High Energy Asymptotics of the Magnetic Spectral Shift Function

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Abstract. We consider the three-dimensional Schrödinger operators H_0 and H where $H_0 = (i\nabla + A)^2 - b$, A is a magnetic potential generating a constant magnetic field of strength $b > 0$, and $H = H_0 + V$ where $V \in L^1(\mathbb{R}^3; \mathbb{R})$ satisfies certain regularity conditions. Then, the spectral shift function $\xi(E; H, H_0)$ for the pair of operators H, H_0 is well-defined for energies $E \neq 2qb$, $q \in \mathbb{Z}_+$. We study the asymptotic behaviour of $\xi(E; H, H_0)$ as $E \to \infty$, $E \in \mathcal{O}_r$, $r \in (0, b)$, where $\mathcal{O}_r := \{E \in (0,\infty) | \text{dist}\,(E, 2b\mathbb{Z}_+) > r\}$. We obtain a Weyl-type formula $\lim_{E\to\infty, E\in\mathcal{O}_r} E^{-1/2}\xi(E; H, H_0) = \frac{1}{4\pi^2}\int_{\mathbb{R}^3} V(x)dx$.

1 Introduction

In this note we study the high energy asymptotics of the spectral shift function (SSF) for the three-dimensional Schrödinger operator with constant magnetic field, perturbed by an electric potential which decays fast enough at infinity. The note could be regarded as a supplement to the articles [5] where the asymptotic behaviour of the SSF in the strong magnetic field regime was considered, and [6] where the singularities of the SSF at the Landau levels were investigated.

Let $H_0 := (i\nabla + A)^2 - b$ be the unperturbed three-dimensional magnetic Schrödinger operator, essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. Here the magnetic potential $A = \left(-\frac{bx_2}{2}, \frac{bx_1}{2}, 0\right)$ generates the constant magnetic field $B = \text{curl}\ A = (0, 0, b)$, $b > 0$. It is well-known that $\sigma(H_0) = \sigma_{\rm ac}(H_0) = [0,\infty)$ (see [1]), where $\sigma(H_0)$ denotes the spectrum of H_0, and $\sigma_{\rm ac}(H_0)$ its absolutely continuous spectrum. Moreover, the so-called Landau levels $2bq$, $q \in \mathbb{Z}_+ := \{0, 1, \ldots\}$, play the role of thresholds in $\sigma(H_0)$.

For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we denote by $X_\perp = (x_1, x_2)$ the variables on the plane perpendicular to the magnetic field. We assume that V satisfies

$$V \not\equiv 0, \quad V \in C(\mathbb{R}^3), \quad |V(\mathbf{x})| \leq C_0\langle X_\perp\rangle^{-m_\perp}\langle x_3\rangle^{-m_3}, \quad \mathbf{x} = (X_\perp, x_3) \in \mathbb{R}^3, \qquad (1.1)$$

with $C_0 > 0$, $m_\perp > 2$, $m_3 > 1$, and $\langle x\rangle := (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^d$, $d \geq 1$. By (1.1) and the diamagnetic inequality (see e.g. [1]), for each $E_0 < 0$ we have

$$|V|^{1/2}(H_0 - E_0)^{-1} \in S_2, \qquad (1.2)$$

$$|V|^{1/2}(H_0 - E_0)^{-1/2} \in S_\infty, \tag{1.3}$$

where S_2 denotes the Hilbert-Schmidt class, while S_∞ denotes the class of linear compact operators. The resolvent identity combined with (1.2), implies that for each $E_0 < \inf \sigma(H) \leq \inf \sigma(H_0)$ we have

$$(H - E_0)^{-1} - (H_0 - E_0)^{-1} \in S_1 \tag{1.4}$$

where S_1 denotes the trace class. Then there exists a unique function $\xi(.; H, H_0) \in L^1(\mathbb{R}; (1 + E^2)^{-1}dE)$ which vanishes identically on $(-\infty, \inf \sigma(H))$, and satisfies the Lifshits-Krein trace formula

$$\mathrm{Tr}(\phi(H) - \phi(H_0)) = \int_{\mathbb{R}} \xi(E; H, H_0)\phi'(E)dE, \quad \phi \in C_0^\infty(\mathbb{R}),$$

(see [14, Theorem 8.9.1]). The function $\xi(.; H, H_0)$ is called the SSF for the pair of the operators H and H_0. For almost every $E > 0$ the SSF $\xi(E; H, H_0)$ is related to the scattering determinant $\det S(E; H, H_0)$ for the pair (H, H_0) by the Birman-Krein formula

$$\det S(E; H, H_0) = e^{-2\pi i \xi(E; H, H_0)}$$

(see [3] or [14, Section 8.4]).

