A Reduced Basis technique for Long-Time Unsteady Turbulent Flows

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Work in collaboration with Lambert Fick (Texas A&M), Yvon Maday (UPMC, Brown University), Anthony T Patera (MIT).

Acknowledgments: Paul F Fischer, Elia Merzari (Argonne)

Objective

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Develop a **model order reduction** procedure for

the parametrized unsteady Navier-Stokes equations

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in the turbulent regime.

We wish to *efficiently* and *accurately* estimate QOIs associated with the velocity field

$$u(\mu) = u(x, t; \mu), \qquad x \in \Omega \subset \mathbb{R}^d, \quad t \in (0, \infty), \quad \mu \in \mathcal{P} \subset \mathbb{R}^P$$

in the limit of many queries.

Efficiency: measured wrt the FOM in terms of

memory requirements;

computational time.

Accuracy: measured wrt the FOM in terms of

the long-time averaged flow $\langle u \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)\,dt,$

the TKE[1] $\mathrm{TKE}(t) = \frac{1}{2}\int_\Omega \|u(x,t) - \langle u \rangle(x)\|_2^2\,dx.$

For chaotic flows, prediction of the *instantaneous* velocity is out of reach.

[1] More precisely, we shall estimate the *moments* of the TKE.

The parametric problem: the offline/online paradigm

We seek an estimate of $u(\mu)$ of the form:

$$\hat{u}(x, t; \mu) = \sum_{n=1}^{N} a_n(t, \mu)\zeta_n(x) \text{ for all } \mu \in \mathcal{P}.$$

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Offline stage: expensive, performed once

given $\{u(\cdot,t^k,\mu^\ell)\}_{k,\ell}$, $\{t^k\}_{k=1}^K \subset (0,T)$, $\{\mu^\ell\}_{\ell=1}^L \subset \mathcal{P}$,

generate the reduced space $\mathcal{Z}^{\mathrm{u}} = \mathrm{span}\{\zeta_n\}_{n=1}^N$, and

formulate the Reduced Order Model

Online stage: inexpensive, performed many times

given $\mu \in \mathcal{P}$,

estimate the coefficients $\{a_n(t;\mu)\}_{n=1}^N$ for $t > 0$.

A simplified task: solution reproduction problem ($\mathcal{P} = \{\bar{\mu}\}$)

We seek an estimate of $u(\bar{\mu})$ s.t. $\hat{u}(x,t) = \sum_{n=1}^{N} a_n(t)\zeta_n(x)$

Offline stage:

given $\{u(\cdot, t^k, \bar{\mu})\}_{k=1}^{K}$,

generate the reduced space $\mathcal{Z}^{\mathrm{u}} = \mathrm{span}\{\zeta_n\}_{n=1}^{N}$, and

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Online stage:

query the ROM for **the same** $\mu = \bar{\mu}$ to estimate $\{a_n(t)\}_{n=1}^{N}$ for $t > 0$.

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Limited practical interest, but

key intermediate step toward the development of the ROM formulation.

Outline of the talk

1. A model lid-driven cavity problem

2. Solution reproduction problem

Galerkin ROM
constrained Galerkin ROM

3. Parametric problem

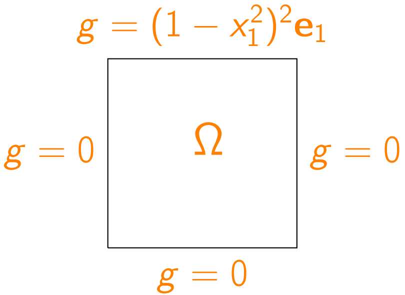
a posteriori error estimation
POD-*h*Greedy

A model lid-driven cavity problem

A lid-driven cavity problem[2]

Consider the problem:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \frac{1}{\mathrm{Re}} \Delta u + \nabla p = 0 & \Omega \times \mathbb{R}_+ \\ \nabla \cdot u = 0 & \Omega \times \mathbb{R}_+ \\ u|_{\partial\Omega} = g, \qquad u(0) = 0 & \end{cases}$$



$$\Omega = (-1, 1)^2$$

$$\mathrm{Re} \in \mathcal{P} = [15000, 25000]$$

(turbulent regime)

[2] Model problem considered in Balajewicz, Dowell, Nonlinear Dyn (2012).

Weak formulation for the lifted equations ($\mathring{u} := u - R_g$)

Given R_g s.t. $R_g|_{\partial\Omega} = g$, $\nabla \cdot R_g = 0$, find $(\mathring{u}, p)$ s.t.

$$\begin{cases} \langle \partial_t \mathring{u}(t), v \rangle_\star + \dfrac{1}{\mathrm{Re}} \displaystyle\int_\Omega \nabla(\mathring{u}(t) + R_g) : \nabla v \, dx \\ \qquad + c(\mathring{u}(t) + R_g, \mathring{u}(t) + R_g, v) + b(v, p(t)) = 0 \\ b(\mathring{u}(t), q) = 0 \qquad \forall\, v \in V, \quad q \in Q, \quad \text{a.e. } t > 0. \end{cases}$$

where $V = [H_0^1(\Omega)]^2$, $Q = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$, and

$$c(w, v, z) = \int_\Omega (w \cdot \nabla) v \cdot z \, dx, \; b(v, q) = -\int_\Omega (\nabla \cdot v) q \, dx.$$

Choice for the lift: R_g = Stokes solution

The choice $R_g = \langle u \rangle$ is not suitable for the parametric case.

