Model order reduction methods for data assimilation; state estimation, and structural health monitoring

T Taddei

Massachusetts Institute of Technology

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Advisor: Prof. AT Patera
Objective
Objective of the present work

Develop **model reduction** techniques to integrate parametrized mathematical models ($\mu$-PDEs), and experimental observations for prediction.

**State estimation:** provide an estimate of the system state (temperature, pressure, displacement...);

**Damage identification:** assess the state of damage of a structure of interest (is the system damaged? which is the type of damage present in the structure?...).
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Model Order Reduction for parametrized PDEs (pMOR)

**pMOR objective:** reduce the marginal computational cost associated with the solution to parametrized models.

**Typical applications:**
- *many-query:* design and optimization, UQ;
- *real-time/interactive:* control, education.

A pMOR procedure should address two separate tasks:

1. data compression (solution manifold $\rightarrow$ linear space) $\Rightarrow$ POD, Greedy, ...
2. offline-online computational decomposition $\Rightarrow$ Galerkin projection, interpolation, ...
Claim: recent advances in pMOR offer new opportunities for the integration of $\mu$PDEs and data.

We rely on pMOR techniques for

1. data compression,

2. offline-online computational decomposition,

as building blocks for our data assimilation strategies.
Contributions

We propose and analyze two computational strategies:

1. Parametrized-Background Data-Weak (PBDW) approach for state estimation.

2. Simulation-Based Classification (SBC) for damage identification.

**PBDW**: Y Maday, AT Patera, JD Penn, M Yano, 2015a, 2015b; T Taddei, 2016 (under review).

**SBC**: T Taddei, JD Penn, M Yano, AT Patera, 2016.
Outline of the presentation

Part I: Simulation-Based Classification (SBC)
  Formulation, role of pMOR.

Part II: PBDW approach
  Formulation, role of pMOR, \textit{a priori} error analysis.

We apply our techniques to two companion experiments.

Topics not covered in this talk (but included in the thesis)
  SBC: error analysis.
  PBDW: \textit{a posteriori} error analysis, localised state estimation, adaptation.
Acknowledgements

James D Penn (MIT)
Conception and implementation of the experiments
Data acquisition
Calibration

Masayuki Yano (University of Toronto)
High-order FE code
Mathematical formulation
Numerical analysis
Simulation-Based Classification

- An example: a microtruss
- Mathematical formulation
- Computational approach
- Application to the microtruss problem
- Perspectives
Simulation-Based Classification

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A target application: monitoring of ship loaders

**Objective:** monitor the integrity of a ship loader during the operations

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1 Photo credit: www.directindustry.com
Our example: the microtruss system
Our example: the microtruss system
Our example: the microtruss system

**Goal:** detect the presence of added mass on top of block \((1, 4)\) and block \((4, 4)\)

**Apparatus:** voice coil actuator; camera&stroboscope

**Input:** \(x_2\)-displacement at prescribed frequencies \(\{f^q\}\);

**Exp data:** \(x_2\)-displacement of blocks’ centers \(\{c_{i,j}^{\exp}(t^\ell, f^q)\}\).

**Data reduction:**

\[
c_{i,j}^{\exp}(t^\ell, f^q) \approx A_{i,j}^{\exp}(f^q) \cos \left(2\pi f^q t^\ell + \phi_{i,j}^{\exp}(f^q)\right)
\]

**Exp outputs:**

\[
A_{i,j}^{\exp}(f^q) := \frac{A_{\text{nom}}^{\exp}}{A_{2,1}^{\exp}(f^q)} \bar{A}_{i,j}^{\exp}(f^q).
\]
Definition of the QOI: damage function

Define $s_L = 1 + \frac{V_{\text{left}}}{V_{\text{nom}}}$, and $s_R := 1 + \frac{V_{\text{right}}}{V_{\text{nom}}}$.

Define $y = \bar{f}^{\text{dam}}(s_L, s_R)$,

$$y = \begin{cases} 
1 & s_L, s_R \leq 1.5, \\
2 & s_L > 1.5, s_R \leq 1.5, \\
3 & s_L \leq 1.5, s_R > 1.5, \\
4 & s_L, s_R > 1.5. 
\end{cases}$$

The QOI $y$ is the state of damage associated with the structure.
Definition of the QOI: damage function

\begin{align*}
y &= 1 \\
y &= 2 \\
y &= 3 \\
y &= 4
\end{align*}
Engineering objective

Generate a decision rule $g$ that maps experimental outputs
\[ \{A_{i,j}^{\text{exp}}(f^q; C)\}_{i,j,q} \]
to the appropriate configuration state of damage
\[ y = \bar{f}^\text{dam}(s_L, s_R) \in \{1, 2, 3, 4\}; \]
for any given system configuration $C = (s_L, s_R, \ldots)$.

Perspective: objective of Structural Health Monitoring (SHM)

**Level I:** is the structure damaged?

**Level II:** where is damage located?

C Farrar, K Worden, 2012
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Simulation-Based Classification

- An example: a microtruss
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- Computational approach
- Application to the microtruss problem
- Perspectives
Mathematical best-knowledge (bk) model

Set
\[ C = (\mu := \left[ s_L = 1 + \frac{V_{\text{left}}}{V_{\text{nom}}}, s_R = 1 + \frac{V_{\text{right}}}{V_{\text{nom}}}, \alpha, \beta, E \right], \ldots ), \]
where \( \alpha, \beta \) Rayleigh-damping coefficients, and \( E \) Young’s modulus.