A priori the SSF $\xi(E; H, H_0)$ is defined only for almost every $E \in \mathbb{R}$. In Subsection 3.3 below we introduce a representative of the equivalence class determined by $\xi(.; H, H_0)$, defined on $\mathbb{R} \setminus 2b\mathbb{Z}_+$, which is bounded on each compact subset of the complement of the Landau levels, and is continuous on $\mathbb{R} \setminus \{2b\mathbb{Z}_+ \cup \sigma_p(H)\}$ where $\sigma_p(H)$ denotes the set of the eigenvalues of the operator H. In this note we will identify the SSF with this particular representative of its equivalence class.

The main goal of the paper is the study of the asymptotics of $\xi(E; H, H_0)$ as $E \to \infty$, $E \in \mathcal{O}_r$, where

$$\mathcal{O}_r := \{E \in (0, \infty) | \text{dist } (E, 2b\mathbb{Z}_+) > r\}, \quad r \in (0, b). \tag{1.5}$$

The paper is organized as follows. In Section 2 we formulate our main result. In Section 3 we obtain some preliminary estimates, while the proof of our main result can be found in Section 4.

2 Formulation of the Main Result

Theorem 2.1. *Assume that V satisfies* (1.1). *Then we have*

$$\lim_{E \to \infty, E \in \mathcal{O}_r} E^{-1/2} \xi(E; H, H_0) = \frac{1}{4\pi^2} \int_{\mathbb{R}^3} V(\mathbf{x})d\mathbf{x}, \quad r \in (0, b). \tag{2.1}$$

Remarks: i) In asymptotic relation (2.1) it is essential that we avoid the Landau levels, i.e. that we suppose $E \in \mathcal{O}_r$, $r \in (0,b)$ as $E \to \infty$, since the SSH has singularities at the Landau levels, at least in the case where V has a fixed sign (see [6]).
ii) For $E \in \mathbb{R}$ set

$$\xi_{\rm cl}(E) := \int_{T^*\mathbb{R}^3} \left(\theta(E - |\mathbf{p} + \mathbf{A}(\mathbf{x})|^2) - \theta(E - |\mathbf{p} + \mathbf{A}(\mathbf{x})|^2 - V(\mathbf{x}))\right) d\mathbf{x} d\mathbf{p} =$$

$$\frac{4\pi}{3} \int_{\mathbb{R}^3} \left(E_+^{3/2} - (E - V(\mathbf{x}))_+^{3/2}\right) d\mathbf{x}$$

where $\theta(t) := \begin{cases} 0 & \text{if} \quad t \leq 0, \\ 1 & \text{if} \quad t > 0, \end{cases}$ is the Heaviside function. Note that $\xi_{\rm cl}(E)$ is independent of the magnetic field $b \geq 0$. Evidently, under the assumptions of Theorem 2.1 we have

$$\lim_{E \to \infty} E^{-1/2} \xi_{\rm cl}(E) = 2\pi \int_{\mathbb{R}^3} V(\mathbf{x}) d\mathbf{x}.$$

Hence, in the case $\int_{\mathbb{R}^3} V(\mathbf{x}) d\mathbf{x} \neq 0$, relation (2.1) is equivalent to

$$\xi(E; H, H_0) = (2\pi)^{-3} \xi_{\rm cl}(E)(1 + o(1)), \quad E \to \infty, \quad E \in \mathcal{O}_r, \quad r \in (0,b). \tag{2.2}$$

Asymptotic relations of the type (2.2) are known in the case of decaying magnetic potentials A (see e.g. [13]). However, in the last case the magnetic part $iA.\nabla + i \,\mathrm{div}\, A + |A|^2$ of the operator H_0 is a relatively compact perturbation of the Laplacian $-\Delta$, so that the resemblance with the case of a constant magnetic field, and hence linear magnetic potential, considered in the present note, is only formal. In particular, the methods of [13] are not directly applicable.
iii) As far as the authors are informed, the high energy asymptotics of the SSF for three-dimensional Schrödinger operators with constant magnetic fields is investigated for the first time in the present note. However, we would like to mention a result contained in [9] where an axisymmetric $V = V(|X_\perp|, x_3)$ is considered. It is well-known (see e.g. [1]) that in this case the operators H and H_0 are unitarily equivalent to the orthogonal sums $\sum_{m \in \mathbb{Z}} \oplus H^{(m)}$ and $\sum_{m \in \mathbb{Z}} \oplus H_0^{(m)}$ respectively, where the operators

