Registration-based model reduction of parameterized PDEs with sharp gradients

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inventeurs du monde numérique

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Acknowledgements

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Objective

Parameterized Model Order Reduction (pMOR) for PDEs

The goal of pMOR is to reduce the **marginal** cost associated with the solution to parameterized problems.

pMOR is motivated by *real-time* and *many-query* problems
design and optimization, UQ, control

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Given the **manifold** $\mathcal{M} = \{u_\mu : \mu \in \mathcal{P}\} \subset \mathcal{X}$,
where $\mathcal{P} \subset \mathbb{R}^P$ is a compact set, and
$(\mathcal{X}, \|\cdot\|)$ is an Hilbert space over $\Omega \subset \mathbb{R}^d$,

the goal of pMOR is to determine a **low-rank approximation** $\widehat{u}_\mu$ of u_μ that can be rapidly computed for any $\mu \in \mathcal{P}$.

pMOR: general recipe

Pb: find $u_\mu \in \mathcal{X} : A_\mu(u_\mu, v) = F(v) \qquad \forall\, v \in \mathcal{Y}\ \mu \in \mathcal{P}$

Approx: $\hat{u}_\mu = \sum_{n=1}^{N} \widehat{\alpha}_\mu^n \zeta_n, \qquad \widehat{\alpha}^n : \mathcal{P} \to \mathbb{R},\ \zeta_n \in \mathcal{X}$

$N \ll N_{\mathrm{hf}} =$ dofs of the Full Order Model $\Big(\underbrace{\mathrm{FOM}}_{=\mathrm{FE},\ldots}\Big)$

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Offline stage: (performed once)

compute $u_{\mu^1}, \ldots, u_{\mu^{n_{\text{train}}}}$ using a FE (or FV...) solver;

construct $\{\zeta_n\}_{n=1}^N$ and define $\mathcal{Z}_N = \mathrm{span}\{\zeta_n\}_{n=1}^N$.

Online stage: (performed for any new $\bar{\mu} \in \mathcal{P}$)

estimate the solution coefficients $\widehat{\alpha}_{\bar{\mu}}^1, \ldots, \widehat{\alpha}_{\bar{\mu}}^N$.

estimate $\| \hat{u}_{\bar{\mu}} - u_{\bar{\mu}} \|$.

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Successful applications of pMOR methods

Past and current research on pMOR focuses on

1. data compression $\mathcal{Z}_N$
2. generalization $\mathcal{Z}_N \Rightarrow \widehat{\boldsymbol{\alpha}}_{\bar{\mu}}$
3. *a posteriori* error estimation $\|\hat{u}_{\bar{\mu}} - u_{\bar{\mu}}\|$

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PR-scRBE: Patera, Huynh, Knezevic, `Akselos S.A.`

Port-**R**educed **s**tatic **c**ondensation **RB E**lement method

component-based structures, solid mechanics.

LRB-Ms: Ohlberger, Schindler, ...

Localized **RB M**ulti**s**cale method

multiscale problems, porous media.

Akselos is a software company that provides a commercial implementation of PR-scRBE.

Data compression

Challenges: turbulence (wide spectrum of scales), approximation of shocks, boundary/internal layers...

nonlinear approximation procedures.

Generalization

Challenges: fragility of Galerkin models, nonlinearities.

stabilized formulations;
hyper-reduction.

Error estimation

Challenge: need for estimates of averaged QOIs.

time-averaged error indicators.

Aim of this talk: focus on data compression

Abstract goal: given snapshots $\{u^k\}_{k=1}^{n_{\rm train}}$, determine a low-rank approximation of $u : \mathcal{P} \to \mathcal{X}$.

Key feature: registration.

Related questions: not covered in this talk.

1. Adaptive sampling; choice of $\{\mu^k\}_k$
2. Online prediction of u_μ for given $\mu \in \mathcal{P}$.

Taddei; *A registration method for model order reduction: data compression and geometry reduction; SISC, 2020*.

Taddei, Zhang; *Space-time registration-based model reduction of parameterized one-dimensional hyperbolic PDEs;* submitted, 2020.

Iollo, Taddei, Zhang; *Registration-based model reduction in complex two-dimensional geometries;* in preparation, 2020.

Registration-based data compression

Linear compression methods

Seek approximations s.t. $\widehat{u}_\mu := Z_N \, \widehat{\boldsymbol{\alpha}}_\mu = \sum_{n=1}^{N} (\widehat{\boldsymbol{\alpha}}_\mu)_n \, \zeta_n$.

Z_N is learned through the snapshots $\{u_{\mu^k}\}_{k=1}^{n_{\mathrm{train}}} \subset \mathcal{M}$.

strong/weak-Greedy, POD,...

