COUNTING FUNCTION OF CHARACTERISTIC VALUES AND MAGNETIC RESONANCES

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1. Introduction

It is well known that several spectral problem for unbounded operators can be reduced to the study of some compact operator $K(z)$. Generally, a complex number $z$ in a domain $D$ is an eigenvalue (or a resonance) of an operator $H$ if and only if $I - K(z)$ is not invertible where $z \mapsto K(z)$ is holomorphic on $D$ with value in $S_\infty$, the space of compact operators. For example under suitable assumptions, according to the Birman-Schwinger principle, the study of the eigenvalues of $H = H_0 + M$ can be related to the compact operator $K(z) = M(H_0 - z)^{-1}M^*$ (see [3], [4], [10], [16], [17], [21], [28], [30], [31], [32]). For more general Birman-Schwinger principle for non-selfadjoint operators we refer to [11]. Similarly, the resonances for $H = H_0 + V$, a perturbation of a free hamiltonian $H_0$, can be analyzed by studying the invertibility of $I - K(z)$ with $K(z)$ a compact operator (see [6], [8], [9], [14], [22], [29], [33], [34], [35], [37]). For example for perturbations of Schrödinger operators $H_0$ by exponentially decreasing potentials $V$, thanks to a resolvent equation like (5.2), we take $K(z) = V^{1/2}(H_0 - z)^{-1}|V|^{1/2}$. In other situations, $K(z)$ is constructed by more sophisticated methods, like Grushin problems or by a representation formula of the scattering matrix.

In the following, as in [13], for $z \mapsto K(z)$ holomorphic on $D$ with value in $S_\infty$, we will say that a complex number $w$ is a characteristic value of $I - K(z)$ if $I - K(w)$ is not invertible. According to the analytic Fredholm theorem, if for some $z_0 \in D$, $I - K(z_0)$ is invertible (which often means that the resolvent set of our operator is non-empty), then $I - K(z)$ has a discrete set of characteristic values in $D$. But the characteristic values could accumulate at some point of the boundary $\partial D$. For example, if $A_0$ is a self-adjoint compact operator of infinite rank, then the characteristic values of $I - A_0/z$ in $C \setminus \{0\}$ are the eigenvalues of $A_0$ which accumulate to 0. If $z \mapsto K(z)$ is holomorphic on a domain $D$, then the number of characteristic values of $I - K(z)$ in each compact subset of $D$ is finite. This property still holds true if $z \mapsto K(z)$ is finite meromorphic on $D$ (see section 6, Proposition 6.2 or [12, Proposition 4.1.4]). We meet this case for the resonances of the 1D Schrödinger operator (see [8]).

In this paper, we consider the case when

$$I - K(z) = I - \frac{A(z)}{z},$$

with $z \mapsto A(z) : D \to S_\infty$ holomorphic on a domain $D \subseteq C$ containing 0 and $A(0)$ self-adjoint of infinite rank. We have such situation for the study of scattering poles on asymptotically hyperbolic manifolds (see [14]) and for the resonances of the magnetic Schrödinger operator in $\mathbb{R}^3$ (see [5]). This type of problem could also arise for the study of resonances near thresholds for other magnetic hamiltonians like these of [1], [15], [26] and [36].

The natural intuition is that the characteristic values of $I - A(z)/z$ accumulate to 0 with the same rate as the spectrum of $A(0)$. In this paper we give situations when this holds true.
Then we apply our abstract result for the distribution of resonances, near the Landau levels, for the magnetic Schrödinger operators with homogeneous magnetic field of strength $b > 0$, pointing at the $x_3$-direction:

$$H(b,eV) := \left(D_{x_1} + \frac{b}{2}x_2\right)^2 + \left(D_{x_2} - \frac{b}{2}x_1\right)^2 - b + D^2_{x_3} + eV, \quad D_{\nu} := -i \frac{\partial}{\partial \nu}, \quad e \in \mathbb{R}.$$ 

We assume that $V$ is the multiplication operator by $V$:

$$V : \mathbb{R}^3 \rightarrow \mathbb{R}, \text{ Lebesgue measurable, and satisfies the estimate}$$

$$V(x) = O((1 + |X_\perp|)^{-m_\perp} \exp(-N|x_3|)), \quad x = (x_1, x_2, x_3) = (X_\perp, x_3),$$

with $m_\perp > 0$, $N > 0$. It was known that for some $V$, $H(b,eV)$ can have infinite negative discrete spectrum and infinitely many embedded eigenvalues below each Landau level (see [2], [24] or [25]). Moreover the spectral shift function associated to $(H(b,eV), H(b,0))$ has singularities at the Landau levels (see [7]). This last property suggests there could be accumulation of resonances of $H(b,eV)$ near the Landau levels. In [5], we defined the resonances and gave an upper bound on their number at a distance $r \rightarrow 0$ of the Landau levels. Moreover, for $e$ sufficiently small and $V$ of definite sign, compactly supported (or decreasing like a Gaussian function) we stated a lower bound. Here in Section 5, we obtain the asymptotic behavior of the counting function of the magnetic resonances near the Landau levels for $V$ of definite sign satisfying the estimate (1.1) and for every $e \in \mathbb{R} \setminus \mathcal{E}$ where $\mathcal{E}$ is a discrete set of $\mathbb{R}$. To our best knowledge, it is the first result giving such asymptotic behaviour for counting function of magnetic resonances (and may be, more generally for resonances which accumulate to a point).

In this article, we first study the distribution of the characteristic values in some small domain of size $r$ at a distance $r$ of $z = 0$, $r \rightarrow 0$ (see Theorem 2.1). Then, we prove a general result for the characteristic values is some domain of size $|1-r|$ at a distance $r$ of $z = 0$, $r \rightarrow 0$ (see Theorem 2.6, Corollary 2.8 and Corollary 2.10). These abstract results are given in Section 2 and respectively proved in Section 3 and in Section 4. We apply these results in Section 5 to magnetic Schrödinger operators. Section 6 is an appendix devoted to the notions of characteristic values and of index of an operator with respect to a contour. This tool avoids the use of the regularized determinants.

2. Asymptotic expansions: abstract results

Let $\mathcal{D}$ be a domain of $\mathbb{C}$ containing 0. We consider a holomorphic operator function

$$A : \mathcal{D} \rightarrow \mathcal{S}_\infty(\mathcal{H}),$$

where $\mathcal{S}_\infty(\mathcal{H})$ is the space of compact operators on the Hilbert space $\mathcal{H}$. For a set $\Omega \subset \mathcal{D} \setminus \{0\}$, we denote by $\mathcal{Z}(\Omega)$ the set of the characteristic values of $I - \frac{A(z)}{z}$:

$$\mathcal{Z}(\Omega) := \left\{ z \in \Omega; \quad I - \frac{A(z)}{z} \text{ is not invertible} \right\},$$

and by $\mathcal{N}(\Omega)$ the number of characteristic values in $\Omega$ counted with their multiplicities:

$$\mathcal{N}(\Omega) := \# \mathcal{Z}(\Omega).$$

We refer to Section 6 for details concerning the characteristic values.

We deduce from Proposition 6.2 that $\mathcal{Z}(\mathcal{D})$ is discrete as soon as $A(0)$ is of finite rank. In this section we assume that $A(0)$ is a self-adjoint operator and we are mainly interested in the case where $A(0)$ is of infinite rank.
Figure 1. The set of characteristic values $Z(D)$ and the setting of Theorem 2.1.

We will give results concerning the number of characteristic values of $I - A(z)$ in two sorts of domains: the small domains which are of the form $s\Omega$ with $\Omega \subseteq \mathbb{C}\setminus\{0\}$ fixed and $s$ tending to 0 and the sectorial domains of the form

$$C_\theta(a,b) := \{x + iy \in \mathbb{C} ; a \leq x \leq b, |y| \leq \theta |x|\},$$

with $b, \theta > 0$ fixed and $a > 0$ tending to 0.

Our goal is to give situations where $N(s\Omega)$, as $s \to 0$, and $N(C_\theta(r,1))$, as $r \to 0$, are related to the number of eigenvalues, counted with their multiplicity, of $A(0)$ in $\Lambda \subset \mathbb{R}$, denoted by $n(\Lambda) := \text{tr}\Lambda(A(0))$.

Let $\Pi_0$ be the spectral projection operator on $\ker A(0)$ and $\pi_0 := I - \Pi_0$. In small domains we have:

**Theorem 2.1.** Let $D$ be a domain of $\mathbb{C}$ containing 0 and let a holomorphic operator function $A : D \to S_\infty(\mathcal{H})$, such that $A(0)$ is self-adjoint and $I - A'(0)\pi_0$ is invertible. Assume that $\Omega \subseteq \mathbb{C}\setminus\{0\}$ is a bounded domain with smooth boundary which is transverse to the real axis at each point of their intersection. Then, for all $\delta > 0$ small enough, there exists $s(\delta) > 0$ such that, for all $0 < s < s(\delta)$, we have

$$N(s\Omega) = n(sJ) + O(n(sI_\delta)|\ln \delta|^2),$$

where $J := \Omega \cap \mathbb{R}$, $I_\delta := \partial \Omega \cap \mathbb{R} + [-\delta, \delta]$ and the $O$ is uniform with respect to $s, \delta$.

