Model order reduction methods for data assimilation; state estimation, and structural health monitoring

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Objective

Objective of the present work

Develop **model reduction** techniques to integrate
- parametrized mathematical models (μPDEs), and
- experimental observations

for prediction.

State estimation: provide an estimate of the system state (temperature, pressure, displacement...);

Damage identification: assess the state of damage of a structure of interest (is the system damaged? which is the type of damage present in the structure?...).

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pMOR objective: reduce the marginal computational cost associated with the solution to parametrized models.

Typical applications:

many-query: design and optimization, UQ;

real-time/interactive: control, education.

A pMOR procedure should address two separate tasks:

1. data compression (solution manifold $\rightarrow$ linear space)
   $\Rightarrow$ POD, Greedy,...

2. offline-online computational decomposition
   $\Rightarrow$ Galerkin projection, interpolation,...

Claim: recent advances in pMOR offer new opportunities for the integration of μPDEs and data.

We rely on pMOR techniques for

1. data compression,
2. offline-online computational decomposition,

as **building blocks** for our data assimilation strategies.

Contributions

We propose and analyze two computational strategies:

1. Parametrized-Background Data-Weak (PBDW) approach for state estimation.

2. Simulation-Based Classification (SBC) for damage identification.

PBDW: Y Maday, AT Patera, JD Penn, M Yano, 2015a, 2015b; T Taddei, 2016 (under review).

SBC: T Taddei, JD Penn, M Yano, AT Patera, 2016.

Part I: Simulation-Based Classification (SBC)

Formulation, role of pMOR.

Part II: PBDW approach

Formulation, role of pMOR, *a priori* error analysis.

We apply our techniques to two companion experiments.

Topics not covered in this talk (but included in the thesis)

SBC: error analysis.

PBDW: *a posteriori* error analysis, localised state estimation, adaptation.

James D Penn (MIT)

Conception and implementation of the experiments

Data acquisition

Calibration

Masayuki Yano (University of Toronto)

High-order FE code

Mathematical formulation

Numerical analysis

Simulation-Based Classification

- An example: a microtruss
- Mathematical formulation
- Computational approach
- Application to the microtruss problem
- Perspectives

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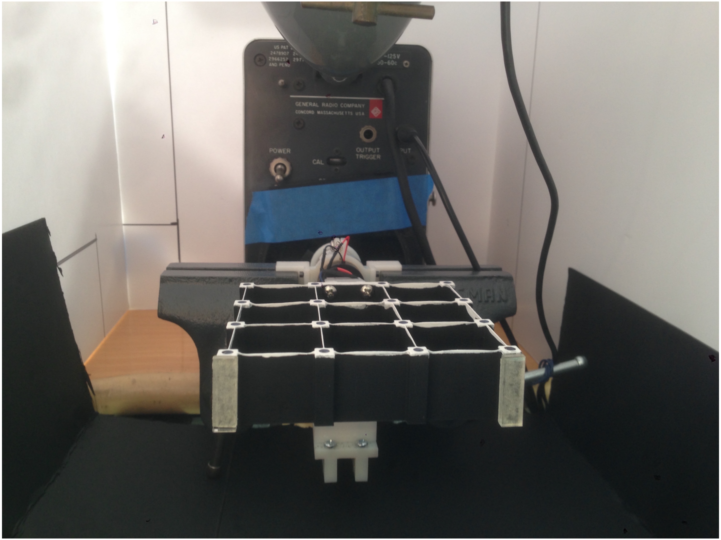
A target application: monitoring of ship loaders[1]

Objective: monitor the integrity of a ship loader during the operations

[1]Photo credit: www.directindustry.com

Our example: the microtruss system



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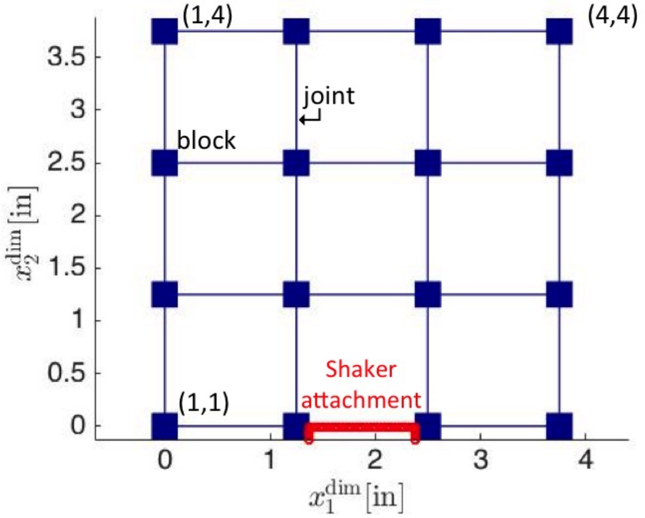


Our example: the microtruss system

Goal: detect the presence of added mass on top of block $(1, 4)$ and block $(4, 4)$

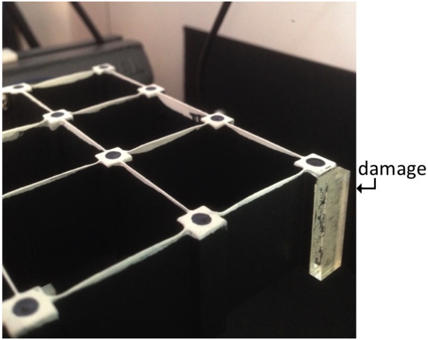
Apparatus: voice coil actuator; camera&stroboscope

Input: x_2-displacement at prescribed frequencies $\{f^q\}$;
Exp data: x_2-displacement of blocks' centers $\{c_{i,j}^{\mathrm{exp}}(t^\ell, f^q)\}$.

Data reduction:

$$c_{i,j}^{\mathrm{exp}}(t^\ell, f^q) \approx \overline{A}_{i,j}^{\mathrm{exp}}(f^q) \cos\left(2\pi f^q\, t^\ell + \overline{\phi}_{i,j}^{\mathrm{exp}}(f^q)\right)$$

Exp outputs: $A_{i,j}^{\mathrm{exp}}(f^q) := \dfrac{A_{\mathrm{nom}}}{\overline{A}_{2,1}^{\mathrm{exp}}(f^q)}\, \overline{A}_{i,j}^{\mathrm{exp}}(f^q).$



Define $s_L = 1 + \frac{V_{\text{left}}}{V_{\text{nom}}}$, and

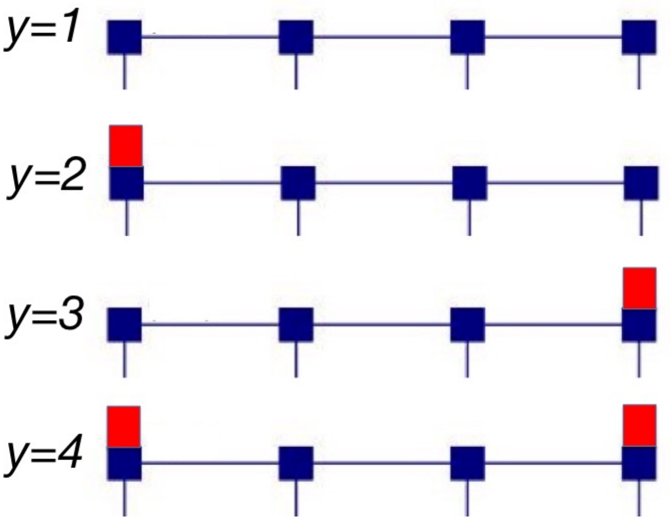
$$s_R := 1 + \frac{V_{\text{right}}}{V_{\text{nom}}}.$$

Define $y = \bar{f}^{\text{dam}}(s_L, s_R)$,

$$y = \begin{cases} 1 & s_L, s_R \leq 1.5, \\ 2 & s_L > 1.5,\ s_R \leq 1.5, \\ 3 & s_L \leq 1.5,\ s_R > 1.5, \\ 4 & s_L, s_R > 1.5. \end{cases}$$

The QOI y is the **state of damage** associated with the structure.

