Simulation-Based Classification; a Model-Order-Reduction Approach for Structural Health Monitoring

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Recent developments in numerical methods for model reduction
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Acknowledgements

Collaborators:

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Sponsors:

Air Force Office of Scientific Research (AFOSR);

Office of Naval Research (ONR).

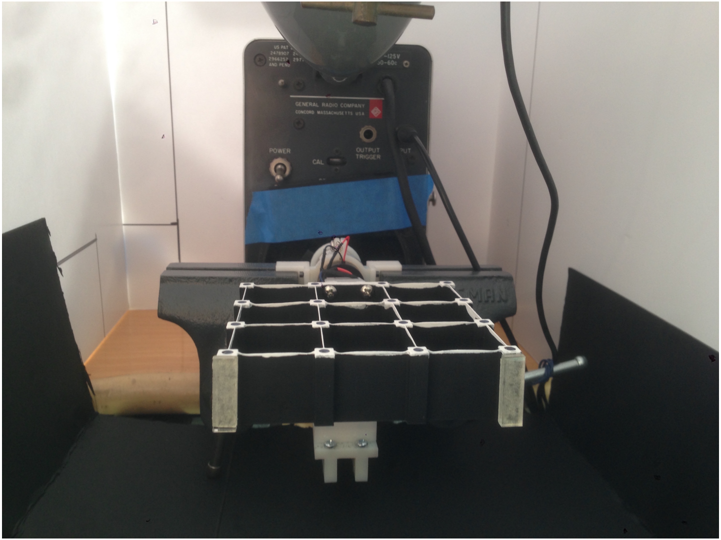
An example: a microtruss

A target application: monitoring of ship loaders[1]

Objective: monitor the integrity of a ship loader during the operations

[1]Photo credit: www.directindustry.com

Our example: the microtruss system



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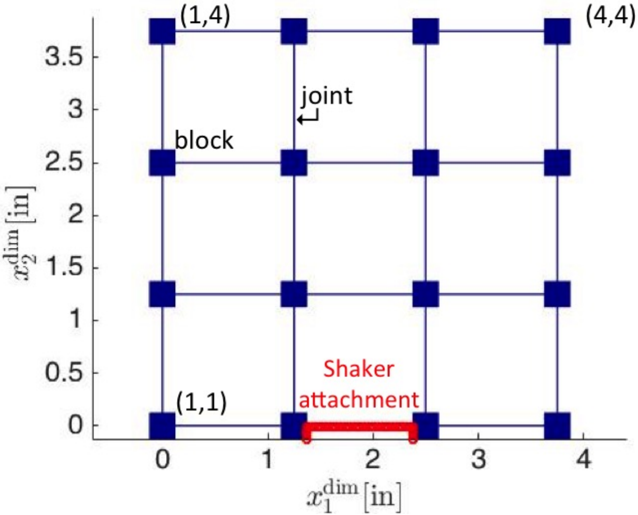


Our example: the microtruss system

Goal: detect the presence of added mass on top of block $(1,4)$ and block $(4,4)$

Apparatus: voice coil actuator; camera&stroboscope

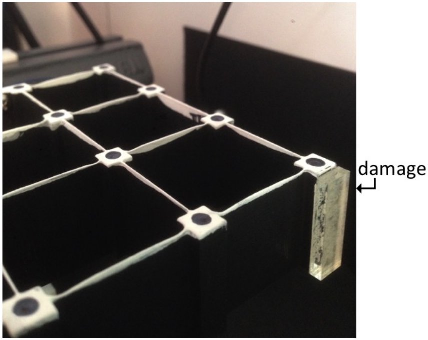
Input: x_2-displacement at prescribed frequencies $\{f^q\}$;
Exp data: x_2-displacement of blocks' centers $\{c^{\mathrm{exp}}_{i,j}(t^\ell, f^q)\}$.

Data reduction:

$$c^{\mathrm{exp}}_{i,j}(t^\ell, f^q) \approx \overline{A}^{\mathrm{exp}}_{i,j}(f^q) \cos\left(2\pi f^q t^\ell + \overline{\phi}^{\mathrm{exp}}_{i,j}(f^q)\right)$$

Exp outputs: $A^{\mathrm{exp}}_{i,j}(f^q) := \dfrac{A_{\mathrm{nom}}}{\overline{A}^{\mathrm{exp}}_{2,1}(f^q)} \overline{A}^{\mathrm{exp}}_{i,j}(f^q)$.