$$H_0^{(m)} := -\frac{1}{\varrho} \frac{\partial}{\partial \varrho} \varrho \frac{\partial}{\partial \varrho} - \frac{\partial^2}{\partial x_3^2} + \left(\frac{b\varrho}{2} + \frac{m}{\varrho}\right)^2, \quad H^{(m)} := H_0^{(m)} + V(\varrho, x_3), \quad m \in \mathbb{Z},$$

are self-adjoint in $L^2(\mathbb{R}_+ \times \mathbb{R}; \varrho d\varrho dx_3)$. For an arbitrary fixed $m \in \mathbb{Z}$ the authors of [9] studied the asymptotics as $E \to \infty$, $E \in \mathcal{O}_r$ of the SSF $\xi(E; H^{(m)}, H_0^{(m)})$. Note that (2.1) cannot be deduced from the results of [9] even in the case of axial symmetry of V.

3 Auxiliary results

3.1. In this subsection we introduce some basic notations used throughout the paper. As above, we denote by S_∞ the class of linear compact operators acting in a fixed Hilbert space. Let $T = T^* \in S_\infty$. Denote by $\mathbb{P}_I(T)$ the spectral projection of T associated with the interval $I \subset \mathbb{R}$. For $s > 0$ set

$$n_\pm(s;T) := \text{rank } \mathbb{P}_{(s,\infty)}(\pm T).$$

For an arbitrary (not necessarily self-adjoint) operator $T \in S_\infty$ put

$$n_*(s;T) := n_+(s^2; T^*T), \quad s > 0. \tag{3.1}$$

If $T = T^*$, then evidently

$$n_*(s;T) = n_+(s,T) + n_-(s;T), \quad s > 0. \tag{3.2}$$

Further, we denote by S_p, $p \in [1,\infty)$, the Schatten-von Neumann class of compact operators for which the norm $\|T\|_p : = \left(p\int_0^\infty s^{p-1} n_*(s;T)\, ds\right)^{1/p}$ is finite. In particular, as already indicated, S_1 stands for the trace class, and S_2 for the Hilbert-Schmidt class. If $T \in S_p$, $p \in [1,\infty)$, then the following elementary inequality

$$n_*(s;T) \leq s^{-p}\|T\|_p^p \tag{3.3}$$

holds for every $s > 0$. If $T = T^* \in S_p$, $p \in [1,\infty)$, then (3.2) and (3.3) imply

$$n_\pm(s;T) \leq s^{-p}\|T\|_p^p, \quad s > 0. \tag{3.4}$$

Finally, we define the self-adjoint operators $\text{Re}\, T := \frac{1}{2}(T + T^*)$ and $\text{Im}\, T := \frac{1}{2i}(T - T^*)$.

3.2. In this subsection we introduce the concepts of index of a Fredholm pair of orthogonal projections, and index for a pair of selfadjoint operators, and describe some of basic properties related to these concepts which will be often used in the sequel. More details can be found in [2].
A pair of orthogonal projections P, Q is said to be Fredholm if

$$\{-1, 1\} \cap \sigma_{\text{ess}}(P - Q) = \emptyset.$$

In particular, if $P - Q \in S_\infty$, then the pair P, Q is Fredholm.
Assume that the pair of orthogonal projections P, Q is Fredholm. Set

$$\text{index}(P,Q) := \dim \text{Ker } (P - Q - I) - \dim \text{Ker } (P - Q + I).$$

Let $\tilde{M}$, M, be bounded self-adjoint operators. If the spectral projections $\mathbb{P}_{(-\infty,0)}(\tilde{M})$ and $\mathbb{P}_{(-\infty,0)}(M)$ form a Fredholm pair, we shall use the short-hand notation

$$\mathrm{ind}(\tilde{M}, M) := \mathrm{index}(\mathbb{P}_{(-\infty,0)}(\tilde{M}), \mathbb{P}_{(-\infty,0)}(M)).$$

A sufficient condition that the pair $\mathbb{P}_{(-\infty,0)}(\tilde{M}), \mathbb{P}_{(-\infty,0)}(M)$ be Fredholm, is $\tilde{M} = M + A$ where M is a bounded self-adjoint operator such that $0 \notin \sigma_{\mathrm{ess}}(M)$, and $A = A^* \in S_\infty$.