Definition of the QOI: damage function



Engineering objective

Generate a *decision rule* g that maps experimental outputs

$$\{A_{i,j}^{\mathrm{exp}}(f^q; \mathcal{C})\}_{i,j,q}$$

to the appropriate configuration state of damage

$$y = \bar{f}^{\mathrm{dam}}(s_L, s_R) \in \{1, 2, 3, 4\};$$

for any given system configuration $\mathcal{C} = (s_L, s_R, \ldots)$.

Perspective: objective of Structural Health Monitoring (SHM)

Level I: is the structure damaged?

Level II: where is damage located?

C Farrar, K Worden, 2012

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Simulation-Based Classification

- An example: a microtruss
- Mathematical formulation
- Computational approach
- Application to the microtruss problem
- Perspectives

Mathematical best-knowledge (bk) model

Set

$\mathcal{C} = \Big(\mu := [s_L = 1 + \frac{V_{\mathrm{left}}}{V_{\mathrm{nom}}}, s_R = 1 + \frac{V_{\mathrm{right}}}{V_{\mathrm{nom}}}, \alpha, \beta, E], \dots \Big)$,

where α, β Rayleigh-damping coefficients, and

E Young's modulus.

Estimate

$$A_{i,j}^{\mathrm{exp}}(f^q; \mathcal{C}) \approx A_{i,j}^{\mathrm{bk}}(f^q; \mu) := A_{\mathrm{nom}} \frac{|u_2^{\mathrm{bk}}(x_{i,j}; f^q, \mu)|}{|u_2^{\mathrm{bk}}(x_{2,1}; f^q, \mu)|}$$

where $x_{i,j}$ is the center of block (i, j), and $u^{\mathrm{bk}}(\cdot; f^q, \mu)$ solves the parametrized PDE:

$$\mathcal{G}_{\mathrm{elast-helmhotz}}\big(u^{\mathrm{bk}}(f^q, \mu); f^q; \mu\big) = 0 + \mathrm{BC}$$

Interpretation:

μ incomplete representation of $\mathcal{C}$;

$\mathcal{G}_{\mathrm{elast-helmhotz}}$ bk-parametrized mathematical model.

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where α, β Rayleigh-damping coefficients, and

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Estimate

$$A^{\rm exp}_{i,j}(f^q; \mathcal{C}) \approx A^{\rm bk}_{i,j}(f^q; \mu) := A_{\rm nom} \frac{|u^{\rm bk}_2(x_{i,j}; f^q, \mu)|}{|u^{\rm bk}_2(x_{2,1}; f^q, \mu)|}$$

where $x_{i,j}$ is the center of block (i, j), and $u^{\rm bk}(\cdot; f^q, \mu)$ solves the parametrized PDE:

$$\mathcal{G}_{\rm elast-helmhotz}\big(u^{\rm bk}(f^q, \mu); f^q; \mu\big) = 0 + \text{BC}$$

Interpretation:

μ incomplete representation of $\mathcal{C}$;

$\mathcal{G}_{\rm elast-helmhotz}$ bk-parametrized mathematical model.

Define the **feature map** $\mathcal{F} : \mathbb{R}^{16Q_f} \to \mathbb{R}^Q$ that

takes as input the experimental (or bk) outputs

$$\{A^{\bullet}_{i,j}(f^q; \star)\}_{i,j,q}, \ (\bullet = \mathrm{exp}, \mathrm{bk}, \ \star = \mathcal{C}, \mu)$$

and returns the Q *features*

$$\mathbf{z}^{\bullet}(\star) = \mathcal{F}(\{A^{\bullet}_{i,j}(f^q; \star)\}_{i,j,q}) \in \mathbb{R}^Q$$

$\mathcal{F} : \mathbb{R}^{16Q_f} \to \mathbb{R}^Q$ should be chosen such that

- $\mathbf{z}^{\bullet}(\star)$ is sensitive to the expected damage;
- $\mathbf{z}^{\bullet}(\star)$ is insensitive to noise.

Mathematical objective

Given the features $\mathbf{z}^{\mathrm{bk}}(\mu) = \mathcal{F}(\{A_{i,j}^{\mathrm{bk}}(f^q;\mu)\}_{i,j,q}) \in \mathbb{R}^Q$, we seek $g : \mathbb{R}^Q \to \{1, \ldots, 4\}$ that minimizes

$$R^{\mathrm{bk}}(g) = \int_{\mathcal{P}^{\mathrm{bk}}} \mathbb{1}\left(g(\mathbf{z}^{\mathrm{bk}}(\mu)) \neq f^{\mathrm{dam}}(\mu)\right) w^{\mathrm{bk}}(\mu)\, d\mu,$$

where

- $\mu = [s_L, s_R, \alpha, \beta, E] \in \mathcal{P}^{\mathrm{bk}}$ anticipated configuration;
- $\mathcal{P}^{\mathrm{bk}}$ anticipated configuration set;
- $\mu \mapsto f^{\mathrm{dam}}(\mu) = \bar{f}^{\mathrm{dam}}(s_L, s_R) \in \{1, \ldots, 4\}$ damage;
- $\mathcal{F} : \mathbb{R}^{16Q_f} \to \mathbb{R}^Q$ feature map (to be defined);
- $\mu \mapsto w^{\mathrm{bk}}(\mu)$ user-defined weight ($\leftrightarrow P_{w^{\mathrm{bk}}}$).

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Simulation-Based Classification

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Offline stage: (before operations)

1. Generate $\mu^1, \ldots, \mu^M \overbrace{\sim}^{iid} P_{w^{\mathrm{bk}}}$
2. Generate $\mathcal{D}_M^{\mathrm{bk}} = \left\{ \mathbf{z}^{\mathrm{bk}}(\mu^m), f^{\mathrm{dam}}(\mu^m) \right\}_{m=1}^M$
3. $[g_M^\star] = \texttt{Supervised-Learning-alg}(\mathcal{D}_M^{\mathrm{bk}})$

Online stage: (during operations)

1. Acquire the new outputs $\{A_{i,j}^{\mathrm{exp}}(f^q; \overline{\mathcal{C}})\}_{i,j,q}$.
2. Compute $\bar{\mathbf{z}}^{\mathrm{exp}} = \mathcal{F}(A_{i,j}^{\mathrm{exp}}(f^q; \overline{\mathcal{C}}))$.
3. Return the label $g_M^\star(\bar{\mathbf{z}}^{\mathrm{exp}})$.

Taddei, Penn, Yano, Patera, 2016.

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Related works: Farrar et al. (based on experiments);
Basudhar, Missoum;
Willcox et al.

Opportunities: no need to estimate $\mu = [s_L, s_R, \alpha, \beta, E]$
(which includes nuisance variables α, β, E)
non-intrusive approach
(it requires only forward solves)

Challenge: generation of $\mathcal{D}_M^{\mathrm{bk}}$

$\Rightarrow$ Exploit pMOR ($\leftrightarrow$ parametric def of damage) to generate $\mathcal{D}_M^{\mathrm{bk}}$.

