + \frac{1}{\varrho}\partial_x^\eta P = 0, \\ \mu(\partial_t w + \epsilon V \partial_x w) + \frac{1}{\varrho}\partial_r^\eta P - \frac{g\varrho'}{\varrho}\eta \approx 0, \\ \partial_t \eta + \epsilon V \cdot \nabla_x \eta - w = 0, \\ \partial_x^\eta V + \partial_r^\eta w = 0, \end{cases} \qquad \text{in } [0,T]\times\mathbb{R}\times[-1,0].$$

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(Bianchini, Duchêne [BD24])

There are more non-linearities than in Eulerian coordinates !

How to recover the structure of the system in Eulerian coordinates?

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Alinhac's good unknown

In Eulerian coordinates, with $\dot{V}^{(n)} := \partial^n V, \dot{w}^{(n)} := \partial^n w$:

$$\partial_x \dot{V}^{(n)} + \partial_r \dot{w}^{(n)} = 0.$$

Alinhac's good unknown

In isopycnal coordinates, with $\dot{V}^{(n)} := \partial^n V, \dot{w}^{(n)} := \partial_n w$:

$$\partial_x^\eta \dot{V}^{(n)} + \partial_r^\eta \dot{w}^{(n)} = [\partial^n, \partial_x^\eta] V + [\partial^n, \partial_r^\eta] w.$$

$$\approx -\epsilon \partial^n \left(\frac{\partial_x \eta}{1 + \epsilon \partial_r \eta} \right) \partial_r V - \epsilon \partial^n \left(\frac{\partial_r \eta}{1 + \epsilon \partial_r \eta} \right) \partial_r w.$$

Problem: "Loss of derivatives"

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Problem: "Loss of derivatives"

Solution: Alinhac's good unkown [Ali89]

$$\tilde{f}^{(n)} := \partial^n f - \epsilon \frac{\partial^n \eta}{1 + \epsilon \partial_r \eta} \partial_r f,$$

$$\partial^n \nabla_{t,x,r}^\eta f \approx \nabla_{t,x,r}^\eta \tilde{f}^{(n)}.$$

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In isopycnal coordinates, with
$\tilde{V}^{(n)} := \partial^n V - \epsilon \frac{\partial^n \eta}{1+\epsilon\partial_r\eta}\partial_r V$, $\tilde{w}^{(n)} := \partial^n w - \epsilon \frac{\partial^n \eta}{1+\epsilon\partial_r\eta}\partial_r w$:

$$\partial_x^\eta \tilde{V}^{(n)} + \partial_r^\eta \tilde{w}^{(n)} \approx 0.$$

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$$\tilde{f}^{(n)} := \partial^n f - \epsilon \frac{\partial^n \eta}{1 + \epsilon \partial_r \eta} \partial_r f,$$

$$\partial^n \nabla^\eta_{t,x,r} f \approx \nabla^\eta_{t,x,r} \tilde{f}^{(n)}.$$

Quasilinear structure in isopycnal coordinates

After applying ∂^n, the stratified Euler equations read

$$\begin{cases} \partial_t \tilde{V}^{(n)} + (\underline{V} + \epsilon V)\partial_x \tilde{V}^{(n)} + \underline{V}' \tilde{w}^{(n)} + \frac{1}{\varrho}\partial_x^\eta \tilde{P}^{(n)} \approx O(1), \\ \mu(\partial_t \tilde{w}^{(n)} + (\underline{V} + \epsilon V)\partial_x \tilde{w}^{(n)}) + \frac{1}{\varrho}\partial_r^\eta \tilde{P}^{(n)} - \frac{g\varrho'}{\varrho}\dot{\eta}^{(n)} \approx O(1 + \epsilon/\sqrt{\mu}), \\ \partial_t \dot{\eta}^{(n)} + (\underline{V} + \epsilon V)\cdot \nabla_x \dot{\eta}^{(n)} - \tilde{w}^{(n)} \approx O(1), \\ \partial_x^\eta \tilde{V}^{(n)} + \partial_r^\eta \tilde{w}^{(n)} \approx 0. \end{cases}$$

Quasilinear structure in isopycnal coordinates

After applying ∂^n, the stratified Euler equations read

$$\begin{cases} \partial_t \tilde V^{(n)} + (\underline{V} + \epsilon V)\partial_x \tilde V^{(n)} + \underline{V}' \tilde w^{(n)} + \frac{1}{\varrho}\partial_x^\eta \tilde P^{(n)} \approx O(1), \\ \mu(\partial_t \tilde w^{(n)} + (\underline{V} + \epsilon V)\partial_x \tilde w^{(n)}) + \frac{1}{\varrho}\partial_r^\eta \tilde P^{(n)} - \frac{g\varrho'}{\varrho}\dot\eta^{(n)} \approx O(1 + \epsilon/\sqrt{\mu}), \\ \partial_t \dot\eta^{(n)} + (\underline{V} + \epsilon V)\cdot \nabla_x \dot\eta^{(n)} - \tilde w^{(n)} \approx O(1), \\ \partial_x^\eta \tilde V^{(n)} + \partial_r^\eta \tilde w^{(n)} \approx 0. \end{cases}$$

$$\mathcal{E}^{(N)} := \sum_{n=0}^{N} \Big(\| \sqrt{\varrho(1+\epsilon\partial_r\eta)} (\tilde V^{(n)}, \sqrt{\mu}\tilde w^{(n)}) \|_{L^2(S)}^2 + \| \sqrt{-g\varrho'}\dot\eta^{(n)} \|_{L^2(S)}^2 \Big)$$

Quasilinear structure in isopycnal coordinates

After applying ∂^n, the stratified Euler equations read

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$$\mathcal{E}^{(N)} := \sum_{n=0}^{N} \Big(\| \sqrt{\varrho(1 + \epsilon\partial_r \eta)} (\tilde{V}^{(n)}, \sqrt{\mu}\tilde{w}^{(n)}) \|_{L^2(S)}^2 + \| \sqrt{-g\varrho'}\dot{\eta}^{(n)} \|_{L^2(S)}^2 \Big)$$

$$\boxed{\frac{d}{dt}\mathcal{E}^{(N)} \lesssim (1 + \frac{|\underline{V}'|_\infty}{\sqrt{\mu}} + \frac{\epsilon}{\sqrt{\mu}})\mathcal{E}^{(N)}.}$$

Conclusion

Recap:

- Well-posedness of the stratified Euler equations.
- The time interval is independent of μ*(under smallness assumptions)*.
- Isoycnal coordinates provide a simple framework for the construction of solutions.
- Alinhac's good unknown is used to solve the "loss of derivatives" problem due to the change of coordinates.

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Open problems:

- Remove the smallness assumptions.
- Study the hydrostatic stratified Euler equations ($\mu = 0$).

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Thank you for your attention !

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