Lemma 3.1. (see [12, Lemma 5.2]) *Let $M = M^*$, $0 \notin \sigma(M)$, $0 \leq A = A^* \in S_\infty$. Then for $t \in (0, \infty)$ we have*

$$\mathrm{ind}(M + tA, M) = -\lim_{\varepsilon \downarrow 0} n_-(1 - \varepsilon; tA^{1/2}M^{-1}A^{1/2}), \tag{3.5}$$

$$\mathrm{ind}(M - tA, M) = n_+(1; tA^{1/2}M^{-1}A^{1/2}). \tag{3.6}$$

Lemma 3.2. (see [5, Subsection 3.2, Property (g)]) *Let M be a bounded self-adjoint operator such that $0 \notin \sigma(M)$. Let A and B be compact self-adjoint operators. Then for $s \in (0, \infty)$ such that $[-s, s] \cap \sigma(M) = \emptyset$ we have*

$$\mathrm{ind}(M+s+B, M+s) - n_+(s; A) \leq \mathrm{ind}(M+A+B, M) \leq \mathrm{ind}(M-s+B, M-s) + n_-(s; A). \tag{3.7}$$

Lemma 3.3. (see [11, Lemma 2.1], [5, Subsection 3.3]) *Let M be a bounded self-adjoint operator such that $0 \notin \sigma(M)$. Let $T_1 = T_1^* \in S_\infty$ and $T_2 = T_2^* \in S_1$. Then for each $s_1 > 0$, $s_2 > 0$ such that $[-s, s] \cap \sigma(M) = \emptyset$ with $s = s_1 + s_2$ we have*

$$\int_{\mathbb{R}} |\mathrm{ind}(M + T_1 + t\,T_2, M)| \; d\mu(t) \leq n_*(s_1; T_1) + \frac{1}{\pi s_2} \|T_2\|_1 \tag{3.8}$$

where $d\mu(t) := \frac{1}{\pi} \frac{dt}{1+t^2}$.

3.3. In this subsection we describe a representation of the SSF $\xi(E; H, H_0)$ which is a special case of the general representation of the SSF for a pair of lower-bounded self-adjoint operators, obtained by F. Gesztesy, K. Makarov and A. Pushnitski (see [11], [8], [12]).

For $z \in \mathbb{C}$, $\mathrm{Im}\, z > 0$, set $T(z) := |V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}$.

Lemma 3.4. (see [5, Lemma 2.4]) *Let* (1.1) *hold. Then for every $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$ the operator-norm limit*

$$T(E) := \mathrm{n} - \lim_{\delta \downarrow 0} T(E + i\delta) \tag{3.9}$$

exists and by (1.3) *we have $T(E) \in S_\infty$. Moreover,* $\mathrm{Im}\, T(E) \in S_1$.

Lemma 3.4 follows easily from Propositions 3.2 and 3.3 below (see Corollary 3.2). Denote by J the multiplier by the function

$$\operatorname{sign} V(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad V(\mathbf{x}) \geq 0, \\ -1 & \text{if} \quad V(\mathbf{x}) < 0. \end{cases}$$

Introduce the function

$$\tilde{\xi}(E; H, H_0) = \int_{\mathbb{R}} \operatorname{ind}(J + \operatorname{Re} T(E) + t \operatorname{Im} T(E), J) \, d\mu(t), \quad E \in \mathbb{R} \setminus 2b\mathbb{Z}_+, \tag{3.10}$$

which is well-defined by Lemmas 3.3 and 3.4.

Proposition 3.1. (see [5, Proposition 2.5]) *The function* $\tilde{\xi}(E; H, H_0)$ *is continuous on* $\mathbb{R} \setminus \{2b\mathbb{Z}_+ \cup \sigma_p(H)\}$, *and is bounded on every compact subset of* $\mathbb{R} \setminus 2b\mathbb{Z}_+$.

Remark: Note that, in contrast to the case $b = 0$, we cannot rule out the possibility of existence of embedded eigenvalues, by imposing short-range assumptions of the type of (1.1): Theorem 5.1 of [1] shows that there are axisymmetric potentials V of compact support such that below each Landau level $2bq$, $q \in \mathbb{Z}_+$, there exists an infinite sequence of eigenvalues of H which converges to $2bq$. On the other hand, generically, the only possible accumulation points of the eigenvalues of the operators H are the Landau levels (see [1, Theorem 4.7], [7, Theorem 3.5.3 (iii)]). Further information of the location of these eigenvalues can be found in [5].