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strong/weak-Greedy, POD,...

Notation: $\mathcal{Z}_N = \operatorname{span}\{\zeta_n\}_{n=1}^{N}$ reduced space;

$Z_N : \boldsymbol{\alpha} \mapsto \sum_n \boldsymbol{\alpha}_n \zeta_n$ reduced operator.

Linear compression methods naturally fit within the standard **variational framework**.

$$A_\mu(Z_N \widehat{\boldsymbol{\alpha}}_\mu, Z_N \boldsymbol{\alpha}) = F(Z_N \boldsymbol{\alpha}) \quad \forall \boldsymbol{\alpha} \in \mathbb{R}^N$$

Galerkin projection.

$$\left[Z_N, \{\boldsymbol{\alpha}^k\}_k\right] = \texttt{POD}\left(\{u^k\}_k, (\cdot,\cdot), N\right), \qquad Z_N \boldsymbol{\alpha}^k = \Pi_{\mathcal{Z}_N} u^k$$

Method of snapshots (Sirovich, 1987)

1. Compute the Gramian matrix $\mathbf{C} \in \mathbb{R}^{n_{\text{train}} \times n_{\text{train}}}$,
$$\mathbf{C}_{k,k'} = (u^k, u^{k'})$$
2. Solve the eigenproblem: $\mathbf{C}\boldsymbol{\zeta}_n = \lambda_n \boldsymbol{\zeta}_n$, $\lambda_1 \geq \lambda_2 \geq \ldots$
3. Return the linear operator $Z_N = [\zeta_1, \ldots, \zeta_N]$, where
$$\zeta_n = \sum_{k=1}^{n_{\text{train}}} (\boldsymbol{\zeta}_n)_k \, u^k, \ \|\zeta_1\| = \ldots = \|\zeta_N\| = 1,$$

and
$$\{\boldsymbol{\alpha}^k\}_{k=1}^{n_{\text{train}}}, \ \text{ s.t. } \ \left(\boldsymbol{\alpha}^k\right)_n := (\zeta_n, u^k), \ \ n = 1, \ldots, N.$$

Equivalence with other methods: SVD, Karhunen–Loève expansion, principal component analysis.

Optimality:[1] The space $\mathcal{Z}_N$ satisfies

$$\mathcal{Z}_N \in \arg \inf_{\mathcal{W}\subset\mathcal{X}, \dim(\mathcal{W})=N} \sum_{k=1}^{n_{\mathrm{train}}} \|\Pi_{\mathcal{W}^\perp} u^k\|^2.$$

Furthermore, $\displaystyle\sum_{k=1}^{n_{\mathrm{train}}} \|\Pi_{\mathcal{Z}_N^\perp} u^k\|^2 = \sum_{n=N+1}^{n_{\mathrm{train}}} \lambda_n.$

Practical performance: if $u \in C^\infty(\mathcal{P};\mathcal{X})$, the eigenvalues λ_n are expected to decay exponentially[2].

[1] Volkwein, 2011. Schmidt-Eckart-Young theorem.

[2] See Cohen, DeVore, Schwab, 2010 for the analysis.

Inadequacy of linear compression methods

Consider the parametric field

$$u_\mu(x) = \mathrm{sign}\,(x-\mu),\ x \in \Omega = (0,1),\ \mu \in \mathcal{P} = \left[\tfrac{1}{3}, \tfrac{2}{3}\right].$$

Then, $\displaystyle \inf_{\mathcal{W}\subset\mathcal{X},\dim(\mathcal{W})=N} \sup_{\mu\in\mathcal{P}} \|\Pi_{\mathcal{W}^\perp} u_\mu\|_{L^2(\Omega)} = \mathcal{O}\left(N^{-1/2}\right).$

Linear methods are ill-suited to deal with traveling fronts.

Taddei, Perotto, Quarteroni, 2015; Ohlberger, Rave, 2015.

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Reformulation based on mappings

If we introduce the affine bijection $\Phi_\mu : \Omega \to \Omega$,

$$\Phi_\mu(X) = X + \left(\mu - \tfrac{1}{2}\right)\left(1 - \left|2X - 1\right|\right)$$

we have that $u_\mu \circ \Phi_\mu = \operatorname{sign}(2X-1)$ is μ-independent .

$\Rightarrow$ Registration-based nonlinear compression

Taddei, Perotto, Quarteroni, 2015; Ohlberger, Rave, 2015.