Definition of the QOI: damage function



Define $s_L = 1 + \frac{V_{\mathrm{left}}}{V_{\mathrm{nom}}}$, and

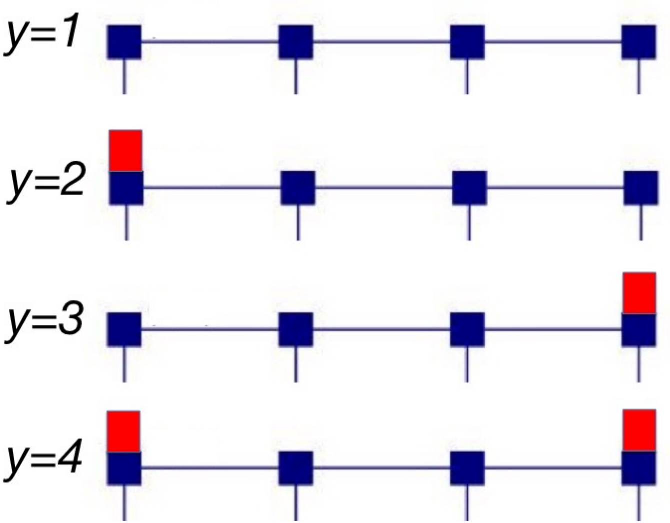
$$s_R := 1 + \frac{V_{\mathrm{right}}}{V_{\mathrm{nom}}}.$$

Define $y = \bar{f}^{\mathrm{dam}}(s_L, s_R)$,

$$y = \begin{cases} 1 & s_L, s_R \leq 1.5, \\ 2 & s_L > 1.5,\ s_R \leq 1.5, \\ 3 & s_L \leq 1.5,\ s_R > 1.5, \\ 4 & s_L, s_R > 1.5. \end{cases}$$

The QOI y is the **state of damage** associated with the structure.

Definition of the QOI: damage function



Engineering objective

Generate a *decision rule* g that maps experimental outputs

$$\{A_{i,j}^{\mathrm{exp}}(f^q; \mathcal{C})\}_{i,j,q}$$

to the appropriate configuration state of damage

$$y = \bar{f}^{\mathrm{dam}}(s_L, s_R) \in \{1, 2, 3, 4\};$$

for any given system configuration $\mathcal{C} = (s_L, s_R, \ldots)$.

Perspective: objective of Structural Health Monitoring (SHM)

Level I: is the structure damaged?

Level II: where is damage located?

C Farrar, K Worden, 2012

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Mathematical formulation

Mathematical best-knowledge (bk) model

Set

$$\mathcal{C} = \left(\mu := [s_L = 1 + \frac{V_{\text{left}}}{V_{\text{nom}}}, s_R = 1 + \frac{V_{\text{right}}}{V_{\text{nom}}}, \alpha, \beta, E], \ldots\right),$$

where α, β Rayleigh-damping coefficients, and

E Young's modulus.

Estimate

$$A^{\text{exp}}_{i,j}(f^q; \mathcal{C}) \approx A^{\text{bk}}_{i,j}(f^q; \mu) := A_{\text{nom}} \frac{|u_2^{\text{bk}}(x_{i,j}; f^q, \mu)|}{|u_2^{\text{bk}}(x_{2,1}; f^q, \mu)|}$$

where $x_{i,j}$ is the center of block (i,j), and $u^{\text{bk}}(\cdot; f^q, \mu)$ solves the parametrized PDE:

$$\mathcal{G}_{\text{elast}-\text{helmhotz}}\left(u^{\text{bk}}(f^q, \mu); f^q; \mu\right) = 0 + \text{BC}$$

Interpretation:

μ incomplete representation of $\mathcal{C}$;

$\mathcal{G}_{\text{elast}-\text{helmhotz}}$ bk-parametrized mathematical model.

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Estimate

$$A^{\text{exp}}_{i,j}(f^q; \mathcal{C}) \approx A^{\text{bk}}_{i,j}(f^q; \mu) := A_{\text{nom}} \frac{|u^{\text{bk}}_2(x_{i,j}; f^q, \mu)|}{|u^{\text{bk}}_2(x_{2,1}; f^q, \mu)|}$$

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Interpretation:

μ incomplete representation of $\mathcal{C}$;

$\mathcal{G}_{\text{elast}-\text{helmhotz}}$ bk-parametrized mathematical model.

Define the **feature map** $\mathcal{F} : \mathbb{R}^{16Q_f} \to \mathbb{R}^Q$ that takes as input the experimental (or bk) outputs

$$\{A^{\bullet}_{i,j}(f^q;\star)\}_{i,j,q}, \ (\bullet = \mathrm{exp}, \mathrm{bk}, \ \star = \mathcal{C}, \mu)$$

and returns the Q *features*

$$\mathbf{z}^{\bullet}(\star) = \mathcal{F}(\{A^{\bullet}_{i,j}(f^q;\star)\}_{i,j,q}) \in \mathbb{R}^Q$$

$\mathcal{F} : \mathbb{R}^{16Q_f} \to \mathbb{R}^Q$ should be chosen such that

- $\mathbf{z}^{\bullet}(\star)$ is sensitive to the expected damage;
- $\mathbf{z}^{\bullet}(\star)$ is insensitive to noise.