**Remark 2.2.** In the previous theorem, the remainder estimate is uniform with respect to some perturbations of the domain $\Omega$. For example, let $\theta > 0$ and $0 < a_- \leq a_+ < b_- \leq b_+ < +\infty$. Then, the conclusion of Theorem 2.1 holds for $\Omega = C_\theta(a,b)$ uniformly with respect to $a, b$ such that $a_- \leq a \leq a_+$ and $b_- \leq b \leq b_+$.

The setting is described in Figure 1. Considering different types of $\Omega$, the previous result implies the following properties on the characteristic values near 0.

**Corollary 2.3.** Under the assumptions of Theorem 2.1, we have

i) If $\Omega \cap \mathbb{R} = \emptyset$ then $N(s\Omega) = 0$ for $s$ small enough. This implies that the characteristic values $z \in Z(D)$ near 0 satisfy

$$|\Im z| = o(|z|).$$

ii) Moreover, if $A(0)$ has a sign (say $\pm A(0) \geq 0$), the characteristic values near 0 satisfy

$$\pm \Re z \geq 0.$$
iii) If \( A(0) \) is of finite rank, then there is no characteristic value in a pointed neighborhood of 0. Moreover, if \( A(0)1_{\pm0,\infty}(A(0)) \) is of finite rank, then there is no characteristic value in a neighborhood of 0 intersected with \( \{ \pm \text{Re } z > 0 \} \).

**Remark 2.4.** According to Proposition 6.2, the first sentence of Corollary 2.3 iii) holds true even if \( A(0) \) is non self-adjoint.

In fact Corollary 2.3 is a consequence of the following result which guarantees that a region has no characteristic values under a spectral condition on \( A(0) \). Since its proof is similar to the one of Lemma 3.1, we omit it.

**Proposition 2.5.** Under the hypotheses of Theorem 2.1, there exists \( C > 0 \) satisfying the following property. Let \( S \subset D \setminus \{0\} \) be a set such that the operator function

\[
z \mapsto \left( I - \frac{A(0)}{z} \right)^{-1},
\]

is well defined and uniformly bounded on \( S \). Then, there exists \( z_0 > 0 \) such that \( \mathcal{Z}(S) \cap \{|z| < z_0\} = \emptyset \) and

\[
\sup_{z \in S \cap \{|z| < z_0\}} \left\| \left( I - \frac{A(z)}{z} \right)^{-1} \right\| \leq C \sup_{z \in S} \left\| \left( I - \frac{A(0)}{z} \right)^{-1} \right\|.
\]

Exploiting Theorem 2.1 with appropriate domains \( \Omega \), we obtain the following result in sectorial domains.

**Theorem 2.6.** Let \( D \) be a domain of \( \mathbb{C} \) containing 0 and let a holomorphic operator function

\[
A : D \longrightarrow S_\infty(\mathcal{H}),
\]

such that \( A(0) \) is self-adjoint and \( I - A'(0)\Pi_0 \) is invertible. For \( \theta > 0 \) fixed, let \( \mathcal{C}_\theta(r, 1) \subset D \) be defined in (2.1). Then, for all \( \delta > 0 \) small enough, there exists \( s(\delta) > 0 \) such that, for all \( 0 < s < s(\delta) \), we have

\[
\mathcal{N}(\mathcal{C}_\theta(r, 1)) = n([r, 1]) \left( 1 + \mathcal{O}(\delta \ln \delta)^2 \right) + \mathcal{O}(\ln \delta^2) n([r(1-\delta), r(1+\delta)]) + \mathcal{O}_\delta(1),
\]

where the \( \mathcal{O}'s \) are uniform with respect to \( s, \delta \) but the \( \mathcal{O}_\delta \) may depend on \( \delta \).
Remark 2.7.  

i) The same result holds mutatis mutandis for $C_\theta(-1,-r)$, $r > 0$, under the same assumptions. To prove this point, it is sufficient to change the variable $z$ by $-z$ which consists to replace $A(z)$ by $-A(-z)$.

ii) The dependence on $\theta$ in Theorem 2.6 is inessential. More precisely, thanks to Corollary 2.3, $N(C_\theta(r,1)) - N(C_\nu(r,1))$ is a constant for $r$ small enough.

Now, let us give conditions on $n([r,1])$ for which Theorem 2.6 implies asymptotic expansion of $N(C_\theta(r,1))$ with main term $n([r,1])$.

**Corollary 2.8.** Under the assumptions of Theorem 2.6, if there exists $\gamma > 0$ such that

$$n([r,1]) = O(r^{-\gamma}) \quad \text{as } r \searrow 0,$$

and if $n([r,1])$ tends to infinity as $r$ tends to $0^+$, then there exists a sequence $(r_k)_k$ which tends to $0^+$ such that

$$N(C_\theta(r_k,1)) = n([r_k,1])(1 + o(1)) \quad \text{as } k \to +\infty.$$ 

**Remark 2.9.** Note that the assumption (2.2) is equivalent to the fact that the operator $A(0)1_{[0,+\infty]}(A(0))$ is in a Schatten class $S_p$ for some $1 \leq p < +\infty$. Such hypothesis is reasonable since it guarantees that, in some sense, $A(0)$ has less eigenvalues near $r$ than in the whole interval $[r,1]$ (see Lemma 4.2).

**Corollary 2.10.** Under the assumptions of Theorem 2.6, if $n([r,1]) = \Phi(r)(1 + o(1))$ as $r \searrow 0$, with $\Phi(r) = Cr^{-\gamma}$ or $\Phi(r) = C|\ln r|^\gamma$ or $\Phi(r) = C|\ln|\ln r||$ for some $\gamma, C > 0$, then

$$N(C_\theta(r,1)) = \Phi(r)(1 + o(1)) \quad \text{as } r \searrow 0.$$ 

These results are proved in Section 4.

3. Proof of Theorem 2.1

We make the assumptions of Theorem 2.1 in all this section. In the following, $C$ will denote a positive constant independent of $\delta, s, z$ which may change from line to line.

3.1. Estimate of the resolvents.

For $0 < \delta \leq 1/2$ and $s > 0$, let us introduce the finite rank operator

$$K_\delta(s) := A(0)1_{I_\delta}\left(\frac{A(0)}{s}\right),$$

where $I_\delta = \partial\Omega \cap \mathbb{R} + [-\delta,\delta]$. This operator will be used to “remove” the eigenvalues of $A(0)$ which are at distance $\delta s$ from $\partial\Omega \cap \mathbb{R}$. We also define the set

$$V_{\nu,\delta} := (\Omega + B(0,\nu)) \cap \{|\text{Im } z| > \varepsilon \delta\} \quad \text{and} \quad W_\delta := (\partial\Omega \cap \mathbb{R}) + B(0,\delta),$$

where $\nu > 0$ is a constant chosen sufficiently small so that $\Omega + B(0,\nu) \subseteq \mathbb{C} \setminus \{0\}$ and $\varepsilon \in ]0, 1/2[$ will be chosen later such that $\partial\Omega \subset V_{\nu,2,\delta} \cup W_\delta$. For $\delta > 0$ small enough, the setting is described in Figure 3. First, we show estimates on the free and perturbed resolvents which hold uniformly with respect to $\delta$. 
Lemma 3.1. There exists $C > 0$ such that, for all $0 < \delta \leq 1/2$, there exists $s(\delta) > 0$ such that, for all $0 < s < s(\delta)$, we have

\begin{equation}
\| (I - A(sz) - K_\delta(s))^{-1} \| \leq \frac{C}{\delta + |\text{Im} z|},
\end{equation}

uniformly for $z \in V_{\nu,\delta} \cup W_{\delta/2}$ and

\begin{equation}
\| (I - A(sz))^{-1} \| \leq \frac{C}{\delta + |\text{Im} z|},
\end{equation}

uniformly for $z \in V_{\nu,\delta}$.

Proof. We begin by proving (3.2). First, we make the following decomposition

\begin{equation}
(I - \frac{A(sz) - K_\delta(s)}{sz})^{-1} = (I - \frac{A(sz) - A(0)}{sz})(I - \frac{A(0) - K_\delta(s)}{sz})^{-1})(I - \frac{A(0) - K_\delta(s)}{sz}).
\end{equation}

From the definition of $K_\delta(s)$ and the spectral theorem, we get

\begin{equation}
\| (I - \frac{A(sz) - K_\delta(s)}{sz})^{-1} \| \leq \frac{C}{\delta + |\text{Im} z|},
\end{equation}

uniformly for $0 < \delta \leq 1/2$, $s \in [0, 1]$ and $z \in V_{\nu,\delta} \cup W_{\delta/2}$.