Two-level approximations

We seek approximations of the form

$$u_\mu \approx \widehat{u}_\mu \circ \underline{\Phi}_\mu^{-1} \text{ where } \widehat{u}_\mu = Z_N \widehat{\boldsymbol{\alpha}}_\mu,\ \underline{\Phi}_\mu = \mathtt{id} + W_M \widehat{\mathbf{a}}_\mu.$$

The mapping $\underline{\Phi}_\mu$ should be a bijection in Ω for all $\mu \in \mathcal{P}$ and should make the mapped manifold

$$\widetilde{\mathcal{M}} := \left\{ u_\mu \circ \underline{\Phi}_\mu : \mu \in \mathcal{P} \right\}$$

more amenable for linear compression methods (e.g., POD).

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A few references:
Ohlberger, Rave, 2013; Iollo, Lombardi, 2014; Taddei, Perotto, Quarteroni, 2015; Mojgani, Balajewicz, 2017; Mowlavi, Sapsis, 2018.

Objective develop a *general* registration-based generalization of POD.

$$\left[Z_N, \{\boldsymbol{\alpha}^k\}_k\right] = \texttt{POD}\left(\{u^k\}_k, (\cdot,\cdot), N\right) \Rightarrow$$

$$\left[Z_N, W_M, \{\boldsymbol{\alpha}^k\}_k, \{\mathbf{a}^k\}_k\right] = \texttt{RePOD}\left(\{u^k\}_k, (\cdot,\cdot), N, M\right).$$

Agenda:

1. Registration for $\Omega = (0,1)^2$.
2. Application to 1D shallow water equations.
3. Beyond rectangular domains.
3. Conclusions and perspectives.

General = independent of the underlying PDE model.

Generalization

Task: given Z_N, W_M, how can we compute $\boldsymbol{\alpha}_\mu, \mathbf{a}_\mu$?

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Consider the problem: find $u_\mu \in \mathcal{X} = H_0^1(\Omega)$ s.t.

$$\int_\Omega \underline{\underline{K}}_\mu \nabla u_\mu \cdot \nabla v d\underline{x} = \int_\Omega f_\mu v \, d\underline{x} \;\; \forall \; v \in \mathcal{X}.$$

Then, $\tilde{u}_\mu = u_\mu \circ \underline{\Phi}_\mu$ solves $\quad (\underline{\underline{G}}_\mu = \nabla \underline{\Phi}_\mu, \; g_\mu = \det(\underline{\underline{G}}_\mu))$

$$\int_\Omega \underline{\underline{\widetilde{K}}}_\mu \nabla \tilde{u}_\mu \cdot \nabla v d\underline{x} = \int_\Omega \widetilde{f}_\mu v \, d\underline{x} \;\; \forall \; v \in \mathcal{X},$$

with $\underline{\underline{\widetilde{K}}}_\mu = g_\mu \underline{\underline{G}}_\mu^{-1} \left(\underline{\underline{K}}_\mu \circ \underline{\Phi}_\mu \right) \underline{\underline{G}}_\mu^{-T}$ and $\widetilde{f}_\mu = g_\mu \left(f_\mu \circ \underline{\Phi}_\mu \right)$.

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Projection-based methods can be used for the approximation of $\widetilde{u}_\mu$ **as is**.

Simultaneous approximation of mapping and solution is also possible. Zahr, Persson, 2018 (DG framework).

Registration for $\Omega = (0, 1)^2$

Plan

Inputs: solution snapshots $\{u^k\}_k$.

Outputs: $Z_N, W_M, \{\boldsymbol{\alpha}^k\}_k, \{\mathbf{a}^k\}_k$ s.t. $u^k \circ \underline{\Phi}^k \approx \widehat{u}^k$,

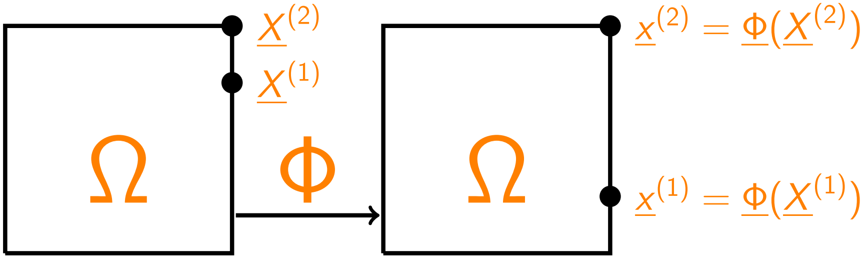
$$\underline{\Phi}^k = \mathtt{id} + \underline{\varphi}^k, \;\; \underline{\varphi}^k = W_M \mathbf{a}^k, \;\; \widehat{u}^k = Z_N \boldsymbol{\alpha}^k.$$

1. Characterize a set of admissible mappings.
2. **Optimization-based registration.** Given $\mathcal{Z}_N$ and u^k, determine $\underline{\Phi}^k$.
3. **Parametric registration.** Use 2 to simultaneously build $\mathcal{Z}_N$ and the mappings $\{\underline{\Phi}^k\}_k$.