Theorem 3.1. (see [11], [8], [12], [5, Subsection 3.3]) *Let* (1.1) *hold. Then for almost every* $E \in \mathbb{R}$ *we have*

$$\xi(E; H, H_0) = \tilde{\xi}(E; H, H_0). \tag{3.11}$$

Remark: As explained in the Introduction, we identify $\xi(E; H, H_0)$ with $\tilde{\xi}(E; H, H_0)$. The identification on the set $\mathbb{R} \setminus \{2b\mathbb{Z}_+ \cup \sigma_p(H)\}$ where $\tilde{\xi}$ is continuous, is natural. On the other hand, the values prescribed to the SSF at the eigenvalues $E \in \sigma_p(H)$ may seem somewhat arbitrary; in any case, as Theorem 2.1 shows, these values are consistent with the asymptotics of $\xi(E; H, H_0)$ as $E \to \infty$, $E \in \mathcal{O}_r$, $r \in (0, b)$, *and* $E \notin \sigma_p(H)$.

3.4. Introduce the Landau Hamiltonian

$$h(b) := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b, \tag{3.12}$$

i.e. the two-dimensional Schrödinger operator with constant scalar magnetic field $b > 0$, essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. It is well-known that $\sigma(h(b)) = \cup_{q=0}^\infty \{2bq\}$, and each eigenvalue $2bq$, $q \in \mathbb{Z}_+$, has infinite multiplicity (see e.g. [1]).

For $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ denote by $\mathcal{P}_{q,b}(\mathbf{x}, \mathbf{x}')$ the integral kernel of the orthogonal projection $p_q(b)$ onto the subspace Ker $(h(b) - 2bq)$, $q \in \mathbb{Z}_+$. It is well-known that

$$\mathcal{P}_{q,b}(\mathbf{x}, \mathbf{x}') = \frac{b}{2\pi} L_q \left(\frac{b|\mathbf{x} - \mathbf{x}'|^2}{2} \right) \exp \left(-\frac{b}{4}(|\mathbf{x} - \mathbf{x}'|^2 + 2i(x_1 x_2' - x_1' x_2)) \right) \tag{3.13}$$

(see [10]) where $L_q(t) := \sum_{k=0}^{q} \binom{q}{k} \frac{(-t)^k}{k!}$, $t \in \mathbb{R}$, $\quad q \in \mathbb{Z}_+$, are the Laguerre polynomials. Note that

$$\mathcal{P}_{q,b}(\mathbf{x}, \mathbf{x}) = \frac{b}{2\pi} \tag{3.14}$$

for each $q \in \mathbb{Z}_+$ and $\mathbf{x} \in \mathbb{R}^2$. Introduce the orthogonal projections $P_q : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, $q \in \mathbb{Z}_+$, by $P_q := p_q \otimes I$ where I denotes the identity operator in $L^2(\mathbb{R}_{x_3})$. For $z \in \mathbb{C}$ with Im $z > 0$, define the operator $R(z) := \left(-\frac{d^2}{dx_3^2} - z \right)^{-1}$ bounded in $L^2(\mathbb{R})$, as well as the operators

$$T_q(z) := |V|^{1/2} P_q (H_0 - z)^{-1} |V|^{1/2}, \quad q \in \mathbb{Z}_+,$$

bounded in $L^2(\mathbb{R}^3)$. The operator $R(z)$ admits the integral kernel $\mathcal{R}_z(x_3 - x_3')$ where $\mathcal{R}_z(x) = ie^{i\sqrt{z}|x|}/(2\sqrt{z})$, $x \in \mathbb{R}$, the branch of $\sqrt{z}$ being chosen so that Im $\sqrt{z} > 0$. Moreover, $T_q(z) = |V|^{1/2} \left(p_q(b) \otimes R(z - 2bq) \right) |V|^{1/2}$.
For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, define $R(\lambda)$ as the operator with integral kernel $\mathcal{R}_\lambda(x_3 - x_3')$ where

$$\mathcal{R}_\lambda(x) := \lim_{\delta \downarrow 0} \mathcal{R}_{\lambda + i\delta}(x) = \begin{cases} \frac{e^{-\sqrt{-\lambda}|x|}}{2\sqrt{-\lambda}} & \text{if} \quad \lambda < 0, \\ \frac{ie^{i\sqrt{\lambda}|x|}}{2\sqrt{\lambda}} & \text{if} \quad \lambda > 0, \end{cases} \qquad x \in \mathbb{R}. \tag{3.15}$$