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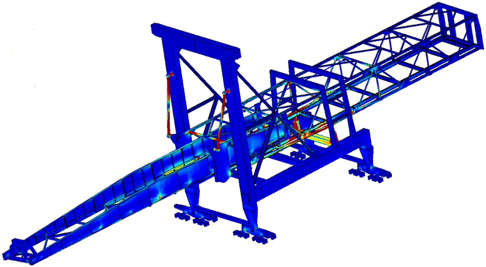
Perspectives: a ship loader model[2]

Cost to build $\mathcal{D}_M^{\mathrm{bk}} = M \times Q_f \times \text{cost per simulation}$



FE model ($\approx 5 \cdot 10^6$ dofs)

cost per simulation $\approx 43'$

$M = 10^4, Q_f = 10 \Rightarrow 8\,\text{years}$

ROM model (PR-scRBE)

cost per simulation $\approx 5''$

$M = 10^4, Q_f = 10 \Rightarrow 6\,\text{days}$

$\Rightarrow$ pMOR enables the use of mathematical models in the simulation-based framework.

[2] Simulations are performed by Akselos S.A. using PR-scRBE.

Offline stage: (before operations)

1. Generate $\mu^1, \ldots, \mu^M \overbrace{\sim}^{iid} P_{w^{\mathrm{bk}}}$

2.a Construct a ROM for $\mu \in \mathcal{P}^{\mathrm{bk}} \mapsto \mathbf{z}^{\mathrm{bk}}(\mu)$

2.b Use the ROM to generate the dataset $\mathcal{D}_M^{\mathrm{bk}}$

3. $[g_M^\star] = \texttt{Supervised-Learning-alg}(\mathcal{D}_M^{\mathrm{bk}})$

pMOR is employed only in the generation of the dataset;

If M is sufficiently large, the cost of 2.a is negligible compared to the cost of 2.b (**many-query context**).

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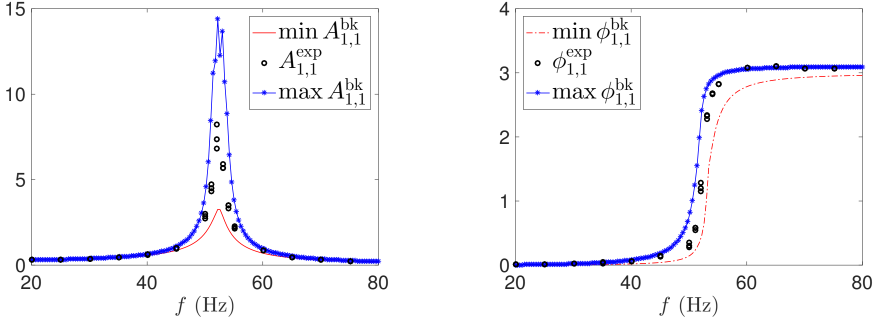
Choice of $\mathcal{P}^{\mathrm{bk}}$

We choose upper bounds for s_L, s_R *a priori*.

We choose lower and upper bounds for α, β, E using textbook values and a preliminary experiment for $s_L = s_R = 1$.



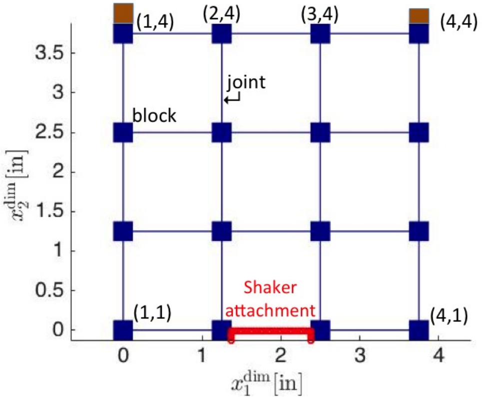
(explanation: $\min A^{\mathrm{bk}}_{1,1} = \min_{\mu=(1,1,\alpha,\beta,E)\in\mathcal{P}^{\mathrm{bk}}} A^{\mathrm{bk}}_{1,1}(\mu, f)$)

Choices of the features

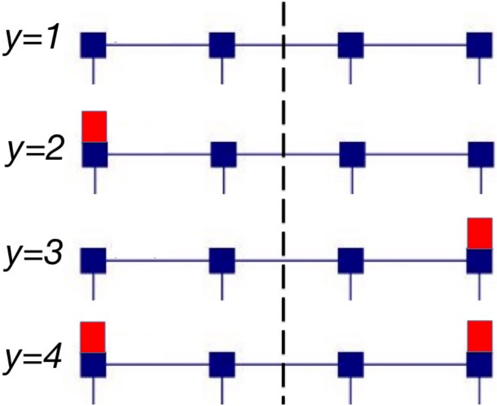
Introduce

$$z_1^{\mathrm{bk}}(\cdot) = \frac{A_{1,4}^{\mathrm{bk}}(\cdot)}{A_{4,4}^{\mathrm{bk}}(\cdot)}, \; z_2^{\mathrm{bk}}(\cdot) = \frac{A_{2,4}^{\mathrm{bk}}(\cdot) + A_{3,4}^{\mathrm{bk}}(\cdot)}{A_{1,1}^{\mathrm{bk}}(\cdot) + A_{4,1}^{\mathrm{bk}}(\cdot)}.$$

and define $\mathbf{z}_\ell^{\mathrm{bk}}(\mu) = [z_\ell^{\mathrm{bk}}(f^1;\mu), \ldots, z_\ell^{\mathrm{bk}}(f^{Q_f};\mu)]$.



Rationale: $z_1^{\cdot}$ detects asymmetry in the structure;
$z_2^{\cdot}$ detects added mass on corners.



Classification procedure

Given $\mathbf{z}_1^{\exp}$, $\mathbf{z}_2^{\exp}$,

Level 1: distinguish between $\{1,4\}$, $\{2\}$ and $\{3\}$ based on $\mathbf{z}_1^{\exp}$;

Level 2: if Level 1 returns $\{1,4\}$, distinguish between $\{1\}$ and $\{4\}$ based on $\mathbf{z}_2^{\exp}$.

From the learning perspective,

Level 1 corresponds to a 3way classification problem;

Level 2 corresponds to a 2way classification problem.

Algorithms used: SVM, ANN, kNN, decision trees, NMC[3].

[3]Implementation is based on off-the-shelf Matlab functions.

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Build a ROM for the state $u^{\mathrm{bk}}(f;\mu)$, $f \in \mathcal{I}_f$, $\mu \in \mathcal{P}^{\mathrm{bk}}$,

Use the ROM to compute $(f^q, \mu^m) \mapsto A^{\mathrm{bk}}_{i,j}(f^q;\mu^m)$ for $m = 1,\ldots,M$ and $q = 1,\ldots,Q_f$ ($= MQ_f$ PDE solves).

Computational summary:

Finite Element (FE): 14670 dof,
$\approx 0.18[\mathrm{s}]$ for each PDE query;

Reduced Basis (RB): 20 dof, pre-processing cost $\approx 24[\mathrm{s}]$,
$\approx 4.4 \cdot 10^{-3}[\mathrm{s}]$ for each PDE query.

$\Rightarrow$ RB is advantageous if $MQ_f \gtrsim 180$
(we consider $MQ_f \approx 10^5$).