Mathematical objective

Given the features $\mathbf{z}^{\mathrm{bk}}(\mu) = \mathcal{F}(\{A_{i,j}^{\mathrm{bk}}(f^q;\mu)\}_{i,j,q}) \in \mathbb{R}^Q$, we seek $g : \mathbb{R}^Q \to \{1,\ldots,4\}$ that minimizes

$$R^{\mathrm{bk}}(g) = \int_{\mathcal{P}^{\mathrm{bk}}} \mathbb{1}\left(g(\mathbf{z}^{\mathrm{bk}}(\mu)) \neq f^{\mathrm{dam}}(\mu)\right) w^{\mathrm{bk}}(\mu)\, d\mu,$$

where

- $\mu = [s_L, s_R, \alpha, \beta, E] \in \mathcal{P}^{\mathrm{bk}}$ anticipated configuration;
- $\mathcal{P}^{\mathrm{bk}}$ anticipated configuration set;
- $\mu \mapsto f^{\mathrm{dam}}(\mu) = \bar{f}^{\mathrm{dam}}(s_L, s_R) \in \{1,\ldots,4\}$ damage;
- $\mathcal{F} : \mathbb{R}^{16Q_f} \to \mathbb{R}^Q$ feature map (to be defined);
- $\mu \mapsto w^{\mathrm{bk}}(\mu)$ user-defined weight ($\leftrightarrow P_{w^{\mathrm{bk}}}$).

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Computational approach

Simulation-Based Classification

Offline stage: (before operations)

1. Generate $\mu^1, \ldots, \mu^M \overbrace{\sim}^{iid} P_{w^{\rm bk}}$
2. Generate $\mathcal{D}_M^{\rm bk} = \left\{ \mathbf{z}^{\rm bk}(\mu^m),\, f^{\rm dam}(\mu^m) \right\}_{m=1}^M$
3. $[g_M^\star] = \texttt{Supervised-Learning-alg}(\mathcal{D}_M^{\rm bk})$

Online stage: (during operations)

1. Acquire the new outputs $\{A_{i,j}^{\rm exp}(f^q; \overline{\mathcal{C}})\}_{i,j,q}$.
2. Compute $\bar{\mathbf{z}}^{\rm exp} = \mathcal{F}(A_{i,j}^{\rm exp}(f^q; \overline{\mathcal{C}}))$.
3. Return the label $g_M^\star(\bar{\mathbf{z}}^{\rm exp})$.

Taddei, Penn, Yano, Patera, 2016.

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Simulation-Based Classification

Related works: Farrar et al. (based on experiments);
Basudhar, Missoum;
Willcox et al.

Opportunities: no need to estimate $\mu = [s_L, s_R, \alpha, \beta, E]$
(which includes nuisance variables α, β, E)
non-intrusive approach
(it requires only forward solves)

Challenge: generation of $\mathcal{D}_M^{\mathrm{bk}}$

$\Rightarrow$ Exploit pMOR ($\leftrightarrow$ parametric def of damage) to generate $\mathcal{D}_M^{\mathrm{bk}}$.

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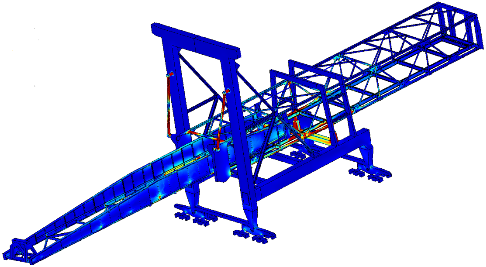
Perspectives: a ship loader model[2]

Cost to build $\mathcal{D}_M^{\text{bk}} = M \times Q_f \times \text{cost per simulation}$



FE model ($\approx 5 \cdot 10^6$ dofs)

cost per simulation $\approx 43'$

$M = 10^4, Q_f = 10 \Rightarrow 8\,\text{years}$

ROM model (PR-scRBE)

cost per simulation $\approx 5''$

$M = 10^4, Q_f = 10 \Rightarrow 6\,\text{days}$

$\Rightarrow$ pMOR enables the use of mathematical models in the simulation-based framework.

---

[2] Simulations are performed by Akselos S.A. using PR-scRBE.

