RESONANCES AND SPECTRAL SHIFT FUNCTION NEAR THE LANDAU LEVELS

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Abstract. We consider the 3D Schrödinger operator $H = H_0 + V$ where $H_0 = (-i\nabla - A)^2 - b$, A is a magnetic potential generating a constant magnetic field of strength $b > 0$, and V is a short-range electric potential which decays superexponentially with respect to the variable along the magnetic field. We show that the resolvent of H admits a meromorphic extension from the upper half plane to an appropriate Riemann surface $\mathcal{M}$, and define the resonances of H as the poles of this meromorphic extension. We study their distribution near any fixed Landau level $2bq$, $q \in \mathbb{N}$. First, we obtain a sharp upper bound of the number of resonances in a vicinity of $2bq$. Moreover, under appropriate hypotheses, we establish corresponding lower bounds which imply the existence of an infinite number of resonances, or the absence of resonances in certain sectors adjoining $2bq$. Finally, we deduce a representation of the derivative of the spectral shift function (SSF) for the operator pair (H, H_0) as a sum of a harmonic measure related to the resonances, and the imaginary part of a holomorphic function. This representation justifies the Breit-Wigner approximation, implies a trace formula, and provides information on the singularities of the SSF at the Landau levels.

2000 AMS Mathematics Subject Classification: 35P25, 35J10, 47F05, 81Q10

Keywords: magnetic Schrödinger operators, resonances, spectral shift function, Breit-Wigner approximation

1. Introduction

Let

$$H_0 := (D_{x_1} + \frac{b}{2}x_2)^2 + (D_{x_2} - \frac{b}{2}x_1)^2 - b + D_{x_3}^2, \quad D_\nu := -i\frac{\partial}{\partial \nu},$$

be the Schrödinger operator with homogeneous magnetic field of strength $b > 0$, pointing at the x_3-direction. Initially, the self-adjoint operator H_0 is defined on $C_0^\infty(\mathbb{R}^3)$, and then is closed in $L^2(\mathbb{R}^3)$. This operator can be written in $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R})$ as

$$H_0 = H_{0,\perp} \otimes I + I \otimes D_{x_3}^2,$$

with $H_{0,\perp} = (D_{x_1} + \frac{b}{2}x_2)^2 + (D_{x_2} - \frac{b}{2}x_1)^2 - b$.

It is well known that the spectrum of the operator $H_{0,\perp}$ consists of the Landau levels $2qb$, $q \in \mathbb{N} := \{0, 1 \dots\}$, and the multiplicity of each eigenvalue $2bq$ is infinite (see e.g. [1]). Consequently, the spectrum of H_0 is absolutely continuous, equals $[0, +\infty[$, and has an infinite set of thresholds $2qb$, $q \geq 0$.

For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we denote by $X_\perp = (x_1, x_2)$ the variables on the plane perpendicular to the magnetic field. We assume that the electric potential $V : \mathbb{R}^3 \to \mathbb{R}$ is Lebesgue measurable, and satisfies the estimates

$$V(\mathbf{x}) = \mathcal{O}(\langle X_\perp \rangle^{-m_\perp} \langle x_3 \rangle^{-m_3}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{1.1}$$

with $m_\perp > 2$, $m_3 > 1$, and $\langle x \rangle := (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^d$, $d \geq 1$.

On the domain of H_0 we introduce the operator $H := H_0 + V$. Since V is a relatively compact perturbation of H_0, it follows from the Weyl criterion that the essential spectra of H and H_0 are the same. Moreover, since V is a relatively trace-class perturbation, the Kato-Rosenblum theorem implies that the absolutely continuous spectrum coincides with $[0, +\infty[$.

It is well known that H can have infinite negative discrete spectrum and, for some special V, it can have infinitely many embedded eigenvalues below each Landau level (see [1], [22] or [23]). On the other hand, it is shown in [10] that in the case of sign-definite V, the spectral shift function (SSF) for the operator pair (H, H_0) has a singularity at each Landau level. Therefore, it is natural to expect that there could be accumulation of the resonances of the operator H near the Landau levels. For a Coulomb potential, some numerical results confirm this conjecture [7]. The goal of this paper is to study the resonances near the Landau levels, and to establish the link between these resonances and the spectral shift function by the so-called Breit-Wigner approximation. Such a representation of the derivative of the spectral shift function related to the resonances, implies trace formulas which have given recently a substantial impetus to the research concerning the upper and lower bounds of the number of resonances in different situations (see [26], [27], [28], [29], [20], [2], [5], [9]).

We consider potentials V which decay super-exponentially with respect to x_3 (or are compactly supported with respect to x_3). Hence, we do not use dilation methods in order to define the resonances near the Landau levels. For a definition using complex dilation, we refer the reader to [31], [13] where precise asymptotics as $b \to \infty$ of the resonances near the real axis is given. In our work, b is fixed, and we study the number of resonances in a domain $2bq + r\Omega$ as r tends to 0. Then we justify the Breit-Wigner approximation for the spectral shift function near the Landau levels.

The paper is organized as follows. In the next section, we define the resonances as the poles of the resolvent, the first step being to introduce a Riemann surface to which the resolvent is extended. Note that the resonances defined as poles of the resolvent, are also zeros of a generalized Krein perturbation determinant with the same multiplicity. In Section 3, we obtain an upper bound of the number of resonances in a domain $2bq + r\Omega$ as r tends to 0 (see Theorem 1). In Section 4, we obtain more information on the localization of the resonances for the case of perturbations of definite sign. In particular, we show that there is an infinite number of resonances near any arbitrary fixed Landau level for small V of sufficiently rapid decay (see Theorem 2), and that there are no embedded eigenvalues for small positive V (see Proposition 7). At last, in Section 5, we represent the derivative of the spectral shift function near the Landau levels as a sum of a harmonic measure related to the resonances and the imaginary part of a holomorphic function (see Theorem 3). Such a representation justifies the Breit-Wigner approximation, implies a trace formula, and for a special class of V sufficiently slowly decaying with respect to the variables perpendicular to the magnetic field, allows us to estimate the remainder in the asymptotic relations obtained in [10].

2. Resonances

In this section we define the resonances of $H = H_0 + V$ for V decaying super-exponentially with respect to x_3, i.e.

$$V(\mathbf{x}) = \mathcal{O}(\langle X_\perp \rangle^{-m_\perp} \exp(-N|x_3|)), \tag{2.1}$$

for $m_\perp \geq 0$ and any $N > 0$. As in [25], the resonances will be defined as the poles of the meromorphic continuation of the resolvent in some weighted L^2 spaces. Since V is not compactly supported with respect to x_3, the cut-off resolvent cannot be used here.

First, we have to prove the existence of a holomorphic extension for the unperturbed operator. Let $\mathbb{C}_+ := \{\lambda \in \mathbb{C};\ \operatorname{Im}\lambda > 0\}$ be the open upper half plane. For $\lambda \in \mathbb{C}_+$ we have

$$(H_0 - \lambda)^{-1} = \sum_{q\in\mathbb{N}} p_q \otimes (D_{x_3}^2 + 2bq - \lambda)^{-1}, \tag{2.2}$$

where p_q is the orthogonal projection onto $\mathcal{H}_q := \ker(H_{0,\perp} - 2bq)$.

Let us recall that for $\operatorname{Im} k > 0$, the integral kernel of the operator $(D_{x_3}^2 - k^2)^{-1}$ is given by

$$\mathcal{R}(x_3, x_3') = \frac{ie^{ik|x_3 - x_3'|}}{2k}. \tag{2.3}$$

Then, for $N > 0$, the operator-valued function

$$t_N(k^2) := e^{-N\langle x_3\rangle}(D_{x_3}^2 - k^2)^{-1}e^{-N\langle x_3'\rangle} \in \mathcal{L}(L^2(\mathbb{R}_{x_3}), H^2(\mathbb{R}_{x_3})), \tag{2.4}$$

can be extended holomorphically from $\mathbb{C}_+$ to $\{k \in \mathbb{C}^*;\ \operatorname{Im} k > -N\}$. Hence, for any $N > 0$ and $q \in \mathbb{N}$,

$$z \mapsto (D_{x_3}^2 + 2bq - z)^{-1} \in \mathcal{L}(e^{-N\langle x_3\rangle}L^2(\mathbb{R}_{x_3}), e^{N\langle x_3\rangle}H^2(\mathbb{R}_{x_3})),$$

has a holomorphic extension from $\mathbb{C}\setminus[2bq, +\infty[$ to the 2-sheeted covering $\pi_q : k \in \mathbb{C}^* \mapsto k^2 + 2bq \in \mathbb{C}\setminus\{2bq\}$ (with $\operatorname{Im} k > -N$). However, this covering depends on q, and therefore it is not suitable for the extension of (2.2).

A natural domain of analytic extension of (2.2) is the universal covering of $\mathbb{C}\setminus 2b\mathbb{N}$:

$$\overline{\pi} : \overline{\mathbb{C}\setminus 2b\mathbb{N}} \to \mathbb{C}\setminus 2b\mathbb{N},$$

but it does not give a maximal analytic continuation. Indeed for some $z \in \mathbb{C}\setminus 2b\mathbb{N}$, there are some points $z_1,\ z_2 \in \overline{\pi}^{-1}(z)$, $z_1 \neq z_2$ such that the germs of $(H_0 - z)^{-1}$ at z_1 and at z_2 are the same. Next, we introduce the equivalence relation $\mathcal{R}$ concerning such pairs of points (z_1, z_2). Let $\pi_1(\mathbb{C}\setminus 2b\mathbb{N})$ be the fundamental group of $\mathbb{C}\setminus 2b\mathbb{N}$, and G be its subgroup generated by $\{a_1^2,\ a_2a_1a_2^{-1}a_1^{-1};$ with $a_1, a_2 \in \pi_1(\mathbb{C}\setminus 2b\mathbb{N})\}$. We will write $z_1\mathcal{R}z_2$ if and only if $\overline{\pi}z_1 = \overline{\pi}z_2$, and for any path γ connecting z_1 with z_2, the class of the closed path $\pi\gamma$ in $\pi_1(\mathbb{C}\setminus 2b\mathbb{N})$ is an element of G (that is $\pi(\gamma)$ goes an even number of times round each Landau level).

Then we define the domain $\mathcal{M}$ of the analytic extension of $(H_0 - z)^{-1}$ as the quotient of $\overline{\mathbb{C}\setminus 2b\mathbb{N}}$ by the relation $\mathcal{R}$. This domain can also be identified with the following covering of $\mathbb{C}\setminus 2b\mathbb{N}$ (see for instance Proposition 13.23 of [14]):

Definition 1. *Let $\pi_1(\mathbb{C}\setminus 2b\mathbb{N})$ be the fundamental group of $\mathbb{C}\setminus 2b\mathbb{N}$. Let G be the subgroup of $\pi_1(\mathbb{C}\setminus 2b\mathbb{N})$ generated by $\{a_1^2,\ a_2a_1a_2^{-1}a_1^{-1};$ with $a_1, a_2 \in \pi_1(\mathbb{C}\setminus 2b\mathbb{N})\}$. We define $\pi_G :\ \mathcal{M} \to \mathbb{C}\setminus 2b\mathbb{N}$ as the connected infinite-sheeted covering such that $\pi_1(\mathcal{M}) = G$.*

From now on, we fix a base point in $\mathcal{M}$, and define the physical plane $\mathcal{F}$ as the connected component of $\pi_G^{-1}(\mathbb{C}\setminus[0, +\infty[)$ containing this base point. By definition, the functions $\mathcal{M} \ni z \mapsto \sqrt{z - 2bq}$ have a positive imaginary part on $\mathcal{F}$. Let $\mathcal{F}_+ = \mathcal{F}\cap\pi_G^{-1}(\mathbb{C}_+)$ be the upper half-plane. In what follows, we identify $\mathcal{F}$ (resp. $\mathcal{F}_+$ and $\partial\mathcal{F}_+$) with $\mathbb{C}\setminus[0, +\infty[$ (resp. $\mathbb{C}_+$ and $\mathbb{R}\setminus 2b\mathbb{N}$), and denote by z the generic point on $\mathcal{M}$.

For $\lambda_0 \in \mathbb{C}$ and $\varepsilon > 0$ put $D(\lambda_0, \varepsilon) := \{\lambda \in \mathbb{C}, |\lambda - \lambda_0| < \varepsilon\}$ and $D(\lambda_0, \varepsilon)^* := \{\lambda \in \mathbb{C}, 0 < |\lambda - \lambda_0| < \varepsilon\}$.

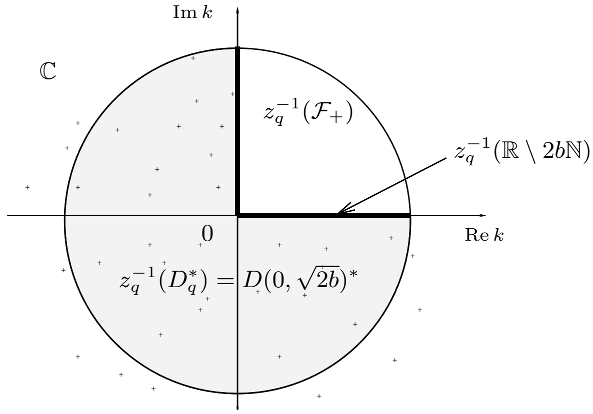

Figure 1. **Pre-images under z_q.**

Definition 2. *We denote by $D_q^* \subset \mathcal{M}$, the connected component of $\pi_G^{-1}(D(2bq, 2b)^*)$ that intersects $\mathcal{F}_+$.*

Since $\pi_G : D_q^* \to D(2bq, 2b)^*$ is a 2-sheeted covering of $D(2bq, 2b)^*$, there exists an analytic bijection

$$z_q : k \in D(0, \sqrt{2b})^* \to z_q(k) \in D_q^*, \tag{2.5}$$

such that $\pi_G(z_q(k)) = 2bq + k^2$ and $z_q^{-1}(D_q^* \cap \mathcal{F}_+)$ is the first quadrant of $D(0, \sqrt{2b})^*$.

For $N > 0$, we denote by $\mathcal{M}_N$ the set of points $m \in \mathcal{M}$ such that for each $q \in \mathbb{N}$, we have $\operatorname{Im} \sqrt{z - 2bq} > -N$. Of course, we have $\cup_{N>0} \mathcal{M}_N = \mathcal{M}$.

Figure 1 summarizes the setting near the Landau level $2bq$. For the free operator, we have the following proposition.

Proposition 1. *For each $N > 0$ the operator $(H_0 - z)^{-1} : e^{-N\langle x_3 \rangle} L^2(\mathbb{R}^3_{\mathbf{x}}) \to e^{N\langle x_3 \rangle} L^2(\mathbb{R}^3_{\mathbf{x}})$ has a holomorphic extension from the open upper half plane to $\mathcal{M}_N$. We denote its holomorphic extension by $R_0(z)$.*

Moreover, for $N > 0$ and $v_\perp(X_\perp) = \langle X_\perp \rangle^{-\alpha}$, $\alpha > 1$, the holomorphic extension of

$$T_{v_\perp} : z \mapsto v_\perp(X_\perp) e^{-N\langle x_3 \rangle} (H_0 - z)^{-1} e^{-N\langle x_3 \rangle},$$

is holomorphic on $\mathcal{M}_N$ with values in the Hilbert-Schmidt class S_2 on $L^2(\mathbb{R}^3_{\mathbf{x}})$.

Proof. Since the kernel of $t_N(k^2)$ is given by

$$e^{-N\langle x_3 \rangle} \frac{i e^{ik|x_3 - x_3'|}}{2k} e^{-N\langle x_3' \rangle},$$

the operator-valued function $k \mapsto t_N(k^2)$ has a holomorphic extension from $\mathbb{C}_+$ to $\{k \in \mathbb{C}^*; \operatorname{Im} k > -N\}$ in the Hilbert-Schmidt class S_2 and in the trace class S_1 (see for instance [12]). For $\operatorname{Im} k > 0$, we have the trace-class estimate

$$\begin{aligned} \|t_N(k^2)\|_1 = \|e^{-N\langle x_3 \rangle} (D_{x_3} - k)^{-1}\|_2 \|(D_{x_3} + k)^{-1} e^{-N\langle x_3 \rangle}\|_2 &\leq \frac{1}{2\pi N} \int_{\mathbb{R}} \frac{d\eta}{\eta^2 + |\operatorname{Im} k|^2} \\ &= \mathcal{O}\left(|\operatorname{Im} k|^{-1}\right), \end{aligned} \tag{2.6}$$

and when moreover $\operatorname{Re} k^2 < 0$, we have the Hilbert-Schmidt estimate

$$\begin{aligned} \|t_N(k^2)\|_2^2 =& \mathrm{tr}\Big(e^{-N\langle x_3\rangle}(D_{x_3}^2 - k^2)^{-1}(D_{x_3}^2 - \overline{k}^2)^{-1}e^{-N\langle x_3\rangle}\Big) \\ \leq& \frac{1}{2\pi N}\int_{\mathbb{R}} \frac{d\eta}{(\eta^2 + |\operatorname{Re} k^2|)^2} = \mathcal{O}\left(|\operatorname{Re} k^2|^{-\frac{3}{2}}\right), \end{aligned} \tag{2.7}$$

where $\|\cdot\|_j$ stands for the norm in S_j, $j = 1, 2$.