High-fidelity model: SEM in Nek5000

We rely on the **spectral element** solver Nek5000 to generate the DNS data.

We refer to nek5000.mcs.anl.gov for details concerning the software.

Simulations were performed by Dr. Lambert Fick (Texas A&M) at Argonne National Lab.

Deville, Fischer, Mund, Cambridge University Press (2002).

Solution reproduction problem

- A first attempt: POD-Galerkin
- Our proposal: POD-constrained Galerkin

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Galerkin ROM (semi-implicit time integration)

Given $\mathcal{Z}^{\mathrm{u}} := \mathrm{span}\{\zeta_n\}_{n=1}^{N} \subset V_{\mathrm{div}} = \{v \in V : \nabla \cdot v = 0\}$, and $\{t^j = j\Delta t\}_{j=0}^{J}$, find $\{\hat{u}^j\}_j \subset \mathcal{Z}^{\mathrm{u}}$ such that

$$\left(\frac{\hat{u}^{j+1} - \hat{u}^j}{\Delta t}, v\right)_{L^2(\Omega)} + \frac{1}{\mathrm{Re}} \int_\Omega \nabla(\hat{u}^{j+1} + R_g) : \nabla v \, dx$$

$$+ c(\hat{u}^j + R_g, \hat{u}^{j+1} + R_g, v) = 0 \quad \forall v \in \mathcal{Z}^{\mathrm{u}}, \quad j = 0, 1, \ldots$$

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$$+ c(\hat{u}^j + R_g, \hat{u}^{j+1} + R_g, v) = 0 \;\; \forall\, v \in \mathcal{Z}^{\mathrm{u}}, \;\; j = 0, 1, \ldots$$

The space $\mathcal{Z}^{\mathrm{u}}$ is built through the DNS data $\{\mathring{u}^k = u(t^k) - R_g\}_{k=1}^{K} \subset V_{\mathrm{div}}$ using POD.

We consider the following choice of the inner product $(\cdot, \cdot)$:

$$(w, v) = \int_\Omega \nabla w : \nabla v \, dx \quad H^1 - \mathrm{POD}$$

Iollo, Lanteri, Désidéri, Theor Comp Fluid Dyn (2000).

Galerkin ROM: algebraic formulation (semi-implicit in time)

The coefficients $\mathbf{a}^j = [a_1^j, \ldots, a_N^j]$ $(\leftrightarrow \{\hat{u}^j\}_j)$ solve

$$\mathbb{A}(\mathbf{a}^j)\,\mathbf{a}^{j+1} = \mathbf{F}(\mathbf{a}^j) \quad j = 0, 1, \ldots, \qquad (\hat{u}^\cdot = \textstyle\sum_n a_n^\cdot \zeta_n)$$

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where $\mathbb{A} : \mathbb{R}^N \to \mathbb{R}^{N,N}$, and $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$ are

$$\mathbb{A}_{m,n}(\mathbf{w}) = \frac{1}{\Delta t}\int_\Omega \zeta_n \cdot \zeta_m \, dx + \frac{1}{\mathrm{Re}}\int_\Omega \nabla\zeta_n : \nabla\zeta_m \, dx + c(R_g, \zeta_n, \zeta_m) + \sum_{i=1}^N w_i \, c(\zeta_i, \zeta_n, \zeta_m)$$

$$\mathbf{F}_m(\mathbf{w}) = \sum_{n=1}^N w_n \left(\frac{1}{\Delta t}\int_\Omega \zeta_n \cdot \zeta_m \, dx - c(\zeta_n, R_g, \zeta_m) \right) - \frac{1}{\mathrm{Re}}\int_\Omega \nabla R_g : \nabla\zeta_m \, dx$$

Computational summary (POD-Galerkin)

Offline stage:

Compute DNS data $\{u^k = u(\cdot, t^k)\}_{k=1}^{K}$, $t^k = T_0 + k$

$T_0 = 500,\ K = 500$

Use POD to build the space $\mathcal{Z}^{\mathrm{u}} = \mathrm{span}\{\zeta_n\}_{n=1}^{N}$

Define $\mathbb{A} : \mathbb{R}^N \to \mathbb{R}^{N,N}$, and $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$

Online stage:

Solve the discrete dynamical system:

$$\mathbb{A}(\mathbf{a}^j)\mathbf{a}^{j+1} = \mathbf{F}(\mathbf{a}^j), \qquad j = 0, \ldots, J-1$$

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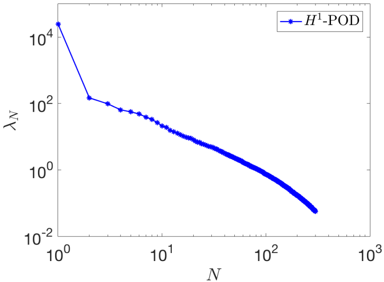
Online memory requirements: $\mathcal{O}(N^3)$.

Online cost: $\mathcal{O}(N^3 J)$.