Estimate
\[ A_{i,j}^{\text{exp}}(f_q; C) \approx A_{i,j}^{\text{bk}}(f_q; \mu) := A_{\text{nom}} \frac{|u_{2}^{\text{bk}}(x_{i,j}; f_q, \mu)|}{|u_{2}^{\text{bk}}(x_{2,1}; f_q, \mu)|}, \]
where \( x_{i,j} \) is the center of block \((i, j)\), and \( u_{2}^{\text{bk}}(\cdot; f_q, \mu) \) solves the parametrized PDE:
\[ \mathcal{G}_{\text{elast–helmhotz}}(u_{2}^{\text{bk}}(f_q, \mu); f_q; \mu) = 0 + \text{BC} \]

Interpretation:
\( \mu \) incomplete representation of \( C \);
\( \mathcal{G}_{\text{elast–helmhotz}} \) bk-parametrized mathematical model.
Mathematical best-knowledge (bk) model

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\( \mathcal{G}_{\text{elast–helmhotz}} \) bk-parametrized mathematical model.
Feature extraction

Define the **feature map** $\mathcal{F}: \mathbb{R}^{16Q_f} \rightarrow \mathbb{R}^{Q}$ that takes as input the experimental (or bk) outputs

$$\{A_{i,j}(f^q; \star)\}_{i,j,q}, (\cdot = \exp, \text{bk}, \star = \mathcal{C}, \mu)$$

and returns the $Q$ **features**

$$z^\star(\star) = \mathcal{F}(\{A_{i,j}(f^q; \star)\}_{i,j,q}) \in \mathbb{R}^{Q}$$

$\mathcal{F}: \mathbb{R}^{16Q_f} \rightarrow \mathbb{R}^{Q}$ should be chosen such that

- $z^\star(\star)$ is sensitive to the expected damage;
- $z^\star(\star)$ is insensitive to noise.
Mathematical objective

Given the features \( z^{bk}(\mu) = \mathcal{F}(\{ A^{bk}_{i,j}(f^q; \mu) \}_{i,j,q}) \in \mathbb{R}^Q \), we seek \( g : \mathbb{R}^Q \rightarrow \{1, \ldots, 4\} \) that minimizes

\[
R^{bk}(g) = \int_{\mathcal{P}^{bk}} 1( g(z^{bk}(\mu)) \neq f^{\text{dam}}(\mu) ) \, w^{bk}(\mu) \, d\mu,
\]

where

- \( \mu = [s_L, s_R, \alpha, \beta, E] \in \mathcal{P}^{bk} \) anticipated configuration;
- \( \mathcal{P}^{bk} \) anticipated configuration set;
- \( \mu \mapsto f^{\text{dam}}(\mu) = \bar{f}^{\text{dam}}(s_L, s_R) \in \{1, \ldots, 4\} \) damage;
- \( \mathcal{F} : \mathbb{R}^{16Q_f} \rightarrow \mathbb{R}^Q \) feature map (to be defined);
- \( \mu \mapsto w^{bk}(\mu) \) user-defined weight (\( \leftrightarrow P_{w^{bk}} \)).
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Simulation-Based Classification

- An example: a microtruss
- Mathematical formulation
- Computational approach
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- Perspectives
Simulation-Based Classification

**Offline stage: ** (before operations)

1. Generate $\mu^1, \ldots, \mu^M \overset{iid}{\sim} P_{w^{bk}}$
2. Generate $D^{bk}_M = \{ z^{bk}(\mu^m), f^{\text{dam}}(\mu^m) \}_{m=1}^M$
3. $[g^*_M] = \text{Supervised-Learning-alg}(D^{bk}_M)$

**Online stage:** (during operations)

1. Acquire the new outputs $\{ A_{i,j}^{\text{exp}}(f^q; \overline{C}) \}_{i,j,q}$.
2. Compute $\overline{z}^{\text{exp}} = \mathcal{F}(A_{i,j}^{\text{exp}}(f^q; \overline{C}))$.
3. Return the label $g^*_M(\overline{z}^{\text{exp}})$.

Taddei, Penn, Yano, Patera, 2016.
Simulation-Based Classification

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Simulation-Based Classification

**Related works:** Farrar et al. (based on experiments); Basudhar, Missoum; Willcox et al.

**Opportunities:** no need to estimate $\mu = [s_L, s_R, \alpha, \beta, E]$ (which includes nuisance variables $\alpha, \beta, E$)
- non-intrusive approach
- (it requires only forward solves)

**Challenge:** generation of $D_{bk}^M$

$\Rightarrow$ Exploit pMOR ($\leftrightarrow$ parametric def of damage) to generate $D_{bk}^M$. 
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**Challenge:** generation of $D_{bk}^M$

$\Rightarrow$ Exploit pMOR (↔ parametric def of damage) to generate $D_{bk}^M$. 
Perspectives: a ship loader model\textsuperscript{2}

\textbf{Cost to build} \[ D_{\text{blk}}^{\text{M}} = M \times Q_f \times \text{cost per simulation} \]

\textbf{FE model} \ ($\approx 5 \cdot 10^6$ dofs)\n\text{cost per simulation} $\approx 43'$\n$M = 10^4$, $Q_f = 10 \Rightarrow 8$ years

\textbf{ROM model} (PR-scRBE)\n\text{cost per simulation} $\approx 5''$\n$M = 10^4$, $Q_f = 10 \Rightarrow 6$ days

$\Rightarrow$ pMOR enables the use of mathematical models in the simulation-based framework.