Offline stage: (before operations)

1. Generate $\mu^1, \ldots, \mu^M \overbrace{\sim}^{iid} P_{w^{\mathrm{bk}}}$

2.a Construct a ROM for $\mu \in \mathcal{P}^{\mathrm{bk}} \mapsto \mathbf{z}^{\mathrm{bk}}(\mu)$

2.b Use the ROM to generate the dataset $\mathcal{D}_M^{\mathrm{bk}}$

3. $[g_M^\star] = \texttt{Supervised-Learning-alg}(\mathcal{D}_M^{\mathrm{bk}})$

pMOR is employed only in the generation of the dataset;

If M is sufficiently large, the cost of 2.a is negligible compared to the cost of 2.b (**many-query context**).

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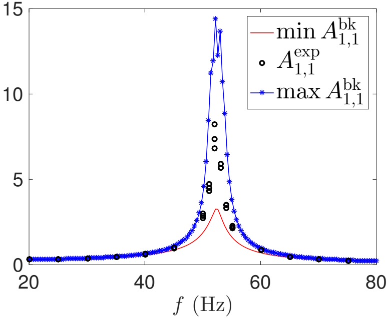
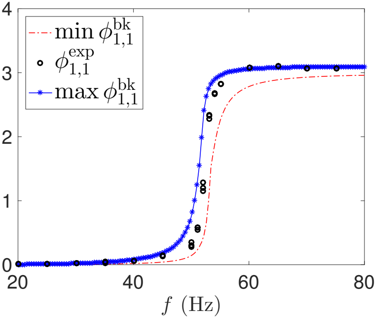
Application to the microtruss problem

Choice of $\mathcal{P}^{\mathrm{bk}}$

We choose upper bounds for s_L, s_R *a priori*.

We choose lower and upper bounds for α, β, E using textbook values and a preliminary experiment for $s_L = s_R = 1$.





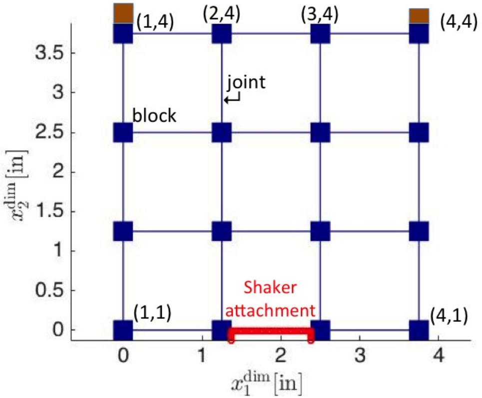
(explanation: $\min A_{1,1}^{\mathrm{bk}} = \min_{\mu=(1,1,\alpha,\beta,E)\in\mathcal{P}^{\mathrm{bk}}} A_{1,1}^{\mathrm{bk}}(\mu, f)$)

Choices of the features

Introduce

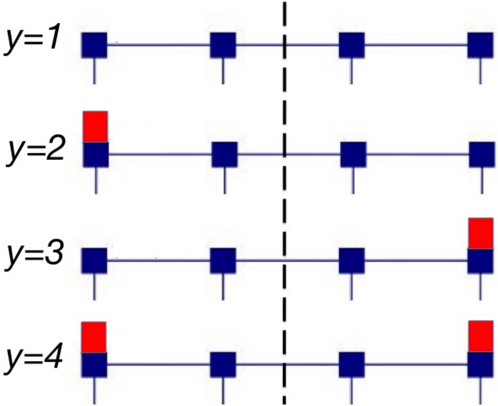
$$z_1^{\mathrm{bk}}(\cdot) = \frac{A_{1,4}^{\mathrm{bk}}(\cdot)}{A_{4,4}^{\mathrm{bk}}(\cdot)}, \; z_2^{\mathrm{bk}}(\cdot) = \frac{A_{2,4}^{\mathrm{bk}}(\cdot) + A_{3,4}^{\mathrm{bk}}(\cdot)}{A_{1,1}^{\mathrm{bk}}(\cdot) + A_{4,1}^{\mathrm{bk}}(\cdot)}.$$

and define $\mathbf{z}_\ell^{\mathrm{bk}}(\mu) = [z_\ell^{\mathrm{bk}}(f^1;\mu), \ldots, z_\ell^{\mathrm{bk}}(f^{Q_f};\mu)]$.



Choices of the features: motivation

Rationale: $z_1^{\cdot}$ detects asymmetry in the structure;
$z_2^{\cdot}$ detects added mass on corners.



Classification procedure

Given $\mathbf{z}_1^{\exp}$, $\mathbf{z}_2^{\exp}$,

Level 1: distinguish between $\{1, 4\}$, $\{2\}$ and $\{3\}$ based on $\mathbf{z}_1^{\exp}$;

Level 2: if Level 1 returns $\{1, 4\}$, distinguish between $\{1\}$ and $\{4\}$ based on $\mathbf{z}_2^{\exp}$.

From the learning perspective,

Level 1 corresponds to a 3way classification problem;

Level 2 corresponds to a 2way classification problem.