A class of admissible mappings: theoretical rationale

Consider $\underline{\Phi} = \texttt{id} + \underline{\varphi}$ where $\underline{\varphi} \in C^1$, $\underline{\varphi} \cdot \underline{n}\big|_{\partial\Omega} = 0$. Then, $\underline{\Phi}$ is bijective in Ω if $\inf_{\underline{x}\in\Omega} g(\underline{x}) := \det(\nabla\underline{\Phi}(\underline{x})) > 0$.

Condition $\underline{\varphi} \cdot \underline{n}\big|_{\partial\Omega} = 0$ allows tangential displacements.



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We consider a space of tensorized polynomials of degree $J+1$, that is $\underline{\varphi} \in \mathcal{W}_{\mathrm{hf}}$, $\dim(\mathcal{W}_{\mathrm{hf}}) = M_{\mathrm{hf}} = 2J^2$.

We replace the constraint $\inf_{\underline{x}\in\Omega} g(\underline{x}) > 0$ with $\mathcal{C}(\underline{\varphi}) :=$

$$\int_\Omega \exp\left(\frac{\epsilon - g(\underline{x})}{C_{\exp}}\right) + \exp\left(\frac{g(\underline{x}) - 1/\epsilon}{C_{\exp}}\right)\, d\underline{x} - \delta \leq 0,$$

which provides a sufficient condition for bijectivity, for $\exp(\frac{\epsilon}{C_{\exp}}) \gg 1$ and moderate $\|\nabla g\|_{L^\infty(\Omega)}$.

Optimization-based registration

Given the target $u \in \mathcal{X}$, the spaces $\mathcal{Z}_N \subset L^2(\Omega)$, $\mathcal{W}_M \subset \mathcal{W}_{\rm hf}$, we seek $\underline{\Phi} = \texttt{id} + \underline{\varphi}$ to minimize

$$\left(\min_{\psi \in \mathcal{Z}_N} \| u \circ \underline{\Phi} - \psi \|^2_{L^2(\Omega)} \right) + \xi \left| \underline{\varphi} \right|^2_{H^2(\Omega)},$$

subject to $\mathcal{C}(\underline{\varphi}) \leq 0$.

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subject to $\mathcal{C}(\underline{\varphi}) \leq 0$.

$$\mathfrak{f}(\underline{\Phi}; u) := \min_{\psi \in \mathcal{Z}_N} \| u \circ \underline{\Phi} - \psi \|^2_{L^2(\Omega)}$$ **proximity measure**

measures approximability of the target in the mapped domain.

$\xi \left| \underline{\varphi} \right|^2_{H^2(\Omega)}$ is a regularization term to bound gradient and Hessian of $\underline{\varphi}$ (and thus ∇g).

Parametric registration $\{\underline{\Phi}^{\star,k}\}_k, \mathcal{Z}_N \leftarrow \{u^k\}_k, \mathcal{Z}_{N_0}, \mathcal{W}_{\rm hf}$

1. Set $\mathcal{Z}_{N=N_0} = \mathcal{Z}_{N_0}$, $\mathcal{W}_M = \mathcal{W}_{\rm hf}$.

`For` $N = N_0, \ldots, N_{\rm max} - 1$

2. $[\underline{\varphi}^{\star,k}, \mathfrak{f}_{N,M}^{\star,k}] = \texttt{registration}\left(u^k, \mathcal{Z}_N, \mathcal{W}_M\right)$ $\quad k = 1, \ldots, n_{\rm train}$.

3. $[\mathcal{W}_M, \ \{\mathbf{a}^k\}_k] = \texttt{POD}\left(\{\underline{\varphi}^{\star,k}\}_{k=1}^{n_{\rm train}}, tol_{\rm pod}, (\cdot,\cdot)_\star\right)$

`if` $\max_k \mathfrak{f}_{N,M}^{\star,k} < \texttt{tol}$, `break`

`else`

4. $\mathcal{Z}_{N+1} = \mathcal{Z}_N \cup {\rm span}\{u_{\mu^{k^\star}} \circ \underline{\Phi}^{\star,k^\star}\}$ $\quad k^\star = \arg\max_k \mathfrak{f}_{N,M}^{\star,k}$.