\textsuperscript{2}Simulations are performed by Akselos S.A. using PR-scRBE.
Offline stage: (before operations)

1. Generate $\mu^1, \ldots, \mu^M \overset{iid}{\sim} P_{w^b_k}$
2.a Construct a ROM for $\mu \in \mathcal{P}_b^k \mapsto z^{b_k}(\mu)$
2.b Use the ROM to generate the dataset $\mathcal{D}_M^{b_k}$
3. $[g^*_M] = \text{Supervised-Learning-alg}(\mathcal{D}_M^{b_k})$

pMOR is employed only in the generation of the dataset;

If $M$ is sufficiently large, the cost of 2.a is negligible compared to the cost of 2.b (many-query context).
Simulation-Based Classification with pMOR

**Offline stage:** (before operations)

1. Generate $\mu^1, \ldots, \mu^M \overset{iid}{\sim} P_{w_{bk}}$
2.a Construct a ROM for $\mu \in P_{bk}^{bk} \mapsto z^{bk}(\mu)$
2.b Use the ROM to generate the dataset $D_M^{bk}$
3. $[g^*_M] = \text{Supervised-Learning-alg}(D_M^{bk})$

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Choice of $\mathcal{P}^{\text{bk}}$

We choose upper bounds for $s_L, s_R$ \textit{a priori}.

We choose lower and upper bounds for $\alpha, \beta, E$ using textbook values and a preliminary experiment for $s_L = s_R = 1$.

(explanation: $\min A^{\text{bk}}_{1,1} = \min_{\mu=(1,1,\alpha,\beta,E)\in\mathcal{P}^{\text{bk}}} A^{\text{bk}}_{1,1}(\mu, f)$)
Choices of the features

Introduce

\[
\begin{align*}
z_{1}^{bk} (\cdot) &= \frac{A_{1,4}^{bk}(\cdot)}{A_{4,4}^{bk}(\cdot)}, \quad z_{2}^{bk} (\cdot) = \frac{A_{2,4}^{bk}(\cdot) + A_{3,4}^{bk}(\cdot)}{A_{1,1}^{bk}(\cdot) + A_{4,1}^{bk}(\cdot)}.
\end{align*}
\]

and define \( z_{\ell}^{bk} (\mu) = [z_{\ell}^{bk}(f^1; \mu), \ldots, z_{\ell}^{bk}(f^{Q_f}; \mu)] \).
Choices of the features: motivation

Rationale: $z_1$ detects asymmetry in the structure; $z_2$ detects added mass on corners.
Classification procedure

Given $z_1^{\text{exp}}, z_2^{\text{exp}}$,

**Level 1:** distinguish between $\{1, 4\}$, $\{2\}$ and $\{3\}$ based on $z_1^{\text{exp}}$;

**Level 2:** if Level 1 returns $\{1, 4\}$, distinguish between $\{1\}$ and $\{4\}$ based on $z_2^{\text{exp}}$.

From the learning perspective,

Level 1 corresponds to a 3way classification problem;
Level 2 corresponds to a 2way classification problem.

**Algorithms used:** SVM, ANN, kNN, decision trees, NMC$^3$.

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$^3$Implementation is based on off-the-shelf Matlab functions.
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Model reduction procedure: Reduced Basis (RB) method

Computational procedure (essential):
Build a ROM for the state $u^{bk}(f; \mu), f \in \mathcal{I}_f, \mu \in \mathcal{P}^{bk}$.
Use the ROM to compute $(f^q, \mu^m) \mapsto A^{bk}_{i,j}(f^q; \mu^m)$ for $m = 1, \ldots, M$ and $q = 1, \ldots, Q_f (= MQ_f$ PDE solves).

Computational summary:
Finite Element (FE): 14670 dof,
≈ 0.18[s] for each PDE query;
Reduced Basis (RB): 20 dof, pre-processing cost ≈ 24[s],
≈ 4.4 \cdot 10^{-3}[s] for each PDE query.

⇒ RB is advantageous if $MQ_f \gtrsim 180$
(we consider $MQ_f \approx 10^5$).
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(we consider \( MQ_f \approx 10^5 \)).
Results (synthetic data)

Test

1. Generate a dataset $\mathcal{D}_{N_{\text{train}}}^{bk}$, $N_{\text{train}} = 10^4$, $Q_f = 9$;
2. Use $M$ points for learning, $N_{\text{train}} - M$ for testing;
3. Average over 100 partitions.

Memo:

$R_{bk}(g) = 0$
$\Rightarrow$ no mistakes.

$R_{bk}(g) = 1$
$\Rightarrow$ always wrong.