Algorithms used: SVM, ANN, kNN, decision trees, NMC[3].

[3] Implementation is based on off-the-shelf Matlab functions.

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Computational procedure (essential):

Build a ROM for the state $u^{\mathrm{bk}}(f;\mu)$, $f \in \mathcal{I}_f$, $\mu \in \mathcal{P}^{\mathrm{bk}}$,

Use the ROM to compute $(f^q, \mu^m) \mapsto A^{\mathrm{bk}}_{i,j}(f^q; \mu^m)$ for $m = 1, \ldots, M$ and $q = 1, \ldots, Q_f$ ($= MQ_f$ PDE solves).

Computational summary:

Finite Element (FE): 14670 dof,
$\approx 0.18[\mathrm{s}]$ for each PDE query;

Reduced Basis (RB): 20 dof, pre-processing cost $\approx 24[\mathrm{s}]$,
$\approx 4.4 \cdot 10^{-3}[\mathrm{s}]$ for each PDE query.

$\Rightarrow$ RB is advantageous if $MQ_f \gtrsim 180$
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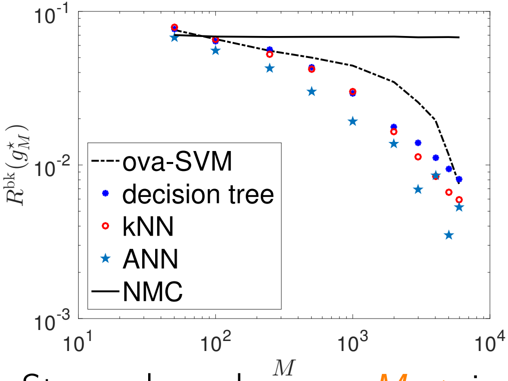
Results (synthetic data)

Test: generate a dataset $\mathcal{D}_K^{\mathrm{bk}}$, $K = 10^4$, $Q_f = 9$.

Then,

1. use M points for learning, $K - M$ for testing;
2. average over 100 partitions learning/testing.



Memo:

$R^{\mathrm{bk}}(g) = 0$

$\Rightarrow$ no mistakes.

$R^{\mathrm{bk}}(g) = 1$

$\Rightarrow$ always wrong.

Strong dependence on $M \Rightarrow$ importance of pMOR.

Results (experimental data)

Test: consider $\mathcal{D}_K^{\rm bk}$ (as before), and
$15 = \underbrace{5}_{\text{microtrusses}} \times \underbrace{3}_{\text{trials}}$ exp datapoints. Then,

1. use $M = 7 \cdot 10^3$ synthetic datapoints for learning;
2. use $3 \cdot 10^3$ synth datapoints and all 15 exp points[4] for testing.

	bk-risk $R^{\rm bk}(g)$	exp risk (5×3)
ova-SVM	0.0059	0.2093
decision tree	0.0072	0.4000
kNN ($k = 5$)	**0.0050**	**0**
ANN (10 layers)	0.0026	0.6000
NMC	0.0661	0

[4] We average over 100 learning/testing partitions of the synthetic dataset.

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Summary and perspectives

Summary

We propose a MOR approach to Simulation-Based Classification for the estimate of discrete-valued QOIs.

The approach exploits

1. pMOR procedures for rapid generation of datasets;
2. ML algorithms for the construction of the classifier.

Challenges

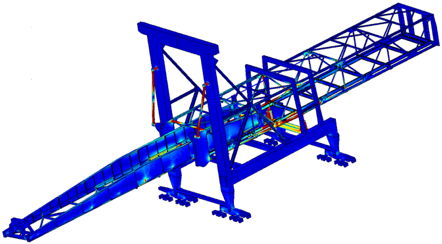
Parametrization of damage

damage is a local phenomenon,

⇒ component-based pMOR

Choice of features

automated feature identification[5].



[5] In collaboration with Prof. D Bertsimas, C Pawlowski (MIT).

Thank you for your attention!

Please visit augustine.mit.edu for further information

Backup slides

- Error analysis
- Comparison with a model-based approach
- Mathematical model
- Choice of the features
- Explanation of the Table

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Define $\mathcal{C} = (\mu, \xi)$, $\xi \in \mathbb{R}^D$

ξ accounts for unmodeled physics, geometry uncertainty...