`EndIf`

`EndFor`

$$\mathfrak{f}_{N,M}^{\star,k} = \mathfrak{f}(\underline{\Phi}^\star; u^k) := \min_{\psi\in\mathcal{Z}_N} \| u^k \circ \underline{\Phi} - \psi \|_{L^2(\Omega)}^2$$

Parametric registration: remarks

The Greedy procedure simultaneously constructs the space $\mathcal{Z}_N$ and the mappings $\{\underline{\Phi}\}_k$.

If $\mathcal{W}_{\rm hf} = \emptyset$ (no registration), $\Rightarrow$ Strong Greedy.

In practice, the algorithm is applied to the modified snapshots $\{s^k = \mathfrak{s}(u^k)\}_k$ where $\mathfrak{s} : \mathcal{X} \to L^2(\Omega)$ is a **registration sensor.** more on s later.

$\Rightarrow$ We cannot use the algorithm to build $\mathcal{Z}_N$. Instead,

$$\left[Z_N, \{\boldsymbol{\alpha}^k\}_k \right] = \texttt{POD}\left(\{u^k \circ \underline{\Phi}^{\star,k}\}_k, (\cdot,\cdot), N \right),$$

POD reduction inside the for loop

preserves the condition $\underline{\varphi} \in \mathcal{W}_M \Rightarrow \underline{\varphi} \cdot \underline{n}|_{\partial\Omega} = 0$;

reduces dramatically the cost of subsequent iterations.

Application to 1D shallow water equations

Problem statement

Consider the problem: find $\underline{u} = [h, q]^T$ such that

$$\begin{cases} \partial_t \underline{u} + \partial_x \underline{f}(\underline{u}) = -gh\partial_x b \underline{e}_2, & (x,t) \in \Omega = (0,L) \times (0,T) \\ q(0,t) = q_{\mathrm{in},\mu}(t), \; h(L,t) = 2, \; \underline{u}(x,0) = \underline{u}_0(x), \end{cases}$$

with $\underline{f}(\underline{u}) = [q, \frac{q^2}{h} + \frac{g}{2} h^2]^T$, $b(x) = -0.2 + e^{-0.125(x-10)^4}$,

$$q_{\mathrm{in},\mu}(t) \; = \; q_0 \left(1 + \mu_1 \, t \, e^{-\frac{1}{2\mu_2^2}(t-0.05)^2} \right), \quad q_0 = 4.4,$$

$\underline{u}_0$ is the steady-state solution obtained for $q_{\mathrm{in},\mu} \equiv q_0$.

$$\mu = [\mu_1, \mu_2] \in \mathcal{P} = [2,8] \times [0.1, 0.2].$$

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$$\mu = [\mu_1, \mu_2] \in \mathcal{P} = [2, 8] \times [0.1, 0.2].$$

The problem shares relevant features with **dam-break studies** with non-constant bathymetry.

Behavior of the free surface $z = h + b$

Application of the registration procedure

We train our model based on $n_{\mathrm{train}} = 10^2$ samples; we assess performance based on $n_{\mathrm{test}} = 20$ samples.

We consider the **registration sensor** $\mathfrak{s}(\underline{u}) = h$.

We initialize the template space $\mathcal{T}_{N_0=2} = \mathrm{span}\{h_0, h_{\bar{\mu}}\}$, we set $\xi = 10^{-4}, M_{\mathrm{hf}} = 128, tol_{\mathrm{pod}} = 10^{-4}$ $\Rightarrow M = 5$

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Generalization (for out-of-sample μ)

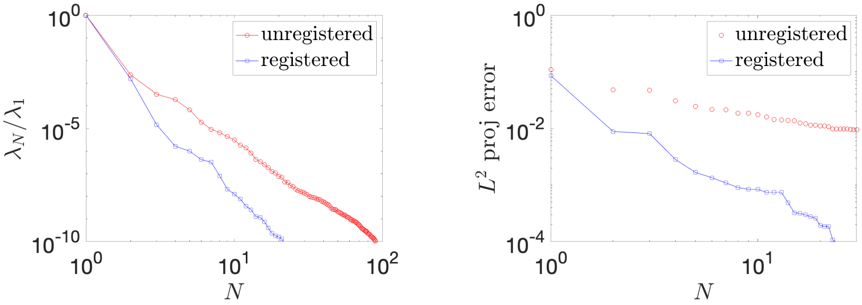
Mapping coefficients: RBF-based regression.
Wendland, 2004.

Solution coefficients: Petrov-Galerkin proj + EQ.
Farhat et al. 2015; Yano, 2019.