Strong dependence on $M \Rightarrow$ importance of pMOR.
Test

1. Consider 5 different experimental system configurations, and perform 3 independent trials ($= 15$ exp datapoints).
2. Train based on $M = 7 \cdot 10^3$ synthetic datapoints.
3. Average over 100 partitions of the synthetic dataset.

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Towards the application to real problems

Challenges

*Parametrization of damage*

damage is a local phenomenon,

⇒ component-based pMOR

*Choice of features*

automated feature identification\(^4\).

\(^4\)In collaboration with Prof. D Bertsimas, C Pawlowski (MIT).
PBDW approach for state estimation

- An example: a thermal patch configuration
- The PBDW approach
- Application to the thermal patch problem
- A priori error analysis
- Application to a synthetic problem
PBDW approach for state estimation

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Thermal patch experiment

**Objective:** estimate the temperature field over the surface $\Omega$. 
Refined goal and experimental apparatus

Practical applications: local probes.

Refined goal: given $\ell_m^{\text{obs}} \approx u^{\text{true}}(x_m^{\text{obs}})$, $x_m^{\text{obs}} \in \Omega$, estimate $u^{\text{true}}$ over $\Omega$.

Our apparatus:
IR camera
Full-field information $\Rightarrow$ performance assessment.
PBDW approach for state estimation

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Mathematical best-knowledge (bk) model

Estimate the steady-state temperature field as

\[
\begin{cases}
-\Delta u^{bk} = 0, & \text{in } \Omega^{bk}, \\
\kappa \partial_n u^{bk} + \gamma (u^{bk} - \Theta^{room}) = C \chi_{\Gamma^{\text{patch}}} & \text{on } \Gamma^{\text{in}}, \\
\kappa \partial_n u^{bk} = 0 & \text{on } \partial\Omega^{bk} \setminus \Gamma^{\text{in}},
\end{cases}
\]

\(\Theta^{room}\) room temperature \((= 20^\circ C)\); 
\(\kappa\) thermal conductivity; 
\(\gamma\) convective heat transfer coefficient; 
\(C\) incoming flux (patch \(\rightarrow\) plate).

\(\Rightarrow \mu := [\gamma/\kappa, C/\kappa] \in \mathcal{P}^{bk}\)
Mathematical best-knowledge (bk) model

\[ \Omega \subset \partial \Omega^{\text{bk}}, \quad \hat{L} = 22.606\text{mm}, \quad \hat{H} = 9.271\text{mm}. \]

\[ \Omega \]

\[ \Gamma^\text{in} \]

\[ \Omega^{\text{bk}} \]

\[ \Gamma^\text{in} \]

\[ \Gamma^\text{patch} \]

\[ \hat{L} \]

\[ \hat{H} \]
Define the bk solution manifold

\[ M^{bk} = \{ u^{bk}(\mu) |_{\Omega} : \mu \in P^{bk} \} \subset U = U(\Omega) \]

\( M^{bk} \) takes into account parametrized uncertainty in the system.

\( M^{bk} \) does not take into account non-parametric uncertainty in the system:
- nonlinear effects due to natural convection,
- heat-exchange between the patch and the sheet.
Given $\mathcal{M}^{\text{bk}}$, define $\mathcal{Z}_N = \text{span}\{\zeta_n\}_{n=1}^N$ such that
\[
\sup_{\mu} \inf_{z} \| u^{\text{bk}}(\mu)|_{\Omega} - z \| \text{ is small.}
\]

Then, given measurements $\ell_1^{\text{obs}}, \ldots, \ell_M^{\text{obs}}$,

**step 1.** find $z^* \in \mathcal{Z}_N$ such that $z^* \approx u^{\text{true}}$

**step 2.** find $\eta^* \in \mathcal{U}$ such that $\eta^* \approx u^{\text{true}} - z^*$

**step 3.** return the state estimate $u^* = z^* + \eta^*$. 
Variational formulation

Given the Hilbert space \( \mathcal{U} = \mathcal{U}(\Omega), \| \cdot \| \), introduce \( \ell_1^o, \ldots, \ell_M^o \in \mathcal{U}' \) such that

\[
\ell_m^{\text{obs}} \approx \ell_m^o(\mathcal{u}^{\text{true}}), \quad m = 1, \ldots, M.
\]

Define \( \mathcal{u}_\xi^* = \mathcal{z}_\xi^* + \mathcal{\eta}_\xi^* \) to minimise

\[
\min_{(\mathcal{z}, \mathcal{\eta}) \in \mathcal{Z}_N \times \mathcal{U}} \xi \| \mathcal{\eta} \|^2 + \frac{1}{M} \sum_{m=1}^{M} \left( \ell_m^o(\mathcal{z} + \mathcal{\eta}) - \ell_m^{\text{obs}} \right)^2.
\]

Computation of \( \mathcal{z}_\xi^* \) corresponds to a weighted LS problem. Computation of \( \mathcal{\eta}_\xi^* \) corresponds to a generalized smoothing problem based on \( \ell_m^{\text{err}} = \ell_m^{\text{obs}} - \ell_m^o(\mathcal{z}_\xi^*) \approx \ell_m^o(\mathcal{u}^{\text{true}} - \mathcal{z}_\xi^*). \)
Variational formulation