POD eigenvalues ($\mathrm{Re} = 15000$)

POD eigenvalues $\{\lambda_N\}_N$ decay slowly with N.

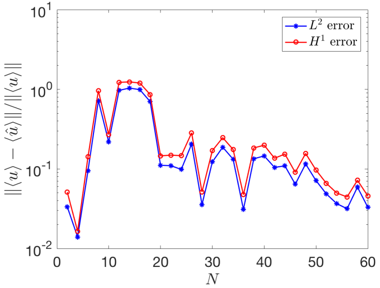


$u^k := u(\cdot, t^k)$
$t^k := 500 + k,$
$k = 1, \ldots, K = 500$

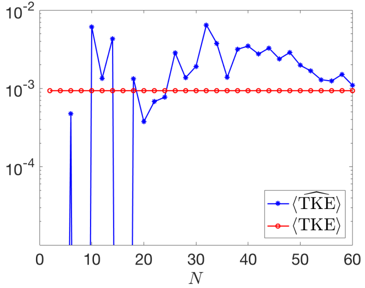
$\zeta_1 \approx C(\langle u \rangle - R_g)$ $\Rightarrow$ no contribution to fluctuating field

$$\frac{\sum_{k'=2}^{N} \lambda_{k'}}{\sum_{k=2}^{K} \lambda_k} = \begin{cases} 16.5\% & N = 2 \\ 73.1\% & N = 20 \\ 79.7\% & N = 30 \\ 87.0\% & N = 50 \end{cases}$$

Numerical results ($\mathrm{Re} = 15000$): performance (I)



Rel. error in $\langle u \rangle$



Mean TKE

We observe several **spurious** effects for moderate N:

false stable steady flows,

overly unstable flows...

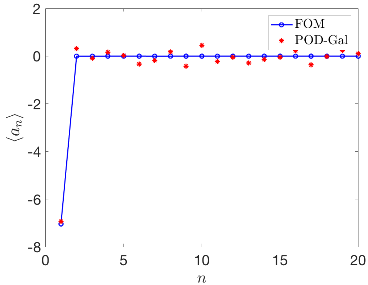
For $N \gtrsim 50$, accuracy improves.

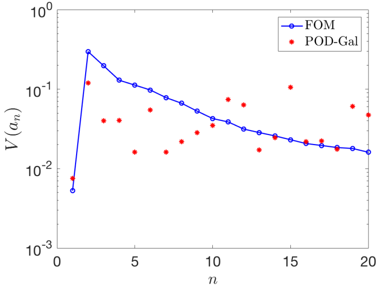
Numerical results ($\mathrm{Re} = 15000$): performance (II)

Moments of $\{a_n\}_n$ ($\mathring{u}(\cdot, t) = \sum_{n=1}^{\infty} a_n(t)\zeta_n$): $\mathbf{N = 20}$

$$E(a_n, \{t^k\}) = \frac{1}{K}\sum_{k=1}^{K} a_n(t^k),$$

$$V(a_n, \{t^k\}) = \frac{1}{K-1}\sum_{k=1}^{K} \left(a_n(t^k) - E(a_n, \{t^k\})\right)^2$$



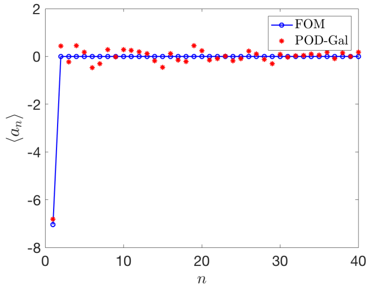


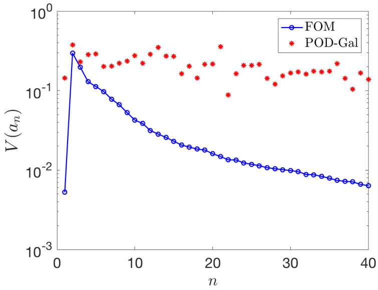
Numerical results ($\mathrm{Re} = 15000$): performance (II)

Moments of $\{a_n\}_n$ ($\mathring{u}(\cdot, t) = \sum_{n=1}^{\infty} a_n(t)\zeta_n$): **N = 40**

$$E(a_n, \{t^k\}) = \frac{1}{K}\sum_{k=1}^{K} a_n(t^k),$$

$$V(a_n, \{t^k\}) = \frac{1}{K-1}\sum_{k=1}^{K} \left(a_n(t^k) - E(a_n, \{t^k\})\right)^2$$





Numerical results ($\mathrm{Re} = 15000$): performance (II)

Moments of $\{a_n\}_n$ ($\mathring{u}(\cdot, t) = \sum_{n=1}^{\infty} a_n(t)\zeta_n$): **N = 60**

$$E(a_n, \{t^k\}) = \frac{1}{K}\sum_{k=1}^{K} a_n(t^k),$$

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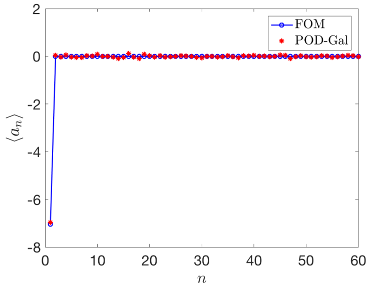

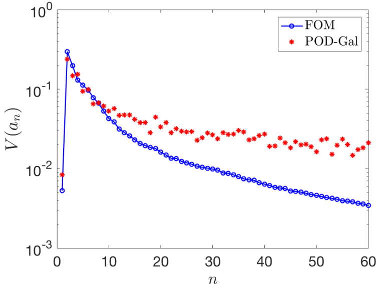

Comments

POD-Galerkin approach does not provide an adequate approximation of the long-time system dynamics, particularly for moderate N.