By the definition of $\mathcal{M}_N$, it follows that for any $q \in \mathbb{N}$, the operator-valued function $z \mapsto e^{-N\langle x_3\rangle}(D_{x_3}^2 - z + 2bq)^{-1}e^{-N\langle x_3\rangle} \in S_1$ can be holomorphically extended from $\mathcal{F}_+$ to $\mathcal{M}_N$. We denote its holomorphic extension to $\mathcal{M}_N$ also by $t_N(z - 2bq)$. Since $\{p_q,\ q \in \mathbb{N}\}$ is a family of orthogonal projectors, we deduce the holomorphic extension of (2.2).

Now, let us prove the existence of a holomorphic extension of $T_{v_\perp}$ in the Hilbert-Schmidt class. Let $z_0 \in \mathcal{M}_N$ be fixed, and Ω_0 be a relatively compact neighborhood of z_0. Since any path on $\mathcal{M}_N$ can enclose only a finite number of Landau levels, there exists q_0 sufficiently large (depending of Ω_0) such that $q \geq q_0$ implies $t_N(z - 2bq) = e^{-N\langle x_3\rangle}(D_{x_3}^2 - z + 2bq)^{-1}e^{-N\langle x_3\rangle}$. Then for $q \geq q_0$ we have $\|t_N(z - 2bq)\| = \mathcal{O}\left(\langle q\rangle^{-1}\right)$, and furthermore, it follows from (2.7) that the identity

$$\|t_N(z - 2bq)\|_2 = \|e^{-N\langle x_3\rangle}(D_{x_3}^2 - z + 2bq)^{-1}e^{-N\langle x_3\rangle}\|_2 = \mathcal{O}\Big(\langle q\rangle^{-\frac{3}{4}}\Big), \tag{2.8}$$

holds for any $q \geq q_0$, uniformly with respect to $z \in \Omega_0$.

Next, we have

$$T_{v_\perp}(z) = \sum_{q=0}^{q_0} v_\perp p_q \otimes t_N(z - 2bq) + \sum_{q>q_0} v_\perp p_q \otimes t_N(z - 2bq), \tag{2.9}$$

where q_0 is chosen as above. It is well known (see [19]) that the orthogonal projection p_q admits an explicit integral kernel

$$\mathcal{P}_{q,b}(X_\perp, X'_\perp) = \frac{b}{2\pi} L_q\left(\frac{b|X_\perp - X'_\perp|^2}{2}\right) \exp\Big(-\frac{b}{4}\big(|X_\perp - X'_\perp|^2 + 2i(x_1x'_2 - x'_1x_2)\big)\Big), \tag{2.10}$$

where $L_q(t) := \frac{1}{q!}e^t\frac{d^q(t^qe^{-t})}{dt^q}$ are the Laguerre polynomials. Note that $\mathcal{P}_{q,b}$ is constant onto the diagonal, i.e.

$$\mathcal{P}_{q,b}(X_\perp, X_\perp) = \frac{b}{2\pi}, \quad X_\perp \in \mathbb{R}^2.$$

Further, if $U \in L^r(\mathbb{R}^2)$, $r \geq 1$, then p_qUp_q is in the Schatten-von Neumann class S_r (see Lemma 5.1 of [21]). In particular, $p_qv_\perp^2p_q \in S_1$, and hence $v_\perp p_q v_\perp \in S_1$, and $v_\perp p_q \in S_2$ with

$$\|v_\perp p_q\|_2^2 = \mathrm{tr}(v_\perp p_q v_\perp) = \frac{b}{2\pi}\int_{\mathbb{R}^2} v_\perp(X_\perp)^2 dX_\perp,$$

which is uniform with respect to q. Combining this with (2.8) and $p_qp_k = \delta_{q,k}p_q$, we deduce that the infinite sum in (2.9) is convergent in S_2, and hence $z \mapsto T_{v_\perp}(z) \in S_2$ has a holomorphic extension to $\mathcal{M}_N$. This concludes the proof of Proposition 1. $\square$

For further references we formulate the following lemma which complements Proposition 1.

Lemma 1. *For V satisfying* (2.1) *with $m_\perp > 2$, the operator*

$$\mathcal{F}_+ \ni z \mapsto \mathcal{T}_V(z) := J|V|^{\frac{1}{2}}(H_0 - z)^{-1}|V|^{\frac{1}{2}} \in S_2,$$

with $J := \operatorname{sign} V$ defined so that $J^2 = 1$, has an analytic extension from $\mathbb{C}_+$ to $\mathcal{M}$, denoted again by $\mathcal{T}_V(z)$. Moreover the operator $d\mathcal{T}_V(z)/dz \in S_1$ is analytic on $\mathcal{M}$.

Proof. The existence of the holomorphic extension in S_2 is a direct consequence of Proposition 1, because for any $N > 0$, we have $|V|^{\frac{1}{2}} = \mathcal{V}(v_\perp \otimes e^{-N\langle x_3\rangle})$, with a bounded operator $\mathcal{V}$. In order to prove $d\mathcal{T}_V(z)/dz \in S_1$, it suffices to check that the series of general term $(v_\perp p_q v_\perp) \otimes dt_N(z-2bq)/dz$ converge in the trace class. Arguing as in the proof of Proposition 1, we find that this convergence follows from (2.8), and the fact that for $z \in \mathbb{C} \setminus [0, +\infty[$, we have $dt_N(z)/dz = e^{-N\langle x_3\rangle}(D_{x_3}^2 - z)^{-2} e^{-N\langle x_3\rangle}$. □

Remark. The assumption $m_\perp > 2$ could be weakened to $m_\perp > 1$ in the first part of the proof of the Lemma while it is necessary for the second part.

Suppose (2.1) with $m_\perp > 0$. Using

$$(H - z)^{-1}\big(1 + V(H_0 - z)^{-1}\big) = (H_0 - z)^{-1},$$

we get

$$e^{-N\langle x_3\rangle}(H - z)^{-1} e^{-N\langle x_3\rangle} = e^{-N\langle x_3\rangle}(H_0 - z)^{-1} e^{-N\langle x_3\rangle}\big(1 + e^{N\langle x_3\rangle} V (H_0 - z)^{-1} e^{-N\langle x_3\rangle}\big)^{-1}, \tag{2.11}$$

for $z \in \mathcal{F}_+$, $\operatorname{Im} z \gg 1$. From Proposition 1, combined with (2.1), the operator $Ve^{N\langle x_3\rangle}(H_0 - z)^{-1} e^{-N\langle x_3\rangle}$ is compact on $L^2(\mathbb{R}^3_{\mathbf{x}})$. Then, the analytic Fredholm theorem proves the meromorphic extension of $\big(1 + e^{N\langle x_3\rangle} V (H_0 - z)^{-1} e^{-N\langle x_3\rangle}\big)^{-1}$ from $\mathcal{F}_+$ to $\mathcal{M}_N$. This now allows us to define the resonances of H.

Proposition 2. *Suppose V satisfies (2.1) with $m_\perp > 0$. Then the operator-valued function*

$$(H - z)^{-1} : e^{-N\langle x_3\rangle} L^2(\mathbb{R}^3_{\mathbf{x}}) \to e^{N\langle x_3\rangle} L^2(\mathbb{R}^3_{\mathbf{x}}),$$

has a meromorphic extension from the open upper half plane to $\mathcal{M}_N$. Moreover, the poles and the range of the residues of this extension do not depend on N.

Definition 3. *We define the resonances of H as the poles of the meromorphic extension of the resolvent $(H - z)^{-1}$, denoted by $R(z)$. The multiplicity of a resonance z_0 is defined by*

$$\operatorname{mult}(z_0) := \operatorname{rank} \frac{1}{2i\pi} \int_\gamma R(z)\, dz, \tag{2.12}$$

where γ is a small positively oriented circle centered at z_0.

In the sequel we will use also the regularized determinant $\det_2(I + A)$ defined for a Hilbert-Schmidt operator A by

$$\det_2(I + A) := \det\big((I + A)e^{-A}\big). \tag{2.13}$$

Proposition 3. *Suppose V satisfies (2.1) with $m_\perp > 0$. The following assertions are equivalent:*

(i) *$z_0 \in \mathcal{M}$ is a resonance of H,*

(ii) *z_0 is a pole of $|V|^{\frac{1}{2}} R(z) |V|^{\frac{1}{2}}$,*

(iii) *-1 is an eigenvalue of $\mathcal{T}_V(z_0) = J|V|^{\frac{1}{2}} R_0(z_0) |V|^{\frac{1}{2}}$.*

Moreover, the rank of the residue of $|V|^{\frac{1}{2}} R(z) |V|^{\frac{1}{2}}$ at z_0 is equal to the multiplicity of the resonance of H.

Assume now that V satisfies (2.1) with $m_\perp > 2$. Then

$$\det_2\big((H - z)(H_0 - z)^{-1}\big) = \det_2\big(I + \mathcal{T}_V(z)\big), \tag{2.14}$$

has a analytic continuation from $\mathcal{F}_+$ to $\mathcal{M}$. Its zeros are the resonances of H, and if z_0 is a resonance, there exists a holomorphic function $f(z)$, for z close to z_0, such that $f(z_0) \neq 0$ and

$$det_2\big(I + \mathcal{T}_V(z)\big) = (z - z_0)^{l(z_0)} f(z), \tag{2.15}$$

with $0 < l(z_0) = \mathrm{mult}(z_0)$ *where* $\mathrm{mult}(z_0)$ *is the multiplicity of the resonance defined by* (2.12).

Remarks. (i) The main part of the proof of Proposition 3 follows the arguments of [26]. To our best knowledge the novelty is the proof of (2.27) (i.e. the equality between the rank of the residue of $|V|^{\frac{1}{2}}R(z)|V|^{\frac{1}{2}}$ at z_0 and the multiplicity $\mathrm{mult}(z_0)$), and the equality $l(z_0) = \mathrm{mult}(z_0)$.

(ii) If H and H_0 are two self-adjoint operators such that $H - H_0 \in S_1$, the perturbation determinant $\det\big((H-\lambda)(H_0-\lambda)^{-1}\big)$, $\mathrm{Im}\,\lambda > 0$, was introduced by M. G. Krein in [18] (see also [15, Section IV.3]). In the case $H - H_0 \in S_r$ with $r > 1$ the generalized perturbation determinant $\det_r\big((H-\lambda)(H_0-\lambda)^{-1}\big)$ was introduced in [16]. In the last work, relatively Hilbert-Schmidt perturbations and the corresponding generalized perturbation determinants $\det_2\big((H-\lambda)(H_0-\lambda)^{-1}\big)$ were considered as well; these determinants are exactly of the same type as the one appearing in (2.13).

(iii) For potentials V compactly supported with respect to x_3, the resonances can be defined as the poles of the meromorphic extension of the resolvent:

$$(H-z)^{-1} : L^2_{\mathrm{comp}}\big(\mathbb{R}_{x_3}, L^2(\mathbb{R}^2_{X_\perp})\big) \to L^2_{\mathrm{loc}}\big(\mathbb{R}_{x_3}, L^2(\mathbb{R}^2_{X_\perp})\big) \tag{2.16}$$

from the open upper half plane to $\mathcal{M}$ (see [26] for the Schrödinger operator $-\Delta + V$).

Proof. Clearly, if z_0 is a pole of $|V|^{\frac{1}{2}}R(z)|V|^{\frac{1}{2}}$ then it is a pole of $R(z)$ and conversely according to the resolvent equation

$$\begin{aligned} R(z) =& R_0(z) - R(z)VR_0(z) \\ =& R_0(z) - R_0(z)VR_0(z) \\ &+ R_0(z)|V|^{\frac{1}{2}}J|V|^{\frac{1}{2}}R(z)|V|^{\frac{1}{2}}J|V|^{\frac{1}{2}}R_0(z), \end{aligned} \tag{2.17}$$

if z_0 is a pole of $R(z)$ then it is a pole of $|V|^{\frac{1}{2}}R(z)|V|^{\frac{1}{2}}$, and (i) is equivalent to (ii).

From the resolvent equation we get

$$\big(I + J|V|^{\frac{1}{2}}R_0(z)|V|^{\frac{1}{2}}\big)\big(I - J|V|^{\frac{1}{2}}R(z)|V|^{\frac{1}{2}}\big) = I. \tag{2.18}$$

From Proposition 2 and Lemma 1, we deduce that z_0 is a resonance if and only if -1 is an eigenvalue of $\mathcal{T}_V(z_0)$, and (ii) is equivalent to (iii).

Now we check the preservation of the multiplicity. Let $z_0 \in \mathcal{M}$ be a resonance and $N > 0$ be large enough to have $z_0 \in \mathcal{M}_N$. For z close to z_0, the resolvents, as operators from $L^2_{-N} := e^{-N\langle x_3\rangle}L^2(\mathbb{R}^3_{\mathbf{x}})$ to L^2_N , can be written

$$R_0(z) = \sum_{j\geq 0} M_j(z-z_0)^j \tag{2.19}$$

$$R(z) = (z-z_0)^{-L}A_{-L} + \cdots + (z-z_0)^{-1}A_{-1} + \mathrm{Hol}(z), \tag{2.20}$$

were the last term is holomorphic in a neighborhood of $z = z_0$. Classically, for γ a small positively oriented circle centered at z_0, we have

$$A_{-j} = \frac{1}{2i\pi}\int_\gamma (z-z_0)^{j-1}R(z)\,dz, \quad j \geq 1, \tag{2.21}$$

$\mathrm{mult}(z_0)$ being the rank of A_{-1}, and

$$A_{-j}(H-z_0)=(H-z_0)A_{-j}=A_{-j-1}. \tag{2.22}$$

Our next goal is to check the identities

$$\mathrm{rank}(A_{-1})=\mathrm{rank}(VA_{-1}), \tag{2.23}$$

$$\mathrm{rank}(A_{-1}^*)=\mathrm{rank}(VA_{-1}^*). \tag{2.24}$$

Let us prove (2.23). If this identity is false, there exists a function f such that $f=A_{-1}g$ and $Vf=0$.

Since f belongs to the range of A_{-1}, the distribution $H^m f$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, for any $m\in\mathbb{N}$. In particular, $Hf=H_0f\in H^2_{\mathrm{loc}}(\mathbb{R}^3)$, and hence $f\in H^4_{\mathrm{loc}}(\mathbb{R}^3)\subset C^2(\mathbb{R}^3)$.

Further, $Vf=0$ easily implies $VHf=0$. By recurrence, we obtain

$$VH^nf=0, \tag{2.25}$$

for any $n\in\mathbb{N}$. Plugging (2.20) into the r.h.s. of the resolvent equation

$$R(z)=R_0(z)-R_0(z)VR(z),$$

and integrating with respect to $z\in\gamma$, we find that (2.22) entails

$$f=A_{-1}g=-\sum_{j=0}^{L-1}M_jVA_{-j-1}g=-\sum_{j=0}^{L-1}M_jV(H-z_0)^jf. \tag{2.26}$$

Using (2.25) and (2.26), we get $f=0$ which immediately yields (2.23). Identity (2.24) can be proved exactly in the same way.

Applying (2.23) – (2.24), we get $\mathrm{rank}(A_{-1})=\mathrm{rank}(A_{-1}V)$, next $\mathrm{rank}\,(A_{-1})=\mathrm{rank}\,(A_{-1}V)=\mathrm{rank}\,(A_{-1}|V|^{\frac12})$, and, moreover, find that $|V|^{\frac12}$ is injective on the range of (A_{-1}). Thus we obtain

$$\mathrm{rank}(A_{-1})=\mathrm{rank}\left(|V|^{\frac12}A_{-1}|V|^{\frac12}\right), \tag{2.27}$$

which implies that the multiplicities agree.