We will now prove that, for $\delta$ fixed,

\begin{equation}
s\text{-lim}_{s \to 0} \left( \Pi_0 \left( I - \frac{A(0) - K_\delta(s)}{sz} \right)^{-1} \right)^* = 0,
\end{equation}

uniformly for $z \in V_{\nu,\delta} \cup W_{\delta/2}$. For $\alpha > 0$ fixed, according to the spectral theorem, there exists $M > 0$ such that

\begin{equation}
\| 1_{\{|\lambda| \geq M\delta\}}(A(0))(I - \frac{A(0) - K_\delta(s)}{sz})^{-1} \| < \alpha,
\end{equation}

uniformly with respect to $s \in [0, 1]$ and $z \in V_{\nu,\delta} \cup W_{\delta/2}$. On the other hand, the projection operator $1_{\{0 < |\lambda| < M\delta\}}(A(0))$ tends strongly to 0 as $s$ tends to 0. Then, using (3.5), we deduce...
that, for δ fixed,
\[ s-lim_{s \to 0} \left( 1_{\{0<|\lambda|<M_s\}}(A(0)) \left( I - \frac{A(0) - K_\delta(s)}{s} \right)^{-1} \right)^* = 0, \]
uniformly with respect to \( z \in V_{\nu,\delta} \cup W_{\delta/2} \). The two last equations imply (3.6).

Since \( A \) is holomorphic near 0, there exists a holomorphic operator function \( R_2 \) such that
\[ \frac{A(sz) - A(0)}{sz} = A'(0) + szR_2(sz). \]

Then
\[ I - \frac{A(sz) - A(0)}{sz} \left( I - \frac{A(0) - K_\delta(s)}{s} \right)^{-1} = I - A'(0)\Pi_0 - \left( A'(0)\Pi_0 + szR_2(sz) \right) \left( I - \frac{A(0) - K_\delta(s)}{s} \right)^{-1}. \] (3.7)

Exploiting (3.5), (3.6), \( szR_2(sz) = O(s) \) and that \( A'(0) \) is a compact operator, we deduce that, for δ fixed, the norm of the last term in the previous equation goes to 0 as \( s \to 0 \) uniformly with respect to \( z \in V_{\nu,\delta} \cup W_{\delta/2} \). At last, \( I - A'(0)\Pi_0 \) is invertible by assumption. Then there exists \( C > 0 \) such that, for all \( 0 < \delta \leq 1/2 \), we can choose \( s(\delta) > 0 \) sufficiently small such that
\[ \left\| \left( I - \frac{A(sz) - A(0)}{sz} \left( I - \frac{A(0) - K_\delta(s)}{s} \right)^{-1} \right) \right\| < C, \] (3.8)
uniformly for \( 0 < s < s(\delta) \) and \( z \in V_{\nu,\delta} \cup W_{\delta/2} \). Then, (3.2) follows from (3.4), (3.5) and (3.8).

The proof of (3.3) is similar. Instead of (3.5), we use
\[ \left\| \left( I - \frac{A(0)}{sz} \right)^{-1} \right\| \leq \frac{s|z|}{s|\text{Im } z|} \leq \frac{1}{\delta + |\text{Im } z|}, \]
uniformly for \( 0 < \delta \leq 1/2, s \in [0,1] \) and \( z \in V_{\nu,\delta} \). \( \square \)

### 3.2. Reduction of the problem.

From the definition of the multiplicity of the characteristic values (6.1) and of the index of an operator (6.2), we have the

**Lemma 3.2.** Assume that \( s\Omega \subset \mathcal{D} \) and that there is no characteristic value of \( (I - A(z)/z) \) on \( s\partial \Omega \). Then,
\[ \mathcal{N}(s\Omega) = \text{Ind}_{s\partial \Omega} \left( I - \frac{A(sz)}{sz} \right). \]

Note that (3.3) implies Corollary 2.3 i). Therefore, since the characteristic values form a discrete set, the assumptions of Lemma 3.2 are satisfied for almost all \( s \) small enough. Moreover, from the statement of Theorem 2.1, it is enough to prove it for almost all \( s \) small enough. Then, we can always assume in the following that the assumptions of Lemma 3.2 are satisfied.

Now, we define
\[ f_\delta(s, z) = \det \left( \left( I - \frac{A(sz)}{sz} \right) \left( I - \frac{A(sz) - K_\delta(s)}{sz} \right)^{-1} \right) \]
\[ = \det \left( I - \frac{K_\delta(s)}{sz} \left( I - \frac{A(sz) - K_\delta(s)}{sz} \right)^{-1} \right) \] (3.9)
the relative determinant which is well defined for \( z \in V_{t_0,\delta} \cup W_{\delta/2} \) from Lemma 3.1 and since \( K_\delta(s) \) has finite rank. In the following, for a function \( f \) which is holomorphic on a neighborhood of an oriented path \( \gamma \), we denote by \( \text{ind}_\gamma f \) the quantity

\[
\text{ind}_\gamma f = \frac{1}{2i\pi} \int_{\gamma} f'(z) \, dz.
\]

Note that, if \( f \) is holomorphic on \( \Omega \), then \( \text{ind}_{\partial\Omega} f \) is the number of zeros of \( f \) in \( \Omega \).

**Lemma 3.3.** For all \( 0 < \delta \leq 1/2 \), there exists \( s(\delta) > 0 \) such that, for all \( 0 < s < s(\delta) \),

\[
\text{Ind}_{\partial\Omega} \left( I - \frac{A(sz)}{sz} \right) = \text{Ind}_{\partial\Omega} \left( I - \frac{A(sz) - K_\delta(s)}{sz} \right) + \text{ind}_{\partial\Omega} f_\delta(s, z).
\]

**Proof.** Remark that all the quantities are well defined on \( \partial\Omega \) since we have assumed the hypotheses of Lemma 3.2. We have

\[
I - \frac{A(sz)}{sz} = I - \frac{A(sz) - K_\delta(s)}{sz} - \frac{K_\delta(s)}{sz},
\]

and we can then write

\[
I - \frac{A(sz)}{sz} = \left( I - \frac{K_\delta(s)}{sz} \right) \left( I - \frac{A(sz) - K_\delta(s)}{sz} \right)^{-1} \left( I - \frac{A(sz) - K_\delta(s)}{sz} \right).
\]

Thus we deduce the lemma using the property \( \text{Ind}_{\partial\Omega} F_1 F_2 = \text{Ind}_{\partial\Omega} F_1 + \text{Ind}_{\partial\Omega} F_2 \) and that for a trace class operator function \( K \), \( \text{Ind}_{\partial\Omega} (I + K) = \text{ind}_{\partial\Omega} \det(I + K) \) (see Section 6). \( \square \)

**Lemma 3.4.** For all \( 0 < \delta \leq 1/2 \), there exists \( s(\delta) > 0 \) such that, for all \( 0 < s < s(\delta) \),

\[
\text{Ind}_{\partial\Omega} \left( I - \frac{A(sz) - K_\delta(s)}{sz} \right) = \text{Ind}_{\partial\Omega} \left( I - \frac{A(0) - K_\delta(s)}{sz} \right).
\]

**Proof.** First, by using the following decomposition

\[
I - \frac{A(sz) - K_\delta(s)}{sz} = \left( I - \frac{A(sz) - A(0)}{sz} \right) \left( I - \frac{A(0) - K_\delta(s)}{sz} \right)^{-1} \left( I - \frac{A(0) - K_\delta(s)}{sz} \right),
\]

we have

\[
\text{Ind}_{\partial\Omega} \left( I - \frac{A(sz) - K_\delta(s)}{sz} \right) = \text{Ind}_{\partial\Omega} \left( I - \frac{A(sz) - A(0)}{sz} \right) \left( I - \frac{A(0) - K_\delta(s)}{sz} \right)^{-1} \left( I - \frac{A(0) - K_\delta(s)}{sz} \right) + \text{Ind}_{\partial\Omega} \left( I - \frac{A(sz) - A(0)}{sz} \right) \left( I - \frac{A(0) - K_\delta(s)}{sz} \right)^{-1}.
\]

(3.10)

From (3.7), we can write

\[
I - \frac{A(sz) - A(0)}{sz} \left( I - \frac{A(0) - K_\delta(s)}{sz} \right)^{-1} = I - A'(0)\Pi_0 - \left( A'(0)\Pi_0 + szR_2(sz) \right) \left( I - \frac{A(0) - K_\delta(s)}{sz} \right)^{-1}.
\]

Moreover the discussion below (3.7) shows that the last term of the above equality tends to 0 as \( s \) tends to 0 uniformly for \( z \in \partial\Omega \) with \( 0 < \delta \ll 1 \) fixed. Then, since \( I - A'(0)\Pi_0 \) is invertible, using the Rouché theorem (see Theorem 6.4 below), we deduce

\[
\text{Ind}_{\partial\Omega} \left( I - \frac{A(sz) - A(0)}{sz} \left( I - \frac{A(0) - K_\delta(s)}{sz} \right)^{-1} \right) = \text{Ind}_{\partial\Omega} (I - A'(0)\Pi_0) = 0.
\]

Combining with (3.10), this concludes the proof. \( \square \)
**Proposition 3.5.** Let us consider the function $f_\delta(s, z)$ introduced in (3.9). For all $\delta > 0$ small enough, there exists $s(\delta) > 0$ such that, for all $0 < s < s(\delta)$,

$$\text{Ind}_{\partial \Omega} f_\delta(s, \cdot) = O(n(sI_\delta) |\ln \delta|^2).$$