Given the Hilbert space \( (\mathcal{U} = \mathcal{U}(\Omega), \| \cdot \|) \), introduce \( \ell^0_1, \ldots, \ell^0_M \in \mathcal{U}' \) such that

\[
\ell^\text{obs}_m \approx \ell^0_m(u^{\text{true}}), \quad m = 1, \ldots, M.
\]

Define \( u^*_\xi = z^*_\xi + \eta^*_\xi \) to minimise

\[
\min_{(z, \eta) \in \mathcal{Z}_N \times \mathcal{U}} \xi \| \eta \|^2 + \frac{1}{M} \sum_{m=1}^{M} \left( \ell^0_m(z + \eta) - \ell^\text{obs}_m \right)^2.
\]

Computation of \( z^*_\xi \) corresponds to a weighted LS problem. Computation of \( \eta^*_\xi \) corresponds to a generalized smoothing problem based on \( \ell^\text{err}_m = \ell^\text{obs}_m - \ell^0_m(z^*_\xi) \approx \ell^0_m(u^{\text{true}} - z^*_\xi) \).
**Terminology:**

- $\mathcal{Z}_N$ background space;
- $z^* \in \mathcal{Z}_N$ deduced background;
- $\eta^*$ update;

$z^*$ addresses parametrized uncertainty in the model, while $\eta^*$ addresses non-parametric uncertainty in the model.

Solution to $\min_{(z,\eta) \in \mathcal{Z}_N \times \mathcal{U}} \cdot$ is simpler than $\min_{(z,\eta) \in \mathcal{M}^{bk} \times \mathcal{U}} \cdot$.

Construction of $\mathcal{Z}_N$ is a pMOR problem.

*data compression*
Interpretation

Terminology:
\[ \mathcal{Z}_N \] background space;
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Construction of \( \mathcal{Z}_N \) is a pMOR problem.

data compression
Solution representation

The update is of the form

\[ \eta^*_{\xi}(\cdot) = \sum_{m=1}^{M} \eta^*_{\xi,m} R_U \ell^o_m(\cdot) \in \mathcal{U}_M : = \text{span}\{R_U \ell^o_m\}_{m=1}^M, \]

where \( R_U : \mathcal{U}' \mapsto \mathcal{U} \) depends on \((\mathcal{U}, \| \cdot \|)\).

For \( \ell^o_m = \delta_{x^o_m} \) and suitable \((\mathcal{U}, \| \cdot \|)\),

\[ R_U \ell^o_m(\cdot) = K_\gamma(\cdot, x^\text{obs}_m) = \phi(\gamma \| \cdot - x^\text{obs}_m \|_2) \Rightarrow \text{connection with Kernel methods}. \]

Contributions

Maday et al, 2015

two-level mechanism to accommodate anticipated/unanticipated uncertainty
use of pMOR to generate $\mathcal{Z}_N$;

This thesis

adaptive selection of $\xi$
    $\Rightarrow$ rigorous treatment of noisy measurements;
adaptive selection of $\| \cdot \|$ for pointwise measurements
    $\Rightarrow$ improved convergence with $M$.

Localized state estimation ($\Omega \subset \Omega^{bk}$, $\mu \in \mathbb{R}^P$, $P \gg 1$); not covered in this talk.
PBDW approach for state estimation

- An example: a thermal patch configuration
- The PBDW approach
- Application to the thermal patch problem
- A priori error analysis
- Application to a synthetic problem
Details

Observations: \( \ell_{m}^{\text{obs}} = u_{m}^{\text{obs}}(x_{i_{m},j_{m}}^{\text{obs}}), \quad (\Rightarrow \ell_{m}^{o} = \delta_{x_{i_{m},j_{m}}^{\text{obs}}}) \) 
\( x_{i_{m},j_{m}}^{\text{obs}} \) center of the \((i_{m},j_{m})\) pixel\(^5\).

Background: \( \{Z_{N}\}_{N} \) generated using the weak-Greedy\(^6\) algorithm;

Kernel:\(^7\) \( K_{\gamma}(x, x') = \phi(\gamma \| x - x' \|_{2}), \quad \phi(r) = (1 - r)^{4}(4r + 1), \quad (U = H^{2.5}(\mathbb{R}^{2})). \)

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\(^5\)The IR camera returns 160 \( \times \) 120 pixel-wise measurements.
\(^6\)G Rozza, DBP Huynh, AT Patera, 2008.
\(^7\)H Wendland, 2004.
Numerical results ($N = 2$, $M = 25$): step 1

step 1. find $z^* \in \mathcal{Z}_N$ such that $z^* \approx u^{\text{true}}$
Numerical results ($N = 2$, $M = 25$): step 2

**step 2.** find $\eta^* \in U$ such that $\eta^* \approx u^{true} - z^*$
Numerical results \((N = 2, \ M = 25)\): step 3

**step 3.** return the state estimate \(u^\star = z^\star + \eta^\star\).
step 3. return the state estimate \( u^* = z^* + \eta^* \).
PBDW approach for state estimation

- An example: a thermal patch configuration
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Suppose
\[ y_m = u^{\text{true}}(x_{m}^{\text{obs}}) + \epsilon_m, \quad m = 1, \ldots, M. \]

Define the fill distance:
\[ h_M := \sup_{x \in \Omega} \min_m \| x - x_{m}^{\text{obs}} \|_2; \]