We now prove the second part of the proposition. Let us recall that, if A is a bounded operator and if B is a trace class operator on some separable Hilbert space, we have $\det(I+AB)=\det(I+BA)$. Moreover, for A bounded and B Hilbert-Schmidt, we have

$$\det{}_2(I+AB)=\det{}_2(I+BA). \tag{2.28}$$

Writing, for $z\in\mathcal{F}_+$,

$$(H-z)(H_0-z)^{-1}=I+V(H_0-z)^{-1},$$

where $J|V|^{\frac12}(H_0-z)^{-1}$ is holomorphic on $\mathcal{F}_+$, with value in the Hilbert-Schmidt class, and using (2.28), we get

$$\det{}_2\big((H-z)(H_0-z)^{-1}\big)=\det{}_2\big(I+|V|^{\frac12}J|V|^{\frac12}(H_0-z)^{-1}\big)=\det{}_2\big(I+\mathcal{T}_V(z)\big).$$

From Lemma 1, this determinant has an analytic extension from $\mathcal{F}_+$ to $\mathcal{M}$ and vanishes if and only if z_0 is a resonance of H. Then, there exists a holomorphic function $f(z)$, for z close to z_0, such that $f(z_0)\neq 0$ and

$$\det{}_2\big(I+\mathcal{T}_V(z)\big)=(z-z_0)^{l(z_0)}f(z_0).$$

In order to prove that $l(z_0)=\mathrm{mult}(z_0)$ where $\mathrm{mult}(z_0)$ is the multiplicity of the resonance defined by (2.12), we need the following

Lemma 2. *The operator*

$$\Pi_{-1} = -A_{-1} \sum_{L-1 \geq j,k \geq 0} (H - z_0)^j V M_{j+k+1} V (H - z_0)^k,$$

is well defined in $\mathcal{L}(\mathcal{H}_N^{L-1})$ for any N where $\mathcal{H}_N^L$ is the Hilbert space

$$\mathcal{H}_N^L := \{u \in L_N^2(\mathbb{R}^3) = L^2(\mathbb{R}^3, e^{-N|x_3|}dx) \ \ \text{such that } H^k u \in L_N^2(\mathbb{R}^3), \ \forall k \leq L\},$$

equiped with the norm $\sum_{0 \leq k \leq L} \|H^k u\|_{L_N^2}$.

For any $k \in \mathbb{N}$, in $\mathcal{L}(L_{-N}^2, \mathcal{H}_N^k)$ we have

$$\Pi_{-1} A_{-1} = A_{-1}. \tag{2.29}$$

Here $\mathcal{L}(A, B)$ (resp. $\mathcal{L}(A)$) denotes the space of linear bounded operator from A to B (resp. A).

Proof of Lemma 2. We recall that from (2.26),

$$A_{-1} = -\sum_{j \geq 0} M_j V A_{-j-1}, \tag{2.30}$$

with the convention that $A_{-j} = 0$ for $j > L$. On the other hand, the resolvent equation

$$R_0(z) = R(z)\big(I + V R_0(z)\big),$$

yields

$$M_j = \sum_{k \leq j} A_k \widetilde{M}_{j-k} \tag{2.31}$$

with $\widetilde{M}_0 = I + V M_0$, $\widetilde{M}_j = V M_j$ for $j \geq 1$, and the equation $R_0(z) = \big(I + R_0(z) V\big) R(z)$ implies for any $k \geq 0$:

$$0 = \sum_{j \geq k} \widetilde{\widetilde{M}}_{j-k} A_{-j-1} \tag{2.32}$$

with $\widetilde{\widetilde{M}}_0 = I + M_0 V$, $\widetilde{\widetilde{M}}_j = M_j V$, $\widetilde{\widetilde{M}}_{-j} = 0$ for $j \geq 1$. In the above equality we use again the convention that $A_{-j} = 0$ for $j > L$. By inserting (2.31) into (2.30), we deduce

$$A_{-1} = -\sum_{j \geq 0} \sum_{k \leq j} A_k \widetilde{M}_{j-k} V A_{-j-1}.$$

Since $\widetilde{M}_j V = V \widetilde{\widetilde{M}}_j$ and $A_{-1}(H - z_0)^j = A_{-1-j}$, relation (2.32) implies

$$A_{-1} = -\sum_{j \geq 0} \sum_{k < 0} A_k V M_{j-k} V A_{-j-1} = \Pi_{-1} A_{-1}.$$

This conclude the proof of Lemma 2. □

Let us now complete the proof of Proposition 3. It follows from Lemma 2 that $\operatorname{rank} \Pi_{-1} = \operatorname{rank} A_{-1}$ and (2.29) implies that $\Pi_{-1}\Pi_{-1} = \Pi_{-1}$. Consequently, we have

$$\operatorname{mult}(z_0) = \operatorname{rank} A_{-1} = \operatorname{tr} \Pi_{-1}. \tag{2.33}$$

On the other hand, by the definition of $l(z_0)$, we have

$$l(z_0) = \frac{1}{2i\pi} \int_\gamma \partial_z \ln \det{}_2 \Big(1 + \mathcal{T}_V(z)\Big) dz. \tag{2.34}$$

Further, we have

$$\partial_z \ln\det\Big(1+T(z)\Big) = \operatorname{tr}\Big((1+T(z))^{-1}\partial_z T(z)\Big), \quad z\in\Omega,$$

for any operator-valued holomorphic function $\Omega \ni z \mapsto T(z) \in S_1$. Therefore,

$$\partial_z \ln\det{}_2\Big(1+\mathcal{T}_V(z)\Big) = \operatorname{tr}\Big((1+\mathcal{T}_V(z))^{-1}\partial_z \mathcal{T}_V(z)\Big) - \operatorname{tr}\Big(\partial_z\mathcal{T}_V(z)\Big).$$

According to Lemma 1, $\partial_z\mathcal{T}_V(z)$ is holomorphic in the trace class, then its integral on γ vanishes and (2.18) yields:

$$l(z_0) = -\frac{1}{2i\pi}\int_\gamma \operatorname{tr}\Big(J|V|^{\frac12}R(z)V\partial_z R_0(z)|V|^{\frac12}\Big)dz.$$

By definition of A_{-k} and M_j, we obtain:

$$l(z_0) = -\operatorname{tr}\Big(\sum_{L\geq k\geq 1} J|V|^{\frac12}A_{-k}kVM_k|V|^{\frac12}\Big),$$

where the trace is on $\mathcal{L}(L^2)$. Thanks to (2.22), we have $A_{-k} = A_{-1}(H-z_0)^{k-1}$ in $\mathcal{L}(L^2_{-N}, L^2_N)$ and using the cyclicity of the trace,

$$l(z_0) = -\operatorname{tr}\Big(\sum_{L\geq k\geq 1} A_{-1}(H-z_0)^{k-1}kVM_kV\Big). \tag{2.35}$$

Here the trace is in $\mathcal{L}(L^2_N)$, but since the range of A_{-1} is in any $\mathcal{H}^j_N$, $j\geq 1$, the last trace is also in any $\mathcal{H}^j_N$. At last, combining the cyclicity of the trace, with (2.22), (2.33) and (2.35) we deduce

$$\operatorname{mult}(z_0) = \operatorname{tr}(\Pi_{-1}) = \operatorname{tr}\Big(-\sum_{0\leq j,k\leq L-1} A_{-1}(H-z_0)^{k+j}VM_{k+j+1}V\Big) = l(z_0).$$

□

3. Resonances near the Landau levels

In this section we assume that V satisfies (2.1) with $m_\perp > 2$, and study the resonances localized in D^*_q, the neighborhood of the Landau level $2bq$ introduced in Definition 2. Recall that D^*_q can be parametrized by $z_q(k)$ defined in (2.5).

According to the previous section, these resonances can be identified with the points z where the determinant $\det_2(I+T_V(z))$ vanishes. Note that $\mathcal{T}_V(z)$ is the holomorphic extension of

$$J|V|^{\frac12}(H_0-z)^{-1}|V|^{\frac12} = \sum_{j\in\mathbb{N}} J|V|^{\frac12}P_j(H_0-z)^{-1}|V|^{\frac12}, \quad z\in\mathcal{F}_+, \tag{3.1}$$

where $P_j = p_j\otimes I_{x_3}$, $j\in\mathbb{N}$.

In order to study the resonances near a Landau level $2bq$ we split $\mathcal{T}_V(z)$ into two parts:

$$\mathcal{T}_V(z) = J|V|^{\frac12}P_qR_0(z)|V|^{\frac12} + \sum_{j\neq q} J|V|^{\frac12}P_jR_0(z)|V|^{\frac12}.$$

By Proposition 1, the second term in the r.h.s. is holomorphic in a neighborhood of $2bq$ with values in S_2. Let us consider the first term for $z = z_q(k)$. The series expansion with respect to k of the kernel of the operator t_N (see (2.3) and (2.4)) allows us to write t_N as the sum

$$t_N(k^2) = \frac{1}{k}t_1 + r_1(k), \tag{3.2}$$

where $t_1 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the rank-one operator defined by

$$t_1 u := \frac{i}{2}\langle u, e^{-N\langle\cdot\rangle}\rangle e^{-N\langle x_3\rangle}, \tag{3.3}$$

and $r_1(k)$ is the Hilbert-Schmidt operator with integral kernel

$$\mathcal{R}_1(x_3, x_3') = e^{-N\langle x_3\rangle}\, i\frac{e^{ik|x_3-x_3'|}-1}{2k} e^{-N\langle x_3'\rangle}. \tag{3.4}$$

Clearly, the operator-valued function $\mathbb{C} \ni k \mapsto r_1(k) \in S_2$ is analytic. Putting together the above considerations, we obtain the following

Proposition 4. *Suppose V satisfies* (2.1) *with $m_\perp > 2$. For $k \in \mathbb{C}^*$, $|k| < \sqrt{2b}$, we have:*

$$\mathcal{T}_V(z_q(k)) = \frac{iJ}{k} B_q + A(k), \tag{3.5}$$

where $J = \operatorname{sign} V$, B_q is the positive self-adjoint operator

$$B_q = \frac{1}{2}|V|^{1/2} P_q |V|^{1/2}, \tag{3.6}$$

and $A(k) \in S_2$ is the holomorphic operator defined on $\{k \in \mathbb{C},\ |k| < \sqrt{2b}\}$ by

$$A(k) = JA_q(k) + J\sum_{j\neq q} |V|^{\frac{1}{2}} P_j R_0(z_q(k))|V|^{\frac{1}{2}}, \tag{3.7}$$

where $A_q(k)$ is the operator with integral kernel

$$\mathcal{K}_{A_q}(X_\perp, x_3; X_\perp', x_3') = |V(X_\perp, x_3)|^{\frac{1}{2}} \mathcal{P}_{q,b}(X_\perp, X_\perp') \frac{1-e^{ik|x_3-x_3'|}}{2ik} |V(X_\perp', x_3')|^{\frac{1}{2}}.$$

Here, $\mathcal{P}_{q,b}$ is the integral kernel of the orthogonal projection p_q written in (2.10).

Since there exists an operator $C : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)$ such that $B_q = C^*C$ and $CC^* = \frac{1}{2}p_q W p_q$ with

$$W(X_\perp) = \int_{\mathbb{R}} |V(X_\perp, x_3)| dx_3, \tag{3.8}$$

(see [30] for $q = 0$, and the proof of Proposition 5.3 of [10] for any $q \in \mathbb{N}$), then for any $s > 0$ we have

$$n_+(s; B_q) = n_+(2s; p_q W p_q), \tag{3.9}$$

where for a compact self-adjoint operator A, we set $n_+(s; A) = \operatorname{rank} \mathbf{1}_{(s,+\infty)}(A)$.

Remark. Using (2.18) and (3.5), we can prove that each Landau level is an essential singularity of the resolvent of H, but it is not sufficient to deduce the existence of an infinite number of resonances near the Landau levels.

In the case where the decay of U at infinity is regular enough, the asymptotic distribution of the eigenvalues of Toeplitz-type operators $p_q U p_q$ is well known. The following three lemmas describe the eigenvalue asymptotics for $p_q U p_q$ in the case of power-like decay, exponential decay, or compact support of U, respectively.

Lemma 3. (Theorem 2.6 of [21]) *Let the function $U \in C^1(\mathbb{R}^2)$ satisfy the estimates*

$$0 \leq U(X_\perp) \leq C_1 \langle X_\perp \rangle^{-\alpha}, \qquad |\nabla U(X_\perp)| \leq C_1 \langle X_\perp \rangle^{-\alpha-1}, \quad X_\perp \in \mathbb{R}^2,$$

for some $\alpha > 0$ and $C_1 > 0$. Assume, moreover that

$$U(X_\perp) = u_0(X_\perp/|X_\perp|)\,|X_\perp|^{-\alpha}(1 + o(1)), \quad |X_\perp| \to \infty,$$

where u_0 is a continuous function on S^1 which is non-negative and does not vanish identically. Then for each $q \in \mathbb{N}$ we have

$$n_+(s; p_q U p_q) = C_\alpha s^{-2/\alpha}(1 + o(1)), \quad s \searrow 0,$$

where

$$C_\alpha := \frac{b}{4\pi} \int_{S^1} u_0(t)^{2/\alpha} dt. \tag{3.10}$$

Lemma 4. (Theorem 2.1 of [24]) *Let $0 \leq U \in L^\infty(\mathbb{R}^2)$. Assume that*

$$\ln U(X_\perp) = -\mu |X_\perp|^{2\beta}(1 + o(1)), \quad |X_\perp| \to \infty,$$

for some $\beta > 0$, $\mu > 0$. Then for each $q \in \mathbb{N}$ we have

$$n_+(s; p_q U p_q) = \varphi_\beta(s)(1 + o(1)), \quad s \searrow 0,$$

where

$$\varphi_\beta(s) := \begin{cases} \frac{b}{2}\mu^{-\frac{1}{\beta}} |\ln s|^{\frac{1}{\beta}} & \text{if} \quad 0 < \beta < 1, \\ \frac{1}{\ln(1+2\mu/b)} |\ln s| & \text{if} \quad \beta = 1, \\ \frac{\beta}{\beta-1} (\ln|\ln s|)^{-1} |\ln s| & \text{if} \quad \beta > 1, \end{cases} \qquad 0 < s < e^{-1}.$$

Lemma 5. (Theorem 2.4 of [24]) *Let $0 \leq U \in L^\infty(\mathbb{R}^2)$. Assume that the support of U is compact and there exists a constant $C > 0$ such that $U \geq C$ on an non-empty open subset of $\mathbb{R}^2$. Then for each $q \in \mathbb{N}$ we have*

$$n_+(s; p_q U p_q) = \varphi_\infty(s)(1 + o(1)), \quad s \searrow 0,$$

where

$$\varphi_\infty(s) := (\ln|\ln s|)^{-1} |\ln s|, \quad 0 < s < e^{-1}.$$

Remark. In the recent preprint [11] a sharper version of the result of Lemma 5 has been obtained, containing three asymptotic terms as $s \searrow 0$ of $n_+(s; p_q U p_q)$ provided that the support of U is compact, and U satisfies some additional technical assumptions. In particular, the asymptotic expansion of $n_+(s; p_q U p_q)$ obtained in [11] recovers *the logarithmic capacity* of the support of U.