We observe several spurious effects

- false stable steady flows,
- overly unstable flows...

This behavior is similar to the one observed for highly-truncated spectral approximations to turbulent flows.

Curry, Herring, Loncaric, Orszag, J Fluid Mech (1984).

Solution reproduction problem

- A first attempt: POD-Galerkin
- Our proposal: POD-constrained Galerkin

cGalerkin formulation (semi-implicit in time)

We propose the following ROM (cGalerkin):

$$\mathbf{a}^{j+1} := \arg \min_{\mathbf{w}\in\mathbb{R}^N} \|\mathbb{A}(\mathbf{a}^j)\mathbf{w} - \mathbf{F}(\mathbf{a}^j)\|_2^2,$$
$$\text{subject to } \alpha_n \leq w_n \leq \beta_n, \; n = 1, \dots, N.$$

$\mathbb{A}$ and $\mathbf{F}$ are the matrix-valued and vector-valued functions introduced for the Galerkin ROM.

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$\mathbb{A}$ and $\mathbf{F}$ are the matrix-valued and vector-valued functions introduced for the Galerkin ROM.

If $\mathbf{a}_{\mathrm{Gal}}^{j+1} := \mathbb{A}(\mathbf{a}^j)^{-1}\,\mathbf{F}(\mathbf{a}^j)$ satisfies the constraints, cGalerkin = Galerkin.

For semi-implicit and explicit time discretizations, cGalerkin corresponds to a **convex quadratic programming** problem, which can be solved using an interior point method.

Estimates of $\{\alpha_n\}_n$ and $\{\beta_n\}_n$

α_n and β_n are lower and upper bounds for[3]

$$a_n(t) := (\mathring{u}(t) = u(t) - R_g, \zeta_n).$$

Given the snapshots $\{u^k\}_{k=1}^K$, we set $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ as

$$\alpha_n = m_n^{\mathrm{u}} - \epsilon(M_n^{\mathrm{u}} - m_n^{\mathrm{u}}), \qquad \beta_n = M_n^{\mathrm{u}} + \epsilon(M_n^{\mathrm{u}} - m_n^{\mathrm{u}});$$

where $\epsilon = 0.01$[4], and

$$m_n^{\mathrm{u}} := \min_{k=1,\dots,K} (\mathring{u}^k, \zeta_n)_V, \qquad M_n^{\mathrm{u}} := \max_{k=1,\dots,K} (\mathring{u}^k, \zeta_n)_V.$$

[3] **NOTE 1:** $(\zeta_m, \zeta_n) = \delta_{m,n}$

[4] **NOTE 2:** $\{t^k\}_k$ sampling times, $\{t^j\}_j$ time grid, $K \ll J$.

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The hyper-parameters $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ of `cGalerkin`

- admit a simple interpretation, and
- can be easily tuned based on **sparse DNS data**.

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Computational summary (constrained POD-Galerkin)

Offline stage:

Compute DNS data $\{u^k = u(\cdot, t^k)\}_{k=1}^{K}$

Use POD to build the space $\mathcal{Z}^{\mathrm{u}} = \mathrm{span}\{\zeta_n\}_{n=1}^{N}$

Define $\mathbb{A} : \mathbb{R}^N \to \mathbb{R}^{N,N}$, and $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$

Define $\{\alpha_n\}_n$ and $\{\beta_n\}_n$ based on the DNS data $\{\mathring{u}^k\}_k$

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Solve the discrete dynamical system:

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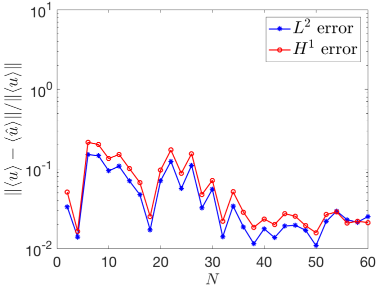
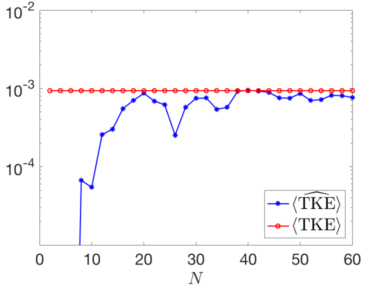
Online memory requirements: $\mathcal{O}(N^3)$.

Online cost: $\mathcal{O}(N^3 \underbrace{J_{\mathrm{pure}}}_{\text{Gal. solves}} + \underbrace{\kappa N^3}_{\text{cost QP}} (J - J_{\mathrm{pure}}))$.

