Please name your python file "familyname.py" and send it to `tommaso.taddei@inria.fr`.

Exercice 1

In full generality, s-stage diagonally-implicit Runge Kutta (DIRK) methods for the equation

$$\begin{cases} y'(t) = f(t, y(t)) \quad t \in (0, T_{\max}), \\ y(0) = y_0, \end{cases} \tag{1}$$

can be written as

$$\begin{cases} k_i = f\left(t_n + c_i \Delta t, y_n + \Delta t \sum_{j=1}^{i-1} a_{i,j} k_j + \Delta t a_{i,i} k_i\right) \quad i = 1, \ldots, s; \\ y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i k_i. \end{cases} \tag{2}$$

Here, $\Delta t > 0$ denotes the time step size; y_n is the estimate of the solution y at time $t_n = \Delta t(n-1)$. Note that, at each time step, we should solve s consecutive nonlinear equations for $k_1, \ldots, k_s$.

1. (**2 points**) Implement the function $f(t, y) = 500y^2(1-y)$ and its derivative.
2. (**5 points**) Implement a function that takes as input the scalar quantities $y', \bar{y}, \bar{t}, \alpha \in \mathbb{R}$, and the functions $f, \partial_y f$, and returns the solution to the equation

$$G(k) = k - f\left(\bar{t}, \bar{y} + \alpha k\right) = 0.$$

3. (**5 points**) Implement the two-stage DIRK method: the Python function should take the matrix A and the vectors b, c as inputs.
4. (**4 points**) Apply the two-stage DIRK method to the ODE (1) with $f(t, y) = 500y^2(1-y)$ and $y_0 = 1/100$, $T_{\max} = 1$. Consider the choices

$$A = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}, \quad c = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1/2 + \sqrt{3}/6 & 0 \\ -\sqrt{3}/3 & 1/2 + \sqrt{3}/6 \end{bmatrix}, \quad b = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \quad c = \begin{bmatrix} 1/2 + \sqrt{3}/6 \\ 1/2 - \sqrt{3}/6 \end{bmatrix}.$$

5. (**4 points**) **Convergence and stability analysis.** Apply the DIRK scheme to the ODE for $\Delta t = 2^{-5}, 2^{-6}, \ldots, 2^{-12}$. We denote by $\{y_n^{(k)}\}_{n=0}^{N_{(k)}}$ the sequence associated with the time step $\Delta t_{(k)} = 2^{-k}$.
   (a) What is the maximum value attained by the sequence $\{y_n^{(k)}\}_{n=0}^{N_{(k)}}$? Do you notice something different between the two schemes?
   (b) We observe that, by construction, $y_n^{(k)}, y_{2n}^{(k+1)}$ approximate the solution at time $t_n^{(k)} = \Delta t^{(k)} n$ or, equivalently, $t_{2n}^{(k+1)} = \Delta t^{(k+2)} 2n$. We can use this formula to estimate the convergence rate.
      i. Compute
      $$e^{(k)} = \max_{n=0,\ldots,N_{(k)}} \left| y_n^{(k)} - y_{2n}^{(k+1)} \right|, \quad k = 1, \ldots,$$
      ii. Compute the estimate of the convergence rate:
      $$\log_2\left(\frac{e^{(k)}}{e^{(k+1)}}\right), \quad \text{for } k = 1, \ldots.$$