The above lemmas imply some useful properties of B_q summarized in the following

Corollary 1. *Let $s > 0$. For V satisfying (1.1) with $m_\perp > 2$, the operator B_q is a trace class operator with $n_+(s, B_q) = \mathcal{O}(s^{-2/m_\perp})$ for $s > 0$ small enough. For $j \in \mathbb{N}^* := \{1, 2, \dots\}$, the operator-valued functions*

$$\mathbb{C} \setminus (\mp i[0, +\infty[) \ni k \mapsto \mathcal{B}(k) = \mathcal{B}^\pm_{q,j}(k) := \frac{iB_q}{k}\Big(I \pm \frac{iB_q}{k}\Big)^{-j} \in S_1 \tag{3.11}$$

are holomorphic. Their Hilbert-Schmidt norms (p=2) and trace-class norms (p=1) satisfy the estimates

$$\|\mathcal{B}(k)\|_p \leq c(\theta)^j \, \sigma_p(|k|)^{\frac{1}{p}}, \tag{3.12}$$

where $\theta = \operatorname{Arg} k$, $c(\theta) = (1 - (\sin\theta)_-)^{-\frac{1}{2}}$, $u_- := \max\{-u, 0\}$ *if* $u \in \mathbb{R}$, *and*

$$\sigma_p(s) := \left\| \frac{B_q}{s}\Big(I + \frac{B_q^2}{s^2}\Big)^{-1/2} \right\|_p^p = \mathcal{O}(s^{-2/m_\perp}), \quad s > 0. \tag{3.13}$$

Further, for $s > 0$, $p \geq 1$, *we have*

$$2^{-p/2}\, \widetilde{n}_p(s) \;\leq\; \sigma_p(s) \;\leq\; \widetilde{n}_p(s) + n_+(s, B_q), \tag{3.14}$$

where

$$\widetilde{n}_p(s) := \left\| \frac{B_q}{s} \mathbf{1}_{[0,s]}(B_q) \right\|_p^p, \quad s > 0, \quad p \geq 1. \tag{3.15}$$

Moreover, for W *defined by (3.8) satisfying the assumptions of Lemma 3 with* $\alpha > 2$, *the estimates*

$$\sigma_p(s) = C_{\alpha,p} s^{-\frac{2}{\alpha}}(1 + o(1)), \quad \widetilde{n}_p(s) = \widetilde{C}_{\alpha,p} s^{-\frac{2}{\alpha}}(1 + o(1)) \quad s \searrow 0, \tag{3.16}$$

hold with some $C_{\alpha,p} > 0$, $\widetilde{C}_{\alpha,p} > 0$, $p = 1, 2$. *Finally, if the assumptions of Lemma 4 or of Lemma 5 hold for* $W = U$, *we have*

$$\sigma_p(s) = \varphi_\beta(s)(1 + o(1)), \quad \widetilde{n}_p(s) = o(\varphi_\beta(s)) \quad s \searrow 0, \tag{3.17}$$

the functions $\varphi_\beta(s)$, $0 < \beta \leq \infty$, *being defined in Lemma 4 or in Lemma 5.*

Proof. By (3.9), $B_q \in S_1$ if and only if $p_q W p_q \in S_1$. For V satisfying (1.1), we have $0 \leq W(X_\perp) \leq C\langle X_\perp\rangle^{-m_\perp}$. Therefore, $W \in L^1(\mathbb{R}^2)$, and hence $p_q W p_q \in S_1$, and then $B_q \in S_1$. According to Lemma 3, $n_+(s, p_q C\langle X_\perp\rangle^{-m_\perp} p_q)$ behaves like $s^{-2/m_\perp}$ as $s \searrow 0$, then $n_+(s, B_q) = \mathcal{O}(s^{-2/m_\perp})$ for $s > 0$ small enough. Taking also into account that $B_q \geq 0$, we conclude that the operator-valued functions $\mathcal{B}$ defined in (3.11) are holomorphic. Let us now estimate their norms in S_1 and in S_2. For $k = |k|e^{i\theta}$, we have

$$\mathcal{B}^*\mathcal{B} = \frac{B_q^2}{|k|^2}\Big(I + \frac{B_q^2}{|k|^2} \pm 2\sin\theta \frac{B_q}{|k|}\Big)^{-j}.$$

Next,

$$\|\mathcal{B}\|_{S_p}^p = -\int_0^\infty f_{j,p}(\frac{u}{|k|}, \pm\theta)dn_+(u; B_q) = -\int_0^\infty f_{j,p}(s, \pm\theta)dn_+(s; \frac{B_q}{|k|}), \tag{3.18}$$

where $f_{j,p}(u,\theta) := u^p(1 + u^2 + 2u\sin\theta)^{-jp/2}$. Evidently, for $\theta \neq -\pi/2$ and $u \geq 0$ we have

$$f_{j,p}(u,\theta) \leq c(\theta)^{jp} f_p(u) \tag{3.19}$$

where $f_p(u) := u^p(1 + u^2)^{-p/2}$, $p = 1, 2$. Finally,

$$\sigma_p(s) = \operatorname{tr}\Big(\frac{B_q^p}{s^p}\Big(I + \frac{B_q^2}{s^2}\Big)^{-p/2}\Big) = -\int_0^\infty f_p(u)dn_+(u; \frac{B_q}{s}). \tag{3.20}$$

Now the combination of (3.18), (3.19), and (3.20), yields (3.12). We have also

$$\widetilde{n}_p(s) = -\int_0^1 u^p dn_+(u; \frac{B_q}{s}). \tag{3.21}$$

Then (3.14) is a consequence of the elementary inequalities

$$2^{-p/2}u^p \mathbf{1}_{[0,1]}(u) \;\leq f_p(u) \;\leq\; u^p \mathbf{1}_{[0,1]}(u) + \mathbf{1}_{]1,+\infty[}(u).$$

In order to prove (3.16) - (3.17), we first note that since $\lim_{u\downarrow 0} u\, n_+(u, B_q) = 0$, relations (3.20), (3.21) and (3.9) imply

$$\sigma_p(s) = \int_0^\infty f_p'(u) n_+(2su; p_q W p_q) du, \tag{3.22}$$

$$\widetilde{n}_p(s) = \int_0^1 p u^{p-1} (n_+(2su; p_q W p_q) - n_+(2s; p_q W p_q)) du. \tag{3.23}$$

Then for W satisfying the assumptions of Lemma 3 or of Lemma 4 or of Lemma 5 we deduce the asymptotic properties claimed. □

Proposition 5. *Suppose that V satisfies (2.1) with $m_\perp > 2$. For $0 < s < |k| < s_0$ with s_0 sufficiently small, $z_q(k) \in D_q^*$ is a resonance of H if and only if k is a zero of*

$$D(k, s) = \det \Big(I + K(k, s) \Big), \tag{3.24}$$

where $K(k, s)$ is a finite-rank operator satisfying

$$\operatorname{rank} K(k, s) = \mathcal{O}\Big(n_+(s; p_q W p_q) + 1 \Big), \quad \| K(k, s) \| = \mathcal{O}(s^{-1}),$$

uniformly with respect to $s < |k| < s_0$.

Moreover, for $\operatorname{Im} k^2 > \delta > 0$, the operator $I + K(k, s)$ is invertible with

$$\| (I + K(k, s))^{-1} \| = \mathcal{O}(\delta^{-1}),$$

uniformly with respect to $s < |k| < s_0$, $\operatorname{Im} k^2 > \delta$.

Proof. By Proposition 3 − 4, for $s < |k| \leq s_0 < \sqrt{2b}$, $z_q(k)$ is a resonance of H if and only if k is a zero of $\det_2(I + \frac{iJ}{k} B_q + A(k))$.

Since $k \mapsto A(k)$ is holomorphic near $k = 0$ with value in S_2, for s_0 sufficiently small, there exist a finite-rank operator A_0 independent of k and $\widetilde{A}(k)$ holomorphic near $k = 0$ in S_2 with $\| \widetilde{A}(k) \| \leq \frac{1}{4}$, $|k| \leq s_0$ such that

$$A(k) = A_0 + \widetilde{A}(k).$$

Further, let us decompose the self-adjoint positive operator B_q into a trace-class operator whose norm is bounded by $s/2$, and an operator of rank $n_+(s/2; B_q)$, namely

$$B_q = B_q \mathbf{1}_{[0, s/2]}(B_q) + B_q \mathbf{1}_{]s/2, +\infty[}(B_q). \tag{3.25}$$

Since $\| \frac{iJ}{k} B_q \mathbf{1}_{[0, s/2]}(B_q) + \widetilde{A}(k) \| \leq \frac{3}{4}$, for $0 < s < |k| < s_0$, we have

$$\det \Big((I + \frac{iJ}{k} B_q \mathbf{1}_{[0, s/2]}(B_q) + \widetilde{A}(k)) \; e^{-\mathcal{T}_V(z_q(k))} \Big) \neq 0.$$

It follows that for $0 < s < |k| < s_0$, the zeros of $\det_2 \Big(I + \mathcal{T}_V(z_q(k)) \Big)$ are the zeros of $D(k, s)$ defined by (3.24) with

$$K(k, s) = \Big(\frac{iJ}{k} B_q \mathbf{1}_{]s/2, +\infty[}(B_q) + A_0 \Big) \Big(I + \frac{iJ}{k} B_q \mathbf{1}_{[0, s/2]}(B_q) + \widetilde{A}(k) \Big)^{-1}.$$

The rank of this operator is bounded by $\mathcal{O}(n_+(s/2; B_q) + 1) = \mathcal{O}(n_+(s; p_q W p_q) + 1)$ (see (3.9)) and its norm is bounded by $\mathcal{O}(|k|^{-1})$.

At last, by the definition of $\mathcal{T}_V(z)$ and of $K(k,s)$, we have

$$I + K(k,s) = \Big(I + \mathcal{T}_V(z_q(k))\Big)\Big(I + \frac{iJ}{k} B_q \mathbf{1}_{[0,s/2]}(B_q) + \widetilde{A}(k)\Big)^{-1},$$

provided that $0 < s < |k| < s_0$. By the resolvent equation (2.18), the operator $I + \mathcal{T}_V(z)$ is invertible for $\operatorname{Im} z > \delta$, and

$$\Big(I + \mathcal{T}_V(z)\Big)^{-1} = I - J|V|^{\frac{1}{2}}(H - z)^{-1}|V|^{\frac{1}{2}}.$$

Then $I + K(k,s)$ is invertible for $\operatorname{Im} k^2 > \delta$, $0 < s < |k| < s_0$, and

$$\|(I + K(k,s))^{-1}\| = \mathcal{O}(1 + \||V|^{\frac{1}{2}}(H - z_q(k))^{-1}|V|^{\frac{1}{2}}\|) = \mathcal{O}(1 + |\operatorname{Im} k^2|^{-1})$$

which concludes the proof of Proposition 5. □

By the properties of $K(k,s)$ (see Proposition 5) for $0 < s < |k| < s_0$, we have:

$$D(k,s) = \prod_{j=1}^{\mathcal{O}(n_+(s;p_qWp_q)+1)} (1 + \lambda_j(k,s)) = \mathcal{O}(1)\exp\Big(\mathcal{O}(n_+(s;p_qWp_q)+1)|\ln s|\Big), \tag{3.26}$$

uniformly with respect to (k,s), where $\lambda_j(k,s)$ are the eigenvalues of $K(k,s)$ which satisfy $\lambda_j(k,s) = \mathcal{O}(|s|^{-1})$.

Moreover, since

$$D(k,s)^{-1} = \det\Big((I+K)^{-1}\Big) = \det\Big(I - K(I+K)^{-1}\Big),$$

for $\operatorname{Im} k^2 > \delta > 0$, and for $0 < s < |k| < s_0$, we have

$$|D(k,s)| \geq C\exp\Big(-C(n_+(s;p_qWp_q)+1)(|\ln\delta| + |\ln s|)\Big), \tag{3.27}$$

uniformly with respect to (k,s).

The following lemma contains a version of the well known Jensen inequality which is suitable for our purposes.

Lemma 6. *Let Ω be a simply connected sub-domain of $\mathbb{C}$ and let g be a holomorphic function in Ω with continuous extension to $\overline{\Omega}$. Assume there exists $z_0 \in \Omega$ such that $g(z_0) \neq 0$ and $g(z) \neq 0$ for $z \in \partial\Omega$. Let $z_1, z_2, ..., z_N \in \Omega$ be the zeros of g repeated according to their multiplicity. For any domain $\Omega' \subset\subset \Omega$, there exists $C' > 0$ such that $N(\Omega', g)$, the number of zeros z_j of g contained in Ω', satisfies*

$$N(\Omega', g) \leq C\Big(\int_{\partial\Omega} \ln|g(z)|dz - \ln|g(z_0)|\Big).$$

Proof. First, let us recall the classical Jensen inequality

$$N(B(0,\nu R), G)|\ln\nu| \leq \frac{1}{2\pi}\int_0^{2\pi} \ln|G(Re^{i\theta})|d\theta - \ln|G(0)|,$$

valid for any $0 < \nu < 1$, and for a function $g = G$ satisfying the assumptions of the lemma in $\Omega = B(0,R) := \{z \in \mathbb{C};\ |z| < R\}$, and $z_0 = 0$.

Now, let $f:\ B(0,R) \to \Omega$ be a bijective analytic function such that $f(0) = z_0$ and $f(\partial B(0,R)) = \partial\Omega$. Then $G = gof$ satisfies the assumptions of the lemma in $\Omega = B(0,R)$, with $z_0 = 0$, and we have the above formula. Since f is a bijection and $f(\partial B(0,R)) = \partial\Omega$, for $\Omega' \subset\subset \Omega$ there exists $0 < \nu < 1$ such that $\Omega' \subset f(B(0,\nu R))$, which implies the claim of the lemma. □

Applying this lemma to the function $g(k) := D(r\,k, r)$, on $\Omega := \{k \in \mathbb{C};\ 1 < |k| < 2,\ \frac{\pi}{3} < \operatorname{Arg} k < 2\pi + \frac{\pi}{6}\}$ with $\operatorname{Im} k_0^2 > \delta > 0$, we deduce from (3.26), (3.27) the following upper bound on the number of resonances near the Landau levels.

Theorem 1. Upper bound. *Suppose that V satisfies (2.1) with $m_\perp > 2$. Then there exists $r_0 > 0$, such that for any $0 < r < r_0$,*

$$\#\{z = z_q(k) \in \operatorname{Res}(H) \cap D_q^*;\ r < |k| < 2r\} = \mathcal{O}(n_+(r, p_q W p_q)|\ln r|),$$

where W is given by (3.8), and $n_+(s; p_q W p_q)$ is the counting function satisfying asymptotic relations depending on the decay of W, described in Lemmas 3, 4, and 5. In particular, under our assumptions we have always $n_+(s; p_q W p_q) = \mathcal{O}(s^{-2/m_\perp})$, and for V compactly supported, we have $n_+(s; p_q W p_q) = \mathcal{O}((\ln|\ln s|)^{-1}|\ln s|)$.

Remark. Instead of the three-dimensional case considered in the present paper, it is possible to consider a general n-dimensional Schrödinger operator with non vanishing constant magnetic field B which can be regarded as a real antisymmetric matrix acting in $\mathbb{R}^n$. Set $2d = \operatorname{rank} B$ and $\tilde{d} := n - 2d$, so that in the three-dimensional case we have $d = 1$ and $\tilde{d} = 1$. Note that the spectrum of the unperturbed Schrödinger operator $H_0(B)$ with magnetic field B is pure point if $\tilde{d} = 0$, and is purely absolutely continuous if $\tilde{d} \geq 1$. For $\tilde{d} \geq 1$ the unperturbed operator can be written in appropriate Cartesian coordinates $(x, y, z) \in \mathbb{R}^n$ with $x, y \in \mathbb{R}^d$ and $z \in \mathbb{R}^{\tilde{d}}$, as

$$\sum_{j=1}^{d} \Big(\Big(D_{x_j} + \frac{b_j}{2} y_j \Big)^2 + \Big(D_{y_j} - \frac{b_j}{2} x_j \Big)^2 \Big) + \sum_{l=1}^{\tilde{d}} D_{z_l}^2,$$

where $B = \sum_{j=1}^{d} b_j dx_j \wedge dy_j$. We believe that the presence of infinitely many resonances is typical for the cases $\tilde{d} = 1$ and $\tilde{d} = 2$. However, the Riemann surfaces where the resonances are defined, and the eigenvalue counting functions for the corresponding Toeplitz operators (see [21]) which occur in the estimates of the resonances, should be of different type in these two cases. On the other hand, if $\tilde{d} \geq 3$, we expect that the number of the resonances near any fixed Landau level should be finite. The qualitative pictures in the cases $\tilde{d} = 1$, $\tilde{d} = 2$, and $\tilde{d} \geq 3$, should be independent of the rank $2d$ of the non-vanishing magnetic field B.

4. Perturbation of definite sign

In this section, we discuss the case $\pm V \geq 0$. We will obtain an upper bound of the number of resonances near the Landau levels outside a semi-axis. Further, for small perturbations, we prove the existence of a region free of resonances, and obtain a lower bound on the number of resonances near a semi-axis. In particular, we show that for small positive perturbations there are no embedded eigenvalues.

In the definite-sign case, we can summarize our results by Figure 2.

Let V have a definite sign, i.e. let $J = \operatorname{sign} V$ be constant, $J = \pm 1$ when $\pm V \geq 0$. In this case, according to Proposition 4, we have

$$\mathcal{T}_V(z_q(k)) = \frac{i}{k} J B_q + A(k),$$

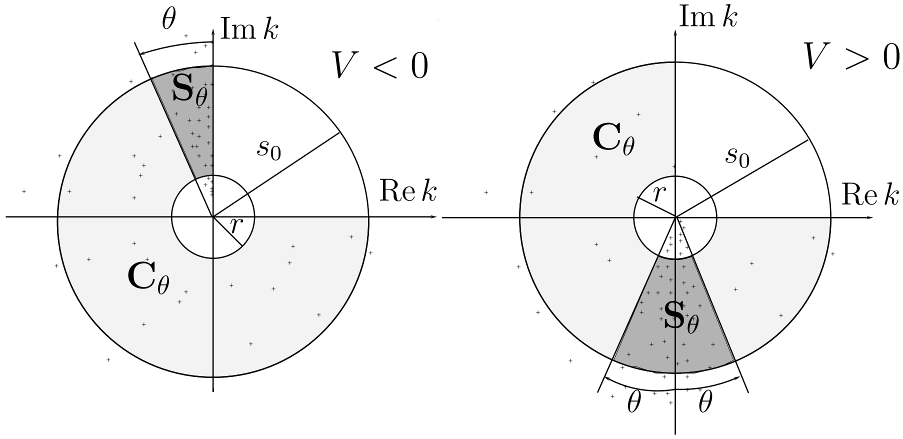

FIGURE 2. **Resonances near a Landau level for V of definite sign.** Resonances $z = z_q(k)$ are concentrated near the semi axis $k = -i(\mathrm{sgn}V)]0, +\infty[$. On the one hand, for any θ, the number of resonances in C_θ is bounded by $\mathcal{O}(|\ln r|)$ for $s_0 = s_0(\theta)$ sufficiently small (Proposition 6). On the other hand, for any $0 < s_0 < \sqrt{2b}$ and any θ, there is no resonance of $H_0 + \varepsilon V$ in C_θ for $\varepsilon \le \varepsilon_0(\theta)$ sufficiently small and for compactly supported V we have lower bound of the number of resonances in S_θ (see Theorem 2).

with B_q a positive self-adjoint operator independent of k, and $A(k)$ holomorphic near $k = 0$ with values in S_2. For $iJk \notin \mathrm{sp}(B_q)$, the operator $I + \frac{i}{k}JB_q$ is invertible with

$$\|(I + \frac{i}{k}JB_q)^{-1}\| \le \frac{|k|}{\sqrt{(J\,\mathrm{Im}\,k)_+^2 + |\,\mathrm{Re}\,k|^2}},$$

and for $-\delta J\,\mathrm{Im}\,k < |\,\mathrm{Re}\,k|$, the estimate $\|(I + \frac{i}{k}JB_q)^{-1}\| \le \sqrt{1+\delta^{-2}}$ holds uniformly with respect to k, $|k| < s_0$, $-\delta J\,\mathrm{Im}\,k < |\,\mathrm{Re}\,k|$.