Numerical results ($\mathrm{Re} = 15000$): performance (I)



Rel. error in $\langle u \rangle$

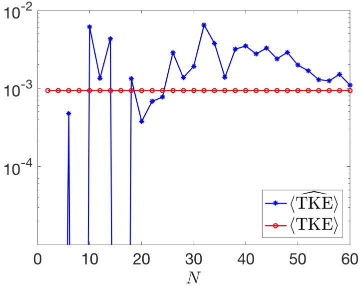


Mean TKE

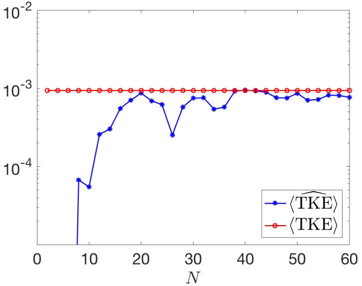
The constrained Galerkin formulation consistently underestimates the TKE.

For $N \gtrsim 40$, $\mathbf{a}^{j+1} = \mathbf{a}^{j+1}_{\mathrm{Gal}}$ for roughly 90% time steps.

Numerical results ($\mathrm{Re} = 15000$): performance (II)



Galerkin



constrained Galerkin

For some values of N, $\langle \mathrm{TKE}_{\mathrm{cGal}} \rangle > \langle \mathrm{TKE}_{\mathrm{Gal}} \rangle$. For some other values $\langle \mathrm{TKE}_{\mathrm{cGal}} \rangle < \langle \mathrm{TKE}_{\mathrm{Gal}} \rangle$.

$\Rightarrow$ `cGalerkin` does **not** add artificial viscosity to `Galerkin`.

Numerical results ($\mathrm{Re} = 15000$): performance (III)

Moments of $\{a_n\}_n$ $\left(\mathring{u}(\cdot, t) = \sum_{n=1}^{\infty} a_n(t)\zeta_n\right)$: **N = 20**

$$E(a_n, \{t^k\}) = \frac{1}{K}\sum_{k=1}^{K} a_n(t^k),$$

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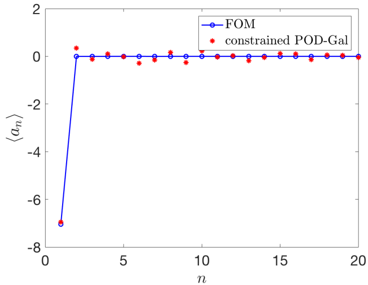
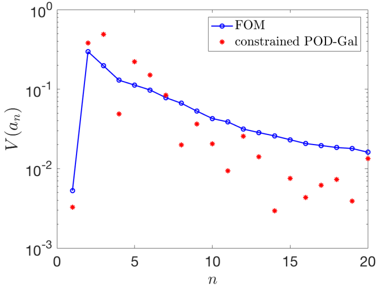


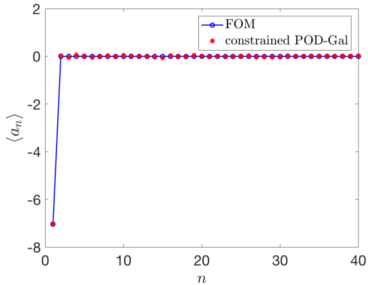


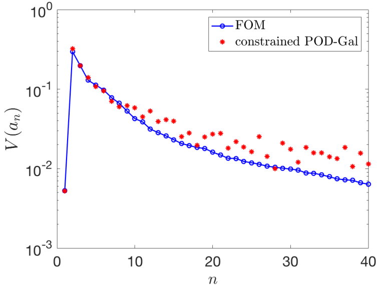
Numerical results ($\mathrm{Re} = 15000$): performance (III)

Moments of $\{a_n\}_n$ ($\mathring{u}(\cdot, t) = \sum_{n=1}^{\infty} a_n(t)\zeta_n$): **N = 40**

$$E(a_n, \{t^k\}) = \frac{1}{K} \sum_{k=1}^{K} a_n(t^k),$$

$$V(a_n, \{t^k\}) = \frac{1}{K-1} \sum_{k=1}^{K} \left(a_n(t^k) - E(a_n, \{t^k\})\right)^2$$



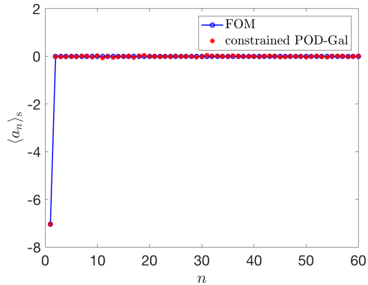


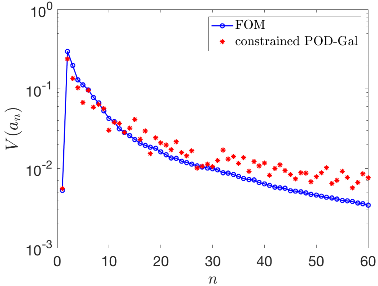
Numerical results ($\mathrm{Re} = 15000$): performance (III)

Moments of $\{a_n\}_n$ $\left(\mathring{u}(\cdot, t) = \sum_{n=1}^{\infty} a_n(t)\zeta_n\right)$: **N = 60**