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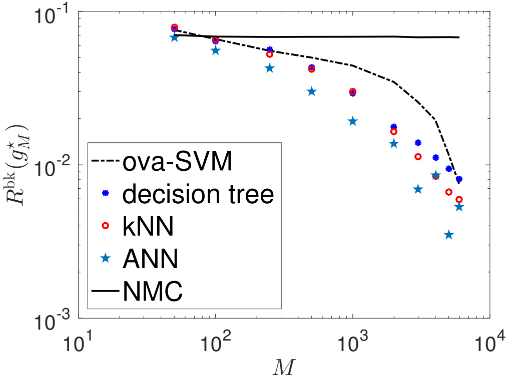
Test

1. Generate a dataset $\mathcal{D}^{\mathrm{bk}}_{N_{\mathrm{train}}}$, $N_{\mathrm{train}} = 10^4$, $Q_f = 9$;
2. Use M points for learning, $N_{\mathrm{train}} - M$ for testing;
3. Average over 100 partitions.



Memo:

$R^{\mathrm{bk}}(g) = 0$

$\Rightarrow$ no mistakes.

$R^{\mathrm{bk}}(g) = 1$

$\Rightarrow$ always wrong.

Strong dependence on $M \Rightarrow$ importance of pMOR.

Test

1. Consider 5 different experimental system configurations, and perform 3 independent trials ($= 15$ exp datapoints).
2. Train based on $M = 7 \cdot 10^3$ synthetic datapoints.
3. Average over 100 partitions of the synthetic dataset.

	bk-risk $R^{\mathrm{bk}}(g)$	exp risk (5×3)
ova-SVM	0.0059	0.2093
decision tree	0.0072	0.4000
kNN ($\mathbf{k = 5}$)	**0.0050**	**0**
ANN (10 layers)	0.0026	0.6000
NMC	0.0661	0

Results (experimental data)

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Challenges

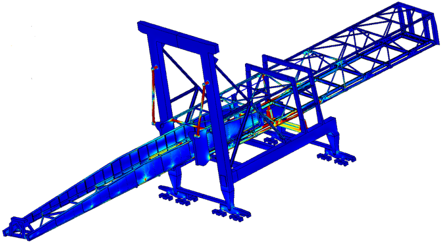
Parametrization of damage

damage is a local phenomenon,

$\Rightarrow$ component-based pMOR

Choice of features

automated feature identification[4].



[4] In collaboration with Prof. D Bertsimas, C Pawlowski (MIT).

PBDW approach for state estimation

- An example: a thermal patch configuration
- The PBDW approach
- Application to the thermal patch problem
- A priori error analysis
- Application to a synthetic problem

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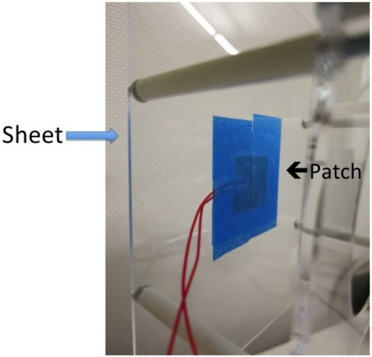
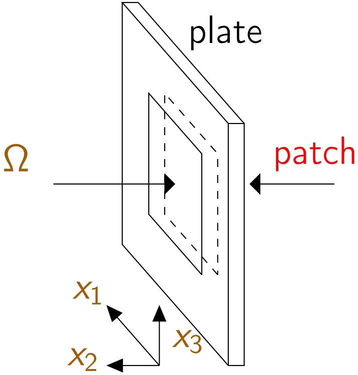
Thermal patch experiment

Objective: estimate the temperature field over the surface Ω.



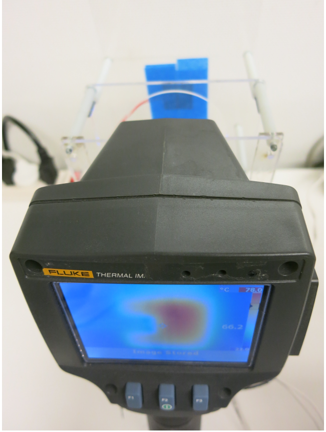


Refined goal and experimental apparatus

Practical applications: local probes.

Refined goal: given $\ell_m^{\mathrm{obs}} \approx u^{\mathrm{true}}(x_m^{\mathrm{obs}})$, $x_m^{\mathrm{obs}} \in \Omega$,

estimate u^{true} over Ω.



Our apparatus:

IR camera

Full-field information

$\Rightarrow$ performance assessment.

PBDW approach for state estimation

- An example: a thermal patch configuration
- The PBDW approach
- Application to the thermal patch problem
- A priori error analysis
- Application to a synthetic problem

Mathematical best-knowledge (bk) model

Estimate the steady-state temperature field as

$$\begin{cases} -\Delta u^{\mathrm{bk}} = 0, & \text{in } \Omega^{\mathrm{bk}}, \\ \kappa \partial_n u^{\mathrm{bk}} + \gamma (u^{\mathrm{bk}} - \Theta^{\mathrm{room}}) = C \chi_{\Gamma^{\mathrm{patch}}} & \text{on } \Gamma^{\mathrm{in}}, \\ \kappa \partial_n u^{\mathrm{bk}} = 0 & \text{on } \partial\Omega^{\mathrm{bk}} \setminus \Gamma^{\mathrm{in}}, \end{cases}$$

Θ^{room} room temperature ($= 20^o C$);

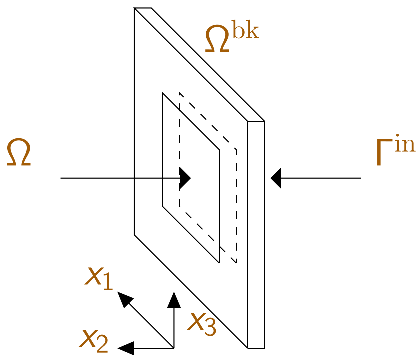
κ thermal conductivity;

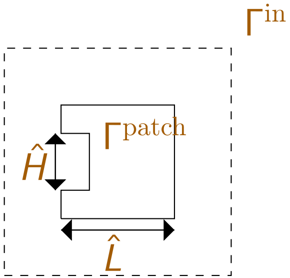
γ convective heat transfer coefficient;

C incoming flux (patch $\rightarrow$ plate).

$\Rightarrow \ \mu := [\gamma/\kappa, \ C/\kappa] \in \mathcal{P}^{\mathrm{bk}}$

Mathematical best-knowledge (bk) model





$\Omega \subset \partial\Omega^{\mathrm{bk}}$, $\hat{L} = 22.606\mathrm{mm}$, $\hat{H} = 9.271\mathrm{mm}$.

Bk solution manifold

Define the bk solution manifold

$$\mathcal{M}^{\mathrm{bk}} = \{u^{\mathrm{bk}}(\mu)|_{\Omega} : \quad \mu \in \mathcal{P}^{\mathrm{bk}}\} \subset \mathcal{U} = \mathcal{U}(\Omega)$$

$\mathcal{M}^{\mathrm{bk}}$ takes into account parametrized uncertainty in the system.

$\mathcal{M}^{\mathrm{bk}}$ does not take into account non-parametric uncertainty in the system:

- nonlinear effects due to natural convection,
- heat-exchange between the patch and the sheet.

General idea

Given $\mathcal{M}^{\mathrm{bk}}$, define $\mathcal{Z}_N = \mathrm{span}\{\zeta_n\}_{n=1}^{N}$ such that

$\sup_\mu \inf_z \|u^{\mathrm{bk}}(\mu)|_\Omega - z\|$ is small.

Then, given measurements $\ell_1^{\mathrm{obs}}, \ldots, \ell_M^{\mathrm{obs}}$,

step 1. find $z^\star \in \mathcal{Z}_N$ such that $z^\star \approx u^{\mathrm{true}}$

step 2. find $\eta^\star \in \mathcal{U}$ such that $\eta^\star \approx u^{\mathrm{true}} - z^\star$

step 3. return the state estimate $u^\star = z^\star + \eta^\star$.