We now prove Theorem 2.1 and postpone the proof of the crucial Proposition 3.5 to the next section. Combining Lemma 3.2, Lemma 3.3, Lemma 3.4 with Proposition 3.5, we obtain

$$\mathcal{N}(s\Omega) = \text{Ind}_{\partial \Omega} \left( I - \frac{A(0) - K_\delta(s)}{sz} \right) + O(n(sI_\delta) |\ln \delta|^2).$$

Then, using the notation $\overline{A}_0 := A(0) - K_\delta(s) = A(0)1_{\mathbb{R} \setminus sI_\delta}(A(0))$, Theorem 2.1 follows from the relation

$$\text{Ind}_{\partial \Omega} \left( I - \frac{A(0) - K_\delta(s)}{sz} \right) = \frac{1}{2i\pi} \text{tr} \int_{\partial \Omega} \frac{\overline{A}_0}{sz} (z - \frac{\overline{A}_0}{s})^{-1} dz = \frac{1}{2i\pi} \text{tr} \int_{\partial \Omega} (z - \frac{\overline{A}_0}{s})^{-1} dz$$

$$= \text{tr} 1_{sJ} \left( \overline{A}_0 \right) = \text{tr} 1_{sJ \setminus I_\delta}(A(0)) = n(sJ) - n(sI_\delta)$$

$$= n(sJ) + O(n(sI_\delta)),$$

where $\overline{I}_\delta = I_\delta \cap \Omega$.

The same way, Remark 2.2 follows from the fact that Proposition 3.5 holds uniformly with respect to $a_- \leq a \leq a_+$ and $b_- \leq b \leq b_+$.

### 3.3. Proof of Proposition 3.5.

We first obtain a factorization of the determinant $f_\delta(s, z)$. This idea, due to Sj¨ ostrand [34], comes from the study of the semi-classical resonances. Since $\partial \Omega$ is transverse to the real axis, the intersection $\partial \Omega \cap \mathbb{R}$ is a finite set of real number $x_j$ with $j = 1, \ldots, J$. Using $W^j_\delta := B(x_j, \delta)$, we have

$$W_\delta = \bigcup_{j=1}^J W^j_\delta.$$ 

Note that this union is disjoint for $\delta$ small enough (see Figure 3).

**Lemma 3.6.** There exists $C > 0$ such that, for all $\delta > 0$ small enough, there exists $s(\delta) > 0$ such that, for all $0 < s < s(\delta)$,

i) for all $j$, the function $f_\delta(s, z)$ can be written in $W^j_\delta/4$

$$f_\delta(s, z) = \prod_{k=1}^{N_\delta(s)} \left( \frac{z - z^k_\delta(s)}{\delta} \right) e^{g_\delta(s, z)},$$

where $z^k_\delta(s) \in W^j_\delta/2$, $z \mapsto g_\delta(s, z)$ is holomorphic in $W^j_\delta/4$ with

$$N_\delta(s) \leq C |\ln \delta| n(sI_\delta) \quad \text{and} \quad |g_\delta(s, z)| \leq C \frac{|\ln \delta|}{\delta} n(sI_\delta), \quad z \in W^j_\delta/4.$$

ii) the function $f_\delta(s, z)$ has no zero in $V_{t_\delta/2, 2t_\delta}$ and, in this set,

$$f_\delta(s, z) = e^{g_\delta(s, z)}$$

with $z \mapsto g_\delta(s, z)$ holomorphic in $V_{t_\delta/2, 2t_\delta}$ and satisfying $|g_\delta(s, z)| \leq C C |\ln |z|| n(sI_\delta).$

To prove Lemma 3.6 i), we will apply a complex-analysis result due to Sjöstrand. To study the distribution of the semi-classical resonances, he has shown the following result.
Theorem 3.7 (see [34], [35]). Let $U$ be a simply connected domain of $\mathbb{C}$ with $U \cap \{\text{Im } \lambda > 0\} \neq \emptyset$. For any $\tilde{U} \Subset U$, there exists $C_{\tilde{U}, U}$ with the following properties:

For all holomorphic function $F : U \rightarrow \mathbb{C}$ such that for some $M \geq 1$,

\[ |F(\lambda)| \leq e^M \quad \text{for } \lambda \in U, \]

\[ |F(\lambda)| \geq e^{-M} \quad \text{for } \lambda \in U \cap \{\text{Im } \lambda > 0\}. \]

Then, there exists $g$ holomorphic in $\tilde{U}$ such that

\[ \forall \lambda \in \tilde{U}, \quad F(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_k)e^{g(\lambda)}, \]

where the $\lambda_k$’s are zeros of $F$ in $U$, $N \leq C_{\tilde{U}, U}M$ and $|g'(\lambda)| \leq C_{\tilde{U}, U}M$ for $\lambda \in \tilde{U}$.

Proof of Lemma 3.6. Since $K(z) := \frac{K_{\delta}(s)}{sz}(I - \frac{A(sz)}{sz}K_{\delta}(s))^{-1}$ is a finite rank operator, we have

\[ f_\delta(s, z) = \det (I - K(z)) = \prod_{j=1}^{\text{rank} K(z)} (1 - \lambda_j(z)), \]

where $\lambda_j(z)$ are the eigenvalues of $K(z)$. On the other hand, $K_{\delta}(s) = A(0)I_k(\frac{A(0)}{s})$ and Lemma 3.1 yield, for $z \in V_\nu, \delta \cup W_\delta/2$,

\[ \text{rank } K(z) = n(sI_\delta) \quad \text{and} \quad \|K(z)\| \leq \frac{C}{\delta + |\text{Im } z|}. \]

Then, there exists $C > 0$ such that, for all $\delta > 0$ small enough and then $s$ small enough,

\[ |f_\delta(s, z)| \leq (1 + \|K(z)\|)^{\text{rank } K(z)} \leq e^{Cn(sI_\delta)|\ln(\delta + |\text{Im } z|)|}, \]

(3.11)

uniformly for $z \in V_\nu, \delta \cup W_\delta/2$. On the other hand, for $z \in V_\nu, \delta$,

\[ f_\delta(s, z)^{-1} = \det \left( I + \frac{K_{\delta}(s)}{sz}(I - \frac{A(sz)}{sz}K_{\delta}(s))^{-1} \right). \]

Applying one more time Lemma 3.1, the previous argument gives

\[ |f_\delta(s, z)| \geq e^{-Cn(sI_\delta)|\ln(\delta + |\text{Im } z|)|}, \]

(3.12)

for $z \in V_\nu, \delta$.

Now we apply Theorem 3.7 to the function $\lambda \mapsto F(\lambda, s, \delta) := f_\delta(s, x_j + \delta \lambda)$ which is holomorphic in $B(0, 1/2)$. Estimates (3.11) and (3.12) give

\[ |F(\lambda, s, \delta)| \leq e^{Cn(sI_\delta)|\ln |\delta||} \quad \text{for } \lambda \in B(0, 1/2), \]

\[ |F(\lambda, s, \delta)| \geq e^{-Cn(sI_\delta)|\ln |\delta||} \quad \text{for } \lambda \in B(0, 1/2) \cap \{\text{Im } z > \varepsilon\}. \]

Then, Theorem 3.7 yields, for all $\lambda \in B(0, 1/4)$,

\[ f_\delta(s, x_j + \delta \lambda) = \prod_{k=1}^{N_\delta(s)} (\lambda - \lambda_k^\delta(s)) e^{g_\delta(s, \lambda)} \]

with $\lambda_k^\delta(s) \in B(0, 1/2)$, $N_\delta(s) \leq \tilde{C}n(sI_\delta)|\ln |\delta||$ and $|g_\delta(s, \lambda)| \leq \tilde{C}n(sI_\delta)|\ln |\delta|$. Making the change of variable $z = x_j + \delta \lambda$, we obtain Lemma 3.6 i).
According to (3.12), for sufficiently small \( s \), \( f_\delta(s, z) \) has no zero in \( V_{\nu, \delta} \). Then there exists \( g_\delta(s, z) \) holomorphic with respect to \( z \in V_{\nu, \delta} \) such that

\[
 f_\delta(s, z) = e^{g_\delta(s, z)}
\]

For \( z \in V_{\nu/2, \delta} \) consider the function

\[
 F : \lambda \mapsto f_\delta\left(s, z + \lambda \frac{\text{Im} z}{4}\right), \quad \lambda \in B(0, 2).
\]

Since \( F \) has no zeroes in \( B(0, 2) \), combining estimates (3.11) and (3.12) with Theorem 3.7, we deduce that for \( \lambda \in B(0, 1) \),

\[
 \left| \frac{\text{Im} z}{4} \right| g_\delta\left(s, z + \lambda \frac{\text{Im} z}{4}\right) \leq \tilde{C} \eta(sI_\delta) \left| \text{Im} z \right|,
\]

with \( \tilde{C} \) independent of \( \lambda, z, s \) and \( \delta \). Then ii) of Lemma 3.6 follows. \( \square \)