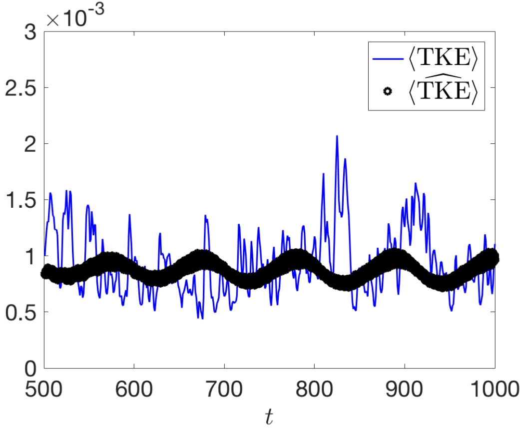
$$E(a_n, \{t^k\}) = \frac{1}{K}\sum_{k=1}^{K} a_n(t^k),$$

$$V(a_n, \{t^k\}) = \frac{1}{K-1}\sum_{k=1}^{K} \left(a_n(t^k) - E(a_n, \{t^k\})\right)^2$$





Behavior of the turbulent kinetic energy ($N = 20$)



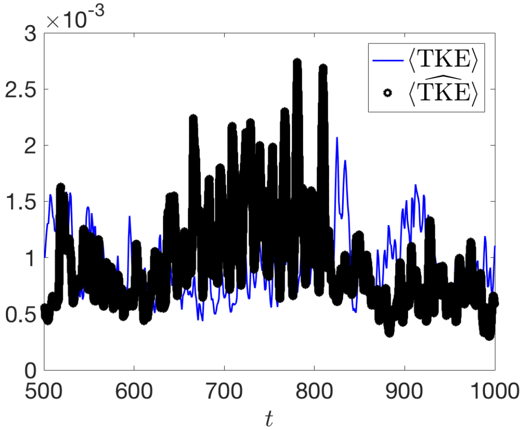
$\langle \mathrm{TKE} \rangle = 9.4 \cdot 10^{-4}$,

$\langle \widehat{\mathrm{TKE}} \rangle = 8.6 \cdot 10^{-4}$;

$\mathbb{V}(\mathrm{TKE}) = 8.5 \cdot 10^{-8}$;

$\mathbb{V}(\widehat{\mathrm{TKE}}) = 5.5 \cdot 10^{-9}$.

Behavior of the turbulent kinetic energy ($N = 40$)



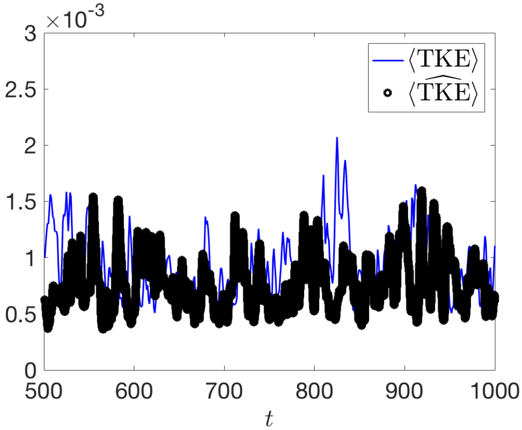
$\langle \text{TKE} \rangle = 9.4 \cdot 10^{-4}$,

$\langle \widehat{\text{TKE}} \rangle = 9.4 \cdot 10^{-4}$;

$\mathbb{V}(\text{TKE}) = 8.5 \cdot 10^{-8}$;

$\mathbb{V}(\widehat{\text{TKE}}) = 1.7 \cdot 10^{-7}$.

Behavior of the turbulent kinetic energy ($N = 60$)



$$\langle \text{TKE} \rangle = 9.4 \cdot 10^{-4},$$

$$\langle \widehat{\text{TKE}} \rangle = 7.65 \cdot 10^{-4};$$

$$\mathbb{V}(\text{TKE}) = 8.5 \cdot 10^{-8};$$

$$\mathbb{V}(\widehat{\text{TKE}}) = 5.8 \cdot 10^{-8}.$$

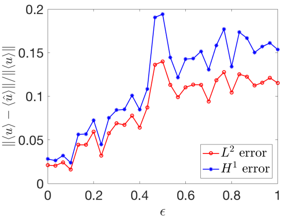
Prediction of instantaneous TKE is out of reach.

Our results suggest that estimation of TKE moments is achievable.

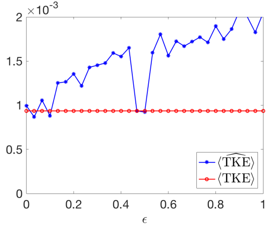
Sensitivity analysis wrt ϵ ($\mathrm{Re} = 15000$, $N = 40$)

ϵ enters in the definition of the bounds α_n and β_n:

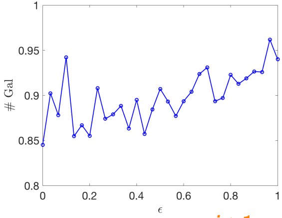
$$\alpha_n = m_n^{\mathrm{u}} - \epsilon(M_n^{\mathrm{u}} - m_n^{\mathrm{u}}), \qquad \beta_n = M_n^{\mathrm{u}} + \epsilon(M_n^{\mathrm{u}} - m_n^{\mathrm{u}});$$