Variational formulation

Given the Hilbert space $(\mathcal{U} = \mathcal{U}(\Omega), \|\cdot\|)$, introduce $\ell_1^o, \ldots, \ell_M^o \in \mathcal{U}'$ such that

$$\ell_m^{\mathrm{obs}} \approx \ell_m^o(u^{\mathrm{true}}),\ m = 1, \ldots, M.$$

Define $u_\xi^\star = z_\xi^\star + \eta_\xi^\star$ to minimise

$$\min_{(z,\eta)\in\mathcal{Z}_N\times\mathcal{U}} \ \xi\|\eta\|^2 + \frac{1}{M}\sum_{m=1}^{M}\left(\ell_m^o(z+\eta) - \ell_m^{\mathrm{obs}}\right)^2.$$

Computation of $z_\xi^\star$ corresponds to a weighted LS problem.
Computation of $\eta_\xi^\star$ corresponds to a generalized smoothing problem based on $\ell_m^{\mathrm{err}} = \ell_m^{\mathrm{obs}} - \ell_m^o(z_\xi^\star) \approx \ell_m^o(u^{\mathrm{true}} - z_\xi^\star)$.

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Terminology:

$\mathcal{Z}_N$ background space;

$z^\star \in \mathcal{Z}_N$ deduced background;

$\eta^\star$ update;

$z^\star$ addresses parametrized uncertainty in the model, while $\eta^\star$ addresses non-parametric uncertainty in the model.

Solution to $\min_{(z,\eta)\in\mathcal{Z}_N\times\mathcal{U}} \cdot$ is simpler than $\min_{(z,\eta)\in\mathcal{M}^{\mathrm{bk}}\times\mathcal{U}} \cdot$.

Construction of $\mathcal{Z}_N$ is a pMOR problem.

data compression

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Construction of $\mathcal{Z}_N$ is a pMOR problem.

data compression

Solution representation

The update is of the form

$$\eta_\xi^\star(\cdot) = \sum_{m=1}^{M} \eta_{\xi,m}^\star R_\mathcal{U} \ell_m^o(\cdot) \in \mathcal{U}_M := \mathrm{span}\{R_\mathcal{U} \ell_m^o\}_{m=1}^{M},$$

where $R_\mathcal{U} : \mathcal{U}' \mapsto \mathcal{U}$ depends on $(\mathcal{U}, \|\cdot\|)$.

For $\ell_m^o = \delta_{x_m^o}$ and suitable $(\mathcal{U}, \|\cdot\|)$,

$$R_\mathcal{U} \ell_m^o(\cdot) = K_\gamma(\cdot, x_m^{\mathrm{obs}}) = \phi(\gamma\|\cdot - x_m^{\mathrm{obs}}\|_2) \Rightarrow \text{connection with Kernel methods.}$$

Bennett, 1985, Kimeldorf, Wahba, 1971;

J Krebs, A Louis, H Wendland, 2009.

Maday et al, 2015

two-level mechanism to accommodate anticipated/ unanticipated uncertainty

use of pMOR to generate $\mathcal{Z}_N$;

This thesis

adaptive selection of ξ
$\Rightarrow$ rigorous treatment of noisy measurements;

adaptive selection of $\|\cdot\|$ for pointwise measurements
$\Rightarrow$ improved convergence with M.

Localized state estimation ($\Omega \subset \Omega^{\mathrm{bk}}$, $\mu \in \mathbb{R}^P$, $P \gg 1$); not covered in this talk.

PBDW approach for state estimation

- An example: a thermal patch configuration
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Details

Observations: $\ell_m^{\mathrm{obs}} = u^{\mathrm{obs}}(x_{i_m,j_m}^{\mathrm{obs}})$, $(\Rightarrow \ell_m^o = \delta_{x_{i_m,j_m}^{\mathrm{obs}}})$
$x_{i_m,j_m}^{\mathrm{obs}}$ center of the (i_m, j_m) pixel[5].

Background: $\{\mathcal{Z}_N\}_N$ generated using the weak-Greedy[6] algorithm;

Kernel:[7] $K_\gamma(x, x') = \phi(\gamma \|x - x'\|_2)$,
$\phi(r) = (1-r)_+^4(4r+1)$, $(\mathcal{U} = H^{2.5}(\mathbb{R}^2))$.

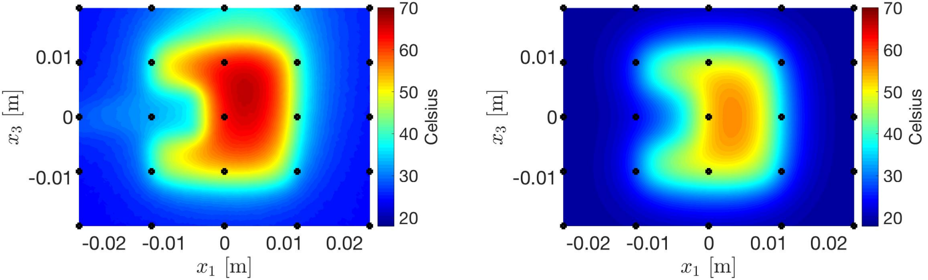
---

[5] The IR camera returns 160×120 pixel-wise measurements.

[6] G Rozza, DBP Huynh, AT Patera, 2008.

[7] H Wendland, 2004.

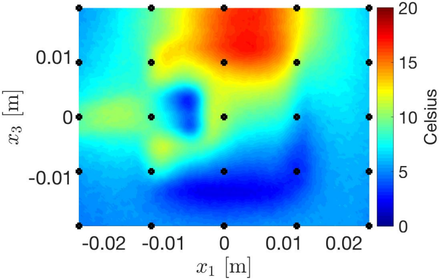
step 1. find $z^\star \in \mathcal{Z}_N$ such that $z^\star \approx u^{\mathrm{true}}$



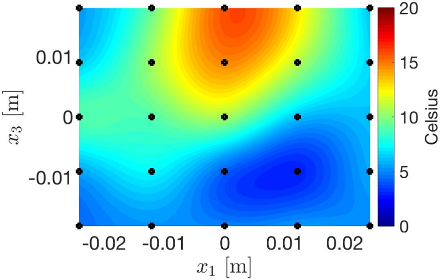
u^{obs} $\quad$ $z^\star_\xi$

Numerical results ($N = 2,\ M = 25$): step 2

step 2. find $\eta^\star \in \mathcal{U}$ such that $\eta^\star \approx u^{\text{true}} - z^\star$



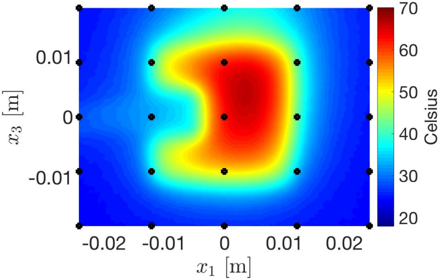
$u^{\text{obs}} - z^\star_\xi$



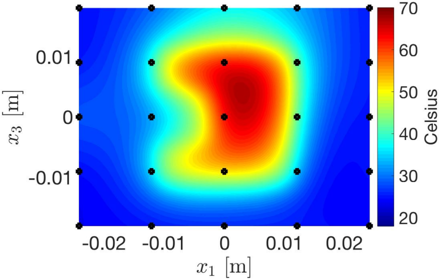
$\eta^\star_\xi$

Numerical results ($N = 2,\ M = 25$): step 3

step 3. return the state estimate $u^\star = z^\star + \eta^\star$.