We can now prove Proposition 3.5. We define, for \( j = 1, \ldots, J \),

\[
 \Gamma^j_\delta := \partial \Omega \cap B(x_j, \delta) \quad \text{and} \quad \Gamma_\delta := \bigcup_{j=1}^J \Gamma^j_\delta.
\]

For \( \delta \) small enough, the \( \Gamma^j_\delta \)'s are segments of size \( \delta \) (see Figure 3). Moreover, from the assumptions on \( \Omega \), there exists \( \varepsilon > 0 \) such that \( \text{Im} z > 2\varepsilon \delta \) for all \( z \in \partial \Omega \setminus \Gamma_{\delta/4} \). Let us write

\[
 \text{ind}_{\partial \Omega} f_\delta(s, z) = \frac{1}{2i\pi} \int_{\partial \Omega} \frac{f'_\delta(s, z)}{f_\delta(s, z)} dz
\]

(3.13)

From Lemma 3.6 ii), the first term of (3.13) verifies

\[
 \left| \frac{1}{2i\pi} \int_{\partial \Omega \setminus \Gamma_{\delta/4}} \frac{f'_\delta(s, z)}{f_\delta(s, z)} dz \right| \leq \frac{1}{2\pi} \int_{\partial \Omega \setminus \Gamma_{\delta/4}} \left| g'_\delta(s, z) \right| |dz|
\]

\[
 \leq \frac{C}{2\pi} \eta(sI_\delta) \int_{\partial \Omega \setminus \Gamma_{\delta/4}} \left| \frac{\ln |\text{Im} z|}{|\text{Im} z|} \right| |dz|
\]

\[
 \leq Cn(sI_\delta) |\ln \delta|^2.
\]

(3.14)

Here, we have used \( \int_{[0,1]} \frac{|\ln x|}{x} dx \leq |\ln \delta|^2 \).

On the other hand, Lemma 3.6 i) gives

\[
 \frac{1}{2i\pi} \int_{\Gamma_{\delta/4}} \frac{f'_\delta(s, z)}{f_\delta(s, z)} dz = \sum_{k=1}^{N_\delta(s)} \frac{1}{2\pi} \int_{\Gamma_{\delta/4}} \frac{1}{z - z_k^\delta(s)} dz + \frac{1}{2\pi} \int_{\Gamma_{\delta/4}} g'_\delta(s, z) dz.
\]

(3.15)

Since \( |\Gamma_{\delta/4}^j| \leq C\delta \), we get

\[
 \left| \frac{1}{2\pi} \int_{\Gamma_{\delta/4}^j} g_\delta(s, z) dz \right| \leq \frac{C}{2\pi} n(sI_\delta) \int_{\Gamma_{\delta/4}^j} \left| \frac{\ln \delta}{\delta} \right| |dz| \leq Cn(sI_\delta) |\ln \delta|.
\]

(3.16)
The proposition 3.5 follows from the equation (3.13) and the estimates (3.14) and (3.17).

exists $\beta$ from $I$ constructed such that the associated intervals

Now, let $\Gamma_{j/4}^j$ be as in Figure 4. Lemma 3.6 yields $z_k^\delta(s) \in B(x_j, \delta/2)$. Moreover, thanks to (3.3), we also have $\text{Im} z_k^\delta(s) = o(\delta)$ as $s$ tends to 0. Therefore, the $z_k^\delta(s)$'s are at distance $\delta$ from $\Gamma_{j/4}^j$ for $s$ small enough. Then,

$$\left| \frac{1}{2\pi} \int_{\Gamma_{j/4}^j} \frac{1}{z - z_k^\delta(s)} \, dz \right| = \left| \frac{1}{2\pi} \int_{\Gamma_{j/4}^j} \frac{1}{z - z_k^\delta(s)} \, dz \right| + \left| \frac{1}{2\pi} \int_{\Gamma_{j/4}^j} \frac{1}{z - z_k^\delta(s)} \, dz \right| 
\leq 1 + \int_{\Gamma_{j/4}^j} \frac{C}{\delta} |dz| \leq C.$$

Combining the previous equation with $N_\delta(s) \leq C|\text{Im} z_k^\delta(s)|$, (3.15) and (3.16), we get

$$(3.17) \quad \left| \frac{1}{2\pi} \int_{\Gamma_{j/4}^j} f_k^j(s, z) \, dz \right| \leq C n(s I_\delta) |\text{Im} z_k^\delta|.$$

The proposition 3.5 follows from the equation (3.13) and the estimates (3.14) and (3.17).

4. Proof of Theorem 2.6 and its corollaries

In this section we prove Theorem 2.6 applying Theorem 2.1 with appropriate domains $s \Omega$ constructed such that the associated intervals $I_\delta$ contain a “small” number of eigenvalues of $A(0)$. First we choose these intervals $I_\delta$ thanks to the following lemma.

**Lemma 4.1.** There exists $C > 0$ such that, for any $\delta > 0$ small enough and $j \in \mathbb{N}$, there

exists $\beta_j \in [j - \frac{1}{4}, j + \frac{1}{4}]$ such that

$$n\left([2^{-\beta_j}(1 - \delta), 2^{-\beta_j}(1 + \delta)]\right) \leq C \delta \, n\left([2^{-\left(j + \frac{1}{4}\right)}, 2^{-\left(j - \frac{1}{4}\right)}]\right).$$

**Proof.** First, there exist $\delta_0 > 0$ such that, for all $0 < \delta < \delta_0$, one can find disjoint intervals

$I_k(\delta) \subset [2^{-\frac{1}{4}}, 2^{\frac{1}{4}}]$, with $0 \leq k \leq \varepsilon_1/\delta$ of the form $[2^{-\beta}(1 - \delta), 2^{-\beta}(1 + \delta)]$ for some $\beta \in [-\frac{1}{4}, \frac{1}{4}]$.

Now assume that the assertion of the lemma is not true. Then, for all $C > 0$ and $\delta_0 > 0$, there exists $0 < \delta < \delta_0$ and $j \in \mathbb{N}$ such that, for all $\beta \in [j - \frac{1}{4}, j + \frac{1}{4}]$,

$$n\left([2^{-\beta}(1 - \delta), 2^{-\beta}(1 + \delta)]\right) > C \delta \, n\left([2^{-\left(j + \frac{1}{4}\right)}, 2^{-\left(j - \frac{1}{4}\right)}]\right).$$

Figure 4. The situation near $x_j$ and the set $\Gamma_{j/4}^j$. 

Now, let $\Gamma_{j/4}^j$ be as in Figure 4. Lemma 3.6 yields $z_k^\delta(s) \in B(x_j, \delta/2)$. Moreover, thanks to (3.3), we also have $\text{Im} z_k^\delta(s) = o(\delta)$ as $s$ tends to 0. Therefore, the $z_k^\delta(s)$'s are at distance $\delta$ from $\Gamma_{j/4}^j$ for $s$ small enough. Then,

$$\left| \frac{1}{2\pi} \int_{\Gamma_{j/4}^j} \frac{1}{z - z_k^\delta(s)} \, dz \right| = \left| \frac{1}{2\pi} \int_{\Gamma_{j/4}^j} \frac{1}{z - z_k^\delta(s)} \, dz \right| + \left| \frac{1}{2\pi} \int_{\Gamma_{j/4}^j} \frac{1}{z - z_k^\delta(s)} \, dz \right| 
\leq 1 + \int_{\Gamma_{j/4}^j} \frac{C}{\delta} |dz| \leq C.$$

Combining the previous equation with $N_\delta(s) \leq C|\text{Im} z_k^\delta(s)|$, (3.15) and (3.16), we get

$$(3.17) \quad \left| \frac{1}{2\pi} \int_{\Gamma_{j/4}^j} f_k^j(s, z) \, dz \right| \leq C n(s I_\delta) |\text{Im} z_k^\delta|.$$

The proposition 3.5 follows from the equation (3.13) and the estimates (3.14) and (3.17).
We choose $C = 2/\varepsilon_1$ and $\delta_0 = \delta_1$. Using the intervals $I_k(\delta)$ constructed previously, we get
\[ n\left(\left[2^{-\left(j+\frac{1}{4}\right)}, 2^{-\left(j-\frac{1}{4}\right)}\right]\right) \geq n\left(\bigcup_{k=0}^{\varepsilon_1/\delta} 2^{-j} I_k(\delta)\right) = \sum_{k=0}^{\varepsilon_1/\delta} n(2^{-j} I_k(\delta)) \]
which gives a contradiction.

Combining this lemma with Theorem 2.1, we prove Theorem 2.6.

**Proof of Theorem 2.6.** We consider the sequence $(\beta_j)_j$ constructed in Lemma 4.1. For $\delta > 0$ small enough, let $r(\delta) > 0$ be such that the conclusion of Theorem 2.1 and Remark 2.2 (with $a_- = a_+ = 1$, $b_- = 2^{\frac{3}{2}}$ and $b_+ = 2^{\frac{3}{2}}$) hold for $0 < r < r(\delta)$. In the sequel, $M(\delta)$ will denote the smaller integer such that $2^{-\beta M(\delta)} < r(\delta)$. For $0 < r < r(\delta)$, $N(r)$ is the unique integer such that $2r < 2^{-N(r)} \leq 4r$.