Write experimental features as

$\mathbf{z}^{\mathrm{exp}}(\mu, \xi) = \mathbf{z}^{\mathrm{bk}}(\mu) + \delta\mathbf{z}(\mu, \xi)$,

Introduce the experimental risk $R^{\mathrm{exp}}(g) = \int_{\mathcal{P}^{\mathrm{bk}}} \mathbb{E}_{\delta\mathbf{z} \sim P_{\delta\mathbf{z},\mu}} \left[\mathcal{L}^{(0,1)}(g(\mathbf{z}^{\mathrm{bk}}(\mu) + \delta\mathbf{z}), f^{\mathrm{dam}}(\mu)) \right] w^{\mathrm{bk}}(\mu) d\mu,$

where $P_{\delta\mathbf{z},\mu}$ is the probability distribution of $\delta\mathbf{z}(\mu, \cdot)$.

Main result

Define the ϵ-uncertainty indicator $E^{\mathrm{bk}} = E^{\mathrm{bk}}(g, \epsilon, \mu)$ as

$$E^{\mathrm{bk}} = \begin{cases} 0 \text{ if } \; g(\mathbf{z}^{\mathrm{bk}}(\mu)) = g(\mathbf{z}^{\mathrm{bk}}(\mu) + \delta\mathbf{z}), \; \forall \, \|\delta\mathbf{z}\|_2 \leq \epsilon; \\ 1 \text{ otherwise.} \end{cases}$$

Then, if $P_{\delta\mathbf{z},\mu}(\|\delta\mathbf{z}(\mu)\|_2 \leq \epsilon^{\mathrm{bk}}) = 1 \; \forall \, \mu \in \mathcal{P}^{\mathrm{bk}}$,

$$R^{\exp}(g) \leq \underbrace{R^{\mathrm{bk}}(g)}_{\text{nominal performance}} + \underbrace{\int_{\mathcal{P}^{\mathrm{bk}}} E^{\mathrm{bk}}(g, \epsilon^{\mathrm{bk}}, \mu)\, w^{\mathrm{bk}}(\mu)\, d\mu}_{\text{robustness to data uncertainty}}.$$

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Comments

Sensitivity to data uncertainty might lead to poor performance on experimental data

Given estimates for ϵ^{bk},

we can explicitly bound $R^{\exp}(g)$ for any g;

we can properly **robustify** the learning procedure.

Ben-Tal, El Ghaoui, Nemirovski, 2009

Bertsimas, Brown, Caramanis, 2011

Backup slides

- Error analysis
- Comparison with a model-based approach
- Mathematical model
- Choice of the features
- Explanation of the Table

Simulation-(Data-) based approach

Farrar et al, . . . , this talk

Offline: Generate $\mathcal{D}_M^{\mathrm{bk}} = \{\mathbf{z}^{\mathrm{bk}}(\mu^m), f^{\mathrm{dam}}(\mu^m)\}_{m=1}^M$

Build $g_M^\star$ based on $\mathcal{D}_M^{\mathrm{bk}}$

Online: Given $\bar{\mathbf{z}}^{\mathrm{exp}}$, return the label $g_M^\star(\bar{\mathbf{z}}^{\mathrm{exp}})$

Model-based approach

Friswell&Mottershead

Online: Estimate the parameter $\mu^\star$ s.t. $\bar{\mathbf{z}}^{\mathrm{exp}} \approx \mathbf{z}^{\mathrm{bk}}(\mu^\star)$

Return $f^{\mathrm{dam}}(\mu^\star)$

Goal: compare performance of SBC with a representative model-based approach.

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Goal: compare performance of SBC with a representative model-based approach.

The approach considered

Formulation: we seek $\mu^\star \in \mathcal{P}^{\mathrm{bk}}$ that minimizes
$J(\mu) := \|\mathbf{z}_1^{\mathrm{bk}}(\mu) - \bar{\mathbf{z}}_1^{\mathrm{exp}}\|_2^2 + \|\mathbf{z}_2^{\mathrm{bk}}(\mu) - \bar{\mathbf{z}}_2^{\mathrm{exp}}\|_2^2$

Computational strategy:

SQP, gradient estimated based on FD (`fmincon`[6]);

4 ICs (one for each region $\mathcal{P}^{\mathrm{bk}}(\kappa) = \{\mu : f^{\mathrm{dam}}(\mu) = \kappa\}$);

Reduced Basis method to speed up calculations.

[6] Matlab R2016a

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Results

Synthetic data: (40 samples) $R^{\mathrm{bk}} = 0$.

Real data:

	$y = 1$	$y = 2$	$y = 3$	$y = 4$
$\hat{y} = 1$	0	0	0	0
$\hat{y} = 2$	2	0	0	0
$\hat{y} = 3$	0	0	6	0
$\hat{y} = 4$	1	0	0	6

Computational cost (for a single IC):

$30 - 50$ SQP iterations

$300 - 500$ evaluations of the objective

($2700 - 4500$ PDE solves)

Comments

The model-based approach considered

- returns an estimate[7] of the full vector μ;
- performs poorly on real data

  $\Rightarrow$ sensitive to model error;
- requires many online PDE solves

  $\Rightarrow$ no real-time response.