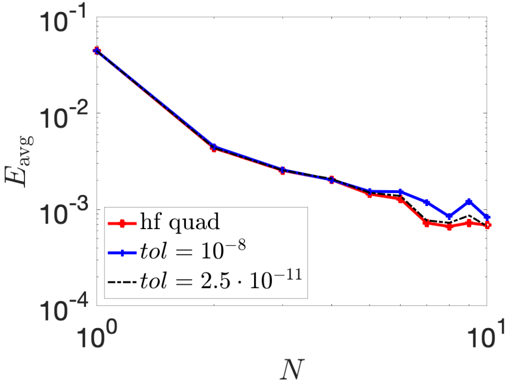
Taddei, Zhang, 2020 (*submitted*).

Behavior of the registered free surface $z = h + b$

Performance of the registration procedure



Performance of the ROM



Beyond rectangular domains

Example: $\Omega = \mathcal{B}_{R=1}(\underline{0})$, consider bijections $\underline{\Phi}_1$, $\underline{\Phi}_2$ and assume that $\underline{\Phi}_1(\underline{x}) \neq \underline{\Phi}_2(\underline{x})$ at $\underline{x} \in \partial\Omega$.

Then, $\underline{\Phi}_t := t\underline{\Phi}_1 + (1-t)\underline{\Phi}_2$ is not a bijection in Ω for any $t \in (0, 1)$.

Extension to non-rectangular domains

Example: $\Omega = \mathcal{B}_{R=1}(\underline{0})$, consider bijections $\underline{\Phi}_1$, $\underline{\Phi}_2$ and assume that $\underline{\Phi}_1(\underline{x}) \neq \underline{\Phi}_2(\underline{x})$ at $\underline{x} \in \partial\Omega$.

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Conclusion: affine mappings — $\underline{\Phi} = \texttt{id} + W_M\mathbf{a}$ — cannot properly capture finite deformations over non-straight edges.

Extension to non-rectangular domains

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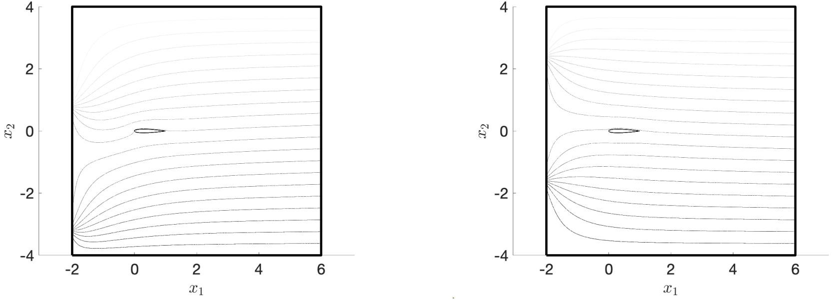
Conclusion: affine mappings — $\underline{\Phi} = \mathtt{id} + W_M\mathbf{a}$ — cannot properly capture finite deformations over non-straight edges.

Question: how can we characterize admissible mappings?

Model problem: potential flow past an airfoil

Consider the problem: $-\Delta u_\mu = 0$ in Ω, $u_\mu|_{\partial\Omega} = h_\mu$,
$\mu = [\mu_1, \mu_2]$, $\Omega = \Omega_{\rm box} \setminus \Omega_{\rm naca}$.



Define $G_{\rm naca}$ s.t. $\partial\Omega_{\rm naca} = \{\underline{x} : G_{\rm naca}(\underline{x}) = 0\}$.

Strategy 1: constrained approach (I)

We consider mappings $\underline{\Phi} = \mathtt{id} + \underline{\varphi}$ over Ω_{box} such that

1. $\underline{\varphi} \cdot \underline{n}|_{\partial\Omega_{\mathrm{box}}} = 0$, $\mathcal{C}(\underline{\varphi}) \leq 0$ — same as before
2. $\sum_i |G_{\mathrm{naca}}(\underline{\Phi}(\underline{x}_i))|^2 - tol \leq 0$. — new
3. $\underline{\Phi}(\underline{x}_j^{\mathrm{fix}}) = \underline{x}_j^{\mathrm{fix}}$. — new

Constraints in 1. enforce bijectivity in Ω_{box}.

Constraint in 2. controls $\max_{\underline{x}\in\partial\Omega} \mathrm{dist}\,(\underline{\Phi}(\underline{x}), \partial\Omega)$.

Constraint in 2 is nonlinear and non-convex $\Rightarrow$
similar per-iteration cost

Constraint in 3. deals with "difficult points".
$\underline{x}_1^{\mathrm{fix}} = [0, 0], [1, 0]$.