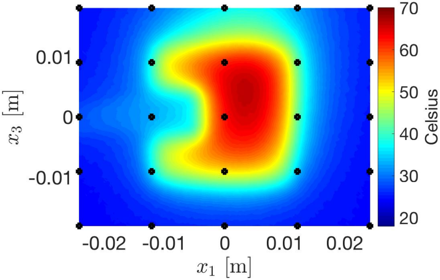
u^{obs}



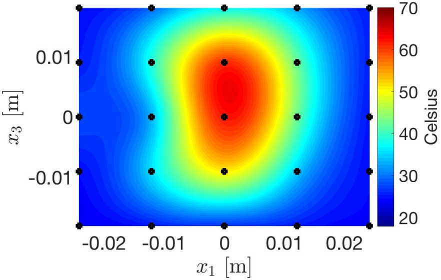
$u^\star_\xi$

Numerical results ($N = 0$, $M = 25$): step 3

step 3. return the state estimate $u^\star = z^\star + \eta^\star$.



u^{obs}



$u_\xi^\star$

PBDW approach for state estimation

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Preliminaries

Suppose

$$y_m = u^{\mathrm{true}}(x_m^{\mathrm{obs}}) + \epsilon_m, \quad m = 1, \dots, M.$$

Define the fill distance:

$$h_M := \sup_{x \in \Omega} \min_m \|x - x_m^{\mathrm{obs}}\|_2;$$

Suppose *quasi-uniform* grid:

$$h_M \sim M^{-1/d}, \qquad \Omega \subset \mathbb{R}^d.$$

Systematic noise: $|\epsilon_m| \leq \delta$

Homoscedastic noise: $\epsilon_m \overbrace{\sim}^{iid} (0, \sigma^2)$

A priori error analysis: $|\epsilon_m| \le \delta$

Suppose: $\mathcal{U} = H^\tau(\mathbb{R}^d)$, $\tau > d/2$, $u^{\mathrm{true}} \in \mathcal{U}, \mathcal{Z}_N \subset \mathcal{U}$;
$h_M \sim M^{-1/d}$;

$$\Rightarrow \|u^{\mathrm{true}} - u^\star_\xi\|^2_{L^2(\Omega)} \le C_N \Big(h_M^{2\tau} \big(2\|\Pi_{\mathcal{Z}_N^\perp} u^{\mathrm{true}}\|_{\mathcal{U}} + \frac{\delta}{2}\frac{1}{\sqrt{\xi}}\big)^2 + \big(\delta + \frac{\sqrt{\xi}}{2}\|\Pi_{\mathcal{Z}_N^\perp} u^{\mathrm{true}}\|_{\mathcal{U}}\big)^2 \Big)$$

$$\xi^{\mathrm{opt}} = \left(\frac{\delta}{\|\Pi_{\mathcal{Z}_N^\perp} u^{\mathrm{true}}\|_{\mathcal{U}}} h_M^{2\tau} \right)^{2/3};$$

$$\text{If } \delta = 0 \Rightarrow \|u^{\mathrm{true}} - u^\star_{\xi,\gamma}\|^2_{L^2(\Omega)} \le C_N \|\Pi_{\mathcal{Z}_N^\perp} u^{\mathrm{true}}\|^2_{\mathcal{U}} \big(h_M^{2\tau} + \xi\big)$$

$\mathcal{Z}_N = \emptyset \Rightarrow$ J Krebs, A Louis, H Wendland, 2009.

A priori error analysis: $|\epsilon_m| \le \delta$

Suppose: $\mathcal{U} = H^\tau(\mathbb{R}^d)$, $\tau > d/2$, $u^{\mathrm{true}} \in \mathcal{U}, \mathcal{Z}_N \subset \mathcal{U}$;
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$$\xi^{\mathrm{opt}} = \left(\frac{\delta}{\|\Pi_{\mathcal{Z}_N^\perp} u^{\mathrm{true}}\|_{\mathcal{U}}} h_M^{2\tau}\right)^{2/3};$$

If $\delta = 0 \Rightarrow \|u^{\mathrm{true}} - u^\star_{\xi,\gamma}\|^2_{L^2(\Omega)} \le C_N \|\Pi_{\mathcal{Z}_N^\perp} u^{\mathrm{true}}\|^2_{\mathcal{U}}\big(h_M^{2\tau} + \xi\big)$

$\mathcal{Z}_N = \emptyset \Rightarrow$ J Krebs, A Louis, H Wendland, 2009.

A priori error analysis: $\epsilon_m \sim (0, \sigma^2)$ i.i.d.

Suppose: $\mathcal{U} = H^\tau(\mathbb{R}^d)$, $\tau > d/2$, $u^{\rm true} \in \mathcal{U}, \mathcal{Z}_N \subset \mathcal{U}$;

$h_M \sim M^{-1/d}$;

$$\Rightarrow \mathbb{E}\left[\|u^{\rm true} - u^\star_\xi\|^2_{L^2(\Omega)}\right] \le C_N\left(h_M^{2\tau} + \xi\right)\|\Pi_{\mathcal{Z}_N^\perp} u^{\rm true}\|^2_{\mathcal{U}} + 2\sigma^2\,\mathcal{T}^\sigma_{N,M}(\xi)$$

where $\mathcal{T}^\sigma_{N,M}(\xi)$ can be computed explicitly.

If $u^{\rm true} \in \mathcal{Z}_N \Rightarrow \mathbb{E}\left[\|u^{\rm true} - u^\star_{\xi,\gamma}\|^2_{L^2(\Omega)}\right] = \sigma^2\,\mathcal{T}^\sigma_{N,M}(\xi)$

Empirical studies show that $\mathcal{T}^\sigma_{N,M}(\xi)$ is monotonic decreasing in ξ.

PBDW approach for state estimation

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An acoustic model problem

Let $u_g(\mu)$ be the solution to

$$\begin{cases} -(1+\epsilon\mu i)\,\Delta u_g(\mu) - \mu^2\, u_g(\mu) = \mu(x_1^2 + e^{x_2}) + \mu g & \text{in } \Omega \\ \partial_n u_g(\mu) = 0 & \text{on } \partial\Omega \end{cases}$$

where $\epsilon = 10^{-2}$ and $\mu \in \mathcal{P}^{\mathrm{bk}} = [2, 10]$.

Perfect model: $u^{\mathrm{true}}(\mu) = u_{g_0}(\mu)$, $u^{\mathrm{bk}}(\mu) = u_{g_0}(\mu)$;

Imperfect model: $u^{\mathrm{true}}(\mu) = u_{\bar{g}}(\mu)$, $u^{\mathrm{bk}}(\mu) = u_{g_0}(\mu)$.

$$g_0 \equiv 0, \; \bar{g}(x) = 0.5(e^{x_1} + \cos(1.3\pi x_2)).$$

Observations: $y_\ell = u^{\mathrm{true}}(x_\ell^{\mathrm{obs}}) + \epsilon_\ell$, $\epsilon_\ell \overbrace{\sim}^{iid} \mathcal{N}(0, \sigma^2)$;

Centers: $\{x_m^{\mathrm{obs}}\}_m$ deterministic (equispaced),
$\{x_i^{\mathrm{obs}}\}_i$ drawn randomly (uniform), $I = M/2$;

Background: $\{\mathcal{Z}_N\}_N$ generated using the weak-Greedy algorithm;

Kernel: $K_\gamma(x, x') = \phi(\gamma \|x - x'\|_2)$,
$\phi(r) = (1 - r)_+^4 (4r + 1)$, $(\mathcal{U} = H^{2.5}(\mathbb{R}^2))$.

G Rozza, DBP Huynh, AT Patera, 2008;

H Wendland, 2004.