Suppose quasi-uniform grid:
\[ h_M \sim M^{-1/d}, \quad \Omega \subset \mathbb{R}^d. \]

Systematic noise: \[ |\epsilon_m| \leq \delta \]

Homoscedastic noise: \[ \epsilon_m \overset{iid}{\sim} (0, \sigma^2) \]
A priori error analysis: $|\epsilon_m| \leq \delta$

Suppose: $\mathcal{U} = H^\tau(\mathbb{R}^d)$, $\tau > d/2$, $u^{\text{true}} \in \mathcal{U}$, $\mathcal{Z}_N \subset \mathcal{U}$;

$h_M \sim M^{-1/d}$;

$$\Rightarrow \left\| u^{\text{true}} - u^*_\xi \right\|_{L^2(\Omega)}^2 \leq C_N \left( h_M^{2\tau} \left( 2 \left\| \Pi_{\mathcal{Z}_N^\perp} u^{\text{true}} \right\|_{\mathcal{U}} + \frac{\delta}{2} \frac{1}{\sqrt{\xi}} \right)^2 \right.$$

$$+ \left( \delta + \frac{\sqrt{\xi}}{2} \left\| \Pi_{\mathcal{Z}_N^\perp} u^{\text{true}} \right\|_{\mathcal{U}}^2 \right) \right)$$

$$\xi^{\text{opt}} = \left( \frac{\delta}{\left\| \Pi_{\mathcal{Z}_N^\perp} u^{\text{true}} \right\|_{\mathcal{U}}} h_M^{2\tau} \right)^{2/3}$$

If $\delta = 0$ $\Rightarrow$ $\left\| u^{\text{true}} - u^*_{\xi,\gamma} \right\|_{L^2(\Omega)}^2 \leq C_N \left\| \Pi_{\mathcal{Z}_N^\perp} u^{\text{true}} \right\|_{\mathcal{U}}^2 \left( h_M^{2\tau} + \xi \right)$

$\mathcal{Z}_N = \emptyset$ $\Rightarrow$ J Krebs, A Louis, H Wendland, 2009.
A priori error analysis: $|\epsilon_m| \leq \delta$

Suppose: $\mathcal{U} = H^\tau(\mathbb{R}^d)$, $\tau > d/2$, $u^{\text{true}} \in \mathcal{U}$, $\mathcal{Z}_N \subset \mathcal{U}$; $h_M \sim M^{-1/d}$;

$\Rightarrow \|u^{\text{true}} - u^{\star}{_\xi}\|^2_{L^2(\Omega)} \leq C_N \left(h_M^{2\tau} (2\|\Pi_{\mathcal{Z}_N^\perp}u^{\text{true}}\|_U + \frac{\delta}{2}\frac{1}{\sqrt{\xi}})^2 + (\delta + \frac{\sqrt{\xi}}{2}\|\Pi_{\mathcal{Z}_N^\perp}u^{\text{true}}\|_U)^2 \right)$

$\xi^{\text{opt}} = \left(\frac{\delta}{\|\Pi_{\mathcal{Z}_N^\perp}u^{\text{true}}\|_U} h_M^{2\tau}\right)^{2/3}$

If $\delta = 0 \Rightarrow \|u^{\text{true}} - u^{\star}{_\xi,\gamma}\|^2_{L^2(\Omega)} \leq C_N \|\Pi_{\mathcal{Z}_N^\perp}u^{\text{true}}\|^2_U (h_M^{2\tau} + \xi)$

$\mathcal{Z}_N = \emptyset \Rightarrow$ J Krebs, A Louis, H Wendland, 2009.
A priori error analysis: $\epsilon_m \sim (0, \sigma^2)$ i.i.d.

Suppose: $\mathcal{U} = H^\tau(\mathbb{R}^d)$, $\tau > d/2$, $u^\text{true} \in \mathcal{U}$, $\mathcal{Z}_N \subset \mathcal{U}$; $h_M \sim M^{-1/d}$.

$$\Rightarrow \mathbb{E} \left[\|u^\text{true} - u_\xi^*\|_{L^2(\Omega)}^2\right] \leq C_N (h_M^{2\tau} + \xi) \|\Pi_{\mathcal{Z}_N^\perp} u^\text{true}\|_{\mathcal{U}}^2 + 2\sigma^2 \mathcal{T}_{N,M}^\sigma(\xi)$$

where $\mathcal{T}_{N,M}^\sigma(\xi)$ can be computed explicitly.

If $u^\text{true} \in \mathcal{Z}_N \Rightarrow \mathbb{E} \left[\|u^\text{true} - u_{\xi,\gamma}^*\|_{L^2(\Omega)}^2\right] = \sigma^2 \mathcal{T}_{N,M}^\sigma(\xi)$

Empirical studies show that $\mathcal{T}_{N,M}^\sigma(\xi)$ is monotonic decreasing in $\xi$. 
PBDW approach for state estimation

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An acoustic model problem

Let \( u_g(\mu) \) be the solution to

\[
\begin{cases}
-(1 + \epsilon \mu i) \Delta u_g(\mu) - \mu^2 u_g(\mu) = \mu(x_1^2 + e^{x_2}) + \mu g & \text{in } \Omega \\
\partial_n u_g(\mu) = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( \epsilon = 10^{-2} \) and \( \mu \in P^{bk} = [2, 10] \).