Since the following decomposition into disjoint sets holds,
\[ C_\theta(r, 1) = C_\theta(r, 2^{-\beta N(r)}) \bigcup_{j = M(\delta)}^{N(r) - 1} C_\theta(2^{-\beta_{j+1}}, 2^{-\beta_j}) \bigcup C_\theta(2^{-\beta M(\delta)}, 1), \]
we have
\[ \mathcal{N}(C_\theta(r, 1)) = \mathcal{N}(C_\theta(r, 2^{-\beta N(r)})) + \sum_{j = M(\delta)}^{N(r) - 1} \mathcal{N}(C_\theta(2^{-\beta_{j+1}}, 2^{-\beta_j})) + \mathcal{N}(C_\theta(2^{-\beta M(\delta)}, 1)). \]
By construction of $\beta_j$, we have $2^{-\beta N(r)} \in r[2^{\frac{3}{4}}, 2^{\frac{3}{2}}]$ and
\[ C_\theta(2^{-\beta_{j+1}}, 2^{-\beta_j}) = 2^{-\beta_{j+1}} C_\theta(1, 2^{\beta_{j+1} - \beta_j}), \quad C_\theta(1, 2^{\frac{3}{2}}) \subset C_\theta(1, 2^{\beta_{j+1} - \beta_j}) \subset C_\theta(1, 2^{\frac{3}{2}}). \]
Then from Theorem 2.1, Remark 2.2 and Lemma 4.1, we get
\[ \mathcal{N}(C_\theta(r, 2^{-\beta N(r)})) = n\left(\left[r, 2^{-\beta N(r)}\right]\right) + \mathcal{O}(\ln \delta^2 n(2^{-\beta N(r)}(1 - \delta), 2^{-\beta N(r)}(1 + \delta))) \]
\[ = n\left(\left[r, 2^{-\beta N(r)}\right]\right) + \mathcal{O}(\ln \delta^2 n(2^{-\beta N(r)}(1 - \delta), r(1 + \delta))) \]
\[ = n\left(\left[r, 2^{-\beta N(r)}\right]\right) + \mathcal{O}(\ln \delta^2 n(2^{-\beta N(r)}(1 - \delta), r(1 + \delta))) \]
\[ = n\left(\left[r, 2^{-\beta N(r)}\right]\right) + \mathcal{O}(\ln \delta^2 n(2^{-N(r) - \frac{3}{4}}, 2^{-N(r) + \frac{1}{2}})), \]
and, for $M(\delta) \leq j \leq N(r) - 1$,
\[ \mathcal{N}(C_\theta(2^{-\beta_{j+1}}, 2^{-\beta_j})) = n\left(\left[2^{-\beta_{j+1}}, 2^{-\beta_j}\right]\right) + \mathcal{O}(\ln \delta^2 n(2^{-\beta_{j+1}(1 - \delta), 2^{-\beta_j}(1 + \delta))) \]
\[ = n\left(\left[2^{-\beta_{j+1}}, 2^{-\beta_j}\right]\right) + \mathcal{O}(\ln \delta^2 n(2^{-j - \frac{1}{4}}, 2^{-j + \frac{1}{2}})) \]
\[ = n\left(\left[2^{-\beta_{j+1}}, 2^{-\beta_j}\right]\right) + \mathcal{O}(\ln \delta^2 n(2^{-j - \frac{1}{4}}, 2^{-j + \frac{1}{2}})) \]
Moreover, we can write
\[ \mathcal{N}(C_\theta(2^{-\beta M(\delta)}, 1)) = n\left(\left[2^{-\beta M(\delta)}, 1\right]\right) + \mathcal{O}(\delta(1)). \]
Combining (4.2) with (4.3)–(4.5), we deduce
\[
\mathcal{N}(\mathcal{C}_\theta(r, 1)) = n([r, 1]) + O(\delta \ln \delta |^2 n([r(1 - \delta), r(1 + \delta)])
\]
\[
+ O(\delta \ln \delta |^2) \sum_{j=M(\delta)}^{N(\delta)} n([2^{-j-\frac{1}{4}}, 2^{-j+\frac{1}{4}}]) + O(1)
\]
\[
(4.6)
\]
\[
= n([r, 1]) (1 + O(\delta \ln \delta |^2)) + O(\delta \ln \delta |^2) n([r(1 - \delta), r(1 + \delta)]) + O(1),
\]
since we have the disjoint inclusion
\[
\bigcup_{j=M(\delta)}^{N(\delta)} [2^{-j-\frac{1}{4}}, 2^{-j+\frac{1}{4}}] \subset [r, 1].
\]
This concludes the proof of Theorem 2.6. \hfill \square

In order to prove Corollary 2.8, let us prove the following Lemma.

**Lemma 4.2.** Let \( \Psi \) be a real decreasing function such that \( \Psi(r) \geq 1 \) and \( \Psi(r) = O(r^{-\gamma}) \), \( \gamma > 0 \), on \( ]0, 1[ \). Then, there exists \( C > 0 \) such that, for any \( \delta > 0 \) small enough and \( \rho > 0 \), there exists \( 0 < r \leq \rho \) satisfying
\[
\Psi(r(1 - \delta)) - \Psi(r(1 + \delta)) \leq C \delta \Psi(r).
\]

**Proof.** Assume that the result is not true. Then, for all \( C, \delta_1 > 0 \), there exists \( \rho > 0 \) and \( 0 < \delta < \delta_1 \) such that, for all \( 0 < r \leq \rho \), we have
\[
\Psi(r(1 - \delta)) - \Psi(r(1 + \delta)) \geq C \delta \Psi(r).
\]
Using the monotonicity of \( \Psi \) and changing variable, we get
\[
\Psi(\sigma r) \geq (1 + C \delta) \Psi(r),
\]
with \( \sigma = \frac{1-\delta}{1+\delta} \) if \( r \leq \rho \). Then, for any \( K \in \mathbb{N} \),
\[
(4.7) \quad \Psi(\sigma^K r) \geq (1 + C \delta)^K \Psi(r).\]
On the other hand, we have, by assumption,
\[
(4.8) \quad \Psi(\sigma^K r) \leq O(\sigma^{-\gamma K}).
\]
Since (4.7) and (4.8) hold (uniformly) for all \( K \in \mathbb{N} \), we deduce
\[
\ln(1 + C \delta) \leq \gamma |\ln \sigma|.
\]
Now, letting \( \delta_1 \) (and then \( \delta \)) tends to 0, the Taylor series in \( \delta \) give
\[
C \leq 2 \gamma,
\]
for all \( C > 0 \). We get a contradiction. \hfill \square

**Proof of Corollary 2.8.** We construct the sequence \((r_k)_k\) in the following way. Let \( \delta > 0 \) be small enough such that
\[
O(\delta \ln \delta |^2) \leq \frac{1}{k} \quad \text{and} \quad C \delta O(\delta \ln \delta |^2) \leq \frac{1}{k},
\]
where the \( O \)'s are the ones appearing in Theorem 2.6 and \( C \) is the constant given in Lemma 4.2. Since \( n([r, 1]) \) tends to infinity as \( r \) tends to \( 0^+ \), one can find \( 0 < \rho \leq 2^{-k} \) such that
\[
O_\delta(1) \leq \frac{n([\rho, 1])}{k}.
\]
Now, applying Lemma 4.2 to the function $\Psi(r) := n([r, 1])$ with $C$ and $\delta$ as before, we deduce that there exists $r_k \leq \rho$ such that

$$n([r_k(1 - \delta), r_k(1 + \delta)]) \leq C\delta \, n([r_k, 1]).$$

Combining the previous estimates with Theorem 2.6, the number $0 < r_k \leq 2^{-k}$ satisfies

$$|N(C_\theta(r_k, 1)) - n([r_k, 1])| \leq \frac{3}{k} n([r_k, 1]).$$

In particular, we have proved

$$N(C_\theta(r_k, 1)) = n([r, 1])(1 + o(1)) \quad \text{as} \quad k \not\to \infty,$$

and $r_k$ tends to $0^+$.