Measure of performances

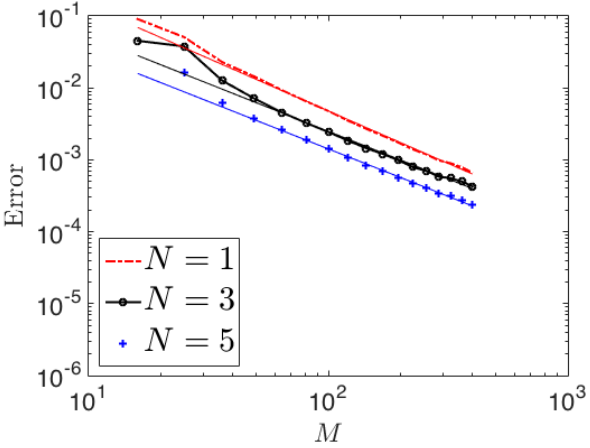
We introduce

$$E_{\text{avg}}^{\text{rel}} = \frac{1}{|\mathcal{P}_{\text{train}}^{\text{bk}}|} \sum_{\mu \in \mathcal{P}_{\text{train}}^{\text{bk}}} \frac{\| u^{\text{true}}(\mu) - u_{\xi}^{\star}(\mu) \|_{L^2(\Omega)}}{\| u^{\text{true}}(\mu) \|_{L^2(\Omega)}},$$

$\mathcal{P}_{\text{train}}^{\text{bk}} \subset [2, 10]$.

if $\sigma > 0$ (noisy measurements), computations of $\| u^{\text{true}}(\mu) - u_{\xi}^{\star}(\mu) \|_{L^2(\Omega)}$ are averaged over $K = 24$ trials.

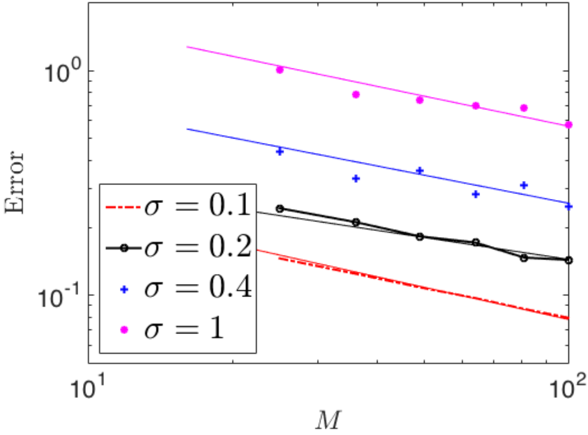
Results: M convergence ($\sigma = 0$, $g = \bar{g}$)



$E^{\text{rel}}_{\text{avg}} \sim M^{-1.3} - M^{-1.5}$, $|\mathcal{P}^{\text{bk}}_{\text{train}}| = 20$

Multiplicative effect between M and N convergence.

Results: M convergence ($N = 5$, $\sigma > 0$, $g = \bar{g}$)



$E_{\rm avg}^{\rm rel} \sim M^{-0.4} - M^{-0.5}$, $|\mathcal{P}_{\rm train}^{\rm bk}| = 1$, $\mu = 6.6$;
Adaptation in ξ allows us to deal with noisy measurements.

Conclusions

Summary

pMOR techniques for

1. data compression and
2. offline/online computational decomposition

offer new opportunities for the integration of μPDEs and data.

We relied on pMOR techniques to develop two Data Assimilation strategies for systems modeled by PDEs.

PBDW for state estimation:

two-level procedure to address parametric and non-parametric uncertainty

pMOR employed to construct $\mathcal{Z}_N$

data compression

SBC for damage identification:

simulation-based approach for discrete-valued QOIs

pMOR procedure for rapid generation of $\mathcal{D}_M^{\mathrm{bk}}$

offline/online decomposition

Thank you for the attention!

Backup slides

- Choice of the features
- Explanation of the Table
- H^1-PBDW vs A-PBDW
- Localised state estimation
- Choice of $\mathcal{P}^{\mathrm{bk}}$ for thermal patch

Backup slides

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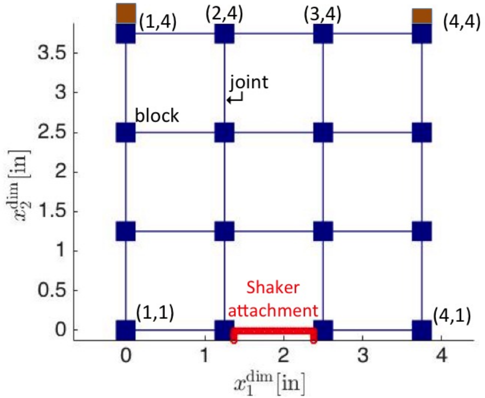
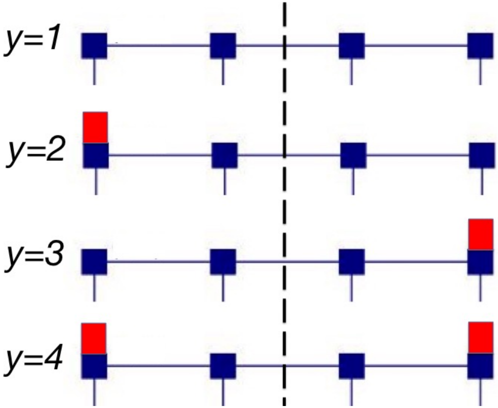
Introduce

$$z_1^{\mathrm{bk}}(\cdot) = \frac{A_{1,4}^{\mathrm{bk}}(\cdot)}{A_{4,4}^{\mathrm{bk}}(\cdot)}, \; z_2^{\mathrm{bk}}(\cdot) = \frac{A_{2,4}^{\mathrm{bk}}(\cdot) + A_{3,4}^{\mathrm{bk}}(\cdot)}{A_{1,1}^{\mathrm{bk}}(\cdot) + A_{4,1}^{\mathrm{bk}}(\cdot)}.$$

and define $\mathbf{z}_\ell^{\mathrm{bk}}(\mu) = [z_\ell^{\mathrm{bk}}(f^1;\mu), \ldots, z_\ell^{\mathrm{bk}}(f^{Q_f};\mu)]$.



Rationale: $z_1^{\cdot}$ detects asymmetry in the structure;
$z_2^{\cdot}$ detects added mass on corners.



Feature visualization: z_1

Rationale: $z_1^{\cdot}$ detects asymmetry in the structure;
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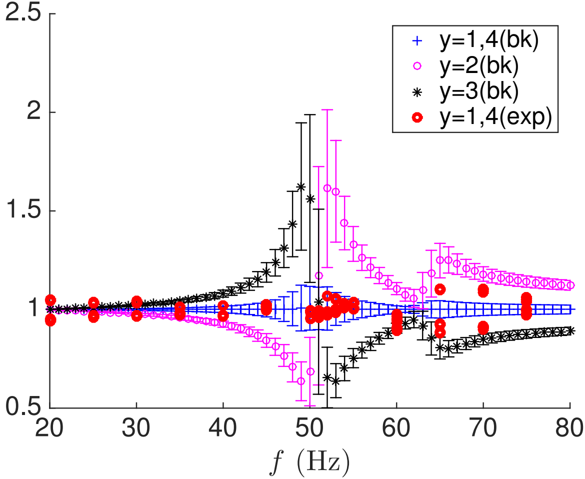