**Perfect model:** \( u^{true}(\mu) = u_{g_0}(\mu), \ u^{bk}(\mu) = u_{g_0}(\mu) \);

**Imperfect model:** \( u^{true}(\mu) = u_{\bar{g}}(\mu), \ u^{bk}(\mu) = u_{g_0}(\mu) \).

\( g_0 \equiv 0, \ \bar{g}(x) = 0.5(e^{x_1} + \cos(1.3\pi x_2)) \).
Observations: $y_\ell = u_\text{true}(x_\ell^{\text{obs}}) + \epsilon_\ell, \epsilon_\ell \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$;

Centers: $\{x_m^{\text{obs}}\}_m$ deterministic (equispaced),
$\{x_i^{\text{obs}}\}_i$ drawn randomly (uniform), $I = M/2$;

Background: $\{Z_N\}_N$ generated using the weak-Greedy algorithm;

Kernel: $K_\gamma(x, x') = \phi(\gamma \|x - x'\|_2)$,
$\phi(r) = (1 - r)^4(4r + 1), (\mathcal{U} = H^{2.5}(\mathbb{R}^2))$.

G Rozza, DBP Huynh, AT Patera, 2008;
Measure of performances

We introduce

\[ E_{\text{avg}}^{\text{rel}} = \frac{1}{|\mathcal{P}_{\text{train}}^{\text{bk}}|} \sum_{\mu \in \mathcal{P}_{\text{train}}^{\text{bk}}} \frac{\| u^{\text{true}}(\mu) - u^{\text{opt}}_\xi(\mu) \|_{L^2(\Omega)}}{\| u^{\text{true}}(\mu) \|_{L^2(\Omega)}} , \]

\[ \mathcal{P}_{\text{train}}^{\text{bk}} \subset [2, 10] . \]

if \( \sigma > 0 \) (noisy measurements), computations of \( \| u^{\text{true}}(\mu) - u^{\text{opt}}_\xi(\mu) \|_{L^2(\Omega)} \) are averaged over \( K = 24 \) trials.
Results: $M$ convergence ($\sigma = 0, \ g = \bar{g}$)

$E_{\text{avg}}^{\text{rel}} \sim M^{-1.3} - M^{-1.5}, \ |P_{\text{train}}^{\text{bk}}| = 20$

Multiplicative effect between $M$ and $N$ convergence.
Results: $M$ convergence ($N = 5$, $\sigma > 0$, $g = \bar{g}$)

$$E_{\text{avg}}^{\text{rel}} \sim M^{-0.4} - M^{-0.5}, \ |P_{\text{train}}^{\text{bk}}| = 1, \ \mu = 6.6;$$

Adaptation in $\xi$ allows us to deal with noisy measurements.
Conclusions
Summary

pMOR techniques for

1. data compression and
2. offline/online computational decomposition

offer new opportunities for the integration of μPDEs and data.

We relied on pMOR techniques to develop two Data Assimilation strategies for systems modeled by PDEs.
Summary

**PBDW for state estimation:**
- two-level procedure to address parametric and non-parametric uncertainty
- pMOR employed to construct $\mathcal{Z}_N$

**SBC for damage identification:**
- simulation-based approach for discrete-valued QOIs
- pMOR procedure for rapid generation of $\mathcal{D}_{bk}^M$
  
  *data compression*

*offline/online decomposition*
Thank you for the attention!
Backup slides

- Choice of the features
- Explanation of the Table
- $H^1$-PBDW vs A-PBDW
- Localised state estimation
- Choice of $P^b_k$ for thermal patch
Backup slides

- Choice of the features
- Explanation of the Table
- $H^1$-PBDW vs A-PBDW
- Localised state estimation
- Choice of $P^{bk}$ for thermal patch
Choices of the features

Introduce

\[ z_{1}^{bk} (\cdot) = \frac{A_{1,4}^{bk} (\cdot)}{A_{4,4}^{bk} (\cdot)}, \quad z_{2}^{bk} (\cdot) = \frac{A_{2,4}^{bk} (\cdot) + A_{3,4}^{bk} (\cdot)}{A_{1,1}^{bk} (\cdot) + A_{4,1}^{bk} (\cdot)}. \]

and define \( z_{\ell}^{bk} (\mu) = [z_{\ell}^{bk} (f^{1}; \mu), \ldots, z_{\ell}^{bk} (f^{Q_{f}}; \mu)] \).

Diagram: 4x4 grid with labels and annotations.
Feature visualization: $z_1$ and $z_2$

Rationale:  
- $z_1$ detects asymmetry in the structure;
- $z_2$ detects added mass on corners.
Feature visualization: $z_1$

**Rationale:** $z_1$ detects asymmetry in the structure; $z_2$ detects added mass on corners.
Feature visualization: $z_2$

**Rationale:**

- $z_1$ detects asymmetry in the structure;
- $z_2$ detects added mass on corners.
Backup slides

- Choice of the features
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- Choice of $\mathcal{P}^{bk}$ for thermal patch
Explanation of the table

For \( i = 1, \ldots, 100 \)

Partition the dataset \( D_{N_{\text{train}}}^{bk} \) into \( D_{M}^{bk} \) and \( D_{N_{\text{train}}-M}^{bk} \)