Rel. error in $\langle u\rangle$



Mean TKE

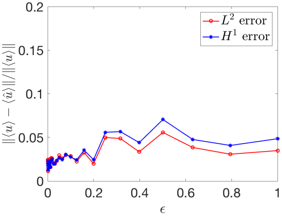


% $\mathbf{a}^{j+1} = \mathbf{a}^{j+1}_{\mathrm{Gal}}$

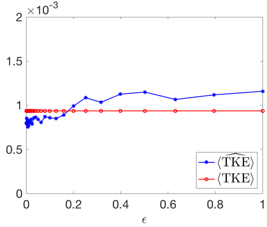
Sensitivity analysis wrt ϵ ($\mathrm{Re} = 15000$, $N = 60$)

ϵ enters in the definition of the bounds α_n and β_n:

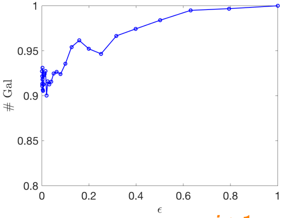
$$\alpha_n = m_n^{\mathrm{u}} - \epsilon(M_n^{\mathrm{u}} - m_n^{\mathrm{u}}), \qquad \beta_n = M_n^{\mathrm{u}} + \epsilon(M_n^{\mathrm{u}} - m_n^{\mathrm{u}});$$



Rel. error in $\langle u \rangle$



Mean TKE



% $\mathbf{a}^{j+1} = \mathbf{a}^{j+1}_{\mathrm{Gal}}$

Interpretation: as N increases, the Galerkin model becomes more and more accurate, and box constraints become less and less important.

Parametric problem

- Error estimation
- POD - *h*Greedy

Offline stage:

Select $\mathrm{Re}_1^\star, \ldots, \mathrm{Re}_L^\star$ in a Greedy fashion based on an error indicator Δ^{u}

Use snapshots $\{u^{k,\ell} = u(\cdot, t^k, \mathrm{Re}_\ell^\star)\}$ to generate the reduced space $\mathcal{Z}^{\mathrm{u}}$

Build the ROM associated with the reduced space $\mathcal{Z}^{\mathrm{u}}$

Haasdonk, Ohlberger, M2AN, (2008).

Offline stage:

Select $\mathrm{Re}_1^\star, \ldots, \mathrm{Re}_L^\star$ in a Greedy fashion based on an error indicator Δ^{u}

Challenge 1: error estimation

Use snapshots $\{u^{k,\ell} = u(\cdot, t^k, \mathrm{Re}_\ell^\star)\}$ to generate the reduced space $\mathcal{Z}^{\mathrm{u}}$

Challenge 2: combination of modes from different regimes

Build the ROM associated with the reduced space $\mathcal{Z}^{\mathrm{u}}$

Haasdonk, Ohlberger, M2AN, (2008).

Challenge 1: error estimation

Error estimators for evolution problems are based on

energy estimates, or

Haasdonk, Ohlberger, 2008

Grepl, Patera, 2005

BRR theory and space-time formulations.

Urban, Patera, 2014

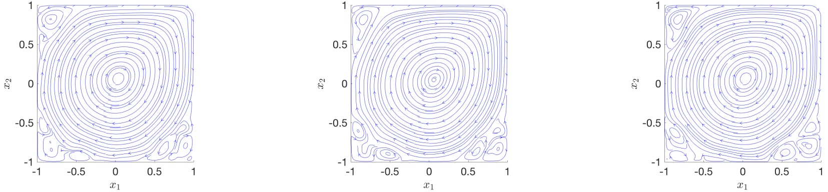
Yano, 2014

These estimators bound the error in the full trajectory

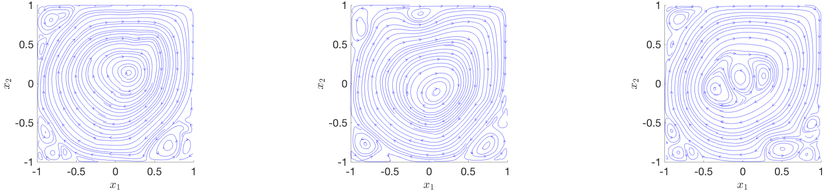
$\Rightarrow$ inappropriate for turbulent flows.

Challenge 2: combination of modes from different regimes

$\mathrm{Re} = 15000$, $t = 501, 600, 700$



$\mathrm{Re} = 20000$, $t = 1252, 1266, 1344$



Combining modes associated with different regimes lead to poor performance.

Parametric problem

- Error estimation
- POD - *h*Greedy

Our proposal: time-averaged error indicator

Given $\{w^j\}_j \subset V_{\mathrm{div}}$, define the residuals

$$e^j(\mathrm{Re}) = \frac{w^{j+1} - w^j}{\Delta t} - \frac{1}{\mathrm{Re}}\Delta(w^{j+1} + R_g) + (w^j + R_g) \cdot \nabla(w^{j+1} + R_g)$$

Then, define the time-averaged residual for all $v \in V_{\mathrm{div}}$:

$$\langle R\rangle(\{w^j\}_j; v; \mathrm{Re}) = \frac{1}{J - J_0} \sum_{j=J_0}^{J-1} \langle e^j(\mathrm{Re}), v\rangle_{V'_{\mathrm{div}} \times V_{\mathrm{div}}},$$

and the error indicator

$$\Delta^{\mathrm{u}}(\{w^j\}_j; \mathrm{Re}) = \|\langle R\rangle(\{w^j\}_j; \cdot; \mathrm{Re})\|_{V'_{\mathrm{div}}}.$$

Comments

If $J - J_0 \to \infty$, $\langle R \rangle$ converges to the discretized residual of RANS

Intuition: Δ^{u} correlated to error in mean flow prediction.