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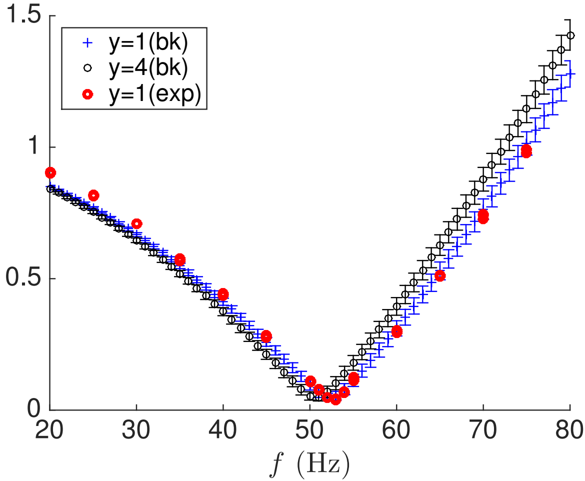

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Explanation of the table

`For` $i = 1, \dots, 100$

Partition the dataset $\mathcal{D}^{\mathrm{bk}}_{N_{\mathrm{train}}}$ into $\mathcal{D}^{\mathrm{bk}}_{M}$ and $\mathcal{D}^{\mathrm{bk}}_{N_{\mathrm{train}}-M}$

Train the learning algorithm based on $\mathcal{D}^{\mathrm{bk}}_{M}$

Test the learning algorithm based on $\mathcal{D}^{\mathrm{bk}}_{N_{\mathrm{train}}-M}$ $\rightarrow R^{\mathrm{bk}}_i$

Test the learning algorithm based on $\mathcal{D}^{\mathrm{exp}}_{15}$ $\rightarrow R^{\mathrm{exp}}_i$

`EndFor`

Return $R^{\mathrm{bk}} = \frac{1}{100}\sum_{i=1}^{100} R^{\mathrm{bk}}_i$

Return $R^{\mathrm{exp}} = \frac{1}{100}\sum_{i=1}^{100} R^{\mathrm{exp}}_i$

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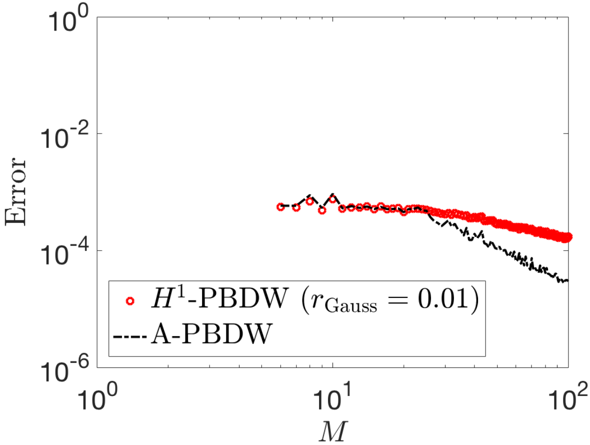
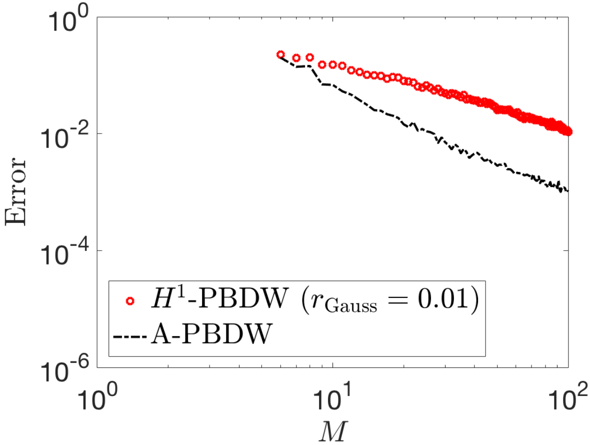
Results ($N = 5$, $\sigma = 0$, $g = g_0$)

H^1-PBDW: $\mathcal{U} = H^1(\Omega)$, $\ell_m^{\mathrm{obs}} = \mathrm{Gauss}(u^{\mathrm{true}}, x_m^{\mathrm{obs}}, r_{\mathrm{Gauss}})$
A-PBDW: $\mathcal{U} = H^1(\Omega)$, $\ell_m^{\mathrm{obs}} = u^{\mathrm{true}}(x_m^{\mathrm{obs}})$



Results ($N = 5$, $\sigma = 0$, $g = \bar{g}$)

H^1-PBDW: $\mathcal{U} = H^1(\Omega)$, $\ell_m^{\rm obs} = \mathrm{Gauss}(u^{\rm true}, x_m^{\rm obs}, r_{\rm Gauss})$
A-PBDW: $\mathcal{U} = H^1(\Omega)$, $\ell_m^{\rm obs} = u^{\rm true}(x_m^{\rm obs})$



Backup slides

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- Explanation of the Table
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Objective: estimate the state in a subregion Ω of the original domain Ω^{pb}.

Strategy: restrict computations to Ω^{bk}, $\Omega \subset \Omega^{\mathrm{bk}} \subset \Omega^{\mathrm{pb}}$.

uncertainty in global inputs $\Rightarrow$ uncertainty at ports.

Solution manifold

$$\mathcal{M}^{\mathrm{bk}} = \{ u_g^{\mathrm{bk}}(\mu)|_{\Omega} : \underbrace{\mu \in \mathcal{P}^{\mathrm{bk}}}_{\text{parameters}} \quad \underbrace{g \in \mathcal{T}}_{\text{boundary conditions}} \}$$

Refined objective: determine rapidly convergent spaces $\mathcal{Z}_N$ to approximate $\mathcal{M}^{bk}$

Fundamental question: is the manifold reducible? ($\leftrightarrow$ evanescence);

Challenge: $\mathcal{P}^{\mathrm{bk}} \times \mathcal{T}$ is infinite-dimensional.

Localised state estimation (Chapter 5)

Strategy: restrict computations to Ω^{bk}, $\Omega \subset \Omega^{\mathrm{bk}} \subset \Omega^{\mathrm{pb}}$.

uncertainty in global inputs $\Rightarrow$ uncertainty at ports.

Solution manifold

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- Choice of the features
- Explanation of the Table
- H^1-PBDW vs A-PBDW
- Localised state estimation
- Choice of $\mathcal{P}^{\mathrm{bk}}$ for thermal patch

Thermal patch: choice of $\mathcal{P}^{\mathrm{bk}}$

$\mu := [\mu_1 = \gamma/\kappa,\ \mu_2 = C/\kappa]$

u^{bk} is linear in $C/\kappa \Rightarrow$ no need to estimate μ_2

$\kappa = 0.2\mathrm{W}/(\mathrm{m}\cdot\mathrm{K})$ thermal conductivity of acrylic,

$\gamma = \frac{Nu\kappa_{\mathrm{air}}}{\widehat{L}} \approx 10 \pm 5\mathrm{W}/\mathrm{m}^2$,

$\kappa_{\mathrm{air}} = 0.0257\mathrm{W}/(\mathrm{m}\cdot\mathrm{K})$ thermal conductivity of air,

$Nu = 0.59(Ra)^{1/4}$ Nusselt number,

$Ra = \frac{\beta g \Delta\Theta \widehat{L}^3}{\nu\alpha}$ Rayleigh number

$g = 9.8\mathrm{m}/\mathrm{s}^2$, $\Delta\Theta = 50^o K$, $\widehat{L} = 22.606\mathrm{mm}$,

$\beta = 1/300\mathrm{K}^{-1}$ thermal expansion coefficient,

$\alpha = 1.9 \cdot 10^{-5}\mathrm{m}^2/\mathrm{s}$ thermal diffusivity coefficient of air,

$\nu = 1.81 \cdot 10^{-5}\mathrm{m}^2/\mathrm{s}$ kinematic viscosity of air.