**Proof of Corollary 2.10.** If $n([r, 1]) = \Phi(r)(1 + o(1))$ with

$$\Phi(r(1 \pm \delta)) = \Phi(r)(1 + o(1) + O(\delta)),$$

then $n([r(1 - \delta), r(1 + \delta)]) = n([r, 1])(o(1) + O(\delta))$. In particular, if in addition $\Phi(r)$ tends to infinity, Theorem 2.6 implies that

$$N(C_\theta(r, 1)) = \Phi(r)(1 + o(1)) \quad \text{as} \quad r \not\to 0.$$

Thus, Corollary 2.10 follows from the following estimates:

- If $\Phi(r) = |r|^{-\gamma}, \gamma > 0$, then $\Phi(r(1 \pm \delta)) = r^{-\gamma}(1 \pm \delta)^{-\gamma} = \Phi(r)(1 + O(\delta))$.
- If $\Phi(r) = |\ln r|^\gamma, \gamma > 0$, then $\Phi(r(1 \pm \delta)) = |\ln r|^\gamma \left(1 + \frac{\ln(1 \pm \delta)}{\ln r}\right)^\gamma = \Phi(r)(1 + o(1))$.
- If $\Phi(r) = \frac{|\ln r|}{\ln |\ln r|},$ then $\Phi(r(1 \pm \delta)) = \frac{|\ln r - \ln(1 \pm \delta)|}{\ln |\ln r| + \ln(1 + \frac{|\ln(1 \pm \delta)|}{\ln r})} = \Phi(r)(1 + o(1)).$ 

\[ \square \]

### 5. Application to the Counting Function of Magnetic Resonances

In this section, we apply the results of Section 2 to the counting function of resonances near the Landau levels. Let

$$H_0 := \left( D_{x_1} + \frac{b}{2} x_2 \right)^2 + \left( D_{x_2} - \frac{b}{2} x_1 \right)^2 - b + D_{x_3}^2, \quad D_{x_\nu} := -i \frac{\partial}{\partial x_\nu},$$

be the Schrödinger operator with homogeneous magnetic field of strength $b > 0$, pointing at the $x_3$-direction. This operator, first defined on $C_0^\infty(\mathbb{R}^3)$, is closed and essentially self-adjoint in $L^2(\mathbb{R}^3)$. Its spectrum is absolutely continuous, equal $[0, +\infty]$ and has an infinite set of thresholds $2bq, q \geq 0$. On the domain of $H_0$, we introduce $H := H_0 + V$ where $V$ is a relatively compact perturbation of $H_0$. More precisely, we assume that $V$ is the multiplication operator by $V : \mathbb{R}^3 \to \mathbb{R}$, Lebesgue measurable, and satisfying the estimate

$$V(x) = O\left( \langle X_\perp \rangle^{-m_\perp} \exp(-N|x_3|) \right), \quad x = (x_1, x_2, x_3) = (X_\perp, x_3),$$

for some $m_\perp > 0$ and $N > 0$. Let $J(x)$ denote the sign of $V(x)$. Under this assumption, the operator $H$ is self-adjoint with essential spectrum $[0, +\infty]$ and the resonances near the real axis are well defined on the following way (for more details see [5]). The resonances are the poles of the meromorphic extension of $z \mapsto (H - z)^{-1}$ considered as an operator in some weight spaces, $L(e^{-N|x_3|}L^2(\mathbb{R}^3), e^{N|x_3|}L^2(\mathbb{R}^3))$

Near a Landau level $2bq$, we parametrize $z$ by $2bq + k^2$ and we have
Proposition 5.1 ([5, Lemma 1]). For $V$ satisfying (5.1), the operator valued function

$$k \mapsto T_V(k) := J|V|^{\frac{1}{2}}(H_0 - 2bq - k^2)^{-1}|V|^{\frac{1}{2}},$$

defined in $]0,\sqrt{2b}|e^{[0,\pi/2]|$, has an analytic extension to the set $D := \{k \in \mathbb{C}; 0 < |k| < \min(\sqrt{2b}, N)\}$.

Then, using the resolvent equation

$$(5.2) \quad (I + J|V|^{\frac{1}{2}}(H - z)^{-1}|V|^{\frac{1}{2}})(I - J|V|^{\frac{1}{2}}(H_0 - z)^{-1}|V|^{\frac{1}{2}}) = I,$$

we have the following characterization of the resonances of $H$.

Definition 5.2. Under the assumption (5.1), $z_0 = 2bq + k_0^2$ is a resonance of $H$ near $2bq$ if and only if $k_0$ is a characteristic value of $I + T_V$ and the multiplicity of the resonance is the multiplicity of the characteristic value defined in Definition 6.3.

According to Proposition 3 of [5] and (6.3), the above definition coincides with the definition of the resonances given in [5]. Moreover, the following proposition shows that we are in the framework of Section 2.

Proposition 5.3 ([5, Proposition 4]). Assume that $V$ satisfies (5.1). For $k \in D$, we have

$$I + T_V(k) = I - \frac{A(ik)}{ik},$$

where $z \mapsto A(z)$ is the holomorphic compact operator valued function given by

$$(5.3) \quad A(z) = J|V|^{\frac{1}{2}}p_q \otimes r(z)|V|^{\frac{1}{2}} + zJ \sum_{j \neq q} |V|^{\frac{1}{2}}(p_j \otimes I)(D^2_{x_j} + 2b(j - q) + z^2)^{-1}|V|^{\frac{1}{2}}.$$

Here, $r(z)$ is the operator on $L^2(\mathbb{R}_{x_q})$ with integral kernel $\frac{1}{2}e^{iz|x_q - x'_q|}$ and, for $j \in \mathbb{N}$, $p_j$ is the orthogonal projection in $L^2(\mathbb{R}^2)$ having the following integral kernel

$$(5.4) \quad P_{j,b}(x_\perp, x'_\perp) = \frac{b}{2\pi}L_j \left(\frac{|x_\perp - x'_\perp|^2}{2}\right) \exp \left(-\frac{b}{4}|(x_\perp - x'_\perp|^2 + 2i(x_1x'_2 - x'_1x_2))\right),$$

where $L_j(t) := \frac{1}{\pi}e^{\frac{t(1+e^{-t})}{2}}$ are the Laguerre polynomials.

Consequently, the resonances of $H$ near a fixed Landau level $2bq$ are the complex number $2bq + k^2$ where $k$ is a characteristic value of $(I - \frac{A(ik)}{ik})$ and $A$ is given by (5.3). In particular, $A(0)$ is the operator $J|V|^{\frac{1}{2}}(p_q \otimes r(0))|V|^{\frac{1}{2}}$ which is self-adjoint as soon as $J$ is $\pm I$ (i.e. for $V$ of definite sign).

Now we assume that $V$ is of definite sign (say $\pm V \geq 0$). In order to apply the results of Section 2, we want to know when

$$(5.5) \quad I - A'(0)\Pi_0$$

(where $\Pi_0$ is the orthogonal projection on the kernel of $A(0)$). Writing $A(0) = \pm L^*L$, a function $f$ is in the kernel of $A(0)$ if and only if $Lf = 0$ with

$$(5.6) \quad Lf(X_\perp) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}_x^2} P_{q,b}(X_\perp, X'_\perp)|V|^{\frac{1}{2}}(X'_\perp, x'_q)f(X'_\perp, x'_q)dX'_{\perp}dx'_q,$$

for $X_\perp \in \mathbb{R}^2$. In general, this set is not reduce to $\{0\}$. Nevertheless, we have the following properties.
Remark 5.4. The assumption (5.5) holds generically on \( V \). More precisely, if the potential \( V \) is fixed, there exists a finite or denumerable discrete set \( \mathcal{E} = \{ e_n \} \) such that the operator \( H_\varepsilon := H_0 + eV \) satisfies (5.5) for all \( e \in \mathbb{R} \setminus \mathcal{E} \). These \( e_n \)'s are in fact the inverses of the real eigenvalues of the compact operator \( A'(0)\Pi_0 \). To prove this point, it is enough to remark that \( \Pi_{0|V} = \Pi_{0\setminus V} \) and \( A'(0)\Pi_0 = eA'(0)\Pi_0 \) for \( e \neq 0 \). Note also that, for \( |e| \) small enough, \( H_\varepsilon \) verifies always (5.5).

Under these assumptions, we can apply Theorem 2.6 and its corollaries. Thus, the distribution of the magnetic resonances near the Landau level is related to the counting function

\[ n_\pm(s; A(0)) = n_+(s; L^*L) = n_+(s; LL^*) = n_+(s; p_q W p_q), \]

where, for a compact self-adjoint operator \( A \), we set \( n_\pm(s; A) = \text{rank} \, 1_{\pm [s, +\infty]}(A) \), \( p_q \) is the orthogonal projection defined by the integral kernel (5.4) and \( W \) is the multiplication operator by the function

\[ W(X_\perp) := \frac{1}{2} \int_\mathbb{R} |V|(X_\perp, x_3) \, dx_3. \]

Let us introduce three types of assumptions for \( W \):

(A1) \( W \in C^1(\mathbb{R}^2) \) satisfy the estimate

\[ \forall X_\perp \in \mathbb{R}^2, \quad |\nabla W(X_\perp)| \leq C\langle X_\perp \rangle^{-m_\perp - 1}, \]

for some \( C > 0 \) and

\[ W(X_\perp) = w_0(X_\perp/|X_\perp|)|X_\perp|^{-m_\perp}(1 + o(1)) \quad \text{as } |X_\perp| \to +\infty, \]

where \( w_0 \) is a continuous function on \( S^1 \) which is non-negative and does not vanish identically.

(A2) There exists \( \beta > 0, \mu > 0 \) such that

\[ \ln W(X_\perp) = -\mu |X_\perp|^{2\beta}(1 + o(1)) \quad \text{as } |X_\perp| \to +\infty. \]

(A3) The support of \( W \) is compact and there exists a constant \( C > 0 \) such that \( W \geq C \) on an non-empty open subset of \( \mathbb{R}^2 \).