Δ^{u} admits an offline/online decomposition, which requires

$\mathcal{O}(N^2)$ Stokes' solves offline

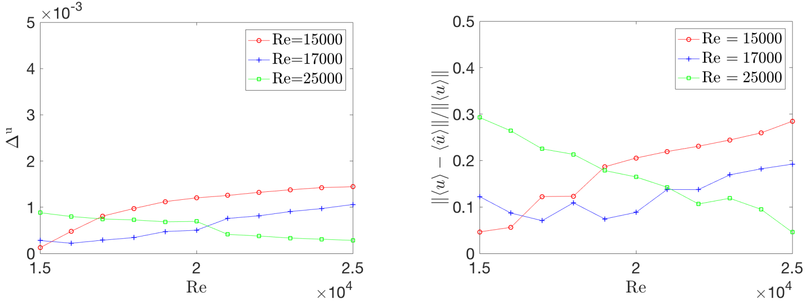
large storage cost if $N^2 > K$;

$\mathcal{O}(N^4 + N^2 J)$ online computational cost

negligible computational cost compared to $\mathcal{O}(N^3 J)$.

Evaluation of the error indicator

Test: generate ROMs for $\mathrm{Re} = 15000, 17000, 25000$;
evaluate Δ^{u} and the relative H^1 error for
$\mathrm{Re} = 15000, 16000, \ldots, 25000$.



Error estimator is good indicator but poor quantitative agreement with true error.

for $10/11$ Re, the same ROM minimizes both the relative error and Δ^{u}.

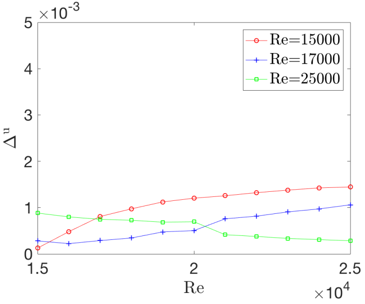
Parametric problem

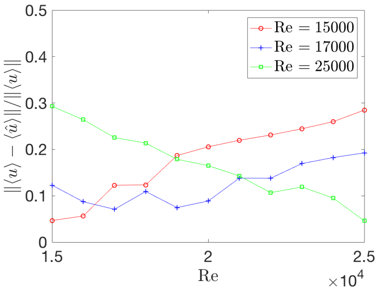
- Error estimation
- POD - hGreedy

Proposal:

given the three ROMs for $\mathrm{Re} = 15000, 17000, 25000$,
for a new value of the Reynolds number,

select the ROM that minimizes the error indicator





Proposal:

given the three ROMs for $\mathrm{Re} = 15000, 17000, 25000$,
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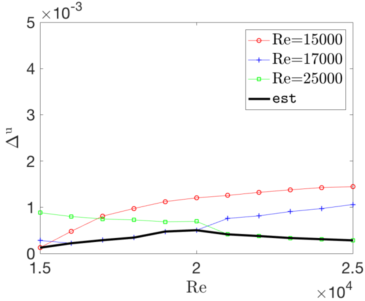

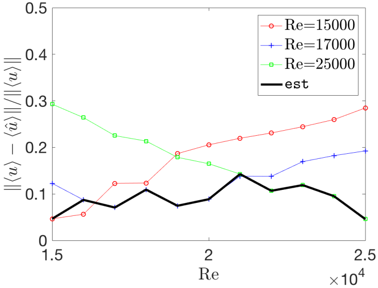

The relative error is less than 13% for all values of the Reynolds number considered.

Comments

The hGreedy partitions the parameter domain $\mathcal{P}$ to deal with different behaviors.

The hGreedy requires the solution to n_{cand} $(= 3$ in this case) ROMs during the online stage.

The anchor points $\mathrm{Re} = 15000, 17000, 25000$ can be chosen in a Greedy fashion based on the error indicator.

Eftang, Knezevic, Patera, Math Comput Model Dyn Syst, (2011).

Conclusions

Challenges

Turbulent flows present several challenges.

Slow decay of the POD eigenvalues λ_N;

Several spurious behaviors

Poor effectivity of traditional error indicators

Difficulty in combining modes from different regimes

Challenges → Our proposals

Turbulent flows present several challenges.

Slow decay of the POD eigenvalues λ_N;

reduce the goal of MOR $\langle u \rangle, \mathrm{TKE}$

Several spurious behaviors

constrained formulation

Poor effectivity of traditional error indicators

time-avg residual indicator

Difficulty in combining modes from different regimes

*h*Greedy

Thank you for your attention!

For more information,
Fick, Maday, Patera, Taddei, *A Reduced Basis Technique for Long-Time Unsteady Turbulent Flows*

available on Arxiv, and ResearchGate