We have

$$I + \mathcal{T}_V(z_q(k)) = (I + K(k))\left(I + \frac{i}{k}JB_q\right),$$

with

$$K(k) := A(k)\left(I + \frac{i}{k}JB_q\right)^{-1}.$$

Note that $K(k) \in S_2$, and its Hilbert-Schmidt norm is uniformly bounded with respect to k, for $|k| < s_0$, $-\delta J\,\mathrm{Im}\,k < |\,\mathrm{Re}\,k|$. Therefore,

$$\det{}_2\Big(I + \mathcal{T}_V(z_q(k))\Big) = \det\left(I + \frac{i}{k}JB_q\right)\det{}_2\Big(I + K(k)\Big)e^{-\mathrm{Tr}(\mathcal{T}_V(z_q(k)) - K(k))}. \tag{4.1}$$

This relation is obtained by approximating the Hilbert-Schmidt operator K by a finite-rank operator, and using the fact that for a trace-class operator B, we have $\det_2(I+B) = \det(I+B)e^{-\mathrm{tr}\,B}$. We exploit moreover, the fact that since B_q is a trace class operator (see Corollary 1), then such is $(\mathcal{T}_V(z_q(k)) - K(k)) = (I + K(k))\frac{i}{k}JB_q$.

According to (4.1), for $|k| < s_0$, $-\delta J\,\mathrm{Im}\,k < |\,\mathrm{Re}\,k|$, the zeros of $\det_2(I + \mathcal{T}_V(z_q(k)))$ are the zeros of $\det_2(I + K(k))$. By the properties of $K(k)$, $\det_2(I + K(k)) = \mathcal{O}(e^{C\|K(k)\|_2^2}) = \mathcal{O}(1)$, uniformly with respect to k. On the other hand, writing

$$(I+K)^{-1} = (I + \frac{i}{k} J B_q)\,(I + \mathcal{T}_V)^{-1},$$

and arguing as in the proof of Proposition 5, we find that

$$\|(I+K)^{-1}\| = \mathcal{O}(|s|^{-1})\mathcal{O}(\delta^{-1})$$

for $\operatorname{Im} k^2 > \delta > 0$, and for $0 < s < |k| < s_0$, uniformly with respect to (k,s). If $(\lambda_j)_j$ denotes the sequence of eigenvalues of $K(k)$, the above estimate implies that for $\operatorname{Im} k^2 > \delta > 0$, and for $0 < s < |k| < s_0$, we have

$$|1+\lambda_j|^{-1} = \mathcal{O}(|s|^{-1})\mathcal{O}(\delta^{-1}). \tag{4.2}$$

Now, we are able to establish a lower bound of $\det_2(I + K(k))$. We have

$$\left| \Big(\det_2(I+K(k)) \Big)^{-1} \right| = \left| \det \Big((I+K(k))^{-1} e^{K(k)} \Big) \right| \leq \prod_{|\lambda_j| \leq \frac{1}{2}} \left| \frac{e^{\lambda_j}}{1+\lambda_j} \right| \times \prod_{|\lambda_j| > \frac{1}{2}} \frac{e^{|\lambda_j|}}{|1+\lambda_j|}.$$

The first product is uniformly bounded because $K(k)$ is uniformly bounded in S_2 and we estimate the second product by $\mathcal{O}(e^{C(|\ln \delta| + |\ln s|)})$ using the fact that it involves a finite number of factors bounded by $\mathcal{O}(|s|^{-1})\mathcal{O}(\delta^{-1})$ (see (4.2)). We get

$$|\det_2(I+K(k))| \geq C e^{-C(|\ln \delta| + |\ln s|)}$$

for $\operatorname{Im} k^2 > \delta > 0$, and for $0 < s < |k| < s_0$. Consequently, from the Jensen inequality (Lemma 6), in the case V of definite sign, we establish upper bounds outside a neighborhood of $\{z_j(k);\ k \in (-iJ)[0,+\infty[\}$:

Proposition 6. Upper bound: special case. *Suppose that V satisfying (2.1) with $m_\perp > 2$, is of definite sign J. For any $\delta > 0$, there exists $s_0 > 0$, such that for any $0 < r < s_0$ we have*

$$\#\{z = z_q(k) \in \operatorname{Res}(H) \cap D_q^*;\ r < |k| < 2r,\ -\delta J \operatorname{Im} k < |\operatorname{Re} k|\} = \mathcal{O}(|\ln r|).$$

In what follows, we prove also that for small perturbations of definite sign the resonances are near $z_q(k)$ with k eigenvalues of $-iJB_q$. In particular, we have a infinite number of resonances close to the Landau levels.

In order to obtain our lower bound of the counting function of resonances, we need the following result deduced from Lemma 4.

Lemma 7. *Let $0 \leq W \in L^\infty(\mathbb{R}^2)$ such that*

$$\ln W(X_\perp) \leq -C \langle X_\perp \rangle^2, \tag{4.3}$$

for some $C > 0$. Let $(\lambda_j)_j$ be the non-increasing sequence of the non-vanishing eigenvalues of $p_q W p_q$, counted with their multiplicity. Then there exists $\nu > 0$ such that

$$\#\{j;\ \lambda_j - \lambda_{j+1} > \nu \lambda_j\} = \infty. \tag{4.4}$$

Proof. By assumption, one can find a function U which satisfies the hypotheses of Lemma 4 with $\beta = 1$ such that $W \leq U$. Then $p_q W p_q \leq p_q U p_q$ and

$$n_+(s; p_q W p_q) \leq n_+(s; p_q U p_q) = \mathcal{O}(|\ln s|), \tag{4.5}$$

by the min-max principle.

Let us assume that the set $\{j;\ \lambda_j - \lambda_{j+1} > \nu\lambda_j\}$ is finite for any $\nu > 0$. Then there exists j_ν such that for any $j \geq j_\nu$, $\lambda_j - \lambda_{j+1} \leq \nu\lambda_j$. This implies that for any $j > j_\nu$, $\lambda_j \geq (1-\nu)^{j-j_\nu}\lambda_{j_\nu}$. In this case for s sufficiently small we would have

$$n_+(s; p_q W p_q) = \#\{j;\ \lambda_j > s\} \geq \#\{j;\ (1-\nu)^{j-j_\nu}\lambda_{j_\nu} > s\},$$

that is

$$n_+(s; p_q W p_q) \geq |\ln(1-\nu)|^{-1}\,|\ln s| - \mathcal{O}_\nu(1).$$

If we choose $\nu > 0$ small enough, this lower bound is in contradiction with the estimate (4.5). $\square$

Theorem 2. Sector free of resonances, upper and lower bound. *Let $0 < s_0 < \sqrt{2b}$ and $q \in \mathbb{N}$. Assume V satisfies (2.1) with $m_\perp > 2$ and is of definite sign J. Then for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that:*

(i) *for any $\varepsilon \leq \varepsilon_0$, $H_\varepsilon := H_0 + \varepsilon V$ has no resonances in $\{z = z_q(k) \in D_q^*;\ 0 < |k| < s_0,\ -J\operatorname{Im} k \leq \frac{1}{\delta}|\operatorname{Re} k|\}$.*

(ii) *there exists $r_0 > 0$, such that for any $0 < r < r_0$ and $\varepsilon \leq \varepsilon_0$, we have*

$$\#\{z = z_q(k) \in \operatorname{Res}(H_\varepsilon) \cap D_q^*;\ r < |k| < 2r\} = \mathcal{O}(n_+(r, \varepsilon p_q W p_q) - n_+(8r, \varepsilon p_q W p_q)). \tag{4.6}$$

(iii) *if W defined by (3.8) satisfies (4.3), then for any $\varepsilon \leq \varepsilon_0$, H_ε has an infinite number of resonances in $\{z = z_q(k) \in D_q^*;\ 0 < |k| < s_0,\ -J\operatorname{Im} k > \frac{1}{\delta}|\operatorname{Re} k|\}$.*

More precisely, there exists a decreasing sequence $(r_l)_{l\in\mathbb{N}}$ of positive numbers, $r_l \searrow 0$ such that,

$$\begin{aligned}\#\{z = z_q(k) \in \operatorname{Res}(H_\varepsilon) \cap D_q^*;\ r_{l+1} < |k| < r_l,\ -J\operatorname{Im}k > \frac{1}{\delta}|\operatorname{Re} k|\}\\ \geq \operatorname{rank} \mathbf{1}_{[2r_{l+1}, 2r_l]}(\varepsilon p_q W p_q).\end{aligned} \tag{4.7}$$

Proof. (i) We have

$$I + \mathcal{T}_{\varepsilon V} = I + \varepsilon \mathcal{T}_V = I + \frac{i}{k}\varepsilon J B_q + \varepsilon A(k). \tag{4.8}$$

Since B_q is self-adjoint and positive, the operator $I + \frac{i}{k}\varepsilon J B_q$ is invertible for $-J\operatorname{Im} k < \frac{1}{\delta}|\operatorname{Re} k|$, and we have

$$\|(I + \frac{i}{k}\varepsilon J B_q)^{-1}\| \leq \sqrt{1+\delta^{-2}}.$$

Moreover, for $|k| \leq s_0$ there exists $C > 0$ such that $\|A(k)\| \leq C$. Consequently, for $\varepsilon < (C\sqrt{1+\delta^{-2}})^{-1}$ and $-J\operatorname{Im} k \leq \frac{1}{\delta}|\operatorname{Re} k|$, the operator $I + \frac{i}{k}\varepsilon J B_q + \varepsilon A(k)$ is invertible and $z_q(k)$ is not a resonance of $H_0 + \varepsilon V$.

(ii) We prove this point like Theorem 1. Let

$$B^+ := \varepsilon B_q \mathbf{1}_{]r/2, 4r[}(\varepsilon B_q) \quad \text{and} \quad B^- := \varepsilon B_q \mathbf{1}_{[0, r/2] \cup [4r, +\infty[}(\varepsilon B_q).$$

For $\frac{2}{3}r < |k| < \frac{3}{2}r$, the spectrum of the self-adjoint operator $\frac{1}{|k|}B^-$ is a subset of $[0, \frac{3}{4}] \cup [\frac{8}{3}, +\infty[$. Then $I + \frac{i}{k}JB^-$ is invertible and

$$\left\| \left(I + \frac{i}{k}JB^-\right)^{-1} \right\| \leq 4.$$

So, if ε_0 is small enough and $0 \leq \varepsilon \leq \varepsilon_0$, the operator $I + \frac{i}{k}JB^- + \varepsilon A(k)$ is invertible for $\frac{2}{3}r < |k| < \frac{3}{2}r$ with a uniformly bounded inverse.

Using (4.8), we can write

$$\begin{aligned} I + \mathcal{T}_{\varepsilon V} &= I + \frac{i}{k} J B^+ + \frac{i}{k} J B^- + \varepsilon A(k) \\ &= \Big(I + \frac{i}{k} J B^- + \varepsilon A(k) \Big) \Big(I + \Big(I + \frac{i}{k} J B^- + \varepsilon A(k) \Big)^{-1} \frac{i}{k} J B^+ \Big), \end{aligned} \tag{4.9}$$

and, from Proposition 3, the resonances of H_ε, with $\frac{2}{3} r < |k| < \frac{3}{2} r$, are the zeros of

$$\widetilde{D}(k,r) := \det \Big(I + \Big(I + \frac{i}{k} J B^- + \varepsilon A(k) \Big)^{-1} \frac{i}{k} J B^+ \Big). \tag{4.10}$$

Moreover, the multiplicity of the resonance, $\mathrm{mult}(k)$, is equal to the order of the zero of $\widetilde{D}(k,r)$. Since B^+/k is uniformly bounded, there exists $C > 0$ such that

$$|\widetilde{D}(k,r)| \leq \exp \big(C \operatorname{rank} \mathbf{1}_{]r/2,4r[}(\varepsilon B_q) \big). \tag{4.11}$$

On the other hand, we have

$$\begin{aligned} \Big(I + \frac{i}{k} J B^- + \varepsilon A(k) \Big)^{-1} \frac{i}{k} J B^+ &= \Big(I + \Big(I + \frac{i}{k} J B^- \Big)^{-1} \varepsilon A(k) \Big)^{-1} \frac{i}{k} J B^+ \\ &= \frac{i}{k} J B^+ + \mathcal{O}(\varepsilon). \end{aligned} \tag{4.12}$$

For $u \in L^2(\mathbb{R}^3)$, $\frac{2}{3} r < |k| < \frac{3}{2} r$ and $k \in \mathbb{R}$, we get

$$\begin{aligned} \operatorname{Re} \Big\langle \Big(I + \Big(I + \frac{i}{k} J B^- + \varepsilon A(k) \Big)^{-1} \frac{i}{k} J B^+ \Big) u, u \Big\rangle &= \operatorname{Re} \Big\langle \Big(I + \frac{i}{k} J B^+ + \mathcal{O}(\varepsilon) \Big) u, u \Big\rangle \\ &= \operatorname{Re} \big\langle \big(I + \mathcal{O}(\varepsilon) \big) u, u \big\rangle \\ &\geq \|u\|^2/2, \end{aligned} \tag{4.13}$$

for $\varepsilon > 0$ small enough. Since we can obtain the same estimate for the adjoint, the operator $I + (I + \frac{i}{k} J B^- + \varepsilon A(k))^{-1} \frac{i}{k} J B^+$ is invertible for $\frac{2}{3} r < |k| < \frac{3}{2} r$, $k \in \mathbb{R}$ with a uniformly bounded inverse. Then, for such k, we get

$$\begin{aligned} (\widetilde{D}(k,r))^{-1} &= \det \Big(I - \Big(I + \frac{i}{k} J B^- + \varepsilon A(k) \Big)^{-1} \frac{i}{k} J B^+ \Big(I + \Big(I + \frac{i}{k} J B^- + \varepsilon A(k) \Big)^{-1} \frac{i}{k} J B^+ \Big)^{-1} \Big) \\ &\leq \exp \big(C \operatorname{rank} \mathbf{1}_{]r/2,4r[}(\varepsilon B_q) \big). \end{aligned}$$

Combining this estimate with (4.11), the Jensen inequality and (3.9), we get (4.6).

(iii) According to Lemma 7, there exists $\nu > 0$ and a decreasing sequence $(r_l)_{l \in \mathbb{N}}$ of positive numbers, $r_l \searrow 0$ such that for any $l \in \mathbb{N}$ we have

$$\operatorname{dist}\Big(r_l, \operatorname{sp}(B_q) \Big) \geq \nu r_l / 2.$$

Then for any $l \in \mathbb{N}$, there exists a path (see Figure 3)

$$\widetilde{\Gamma}_l \subset \{ \widetilde{k} \in \mathbb{C}^*;\ |\widetilde{k}| \leq s_0,\ |\operatorname{Im} \widetilde{k}| \geq \delta \operatorname{Re} \widetilde{k},\ r_l \geq \operatorname{Re} \widetilde{k} \geq r_{l+1} \}$$

enclosing the eigenvalues of B_q contained in the interval $[r_{l+1}, r_l]$, and such that for $\widetilde{k} \in \widetilde{\Gamma}_l$, the operator $(\widetilde{k} - B_q)$ is invertible with

$$\|(\widetilde{k} - B_q)^{-1}\| = \sup_{\lambda_j \in \operatorname{sp}(B_q)} \frac{1}{|\widetilde{k} - \lambda_j|} \leq C / |\widetilde{k}|$$

for some $C = C(\delta, \nu)$, uniformly with respect to $\widetilde{k} \in \widetilde{\Gamma}_l$.