Evidently, if $w \in L^2(\mathbb{R})$ and $\lambda \neq 0$, then $wR(\lambda)\bar{w} \in S_2$. For $E \in \mathbb{R}$, $E \neq 2bq$, $q \in \mathbb{Z}_+$, set

$$T_q(E) := |V|^{1/2} \left(p_q(b) \otimes R(E - 2bq) \right) |V|^{1/2}.$$

Proposition 3.2. [6, Proposition 4.1, Corollary 4.1, Corollary 4.2] *Let $E \in \mathbb{R}$, $q \in \mathbb{Z}_+$, $E \neq 2bq$. Assume that* (1.1) *holds.*
i) We have $T_q(E) \in S_2$, and $\lim_{\delta \downarrow 0} \|T_q(E + i\delta) - T_q(E)\|_2 = 0$.
ii) We have Im $T_q(E) \geq 0$, *and if $E < 2bq$, then* Im $T_q(E) = 0$. *Moreover,* Im $T_q(E) \in S_1$.

Proposition 3.3. (see [6, Proposition 4.2]) *Let $b > 0$, $E \notin 2b\mathbb{Z}_+$. Assume that V satisfies* (1.1). *Then the operator series $T_+(E + i\delta) := \sum_{l=[E/2b]+1}^{\infty} T_l(E + i\delta)$, $\delta > 0$, and $T_+(E) := \sum_{l=[E/2b]+1}^{\infty} T_l(E)$, where $[x]$ denotes the integer part of the real number x, are convergent in S_2. Moreover,*

$$\|T_+(E)\|_2^2 \leq \frac{C_0 b}{8\pi} \sum_{l=[E/2b]+1}^{\infty} (2bl - E)^{-3/2} \int_{\mathbb{R}^3} |V(\mathbf{x})| d\mathbf{x}. \tag{3.16}$$

Finally, $\lim_{\delta \downarrow 0} \|T_+(E + i\delta) - T_+(E)\|_2 = 0$.

Corollary 3.1. *Let $r \in (0,b)$. Then we have*

$$\|\mathrm{Re}\,T_+(E)\|_2^2 = O(1), \quad E \to \infty, \quad E \in \mathcal{O}_r. \tag{3.17}$$

Proof. Estimate (3.17) follows immediately from (3.16) since we have $\sum_{l=[E/2b]+1}^{\infty}(2bl - E)^{-3/2} \leq \sum_{p=1}^{\infty}(2bp)^{-3/2} + r^{-3/2}$. □

For sufficiently large $E \in \mathcal{O}_r$ with $r \in (0,b)$, set $T_-(E) := \sum_{l=0}^{[E/2b]} T_l(E)$. Propositions 3.2 and 3.3 imply the following

Corollary 3.2. *For $E \in \mathcal{O}_r$ with $r \in (0,b)$ the operator-norm limit* (3.9) *exists, and $T(E) = T_-(E) + T_+(E)$. Moreover,* $\mathrm{Re}\ T(E) = \mathrm{Re}\ T_-(E) + T_+(E)$, *and* $\mathrm{Im}\ T(E) = \mathrm{Im}\ T_-(E) \in S_1$.

For $n = 0, 1$ and $E \in \mathcal{O}_r$, $r \in (0,b)$, set $\varphi_n(E) := \sum_{q=0}^{[E/2b]}(E - 2bq)^{-1+n/2}$.

Lemma 3.5. *Let $r > 0$. Then the asymptotic relations*

$$\varphi_0(E) = O(\ln E). \tag{3.18}$$

$$\varphi_1(E) = E^{1/2}\frac{1}{b}\,(1 + o(1)), \tag{3.19}$$

hold as $E \to \infty$, $E \in \mathcal{O}_r$.

Proof. Evidently, for $E > 0$ large enough

$$\varphi_n(E) = E^{-1+n/2}\sum_{q=0}^{[E/2b]-1}\left(1 - \frac{2bq}{E}\right)^{-1+n/2} + (E - 2b[E/2b])^{-1+n/2}, \quad n = 0, 1. \tag{3.20}$$

Since the functions $(0, E/2b) \ni x \mapsto \left(1 - \frac{2bx}{E}\right)^{-1+n/2}$, $n = 0, 1$, are increasing, and $E \in \mathcal{O}_r$, we have

$$\sum_{q=0}^{[E/2b]-1}\left(1 - \frac{2bq}{E}\right)^{-1+n/2} \leq \int_0^{[E/2b]}\left(1 - \frac{2bx}{E}\right)^{-1+n/2} dx \leq$$

$$\int_0^{(E-r)/(2b)}\left(1 - \frac{2bx}{E}\right)^{-1+n/2} dx = \frac{E}{2b}\int_0^{1-r/E}(1-t)^{-1+n/2}dt, \quad n = 0, 1. \tag{3.21}$$