Simulation-based approaches are preferable if we do not need to estimate μ.

[7] Bayesian methods might also provide credible regions for the estimate μ.

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- **Mathematical model**
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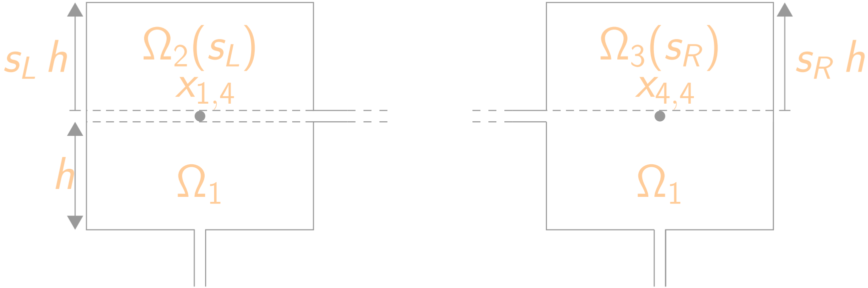
Set $\mathcal{C} = \left(\mu := [s_L, s_R, \alpha, \beta, E], \dots\right)$,

where α, β Rayleigh-damping coefficients, and
E Young's modulus.

Define the reference domain $\Omega_s = \Omega_1 \cup \Omega_2(s_L) \cup \Omega_2(s_R)$.



Assumptions: depths of blocks and masses are uniform; width of the blocks is known exactly.

Mathematical best-knowledge (bk) model (I)

Set $\mathcal{C} = (\mu := [s_L, s_R, \alpha, \beta, E], \dots)$,

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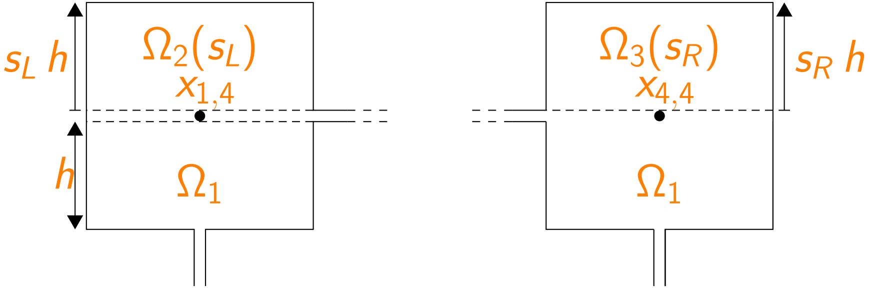

Assumptions: depths of blocks and masses are uniform;
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Mathematical best-knowledge (bk) model (II)

Define the bk time-harmonic displacement $u^{\mathrm{bk}}(\cdot; f^q, \mu)$ as

$$\mathcal{G}(f^q; \mu) u^{\mathrm{bk}}(\cdot; f^q, \mu) = 0 \ \text{ in } \ \Omega_s + \text{BC}$$

where $\mathcal{G}(f^q; \mu) \leftrightarrow$ linear damped elastodynamics .

Given $\mathcal{C} = (\mu, \ldots)$, estimate

$$A_{i,j}^{\exp}(f^q; \mathcal{C}) \approx A_{i,j}^{\mathrm{bk}}(f^q; \mu) := A_{\mathrm{nom}} \frac{|u_2^{\mathrm{bk}}(x_{i,j}; f^q, \mu)|}{|u_2^{\mathrm{bk}}(x_{2,1}; f^q, \mu)|}$$

where $x_{i,j}$ is the center of block (i, j).

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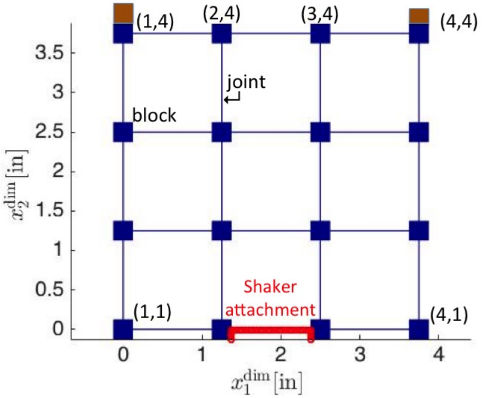
Introduce