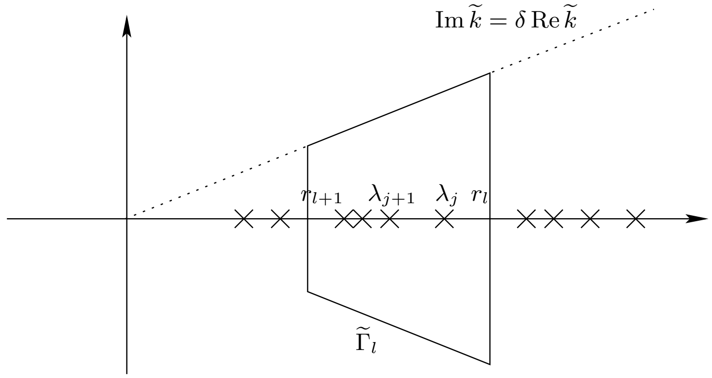

FIGURE 3. The path $\widetilde{\Gamma}_l$

Now, let us consider the path $\Gamma_l := i\varepsilon J\widetilde{\Gamma}_l$, and estimate from below the number of the zeros of $\det_2(I + \frac{i}{k}\varepsilon JB_q + \varepsilon A(k))$ counted with their multiplicity, enclosed in $\{z = z_q(k) \in D_q^*;\ k \in \Gamma_l\}$. By construction of $\widetilde{\Gamma}_l$, for $k \in \Gamma_l$, the operator $I + ik^{-1}\varepsilon JB_q$ is invertible with $\|(I + ik^{-1}\varepsilon JB_q)^{-1}\| \leq C(\delta, \nu)$ uniformly with respect to $k \in \Gamma_l$. Then choosing ε_0 so small that

$$\|\varepsilon_0 A(k)(I + ik^{-1}\varepsilon JB_q)^{-1}\|_2 < 1/2,$$

and using that $\det_2(I + A) \leq e^{\|A\|_2/2}$, we obtain that for $k \in \Gamma_l$,

$$\left|\det_2\Big(I + \varepsilon_0 A(k)(I + ik^{-1}\varepsilon JB_q)^{-1}\Big) - 1\right| < 1.$$

Applying the Rouché theorem we deduce that the number of zeros of $\det_2(I + \frac{i}{k}\varepsilon JB_q + \varepsilon A(k))$ enclosed in $\{z = z_q(k) \in D_q^*;\ k \in \Gamma_l\}$ is equal to the number of zeros of $\det_2(I + \frac{i}{k}\varepsilon JB_q)$ and using (3.9) it is given by $n_+(2r_{l+1}; p_qWp_q) - n_+(2r_l; p_qWp_q)$. Since each zero of $\det_2(I + \frac{i}{k}\varepsilon JB_q + \varepsilon A(k))$ is a resonance, with the same multiplicity, we deduce (4.7), and since the sequence $(r_l)_l$ is infinite, we conclude that the number of the resonances is infinite. $\square$

Since the embedded eigenvalues in $\mathbb{R} \setminus 2b\mathbb{N}$ are resonances $z_q(k)$ with $k \in e^{i\{0,\frac{\pi}{2}\}}]0, \sqrt{2b}[$, a simple consequence of the previous theorem is the absence of embedded eigenvalues in $]2bq - s_0^2, 2bq[\cup]2bq, 2bq + s_0^2[$ for small positive V and in $]2bq, 2bq + s_0^2[$ for small negative V. In fact, by more precise estimates with respect to q, for small positive V, we prove absence of embedded eigenvalues in $\mathbb{R}^+ \setminus 2b\mathbb{N}$ and for small negative V, we obtain information about the localization of the embedded eigenvalues on the left of the Landau levels.

Proposition 7. Absence of embedded eigenvalues. *Assume that V satisfies* (1.1) *with $m_\perp > 0$ and $m_3 > 2$. For a positive potential V, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, $H_\varepsilon := H_0 + \varepsilon V$ has no embedded eigenvalues in $\mathbb{R}^+ \setminus 2b\mathbb{N}$. For a negative potential V, there exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \leq \varepsilon_0$, $H_\varepsilon := H_0 + \varepsilon V$ has no embedded eigenvalues in $\mathbb{R}^+ \setminus (2b\mathbb{N} +] - \varepsilon C, 0[)$.*

Proof. According to Proposition 2.6 of [6], for V satisfying (1.1) with $m_\perp > 0$ and $m_3 > 1$ there exists $C > 0$ such that H_ε has no embedded eigenvalues in $\mathbb{R}^+ \setminus (2b\mathbb{N} +] - \varepsilon C, \varepsilon C[)$. Then, following the proof of Theorem 2 (i) (or see proof of Proposition 2.5 of [6]), we have only to check that $\varepsilon_0 > 0$ can be chosen independently of $\lambda \in \mathbb{R}^+ \setminus 2b\mathbb{N}$ such that for any $\varepsilon \leq \varepsilon_0$ and $\lambda \in \mathbb{R}^+ \setminus 2b\mathbb{N}$, $I + \varepsilon\mathcal{T}_V(\lambda)$

is invertible when V is positive. For negative V, we have to choose $\varepsilon_0 > 0$ such that for any $\varepsilon \le \varepsilon_0$ and $\lambda \in \mathbb{R}^+ \setminus (2b\mathbb{N}+]-b,0[)$, $I+\varepsilon \mathcal{T}_V(\lambda)$ is invertible.

Let $\lambda \in \mathbb{R}^+ \setminus 2b\mathbb{N}$, then there exists $q \in \mathbb{N}$ and $k \in \mathbb{C}$, $|k| \le \sqrt{b}$ such that $\lambda = 2bq+k^2$ ($k \in]0,\sqrt{b}]$ or $k \in i]0,\sqrt{b}]$).

We have:

$$\mathcal{T}_V(\lambda) = J|V|^{\frac12} R_0(\lambda)|V|^{\frac12} = J|V|^{\frac12}\langle x_3\rangle^{\frac{m_3}{2}}\Big(\sum_{j\in\mathbb{N}} p_j \otimes t_{m_3}(\lambda-2bj)\Big)\langle x_3\rangle^{\frac{m_3}{2}}|V|^{\frac12},$$

where t_{m_3} is the continuous extension of $z \mapsto \langle x_3\rangle^{-\frac{m_3}{2}}(D_{x_3}^2 - z)^{-1}\langle x_3\rangle^{-\frac{m_3}{2}}$ from $\operatorname{Im} z > 0$ to $z \in \mathbb{R}\setminus\{0\}$. For $\mu \in \mathbb{R}\setminus\{0\}$, the integral kernel of $t_{m_3}(\mu)$ is given by:

$$\langle x_3\rangle^{-\frac{m_3}{2}}\; i\frac{e^{ik|x_3-x_3'|}}{2k}\;\langle x_3'\rangle^{-\frac{m_3}{2}},$$

where $k=\sqrt{\mu}$ if $\mu>0$ and $k = i\sqrt{-\mu}$ if $\mu<0$.

It is clear that for $\mu<0$, $\|t_{m_3}(\mu)\| \le |\mu|^{-1}$ and for $\mu>0$, $\|t_{m_3}(\mu)\| \le C(m_3)|\mu|^{-\frac12}$ with $C(m_3) = \frac12\int\langle x_3\rangle^{-m_3}dx_3$ (for more details, see [6]).

From the above estimates, we immediately get:

$$\|\sum_{j\neq q} p_j\otimes t_{m_3}(\lambda-2bj)\| \le \sup_{j\neq q}\|t_{m_3}(\lambda-2bj)\| \le \max(b^{-1}, C(m_3)b^{-\frac12}).$$

Moreover, the series expansion with respect to k of the kernel of the operator t_{m_3} allows us to write $p_q\otimes t_{m_3}(\lambda-2bq)$ as the sum

$$p_q\otimes t_{m_3}(\lambda-2bq) = \frac{i}{k}p_q\otimes\tau + p_q\otimes\rho(k),$$

where $\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the rank-one operator defined by $\tau u := \frac12\big\langle u, \langle . \rangle^{-\frac{m_3}{2}}\big\rangle\langle x_3\rangle^{-\frac{m_3}{2}}$, and $\rho(k)$ is the Hilbert-Schmidt operator with integral kernel

$$\langle x_3\rangle^{-\frac{m_3}{2}}\; i\frac{e^{ik|x_3-x_3'|}-1}{2k}\langle x_3'\rangle^{-\frac{m_3}{2}}.$$

Since this integral kernel is bounded by $\mathcal{O}(\langle x_3\rangle^{-\frac{m_3}{2}}\;|x_3-x_3'|\;\langle x_3'\rangle^{-\frac{m_3}{2}})$, uniformly with respect to k, $|k|\le b$, it follows that for $m_3>2$, $p_q\otimes\rho(k)$ is uniformly bounded independently of q and λ.

Consequently, for $B := |V|^{\frac12}\langle x_3\rangle^{\frac{m_3}{2}}\Big(p_q\otimes\tau\Big)\langle x_3\rangle^{\frac{m_3}{2}}|V|^{\frac12}$, we have

$$\|\mathcal{T}_V(\lambda) - \frac{iJ}{k}B\| \le M,$$

with M independent of λ. At last, for $Jk\in\mathbb{R}$ or $Jk \in i\mathbb{R}^+$, since B is a positive self-adjoint operator, $\|(I+i\varepsilon Jk^{-1}B)^{-1}\| \le 1$. Then taking $\varepsilon_0 < M^{-1}$, for $\varepsilon\le\varepsilon_0$, $I+\varepsilon\mathcal{T}_V(\lambda)$ is invertible for any $\lambda\in\mathbb{R}^+\setminus 2b\mathbb{N}$ when $J=1$ (i.e. $V\ge 0$) and for any $\lambda \in \mathbb{R}^+\setminus(2b\mathbb{N}+]-b,0[)$ when $J=-1$ (i.e. $V\le 0$). This concludes the proof of Proposition 7. $\square$

Remark. Further information concerning the localization of the eigenvalues of the operator H for non-sign-definite potentials V is contained in [6, Proposition 2.6].

5. Spectral shift function and resonances

In this section we represent the derivative of the spectral shift function (SSF) near the Landau levels as a sum of a harmonic measure related to resonances, and the imaginary part of a holomorphic function. As in [20], [5], [9], such representation justifies the Breit-Wigner approximation and implies a trace formula. We deduce also an asymptotic expansion of the SSF near a given Landau level; in the case of positive potentials V which decay slowly enough as $|X_\perp| \to \infty$ this expansion yields a remainder estimate for the corresponding asymptotic relations obtained in [10].

In the case of a relative trace class perturbation, the SSF is related to the perturbation determinant by the Krein formula

$$\xi(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Arg} \det \Big((H - \lambda - i\varepsilon)(H_0 - \lambda - i\varepsilon)^{-1} \Big). \tag{5.1}$$

In our case, $|V|^{\frac{1}{2}}(H_0 + i)^{-1}$ is in the Hilbert-Schmidt class, and the distribution

$$\xi' : \ f \in C_0^\infty(\mathbb{R}) \longmapsto -\operatorname{tr}\Big(f(H) - f(H_0) \Big) \tag{5.2}$$

is still well defined, but not the above perturbation determinant. Since $V(H_0+i)^{-2}$ is of trace class, we could give a meaning to (5.1) using meromorphic extension of the regularized Zeta function (see [4]), but it will be more convenient to introduce the regularized spectral shift function

$$\xi_2(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Arg} \operatorname{det}_2 \Big((H - \lambda - i\varepsilon)(H_0 - \lambda - i\varepsilon)^{-1} \Big), \tag{5.3}$$

(see (2.13) for the definition of $\det_2$) whose derivative is the following distribution

$$\xi_2' : \ f \in C_0^\infty(\mathbb{R}) \longmapsto -\operatorname{tr}\Big(f(H) - f(H_0) - \frac{d}{d\varepsilon} f(H_0 + \varepsilon V) \,|_{\varepsilon=0} \Big) \tag{5.4}$$

(see [16] or [4]). Let us note that in [4], these quantities are defined with the opposite sign. We will deduce the properties of the SSF from those of the regularized SSF by using the following lemma which is well known for perturbation of the Laplacian (see [17], [3]).

Lemma 8. *Let V satisfies* (2.1) *with $m_\perp > 2$. On $\mathbb{R} \setminus 2b\mathbb{N}$, we have*

$$\xi' = \xi_2' + \frac{1}{\pi} \operatorname{Im} \operatorname{tr}\Big(\partial_z \mathcal{T}_V(.) \Big), \tag{5.5}$$

$\mathcal{T}_V(z)$ being defined in Lemma 1.

Proof. According to Lemma 1, $\operatorname{tr}(\partial_z \mathcal{T}_V)$ is analytic on $\overline{\mathcal{F}_+}$. Then, exploiting (5.2) and (5.4), we have only to prove:

$$\operatorname{tr}\Big(\frac{d}{d\varepsilon} f(H_0 + \varepsilon V) \,|_{\varepsilon=0} \Big) = -\frac{1}{\pi} \int_{\mathbb{R}} f(\lambda) \operatorname{Im} \operatorname{tr}\Big(\partial_z T(\lambda) \Big) d\lambda, \tag{5.6}$$

for any $f \in C_0^\infty(\mathbb{R} \setminus 2b\mathbb{N})$. By the Helffer-Sjöstrand formula (see for instance [8]), for $\widetilde{f} \in C_0^\infty(\mathbb{R}^2)$ an almost analytic extension of f, (i.e., $\widetilde{f}_{|\mathbb{R}} = f$ and $\overline{\partial}_\lambda \widetilde{f}(\lambda) = \mathcal{O}(|\operatorname{Im} \lambda|^\infty)$), we have

$$f(H_0 + \varepsilon V) = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} \widetilde{f}(z) (z - H_0 - \varepsilon V)^{-1} L(dz), \tag{5.7}$$

where $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}$. This quantity is differentiable with respect to ε and its derivative at $\varepsilon = 0$ is given by

$$\frac{d}{d\varepsilon} f(H_0 + \varepsilon V)\,|_{\varepsilon=0} = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} \widetilde{f}(z)(z - H_0)^{-1} V (z - H_0)^{-1} L(dz).$$

Following the proof of Lemma 1, we check easily that for $\pm \operatorname{Im} z > 0$ the operator $(z - H_0)^{-1} V (z - H_0)^{-1}$ is of trace class with trace norm bounded by $\mathcal{O}(|\operatorname{Im} z|^{-2})$ and by cyclicity of the trace, for $\operatorname{Im} z > 0$ we have

$$\operatorname{tr}\Big((z - H_0)^{-1} V (z - H_0)^{-1}\Big) = \operatorname{tr}\Big(J|V|^{\frac{1}{2}} (z - H_0)^{-2} |V|^{\frac{1}{2}}\Big) = -\operatorname{tr}\Big(\partial_z \mathcal{T}_V(z)\Big)$$

and for $\operatorname{Im} z < 0$

$$\operatorname{tr}\Big((z - H_0)^{-1} V (z - H_0)^{-1}\Big) = -\overline{\operatorname{tr}\Big(\partial_z \mathcal{T}_V(\overline{z})\Big)}.$$

Hence, $\frac{d}{d\varepsilon} f(H_0 + \varepsilon V)\,|_{\varepsilon=0}$ is of trace class, and

$$\operatorname{tr}\Big(\frac{d}{d\varepsilon} f(H_0 + \varepsilon V)\,|_{\varepsilon=0}\Big) = \frac{1}{\pi} \int_{\operatorname{Im} z > 0} \overline{\partial} \widetilde{f}(z) \operatorname{tr}\Big(\partial_z \mathcal{T}_V(z)\Big) L(dz) + \frac{1}{\pi} \int_{\operatorname{Im} z < 0} \overline{\partial} \widetilde{f}(z) \overline{\operatorname{tr}\Big(\partial_z \mathcal{T}_V(\overline{z})\Big)} L(dz).$$

Then the Green formula yields (5.6). □

Let $\mathcal{W}_\pm \subset\subset \Omega_\pm$ be open relatively compact subsets of $\pm e^{\pm i]-2\theta_0, \varepsilon_0]}]0, 2b[$ with $\varepsilon_0 > 0$, $2\theta_0 + \varepsilon_0 < 2\pi$. We assume that these sets are independent of r and that $\mathcal{W}_\pm$ is simply connected. Also assume that the intersections between $\Omega_\pm$, $\mathcal{W}_\pm$ and $\pm]0, 2b[$ are intervals. Set $I_\pm = \mathcal{W}_\pm \cap \pm\,]0, 2b[$ and choose $0 < s_1 < \sqrt{\operatorname{dist}(0, \Omega_\pm)}$. In the following, we will identify $2bq + r\bullet$ with $\pi_G^{-1}(2bq + r\bullet) \cap D_q^*$, for $\bullet = \mathcal{W}_\pm$, $\Omega_\pm$, $I_\pm$.