Further,

$$\int_0^{1-r/E}(1-t)^{-1+n/2}dt = \begin{cases} \ln(E/r) & \text{if} \quad n = 0, \\ 2\left(1 - (r/E)^{1/2}\right) & \text{if} \quad n = 1. \end{cases} \tag{3.22}$$

Finally, we estimate the second term on the r.h.s. of (3.20)

$$(E - 2b[E/2b])^{-1+n/2} \leq r^{-1+n/2}, \quad n = 0, 1. \tag{3.23}$$

Putting together (3.20) – (3.23), we obtain (3.18), as well as $\limsup_{E\to\infty, E\in\mathcal{O}_r} E^{-1/2}\varphi_1(E) \leq \frac{1}{b}$. In order to prove (3.19), it remains to show that $\liminf_{E\to\infty, E\in\mathcal{O}_r} E^{-1/2}\varphi_1(E) \geq \frac{1}{b}$ which follows immediately from

$$\varphi_1(E) \geq E^{-1/2} \int_{-1}^{[E/2b]} \left(1 - \frac{2bx}{E}\right)^{-1/2} dx \geq E^{-1/2} \int_0^{E/2b-1} \left(1 - \frac{2bx}{E}\right)^{-1/2} dx =$$

$$\frac{E^{1/2}}{2b} \int_0^{1-2b/E} (1-t)^{-1/2}\, dt = \frac{E^{1/2}}{b}\left(1 - \left(\frac{2b}{E}\right)^{1/2}\right).$$

□

Corollary 3.3. *Let $r \in (0, b)$. Then the asymptotic estimate*

$$\|T_-(E)\|_2^2 = O(\ln E) \tag{3.24}$$

holds as $E \to \infty$, $E \in \mathcal{O}_r$.

Proof. We have

$$\|T_-(E)\|_2^2 = \left\| |V|^{1/2} \sum_{q=0}^{[E/2b]} (p_q \otimes R(E-2bq))|V|^{1/2} \right\|_2^2 \leq$$

$$C_0^2 \left\| \sum_{q=0}^{[E/2b]} (p_q \langle X_\perp\rangle^{-m_\perp/2}) \otimes \left(\langle x_3\rangle^{-m_3/2} R(E-2bq) \langle x_3\rangle^{-m_3/2}\right) \right\|_2^2 =$$

$$C_0^2 \sum_{q=0}^{[E/2b]} \left\| p_q \langle X_\perp\rangle^{-m_\perp/2} \right\|_2^2 \left\| \langle x_3\rangle^{-m_3/2} R(E-2bq) \langle x_3\rangle^{-m_3/2} \right\|_2^2 =$$

$$C_0^2 \frac{b}{8\pi} \int_{\mathbb{R}^2} \langle X_\perp\rangle^{-m_\perp} dX_\perp \left(\int_{\mathbb{R}} \langle x_3\rangle^{-m_3} dx_3\right)^2 \varphi_0(E) \tag{3.25}$$

(see (3.13) for the definition of the integral kernel of p_q, (3.14) for its value on the diagonal, and (3.15) for the definition of the integral kernel of $R(E-2bq)$). Bearing in mind (3.18), we find that (3.25) implies (3.24). □

Proposition 3.4. *Let $r \in (0, b)$. Then we have*

$$\|\mathrm{Im}\, T(E)\| = O(1), \quad E \to \infty, \quad E \in \mathcal{O}_r. \tag{3.26}$$

Proof. Estimate (3.26) follows immediately from [5, Lemma 4.2] according to which we have $\|T(E)\| \leq r^{-1/2} \frac{C_0}{2} \int_{\mathbb{R}} \langle x_3\rangle^{-m_3} dx_3$. □

4 Proof of the Main Result

Fix an arbitrary $\varepsilon \in (0,1)$. Applying (3.5), (3.6), and (3.7), and arguing as in the proof of [5, Lemma 5.1], we easily get

$$\frac{1}{\pi}\mathrm{Tr}\arctan\,\left((\mathrm{Im}\,T(E))^{1/2}(J+\varepsilon)^{-1}(\mathrm{Im}\,T(E))^{1/2}\right) - n_+(\varepsilon;\mathrm{Re}\,T(E)) \leq$$

$$\int_{\mathbb{R}} \mathrm{ind}(J + \mathrm{Re}\,T(E) + t\,\mathrm{Im}\,T(E); J)\,d\mu(t) \leq$$

$$\frac{1}{\pi}\mathrm{Tr}\arctan\,\left((\mathrm{Im}\,T(E))^{1/2}(J-\varepsilon)^{-1}(\mathrm{Im}\,T(E))^{1/2}\right) + n_-(\varepsilon;\mathrm{Re}\,T(E)). \qquad (4.1)$$