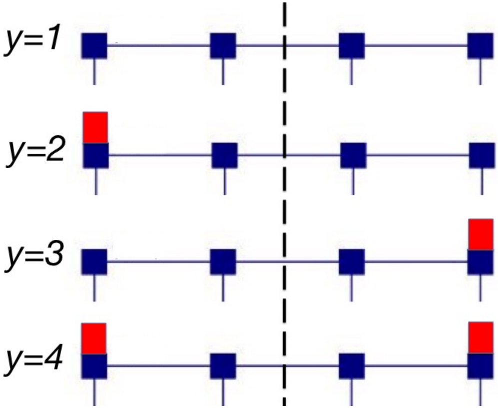
$$z_1^{\mathrm{bk}}(\cdot) = \frac{A_{1,4}^{\mathrm{bk}}(\cdot)}{A_{4,4}^{\mathrm{bk}}(\cdot)}, \; z_2^{\mathrm{bk}}(\cdot) = \frac{A_{2,4}^{\mathrm{bk}}(\cdot) + A_{3,4}^{\mathrm{bk}}(\cdot)}{A_{1,1}^{\mathrm{bk}}(\cdot) + A_{4,1}^{\mathrm{bk}}(\cdot)}.$$

and define $\mathbf{z}_\ell^{\mathrm{bk}}(\mu) = [z_\ell^{\mathrm{bk}}(f^1;\mu), \ldots, z_\ell^{\mathrm{bk}}(f^{Q_f};\mu)]$.



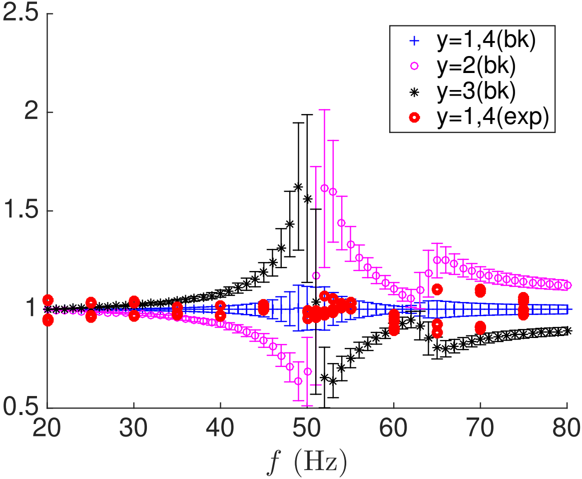
Feature visualization: z_1 and z_2

Rationale: z_1: detects asymmetry in the structure;
z_2: detects added mass on corners.



Feature visualization: z_1

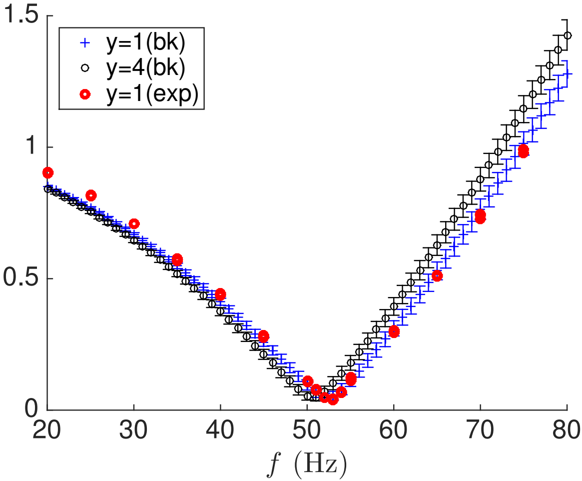
Rationale: $z_1^{\cdot}$ detects asymmetry in the structure;
$z_2^{\cdot}$ detects added mass on corners.



Feature visualization: z_2

Rationale: $z_1^{\cdot}$ detects asymmetry in the structure;

$z_2^{\cdot}$ detects added mass on corners.



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Explanation of the table

```
For i = 1, ..., 100
    Partition the dataset D_K^bk into D_M^bk and D_{K-M}^bk
    Train the learning algorithm based on D_M^bk
    Test the learning algorithm based on D_{K-M}^bk    → R_i^bk
    Test the learning algorithm based on D_15^exp      → R_i^exp
EndFor
```

For $i = 1, \ldots, 100$

Partition the dataset $\mathcal{D}_K^{\mathrm{bk}}$ into $\mathcal{D}_M^{\mathrm{bk}}$ and $\mathcal{D}_{K-M}^{\mathrm{bk}}$

Train the learning algorithm based on $\mathcal{D}_M^{\mathrm{bk}}$

Test the learning algorithm based on $\mathcal{D}_{K-M}^{\mathrm{bk}}$ $\rightarrow R_i^{\mathrm{bk}}$

Test the learning algorithm based on $\mathcal{D}_{15}^{\mathrm{exp}}$ $\rightarrow R_i^{\mathrm{exp}}$

EndFor

Return $R^{\mathrm{bk}} = \frac{1}{100} \sum_{i=1}^{100} R_i^{\mathrm{bk}}$

Return $R^{\mathrm{exp}} = \frac{1}{100} \sum_{i=1}^{100} R_i^{\mathrm{exp}}$