Train the learning algorithm based on \( D_{M}^{bk} \)

Test the learning algorithm based on \( D_{N_{\text{train}}-M}^{bk} \rightarrow R_{i}^{bk} \)

Test the learning algorithm based on \( D_{15}^{exp} \rightarrow R_{i}^{exp} \)

EndFor

Return \( R^{bk} = \frac{1}{100} \sum_{i=1}^{100} R_{i}^{bk} \)

Return \( R^{exp} = \frac{1}{100} \sum_{i=1}^{100} R_{i}^{exp} \)
Backup slides

- Choice of the features
- Explanation of the Table
- $H^1$-PBDW vs A-PBDW
- Localised state estimation
- Choice of $P^b_k$ for thermal patch
Results \((N = 5, \sigma = 0, g = g_0)\)

\(H^1\)-PBDW: \(U = H^1(\Omega), \ell_{m}^{\text{obs}} = \text{Gauss}(u^{\text{true}}, x_{m}^{\text{obs}}, r_{\text{Gauss}})\)

A-PBDW: \(U = H^1(\Omega), \ell_{m}^{\text{obs}} = u^{\text{true}}(x_{m}^{\text{obs}})\)
Results \( (N = 5, \sigma = 0, g = \bar{g}) \)

\[ H^1-\text{PBDW}: \mathcal{U} = H^1(\Omega), \ell_m^{\text{obs}} = \text{Gauss}(u_{\text{true}}, x_m^{\text{obs}}, r_{\text{Gauss}}) \]

\[ \text{A-PBDW}: \mathcal{U} = H^1(\Omega), \ell_m^{\text{obs}} = u_{\text{true}}(x_m^{\text{obs}}) \]
Backup slides

- Choice of the features
- Explanation of the Table
- $H^1$-PBDW vs A-PBDW
- Localised state estimation
- Choice of $\mathcal{P}^{bk}$ for thermal patch
Objective: estimate the state in a subregion $\Omega$ of the original domain $\Omega^{pb}$. 
Localised state estimation (Chapter 5)

**Strategy:** restrict computations to $\Omega^{bk}, \Omega \subset \Omega^{bk} \subset \Omega^{pb}$.

uncertainty in global inputs $\Rightarrow$ uncertainty at ports.

Solution manifold

$$\mathcal{M}^{bk} = \{ u^{bk}_g(\mu) |_{\Omega} : \mu \in \mathcal{P}^{bk}, g \in \mathcal{T} \}$$

**Refined objective:** determine rapidly convergent spaces $\mathcal{Z}_N$ to approximate $\mathcal{M}^{bk}$

**Fundamental question:** is the manifold reducible? (↔ evanescence);

**Challenge:** $\mathcal{P}^{bk} \times \mathcal{T}$ is infinite-dimensional.
Localised state estimation (Chapter 5)

Strategy: restrict computations to $\Omega^{bk}$, $\Omega \subset \Omega^{bk} \subset \Omega^{pb}$.

uncertainty in global inputs $\Rightarrow$ uncertainty at ports.

Solution manifold

$$\mathcal{M}^{bk} = \left\{ u^{bk}_g(\mu) \big| \Omega : \begin{array}{c} \mu \in \mathcal{P}^{bk} \\ g \in \mathcal{T} \end{array} \right\}$$

parameters boundary conditions

Refined objective: determine rapidly convergent spaces $\mathcal{Z}_N$ to approximate $\mathcal{M}^{bk}$

Fundamental question: is the manifold reducible? (↔ evanescence);

Challenge: $\mathcal{P}^{bk} \times \mathcal{T}$ is infinite-dimensional.
Backup slides

- Choice of the features
- Explanation of the Table
- $H^1$-PBDW vs A-PBDW
- Localised state estimation
- Choice of $P^{bk}$ for thermal patch
Thermal patch: choice of $\mathcal{P}^{bk}$

$$\mu := [\mu_1 = \gamma / \kappa, \mu_2 = C / \kappa]$$

$u^{bk}$ is linear in $C / \kappa \Rightarrow$ no need to estimate $\mu_2$

$$\kappa = 0.2 \text{W}/(\text{m} \cdot \text{K})$$ thermal conductivity of acrylic,

$$\gamma = \frac{Nu \kappa_{\text{air}}}{\hat{L}} \approx 10 \pm 5 \text{W/m}^2,$$

$$\kappa_{\text{air}} = 0.0257 \text{W}/(\text{m} \cdot \text{K})$$ thermal conductivity of air,

$$Nu = 0.59 (Ra)^{1/4}$$ Nusselt number,

$$Ra = \frac{\beta g \Delta \Theta \hat{L}^3}{\nu \alpha}$$ Rayleigh number

$$g = 9.8 \text{m/s}^2, \Delta \Theta = 50^\circ \text{K}, \hat{L} = 22.606 \text{mm},$$

$$\beta = 1/300 \text{K}^{-1}$$ thermal expansion coefficient,

$$\alpha = 1.9 \cdot 10^{-5} \text{m}^2/\text{s}$$ thermal diffusivity coefficient of air,

$$\nu = 1.81 \cdot 10^{-5} \text{m}^2/\text{s}$$ kinematic viscosity of air.