Set

$$G_s = G_s(E) := (\mathrm{Im}\,T(E))^{1/2}(J+s)^{-1}(\mathrm{Im}\,T(E))^{1/2}, \quad s \in (-1,1).$$

Evidently, for each $s \in (-1,1)$ we have

$$|\mathrm{Tr}\arctan G_s(E) - \mathrm{Tr}\;G_s(E)| \leq \frac{1}{3}\|G_s(E)\|_3^3 \leq \frac{1}{3}\|G_s(E)\|_2^2\,\|G_s(E)\| \leq$$

$$\frac{1}{3}\|(J+s)^{-1}\|^3\,\|\mathrm{Im}\,T(E)\|_2^2\,\|\mathrm{Im}\,T(E)\| \leq \frac{1}{3}(1-|s|)^{-3}\|\mathrm{Im}\,T(E)\|_2^2\,\|\mathrm{Im}\,T(E)\|. \qquad (4.2)$$

The operator $(J+s)^{-1}\,\mathrm{Im}\,T(E)$ admits an explicit kernel

$$\frac{1}{2}\sum_{q=0}^{[E/2b]}(E-2bq)^{-1/2}\mathcal{P}_q(X_\perp, X'_\perp)\cos\left(\sqrt{E-2bq}(x_3 - x'_3)\right)\times$$

$$(\mathrm{sign}\,(V(X_\perp,x_3)) + s)^{-1}|V(X_\perp,x_3)|^{1/2}|V(X'_\perp,x'_3)|^{1/2}, \quad (X_\perp,x_3)\in\mathbb{R}^3, \quad (X'_\perp,x'_3)\in\mathbb{R}^3,$$

(see (3.13) for the definition of $\mathcal{P}_q$). Therefore,

$$\mathrm{Tr}\,G_s(E) = \mathrm{Tr}((J+s)^{-1}\,\mathrm{Im}\,T(E)) =$$

$$\frac{b}{4\pi}\sum_{q=0}^{[E/2b]}(E-2bq)^{-1/2}\int_{\mathbb{R}^3}(\mathrm{sign}\,(V(\mathbf{x})) + s)^{-1}|V(\mathbf{x})|d\mathbf{x} =$$

$$\frac{b}{4\pi}\varphi_1(E)\int_{\mathbb{R}^3}(\mathrm{sign}\,(V(\mathbf{x})) + s)^{-1}|V(\mathbf{x})|d\mathbf{x} \qquad (4.3)$$

(see (3.14)). Finally, we estimate the second terms in the first and the third line in (4.1):

$$n_\pm(\varepsilon;\mathrm{Re}\,T(E)) \leq \varepsilon^{-2}\|\mathrm{Re}\,T(E)\|_2^2 \leq 2\varepsilon^{-2}\left(\|\mathrm{Re}\,T_-(E)\|_2^2 + \|\mathrm{Re}\,T_+(E)\|_2^2\right), \qquad (4.4)$$

using (3.4) with $p = 2$. Combining (4.1) - (4.4) with (3.10), making use of (3.19), (3.24), (3.17), and (3.26), and applying our convention to identify $\tilde{\xi}(E; H, H_0)$ with $\xi(E; H, H_0)$ we find that for each $\varepsilon \in (0, 1)$ we have

$$\limsup_{E\to\infty,\, E\in\mathcal{O}_r} E^{-1/2}\xi(E; H, H_0) \leq \frac{1}{4\pi^2}\int_{\mathbb{R}^3} (\mathrm{sign}\,(V(\mathbf{x})) - \varepsilon)^{-1}|V(\mathbf{x})|d\mathbf{x},$$

$$\liminf_{E\to\infty,\, E\in\mathcal{O}_r} E^{-1/2}\xi(E; H, H_0) \geq \frac{1}{4\pi^2}\int_{\mathbb{R}^3} (\mathrm{sign}\,(V(\mathbf{x})) + \varepsilon)^{-1}|V(x)|d\mathbf{x}.$$

Letting $\varepsilon \downarrow 0$, we obtain (2.1).

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