Strategy 1: constrained approach (II)

The additional constraints should ultimately control the Hausdorff distance $\mathrm{dist_H}(\underline{\Phi}(\partial\Omega), \partial\Omega) =$

$$\max\left\{ \max_{\underline{x}\in\partial\Omega} \mathrm{dist}\left(\underline{\Phi}(\underline{x}), \partial\Omega\right), \quad \max_{\underline{x}\in\partial\Omega} \mathrm{dist}\left(\underline{x}, \underline{\Phi}(\partial\Omega)\right) \right\}.$$

Theoretical rationale: under proper assumptions on the domain Ω and the mapping $\underline{\Phi}$, we can control $\mathrm{dist_H}(\underline{\Phi}(\partial\Omega), \partial\Omega)$ in terms of $\max_{\underline{x}\in\partial\Omega} \mathrm{dist}\left(\underline{\Phi}(\underline{x}), \partial\Omega\right)$.

Observation: constraint 3 plays a decisive role when there are corners.

Iollo, Taddei, Zhang, (*in preparation*)

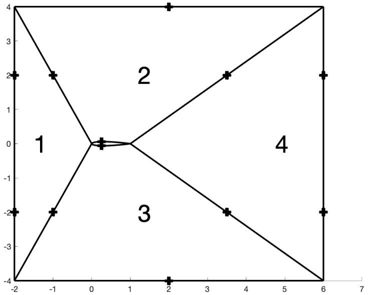
Strategy 2: partitioned approach (I)

Introduce a partition of Ω, $\{\Omega_q\}_{q=1}^{N_{\mathrm{dd}}}$ such that Ω_q is isomorphic to $\widehat{\Omega} = (0,1)^2$.

Consider mappings of the form

$$\underline{\Phi} = \sum_{q=1}^{N_{\mathrm{dd}}} \underline{\Psi}_q \circ \underline{\Phi}_q \circ \underline{\Psi}_q^{-1} \mathbb{1}_{\Omega_q}$$

where $\underline{\Psi}_q : \widehat{\Omega} \to \Omega_q$, and $\underline{\Phi}_q = \mathtt{id} + W_M^q \mathbf{a}$.



$\underline{\Phi}$ should be (i) globally continuous, and (ii) locally bijective, $\underline{\Phi}(\Omega_q) = \Omega_q$, $q = 1, \ldots, N_{\mathrm{dd}}$.

Strategy 2: partitioned approach (II)

Local bijectivity is equivalent to bijectivity of $\underline{\Phi}_q$ in $\widehat{\Omega}$.

admissible class naturally defined.

Implementation borrows several elements from classic isoparametric spectral element discretizations.

KZ Korczak, AT Patera, 1986.

Strategy 2: partitioned approach (II)

Local bijectivity is equivalent to bijectivity of $\underline{\Phi}_q$ in $\widehat{\Omega}$.

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Pro: possibility to approximate exactly the geometry with polynomials of moderate order.

Con: local bijectivity implies global bijectivity but it is a much stronger condition $\Rightarrow$ limited approximation power.

Numerical results: description of the test

Consider $n_{\text{train}} = 50$ snapshots for training and $n_{\text{test}} = 100$ snapshots for testing.

Consider a fully non-intrusive approach (RBF regression for $\widehat{\boldsymbol{\alpha}}_\mu, \widehat{\mathbf{a}}_\mu$).

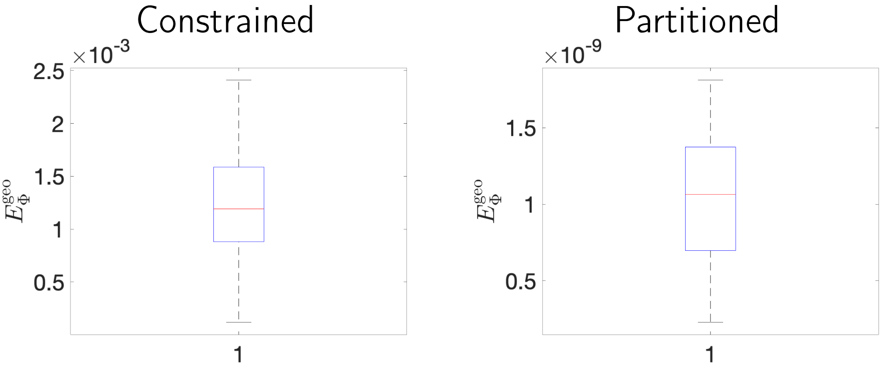
We measure performance using

$$E_{\Phi}^{\text{geo}}(\mu) = \max_{i \in \mathcal{I}_{\text{naca}}} \left| G_{\text{naca}}\left(\underline{\Phi}_\mu(\underline{x}_i^{\text{hf}})\right) \right|,$$

$$E_{\Phi}^{\text{sol}}(\mu) = \frac{\| u_\mu - \widehat{u}_\mu \circ \underline{\Phi}_\mu^{-1} \|_{H^1(\Omega)}}{\| u_\mu \|_{H^1(\Omega)}}.$$

$\{\underline{x}_i^{\text{hf}}\}_{i \in \mathcal{I}_{\text{naca}}}$ nodes of the FE mesh on the airfoil.