Proposition 8. *Let V satisfies (2.1) with $m_\perp > 2$. For $\mathcal{W}_\pm \subset\subset \Omega_\pm$ and $I_\pm$ as above, there exist functions $g_\pm$ holomorphic in $\Omega_\pm$, such that for $\mu \in 2bq + rI_\pm$, we have*

$$\begin{aligned} \xi_2'(\mu) = \frac{1}{r\pi} \operatorname{Im} g_\pm'\Big(\frac{\mu - 2bq}{r}, r\Big) &+ \sum_{\substack{w \in \operatorname{Res}(H) \cap 2bq + r\Omega_\pm \\ \operatorname{Im} w \neq 0}} \frac{\operatorname{Im} w}{\pi |\mu - w|^2} - \sum_{w \in \operatorname{Res}(H) \cap 2bq + rI_\pm} \delta(\mu - w) \\ &- \frac{1}{\pi} \operatorname{Im} \operatorname{tr}\Big(\partial_z \mathcal{T}_V(\mu)\Big) \end{aligned} \tag{5.8}$$

where $g_\pm(z, r)$ satisfies the estimate

$$g_\pm(z, r) = \mathcal{O}\left(\left(n_+(s_1\sqrt{r}; p_q W p_q)|\ln r| + \widetilde{n}_1(s_1\sqrt{r}/2) + \widetilde{n}_2(s_1\sqrt{r}/2)\right)\right) = \mathcal{O}(|\ln r|\, r^{-\frac{1}{m_\perp}}) \tag{5.9}$$

uniformly with respect to $0 < r < r_0$ and $z \in \mathcal{W}_\pm$, with $\widetilde{n}_p$, $p = 1, 2$, defined by (3.23) or (3.15).

In order to obtain such a representation formula, the first step is the factorization of the generalized perturbation determinant. To this end, we need some complex-analysis results due to J. Sjöstrand, summarized in the following

Proposition 9. ([27], [29]) *Let I be a bounded interval in $]0, +\infty[$ and Ω be a complex neighborhood of I in $e^{i]-2\theta_0, \varepsilon_0]}]0, +\infty[$ with $\varepsilon_0 > 0$, $2\theta_0 + \varepsilon_0 < 2\pi$ such that $\Omega \cap \{\operatorname{Im} z > 0;\ \operatorname{Re} z > 0\} \neq \emptyset$. Let $z \mapsto F(z, h)$, $0 < h < h_0$, be a family of holomorphic functions in Ω, having at most a finite number $N(h)$ of zeros in Ω. Assume that*

$$F(z, h) = \mathcal{O}(1) e^{\mathcal{O}(1) N(h)}, \quad z \in \Omega,$$

and that for any $\delta > 0$ small enough, there exists $C > 0$ such that for $z \in \Omega_\delta := \Omega \cap \{\operatorname{Im} z > \delta;\ \operatorname{Re} z > 0\ \}$ we have

$$|F(z,h)| \geq e^{-CN(h)}.$$

Then for any simply connected $\widetilde{\Omega} \subset\subset \Omega$ there exists $g(.,h)$ holomorphic on $\widetilde{\Omega}$ such that

$$F(z,h) = \prod_{j=1}^{N(h)} (z - z_j) e^{g(z,h)}, \quad \frac{d}{dz} g(z,h) = \mathcal{O}(N(h)), \quad z \in \widetilde{\Omega}.$$

Proof of Proposition 8. Fix $\mathcal{W}_\pm \subset\subset \Omega_\pm$, and consider the functions

$$F_\pm :\ z \in \Omega_\pm \longmapsto D(\sqrt{r}\sqrt{z}, \sqrt{r} s_1),$$

where

$$\sqrt{z} = \begin{cases} \sqrt{\rho} e^{i\theta/2} & \text{for} \quad z = \rho e^{i\theta} \in \Omega_+ \\ \sqrt{z} = i\sqrt{\rho} e^{-i\theta/2} & \text{for} \quad z = -\rho e^{-i\theta} \in \Omega_-, \end{cases} \tag{5.10}$$

and $D(k,s)$ is defined by (3.24). The functions $F_\pm$ are holomorphic in $\Omega_\pm$ and $\widetilde{w} \in \Omega_\pm$ is a zero of $F_\pm$ if and only if $w = 2bq + \widetilde{w} r$ is a resonance of H. Then applying Proposition 9 to $F = F_+$ and to $F(z) = \overline{F_-(-\overline{z})}$ with $h = r$, $N(r) = n_+(s_1\sqrt{r}, p_q W p_q)|\ln r|$, we obtain existence of functions $g_{0,\pm}$ holomorphic in $\Omega_\pm$ such that for $z \in \Omega_\pm$, we have the following factorization:

$$D_\pm(\sqrt{r}\sqrt{z}, \sqrt{r} s_1) = \prod_{w \in \operatorname{Res}(H) \cap 2bq + r\Omega_\pm} \Big(\frac{zr + 2bq - w}{r} \Big) e^{g_{0,\pm}(z,r)}, \tag{5.11}$$

with

$$\frac{d}{dz} g_{0,\pm}(z,r) = \mathcal{O}(n_+(s_1\sqrt{r}, p_q W p_q)|\ln r|), \tag{5.12}$$

uniformly with respect to $z \in \mathcal{W}_\pm$.

On the other hand, with the notations of Section 3 (see Proposition 4 and the proof of Proposition 5), for $z = z_q(\sqrt{r}k)$, $0 < s_1 < |k| < s_0$ we have

$$\det{}_2\Big((H-z)(H_0-z)^{-1}\Big) = \det{}_2\Big(I + \mathcal{T}_V(z)\Big)$$

$$= D(\sqrt{r}k, \sqrt{r}s_1))\ \det\Big(\Big(I + \frac{iJ}{\sqrt{r}k} B_q \mathbf{1}_{[0,s_1\sqrt{r}/2]}(B_q) + \widetilde{A}(k\sqrt{r})\Big) e^{-\mathcal{T}_V(z)}\Big).$$

By the properties of $\widetilde{A}(k)$ (see the proof of Proposition 5), for $\widetilde{K}(k) = \frac{iJ}{\sqrt{r}k} B_q \mathbf{1}_{[0,s_1\sqrt{r}/2]}(B_q) + \widetilde{A}(k\sqrt{r})$, the difference $\mathcal{T}_V(z) - \widetilde{K}(k)$ is a finite-rank operator and as in the proof of (4.1), we have

$$\det\Big((I + \frac{iJ}{\sqrt{r}k} B_q \mathbf{1}_{[0,s_1\sqrt{r}/2]}(B_q) + \widetilde{A}(k\sqrt{r})) e^{-\mathcal{T}_V(z)}\Big) = \det{}_2(I + \widetilde{K}(k))\ e^{-\operatorname{Tr}(\mathcal{T}_V(z) - \widetilde{K}(k))},$$

where $\det_2(I + \widetilde{K}(k))$ is a non-vanishing holomorphic function, for $0 < s_1 < |k| < s_0$. Since $\widetilde{A}(k)$ is holomorphic in S_2 and

$$\|\frac{B_q}{s} \mathbf{1}_{[0,s]}(B_q)\|_2^2 = -\int_0^s \frac{u^2}{s^2} dn_+(u, B_q) = \widetilde{n}_2(s),$$

we have:

$$\|\widetilde{K}(k)\|_2^2 = \mathcal{O}(\widetilde{n}_2(s_1\sqrt{r}/2)),$$

which implies that $|\det_2(1+\widetilde{K}(k))| = \mathcal{O}(\exp(\widetilde{n}_2(s_1\sqrt{r}/2)))$. Using moreover that $\|\widetilde{K}(k)\| < 1$, we have also $|\det_2(1+\widetilde{K}(k))|^{-1} = \mathcal{O}(\exp(\widetilde{n}_2(s_1\sqrt{r}/2)))$. Then there exists $g_1(.,r)$ holomorphic on $\Omega_\pm$ such that, $\frac{d}{dz}g_1(z,r) = \mathcal{O}(\widetilde{n}_2(s_1\sqrt{r}/2))$, on $\mathcal{W}_\pm$, and

$$\det_2(1+\widetilde{K}(k)) = e^{g_1(z,r)}.$$

Then by definition of ξ_2 (see (5.3)), for $\mu = z_q(k) \in 2bq + r(\Omega_\pm \cap \mathbb{R})$ we obtain

$$\begin{aligned}\xi_2'(\mu) =& \frac{1}{\pi r}\operatorname{Im}\partial_\lambda(g_{0,\pm}+g_1)(\frac{\mu-2bq}{r},r) - \sum_{\substack{w\in\operatorname{Res}(H)\cap 2bq+r\Omega_\pm \\ \operatorname{Im} w\neq 0}} \frac{-\operatorname{Im} w}{\pi|\mu-w|^2} - \sum_{w\in\operatorname{Res}(H)\cap 2bq+rI_\pm}\delta(\mu-w) \\ &+ \frac{1}{\pi}\operatorname{Im}\operatorname{tr}\Big(\frac{1}{2k}\partial_k(\frac{iJ}{k}B_q\mathbf{1}_{[0,s_1\sqrt{r}/2]}(B_q)+\widetilde{A}(k)) - \partial_z\mathcal{T}_V(\mu+i0)\Big),\end{aligned}$$

where

$$k = \begin{cases} \sqrt{\mu-2bq} & \text{if} \quad \mu-2bq>0, \\ i\sqrt{2bq-\mu} & \text{if} \quad \mu-2bq<0. \end{cases}$$

According to Lemma 1 and since $B_q \in S_1$, the operators $\partial_z\mathcal{T}_V(z)$ and $\partial_k\widetilde{A}(k) = \partial_k A(k) = \partial_k(\mathcal{T}_V(z_q(k)) - \frac{i}{k}JB_q)$ are of trace class. The trace of $\partial_k A(k)$ is given by the integral of its kernel on the diagonal:

$$\operatorname{tr}\Big(\partial_k A(k)\Big) = \frac{2kb}{8\pi}\int_{\mathbb{R}^3}V(x)dx\Big(\sum_{j>q}(2b(j-q)-k^2)^{-3/2} - i\sum_{j<q}(k^2+2b(q-j))^{-3/2}\Big), \tag{5.13}$$

and by definition of $\widetilde{n}_1$ (see (3.15))

$$\operatorname{tr}\Big(\frac{1}{2k}\partial_k(\frac{iJ}{k}B_q\mathbf{1}_{[0,s_1\sqrt{r}/2]}(B_q)\Big) = -\frac{iJs_1\sqrt{r}}{8k^3}\widetilde{n}_1(s_1\sqrt{r}/2).$$

Then, we conclude the proof of Proposition 8 with $g_\pm = g_{0,\pm}+g_1+g_2$ taking

$$g_2(z,r) := \frac{ib}{4\pi}\int_{\mathbb{R}^3}V(x)dx\sum_{j<q}(zr+2b(q-j))^{-1/2} + \frac{iJs_1}{4\sqrt{z}}\widetilde{n}_1(s_1\sqrt{r}/2),$$

where $\sqrt{z}$ is defined on $\Omega_\pm$ by (5.10). □

In the definite-sign case ($J = \operatorname{sign} V$) we can specify the representation of the regularized spectral shift function when for $z = z_q(k)$, the operator $1+\frac{iJ}{k}B_q$ is invertible, that is for $\operatorname{Arg} k \neq -J\pi/2$. Then we consider $\mathcal{W}_\pm \subset\subset \Omega_\pm$ open relatively compact subsets of $\pm e^{\pm i]-2\theta_0,\varepsilon_0]}]0,2b[$ with $\varepsilon_0>0$, $2\theta_0+\varepsilon_0<2\pi$ as above, and we have the assumption that

$$-J\frac{\pi}{2} \notin (\frac{\pi}{2})_\mp \pm [-\theta_0,\varepsilon_0/2], \tag{5.14}$$

where $(\frac{\pi}{2})_- = 0$ and $(\frac{\pi}{2})_+ = \frac{\pi}{2}$. The main restriction is in the case "-" for $V\leq 0$ (i.e. $J=-1$), where we can consider $\Omega_- \subset -e^{-i]-2\theta_0,\varepsilon_0]}]0,2b[$, only with $\theta_0<0$, that is where there are no resonances! So in the definite-sign case we discuss the three following situations:

For $V\geq 0$, we consider $\mathcal{W}_+ \subset\subset \Omega_+ \subset e^{i]-2\theta_0,\varepsilon_0]}]0,2b[$ with $\varepsilon_0>0$, $2\theta_0<\pi$ or $\mathcal{W}_- \subset\subset \Omega_- \subset -e^{-i]-2\theta_0,\varepsilon_0]}]0,2b[$ with $\varepsilon_0>0$, $2\theta_0+\varepsilon_0<2\pi$.

For $V\leq 0$ we consider $\mathcal{W}_+ \subset\subset \Omega_+ \subset e^{i]-2\theta_0,\varepsilon_0]}]0,2b[$ with $\varepsilon_0>0$, $2\theta_0+\varepsilon_0<2\pi$.

Proposition 10. *Assume V satisfies (2.1) with $m_\perp > 2$, and is of definite sign $J = \mathrm{sign}V$. Let $\mathcal{W}_\pm \subset\subset \Omega_\pm$ open relatively compact subsets of $\pm e^{\pm i]-2\theta_0,\varepsilon_0]}]0,2b[$ as above and let the interval $I_\pm = \mathcal{W}_\pm \cap \pm\,]0,2b[$. Assume θ_0 satisfies (5.14).*

Then for $\lambda = \mu - 2bq \in rI_\pm$, the representation (5.8) holds with

$$\frac{1}{r}\operatorname{Im} g'_\pm(\frac{\lambda}{r}, r) = \frac{1}{r}\operatorname{Im}\widetilde{g}'_\pm(\frac{\lambda}{r}, r) + \operatorname{Im}\widetilde{g}'_{1,\pm}(\lambda) + \mathbf{1}_{(0,2b)}(\lambda) J\,\Phi'(\lambda),$$

where

$$\Phi(\lambda) := \mathrm{tr}\Big(\arctan\frac{B_q}{\sqrt{\lambda}}\Big) = \mathrm{tr}\Big(\arctan\frac{p_q W p_q}{2\sqrt{\lambda}}\Big),$$

$z \mapsto \widetilde{g}_\pm(z,r)$ is holomorphic in $\Omega_\pm$ and satisfies

$$\widetilde{g}_\pm(z,r) = \mathcal{O}(|\ln r|) \tag{5.15}$$

uniformly with respect to $0 < r < r_0$ and $z \in \mathcal{W}_\pm$ while the function $\widetilde{g}_{1,\pm}$ is holomorphic on $\pm e^{\pm i]-2\theta_0,\varepsilon_0]}]0,2b[$ and for $z \in \pm e^{\pm i]-2\theta_0,\varepsilon_0]}]0,2b[$, there exists C_{θ_0} such that:

$$|\widetilde{g}_{1,\pm}(z)| \leq C_{\theta_0}\sigma_2(\sqrt{|z|})^{\frac{1}{2}},$$

σ_2 being defined in Corollary 1.

Proof. With the notations of Section 3 (see Proposition 4), and according to relation (4.1), for $z = z_q(k)$, $0 < s_1\sqrt{r} < |k| < s_0$, $-J\operatorname{Im} k < \frac{1}{\delta}|\operatorname{Re} k|$ we have

$$\begin{aligned}\det{}_2\Big((H-z)(H_0-z)^{-1}\Big) &= \det{}_2\Big(I + \mathcal{T}_V(z)\Big)\\ &= \det{}_2\Big(I + K(k)\Big)\det\Big(I + \frac{i}{k}JB_q\Big)e^{-\operatorname{Tr}(\mathcal{T}_V(z) - K(k))}.\end{aligned}$$

with $K(k) = A(k)\Big(I + \frac{i}{k}JB_q\Big)^{-1}$. As in the proof of Proposition 8, applying Proposition 9 and the results of Section 4, we obtain existence of functions $\widetilde{g}_\pm$ holomorphic in $\Omega_\pm$ such that for $z \in \Omega_\pm$, we have the following factorization:

$$\det{}_2\Big(I + K(\sqrt{r}\sqrt{z})\Big) = \prod_{w\in\mathrm{Res}(H)\cap 2bq + r\Omega_\pm}\Big(\frac{zr + 2bq - w}{r}\Big)\, e^{\widetilde{g}_\pm(z,r)}, \tag{5.16}$$

with

$$\frac{d}{d\lambda}\widetilde{g}_\pm(z,r) = \mathcal{O}(|\ln r|), \tag{5.17}$$

uniformly with respect to $z \in \mathcal{W}_\pm$. Then by definition of ξ_2 (see (5.3)), for $\mu = z_q(k) \in 2bq + rI_\pm$ we obtain:

$$\begin{aligned}\xi'_2(\mu) = &\frac{1}{\pi r}\operatorname{Im}\widetilde{g}'_\pm(\frac{\mu - 2bq}{r}, r) + \sum_{\substack{w\in\mathrm{Res}(H)\cap 2bq + r\Omega_\pm\\ \operatorname{Im} w\neq 0}}\frac{\operatorname{Im} w}{\pi|\mu - w|^2} - \sum_{w\in\mathrm{Res}(H)\cap 2bq + rI_\pm}\delta(\mu - w)\\ &+ \frac{1}{2k\pi}\operatorname{Im}\mathrm{tr}\Big((I + \frac{iJ}{k}B_q)^{-1}\,\partial_k(\frac{iJ}{k}B_q)\Big) - \frac{1}{\pi}\operatorname{Tr}\Big(\partial_z\mathcal{T}_V(\mu) - \frac{1}{2k}\partial_k K(k)\Big),\end{aligned}$$

where

$$k = \begin{cases}\sqrt{\mu - 2bq} & \text{if} \quad \mu - 2bq > 0,\\ i\sqrt{2bq - \mu} & \text{if} \quad \mu - 2bq < 0.\end{cases}$$

On the other hand, Lemma 1 and Corollary 1 entails that the operators $\partial_z \mathcal{T}_V(\mu)$ and

$$\partial_k K(k) = \partial_k A(k) - \partial_k \Big(A(k) \frac{iJ}{k} B_q (I + \frac{iJ}{k} B_q)^{-1} \Big),$$

are trace-class. Moreover, from (5.13) and since A is holomorphic in S_2, we have

$$\operatorname{Im} \frac{1}{2k} \operatorname{Tr}(\partial_k K(k)) = \operatorname{Im} \frac{1}{2k} \partial_k \Big(\widetilde{g}_{1,\pm}(k^2) \Big),$$

with $\widetilde{g}_{1,\pm}$ being the holomorphic function:

$$\widetilde{g}_{1,\pm}(z) := \frac{ib}{4\pi} \int_{\mathbb{R}^3} V(x) dx \sum_{j<q} (z + 2b(q-j))^{-1/2} - \operatorname{Tr} \Big(A(\sqrt{z}) \, \frac{iJ}{\sqrt{z}} B_q (I + \frac{iJ}{\sqrt{z}} B_q)^{-1} \Big),$$

which satisfies the claimed estimates thanks to (3.12). Finally, we have

$$\begin{aligned} \frac{1}{2k} \operatorname{Im} \operatorname{tr} \Big((I + \frac{iJ}{k} B_q)^{-1} \partial_k (\frac{iJ}{k} B_q) \Big) &= \frac{-1}{2k^2} \operatorname{Im} \operatorname{tr} \Big(\frac{iJ}{k} B_q (I + \frac{iJ}{k} B_q)^{-1} \Big) \\ &= \begin{cases} 0 & \text{for } Jk \in i\mathbb{R}^+ \\ \dfrac{1}{2k^2} \operatorname{tr} \Big(\dfrac{J}{k} B_q (I + \dfrac{B_q^2}{k^2})^{-1} \Big) = J \Phi'(k^2) & \text{for } k \in \mathbb{R}, \end{cases} \end{aligned}$$

which prove Proposition 10. $\square$

Combining Lemma 8, Proposition 8 and Proposition 10, we deduce Breit-Wigner approximation of the SSF:

Theorem 3. Breit-Wigner approximation. *Assume V satisfies (2.1) with $m_\perp > 2$. Let $\mathcal{W}_\pm \subset\subset \Omega_\pm$ be open relatively compacts subsets of $\pm e^{\pm i]-2\theta_0, \varepsilon_0]}]0, 2b[$ as before Proposition 8 and let $0 < s_1 < \sqrt{dist(\Omega_\pm, 0)}$. Then there exists $r_0 > 0$ and functions $g_\pm$ holomorphic in $\Omega_\pm$, such that for $\mu \in 2bq + rI_\pm$, we have*

$$\xi'(\mu) = \frac{1}{r\pi} \operatorname{Im} g'_\pm(\frac{\mu - 2bq}{r}, r) + \sum_{\substack{w \in \operatorname{Res}(H) \cap 2bq + r\Omega_\pm \\ \operatorname{Im} w \neq 0}} \frac{\operatorname{Im} w}{\pi |\mu - w|^2} - \sum_{w \in \operatorname{Res}(H) \cap 2bq + rI_\pm} \delta(\mu - w) \tag{5.18}$$

where $g_\pm(z, r)$ satisfies the estimate

$$g_\pm(z, r) = \mathcal{O}\left(n_+(s_1\sqrt{r}; p_q W p_q) |\ln r| + \widetilde{n}_1(s_1\sqrt{r}/2) + \widetilde{n}_2(s_1\sqrt{r}/2) \right) = \mathcal{O}(|\ln r| \, r^{-\frac{1}{m_\perp}}) \tag{5.19}$$

uniformly with respect to $0 < r < r_0$ and $z \in \mathcal{W}_\pm$.

Moreover for potentials of definite sign, $J := \operatorname{sign} V$, assuming that θ_0 satisfies

$$-J\frac{\pi}{2} \notin (\frac{\pi}{2})_\mp \pm [-\theta_0, \varepsilon_0/2],$$

($(\frac{\pi}{2})_- = 0$, $(\frac{\pi}{2})_+ = \frac{\pi}{2}$) for $\lambda \in rI_\pm$ we have

$$\frac{1}{r} \operatorname{Im} g'_\pm(\frac{\lambda}{r}, r) = \frac{1}{r} \operatorname{Im} \widetilde{g}'_\pm(\frac{\lambda}{r}, r) + \operatorname{Im} \widetilde{g}'_{1,\pm}(\lambda) + \mathbf{1}_{(0,2b)}(\lambda) J \, \Phi'(\lambda),$$

where

$$\Phi(\lambda) := \operatorname{tr} \Big(\arctan \frac{B_q}{\sqrt{\lambda}} \Big) = \operatorname{tr} \Big(\arctan \frac{p_q W p_q}{2\sqrt{\lambda}} \Big),$$

$z \mapsto \widetilde{g}_\pm(z,r)$ is holomorphic in $\Omega_\pm$ and satisfies

$$\widetilde{g}_\pm(z,r) = \mathcal{O}(|\ln r|) \tag{5.20}$$

uniformly with respect to $0 < r < r_0$ and $z \in \mathcal{W}_\pm$. The function $\widetilde{g}_{1,\pm}$ is holomorphic on $\pm e^{\pm i]-2\theta_0,\varepsilon_0]}$ $]0,2b[$ and for $z \in \pm e^{\pm i]-2\theta_0,\varepsilon_0]}]0,2b[$, there exists C_{θ_0} such that:

$$|\widetilde{g}_{1,\pm}(z)| \le C_{\theta_0}\sigma_2(\sqrt{|z|})^{\frac{1}{2}}, \tag{5.21}$$

σ_2 being defined in Corollary 1.

The following corollary describes the asymptotic behavior of the SSF on the right of a given Landau level.

Corollary 2. Singularities at the Landau levels. *Assume that V satisfies (2.1) with $m_\perp > 2$ and is of definite sign $J = \operatorname{sign} V$. Then the asymptotic relation*

$$\xi(2bq+\lambda) = \frac{J}{\pi}\Phi(\lambda) + \mathcal{O}\left(\Phi(\lambda)^{\frac{1}{2}}\right) + \mathcal{O}\left(|\ln\lambda|^2\right), \tag{5.22}$$

holds as $\lambda \searrow 0$.

Proof. Let us apply Theorem 3 on intervals $2bq+r_n[1,2]$, with $r_n = \lambda 2^n$, $\lambda > 0$. For $\mu \in 2bq+r_n[1,2]$ and Ω_+ a complex neighborhood of $[1,2]$, we have

$$\begin{aligned}\xi'(\mu) =& \frac{1}{r_n\pi}\operatorname{Im}\widetilde{g}'_\pm\left(\frac{\mu-2bq}{r_n}, r_n\right) + \sum_{\substack{w\in\operatorname{Res}(H)\cap 2bq+r_n\Omega_+\\ \operatorname{Im} w\neq 0}} \frac{\operatorname{Im} w}{\pi|\mu-w|^2} - \sum_{w\in\operatorname{Res}(H)\cap 2bq+r_n[1,2]} \delta(\mu-w)\\ &+ \frac{1}{\pi}\left(J\Phi' + \operatorname{Im}\widetilde{g}'_{1,\pm}\right)(\mu-2bq).\end{aligned} \tag{5.23}$$

Using that $\int_{\mathbb{R}} \frac{-\operatorname{Im} w}{\pi|\mu-w|^2}d\mu \le 1$ and that $2bq+r_n\Omega_+$ contains at the most $\mathcal{O}(|\ln r_n|)$ resonances (see Proposition 6), integration of (5.23) on $2bq+r_n[1,2]$ yields

$$\xi(2bq+r_{n+1}) - \xi(2bq+r_n) = \frac{1}{\pi}\left[\operatorname{Im}\widetilde{g}_\pm(\,.\,,r_n)\right]_1^2 + \mathcal{O}(|\ln r_n|) + \frac{1}{\pi}\left[J\Phi + \operatorname{Im}\widetilde{g}_{1,\pm}\right]_{r_n}^{r_{n+1}}. \tag{5.24}$$

Let $N \in \mathbb{N}$ such that $\frac{b}{2} \le \lambda 2^{N+1} \le b$ (then $N = \mathcal{O}(|\ln\lambda|)$). Since ξ, Φ and $\widetilde{g}_{1,\pm}$ are uniformly bounded on $2bq + b[1/2,1]$ (b the fixed strength of the magnetic field), and since $\widetilde{g}_\pm(\,.\,,r_n) = \mathcal{O}(|\ln r_n|)$, taking the sum of (5.24) from $n=0$ to $n=N$, we have:

$$\xi(2bq+\lambda) = \frac{J}{\pi}\Phi(\lambda) + \frac{1}{\pi}\operatorname{Im}\widetilde{g}_{1,\pm}(\lambda) + \sum_{n=0}^{N}\mathcal{O}(|\ln(2^n\lambda)|) + \mathcal{O}(1).$$

Using (5.21) and exploiting that $N = \mathcal{O}(|\ln\lambda|)$, we obtain existence of $C > 0$ such that

$$\left|\xi(2bq+\lambda) - \frac{J}{\pi}\Phi(\lambda)\right| \le C|\ln\lambda|^2 + C\sigma_2(\sqrt{\lambda})^{\frac{1}{2}}.$$

Then Corollary 2 follows from the elementary inequality

$$\frac{u^2}{1+u^2} \le \arctan u, \quad u \ge 0,$$

which implies $\sigma_2(\sqrt{\lambda}) \le \Phi(\lambda)$. □

Let us compare our results with those of [10] where the singularities of the SSF at a given Landau level were investigated.

If W satisfies the assumptions of Lemmas 3, 4 or 5 (plus some generic technical assumptions in the case of a rapid decay), then it is shown in [10] that

$$\xi(2bq+\lambda) = \frac{J}{\pi}\Phi(\lambda)\,(1+o(1)), \quad \lambda \searrow 0. \tag{5.25}$$

In the case of slowly decaying W satisfying the assumptions of Lemma 3, we see that (5.22) provides a remainder estimate of asymptotic relation (5.25). In the case of rapidly decaying W (see Lemma 4 or Lemma 5) we have

$$\Phi(\lambda) = \frac{1}{2}\varphi_\beta(\lambda)(1+o(1)), \quad \lambda \searrow 0.$$

Hence, in the case $\beta \in]0, \frac{1}{2}[$ relation (5.22) again provides a remainder estimate of (5.25). However, in the case $\beta \in [\frac{1}{2}, \infty]$ it does not even recover the first asymptotic term of (5.25). Note also that in [10] the decay of V is assumed to be isotropic in all three directions while here we assume that V is super-exponentially decaying with respect to x_3.

On the left of Landau level, for $V \geq 0$, the results of [10] imply $\xi(2bq-\lambda) = \mathcal{O}(1)$. Here our estimates are not accurate to see this. For $V \leq 0$, it is shown in [10] that

$$\xi(2bq-\lambda) = n_+(2\sqrt{\lambda}; p_q W p_q)(1+o(1)), \quad \lambda \searrow 0.$$

In this case, we have only general representation formula (5.18) with estimate (5.19).

As in [20], (or [5]), from Theorem 3 we deduce also the following trace formula.

Corollary 3. Trace formula. *Let $\mathcal{W}_\pm \subset\subset \Omega_\pm$ be as in Theorem 3. Suppose that $f_\pm$ is holomorphic on a neighborhood of $\Omega_\pm$ and that $\psi_\pm \in C_0^\infty(\Omega_\pm \cap \mathbb{R})$ satisfies $\psi_\pm(\lambda) = 1$ near $\mathcal{W}_\pm$. Then under the assumptions of Theorem 3 we have the following trace formula*

$$\operatorname{tr}\Big((\psi_\pm f_\pm)(\frac{H-2bq}{r}) - (\psi_\pm f_\pm)(\frac{H_0-2bq}{r})\Big) = \sum_{w\in \operatorname{Res}(H)\cap 2bq + r\mathcal{W}_\pm} f_\pm(\frac{w-2bq}{r}) + E_{f_\pm,\psi_\pm}(r) \tag{5.26}$$

with

$$|E_{f_\pm,\psi_\pm}(r)| \leq M(\psi_\pm)\sup\{|f_\pm(z)| \;:\; z\in\Omega_\pm\setminus\mathcal{W}_\pm\,,\ \operatorname{Im} z\leq 0\}\times N_q(r)\,.$$

where $N_q(r) = n_+(s_1\sqrt{r}; p_q W p_q)|\ln r| + \widetilde{n}_1(s_1\sqrt{r}/2) + \widetilde{n}_2(s_1\sqrt{r}/2) = \mathcal{O}(|\ln r|\, r^{-\frac{1}{m_\perp}})$.

Proof. In the proof we omit the subscript $\pm$. Choose an almost analytic extension $\widetilde{\psi}$ of ψ so that $\widetilde{\psi}\in C_0^\infty(\Omega)$, $\widetilde{\psi}=1$ on $\mathcal{W}$ and

$$\operatorname{supp}\, \overline{\partial}_z\widetilde{\psi} \subset \Omega\setminus\mathcal{W}.$$

Applying Theorem 3, we have

$$\begin{aligned}
\operatorname{tr}\Big((\psi f)(\frac{H-2bq}{r}) - (\psi f)(\frac{H_0-2bq}{r})\Big) &= -\Big\langle \xi'(\lambda), (\psi f)(\frac{\lambda-2bq}{r})\Big\rangle \\
&= \sum_{w\in\operatorname{Res}(H)\cap 2bq + r\operatorname{supp}\psi} (\psi f)(\frac{w-2bq}{r}) - \frac{1}{r\pi}\int (\psi f)(\frac{\lambda-2bq}{r})\operatorname{Im} g'(\frac{\lambda-2bq}{r}, r)d\lambda \\
&\quad + \frac{1}{2\pi i}\int (\psi f)(\frac{\lambda-2bq}{r}) \sum_{\substack{w\in\operatorname{Res}(H)\cap 2bq+r\operatorname{supp}\widetilde{\psi} \\ \operatorname{Im} w\neq 0}} \Big(\frac{1}{\lambda-w} - \frac{1}{\lambda-\overline{w}}\Big)d\lambda\,.
\end{aligned}$$

The integral involving g' can be estimated using (5.19) on supp $\widetilde{\psi}$. For the integral related to the resonances, we apply the Green formula and we get the term

$$\sum_{w\in \mathrm{Res}(H), \mathrm{Im}\, w\neq 0} (\widetilde{\psi} f)(\frac{w-2bq}{r}) \\ + \frac{1}{r\pi}\int_{\mathbb{C}_-} (\overline{\partial}_z \widetilde{\psi})(\frac{z-2bq}{r}) f(\frac{z-2bq}{r}) \sum_{\substack{w\in \mathrm{Res}(H)\cap 2bq + r\,\mathrm{supp}\,\widetilde{\psi} \\ \mathrm{Im}\, w\neq 0}} \Big(\frac{1}{z-\overline{w}} - \frac{1}{z-w}\Big) L(dz)\,.$$

We apply the inequality

$$\int_{\Omega_1} \frac{1}{|z-w|} L(dz) \leq 2\sqrt{2\pi|\Omega_1|},$$

and the upper bound of the resonances in Ω contained in Theorem 1, to obtain the result. □

Acknowledgments. The authors would like to thank the referee for the helpful suggestions concerning the descriptions of the subsets of $\mathcal{M}$. Jean-François Bony and Vincent Bruneau are partially supported by the French ANR Grant no. JC0546063. Vincent Bruneau and Georgi Raikov are partially supported by the Chilean Science Foundation *Fondecyt* under Grants 7050263 and 1050716.

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