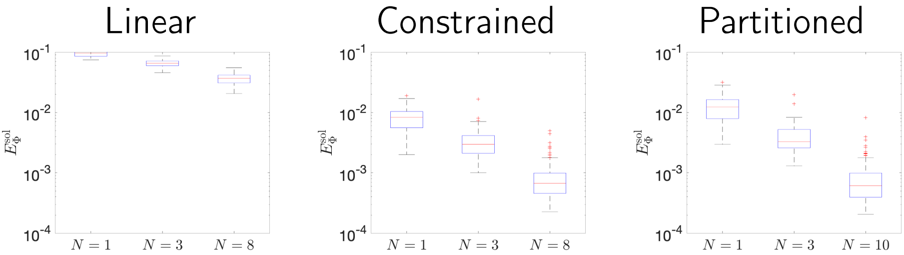
Numerical results: geometrical error



Constrained approach: $tol = 10^{-4}$, $|\mathcal{I}_{\text{naca}}| = 100$, $M_{\text{hf}} = 1250$.

Partitioned approach: $M_{\text{hf}} = 600$.

Numerical results: solution error



Conclusions and perspectives

Summary

We propose a

general = independent of the underlying PDE model

registration-based compression strategy for pMOR,

$$u_\mu \approx \widehat{u}_\mu \circ \underline{\Phi}_\mu^{-1} \text{ with } \widehat{u}_\mu = Z_N \boldsymbol{\alpha}_\mu \text{ and } \underline{\Phi}_\mu = \texttt{id} + W_M \mathbf{a}_\mu$$

We illustrate the application to one-dimensional systems of hyperbolic PDEs (shallow-water equations).

We illustrate the extension to non-rectangular domains (potential flow).

Perspectives

Several theoretical and methodological challenges need to be addressed.

1. Development of fully-intrusive schemes for the simultaneous prediction of $\widehat{\boldsymbol{\alpha}}_{\mu}$ and $\widehat{\mathbf{a}}_{\mu}$.
   Link with Zahr, Persson, JCP, 2018.
2. Investigation of performance for relevant problems. in CFD
   self-similarity, transport.
3. Mathematical analysis.
   for what problems shall registration help?

Geometry reduction

Given the parameterized domains $\{\Omega_\mu\}_{\mu\in\mathcal{P}} \subset \mathbb{R}^d$, the goal of **geometry reduction** is to determine a **low-rank mapping** $\underline{\Phi}$ and a domain $\widehat{\Omega}$ such that

$$\underline{\Phi}_\mu \text{ is invertible in } \widehat{\Omega} \text{ and } \underline{\Phi}_\mu(\widehat{\Omega}) \approx \Omega_\mu, \quad \forall\, \mu \in \mathcal{P}.$$

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pMOR techniques in parameterized domains:

AE Løvgren, Y Maday, and EM Rønquist. M2AN, 2006;

G Rozza, DBP Huynh, AT Patera, ARCME, 2008;

A Manzoni, A Quarteroni, G Rozza, IJNME, 2012.

and many others.

Given the parameterized domains $\{\Omega_\mu\}_{\mu\in\mathcal{P}} \subset \mathbb{R}^d$,
the goal of **geometry reduction** is to determine a **low-rank mapping** $\underline{\Phi}$ and a domain $\widehat{\Omega}$ such that

$$\underline{\Phi}_\mu \text{ is invertible in } \widehat{\Omega} \text{ and } \underline{\Phi}_\mu(\widehat{\Omega}) \approx \Omega_\mu, \quad \forall\, \mu \in \mathcal{P}.$$

This is equivalent to reducing $\mathcal{M}^{\text{geo}} := \{u_\mu := \mathbb{1}_{\Omega_\mu}\}_{\mu\in\mathcal{P}}$ to a singleton $\widehat{u} = \mathbb{1}_{\widehat{\Omega}}$.

Joint work with F Ballarin, E Delgado, A Mola, and G Rozza.

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Data Assimilation and Model Reduction in high-dimensional problems

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Thank you for your attention!

Please visit math.u-bordeaux.fr/~ttaddei/ for further information.