Theorem 5.5. Let \( V : \mathbb{R}^3 \to \mathbb{R} \) be a Lebesgue measurable function of definite sign \( \pm \) satisfying (5.1) and (5.5). Let \( 0 < r_0 < \min(\sqrt{2b}, N) \) be fixed. Then,

i) The resonances \( z_\ell(k) = 2bq + k^2 \) of \( H = H_0 + V \) with \( |k| \) sufficiently small satisfy

\[ \pm \text{Im} \, k \leq 0, \quad \text{Re} \, k = o(|k|). \]

ii) There exists a sequence \( (r_\ell)_{\ell} \in \mathbb{R} \) which tends to 0 such that

\[ \# \{ z = 2bq + k^2 \in \text{Res}(H) : r_\ell < |k| \leq r_0 \} = n_+(r_\ell, p_q W p_q)(1 + o(1)) \quad \text{as } \ell \to +\infty. \]

iii) Eventually, if \( W \) satisfies (A1), (A2) or (A3), then

\[ \# \{ z = 2bq + k^2 \in \text{Res}(H) : r < |k| \leq r_0 \} = n_+(r, p_q W p_q)(1 + o(1)) \quad \text{as } r \searrow 0, \]

with

a) under the assumption (A1):

\[ n_+(r, p_q W p_q) = C_\perp r^{-2/m_\perp}(1 + o(1)) \quad \text{as } r \searrow 0, \]

where

\[ C_\perp := \frac{b}{4\pi} \int_{S^1} w_0(t)^{2/m_\perp} \, dt. \]
b) under the assumption (A2):

\[ n_+(r, p_q W p_q) = \varphi_\beta(r)(1 + o(1)) \quad \text{as } r \searrow 0, \]

where, for \( 0 < r \ll 1 \),

\[ \varphi_\beta(r) := \begin{cases} \frac{b}{2\mu} \frac{1}{|\ln r|^2} & \text{if } 0 < \beta < 1, \\ \frac{1}{|\ln(1 + 2\mu/b)|} |\ln r| & \text{if } \beta = 1, \\ \frac{\beta}{\beta - 1}(|\ln|\ln r||)^{-1} |\ln r| & \text{if } \beta > 1. \end{cases} \]

c) under the assumption (A3):

\[ n_+(r, p_q W p_q) = \varphi_\infty(r)(1 + o(1)) \quad \text{as } r \searrow 0, \]

where, for \( 0 < r \ll 1 \),

\[ \varphi_\infty(r) := (|\ln|\ln r||)^{-1} |\ln r|. \]

**Remark 5.6.**

i) Under the assumption (A1), we can also apply Theorem 2.1 to obtain asymptotics in small domains.

ii) In [5], we have proved that \( H \) has an infinite number of resonances in a vicinity of 0 for small potentials \( V \) of definite sign such that \( W, \) defined in (5.7), satisfies, for some \( C > 0, \)

\[ \ln W(X_\perp) \leq -C\langle X_\perp \rangle^2. \]

**Proof of Theorem 5.5.** According to Definition 5.2 and Proposition 5.3, in order to study the resonances \( z_q(k) = 2bq + k^2 \) of \( H \), it is enough to analyze the characteristic values of \( (I - \frac{A(ik)}{k}) \) for \( A \) given by (5.3). Since \( \pm A(0) \) is non negative, i) is a consequence of Corollary 2.3 with \( z = ik. \)

From i) we deduce that the resonances \( z_q(k) = 2bq + k^2 \) are concentrated in the sector \( k \in \mp iC_\theta \cap D \) for every \( \theta > 0 \) with \( C_\theta \) defined by (2.1). In particular, as \( r \) tends to 0, we have:

\[ \# \{ z = 2bq + k^2 \in \text{Res}(H); \ r_\ell < |k| \leq r_0 \} = \# \{ z = 2bq + k^2 \in \text{Res}(H); \pm ik \in C_\theta(r, r_0) \} + O(1). \]

Since the non-zero eigenvalues of \( \pm A(0) = L^*L \) (see (5.6) for the definition of \( L \)) coincide with these of \( LL^* = p_q W p_q, \) we have \( n(|r, r_0|) = n_+(r, p_q W p_q) + O(1). \) Then, ii) and iii) follow from Corollary 2.8 and Corollary 2.10 together with results of [23] and [27] concerning asymptotic behaviors of \( n_+(r, p_q W p_q) \) under the assumptions (A1)–(A3). In particular, note that, for \( V \neq 0, \) \( n_+(r, p_q W p_q) \) tends to infinity as \( r \) tends to 0.

\[ \square \]

6. **Characteristic values of holomorphic operators**

In this appendix, we define the notions of characteristic values of an operator valued holomorphic function and their multiplicities. For more details, we refer to [13] and to Section 4 of [12].

**Definition 6.1.** Let \( w \in \mathbb{C}, \) let \( U \) be a neighborhood of \( w, \) and let \( F : U \setminus \{w\} \longrightarrow L(E) \) (\( E \) a Banach space) be a holomorphic function which is meromorphic on \( U \) (i.e. \( w \) is an isolated
We say that $F$ is finite meromorphic at $w$ if the Laurent expansion of $F$ at $w$ is of the form

$$F(z) = \sum_{n=m}^{+\infty} (z - w)^n A_n,$$

where, for $m < 0$, the operators $A_m, \cdots, A_{-1}$ are of finite rank.

If in addition $A_0$ is a Fredholm operator, then $F$ is called finite meromorphic and Fredholm at $w$. The index of $A_0$ will be called the index of $F$ at $w$.

**Proposition 6.2** ([12, Proposition 4.1.4]). Let $D \subset \mathbb{C}$ be a connected open set and let $Z \subset D$ be a discrete and closed subset of $D$. If $F : D \rightarrow \mathbb{L}(E)$ is holomorphic on $D \setminus Z$, finite meromorphic on $D$ and Fredholm at each point of $D$ such that for some $z_0 \in D \setminus Z$, $F(z_0)$ is invertible. Then there exists a discrete and closed subset $Z'$ of $D$, $Z' \supset Z$ such that $F(z)$ is invertible for $z \in D \setminus Z'$ and $F^{-1} : D \setminus Z' \rightarrow GL(E)$ is finite meromorphic and Fredholm at each point of $D$.

Then we can define the characteristic values and their multiplicity.

**Definition 6.3.** With the notations of Proposition 6.2, each point of $Z'$ is called a characteristic value of $F$. The multiplicity of a characteristic value $w_0$ is defined by

$$\text{mult}(w_0) := \frac{1}{2i\pi} \text{tr} \int_{|w - w_0| = \rho} F'(z)F(z)^{-1}dz,$$

where $\rho > 0$ is sufficiently small such that $\{w; |w - w_0| \leq \rho \} \cap Z' = \{w_0\}$.

By definition, a characteristic value is a complex number $w$ for which $F(w)$ is not invertible and according to results of [13] and [12, Section 4], mult($w$) is an integer. Moreover, this definition of multiplicity coincides with a definition of the order of $w$ as zero of $F$ (see [13] for more details).

If $\Omega \subset D$ is a connected domain such that $\partial \Omega \cap Z' = \emptyset$ then the sum of the multiplicities of the characteristic values of $F$ inside $\Omega$ is the so-called index of $F$ with respect to the contour $\partial \Omega$, given by

$$\text{Ind}_{\partial \Omega} F := \frac{1}{2i\pi} \text{tr} \int_{\partial \Omega} F'(z)F(z)^{-1}dz = \frac{1}{2i\pi} \text{tr} \int_{\partial \Omega} F(z)^{-1}F'(z)dz.$$

We easily check the following property for two operator valued functions $F_1$ and $F_2$ satisfying assumptions of Proposition 6.2:

$$\text{Ind}_{\partial \Omega}(F_1F_2) = \text{Ind}_{\partial \Omega} F_1 + \text{Ind}_{\partial \Omega} F_2.$$

Let us remark also that if $(I - F)$ is trace class valued, then $F'(z)F(z)^{-1}$ is a trace class operator for $z \in D \setminus Z'$ and in this case we have

$$\text{Ind}_{\partial \Omega} F = \frac{1}{2i\pi} \int_{\partial \Omega} \text{tr}(F'(z)F(z)^{-1})\ dz = \text{ind}_{\partial \Omega} f,$$

where $f(z)$ is the Fredholm determinant of $F(z)$ and $\text{ind}_{\partial \Omega} f$ is the standard index of a holomorphic function giving the number of zeroes:

$$f(z) = \det(F(z)), \quad \text{ind}_{\partial \Omega} f := \frac{1}{2i\pi} \int_{\partial \Omega} \frac{f'(z)}{f(z)}\ dz.$$
More generally, if \((I - F)\) is with value in the Schatten class \(S_p\), then the regularized determinant of \(F\) is well defined (see [18], [19], [20]):

\[
f_p(z) = \det_p(F(z)) := \det \left( F(z) \exp \left( \sum_{k=1}^{p-1} \frac{1}{k} (I - F(z))^k \right) \right),
\]

and if \(\partial \Omega \cap Z' = \emptyset\), we have:

\[(6.3)\]

\[
\text{ind}_{\partial \Omega} f_p = \frac{1}{2i\pi} \int_{\partial \Omega} \text{tr} \left( F'(z)(F(z))^{-1} - \sum_{k=1}^{p-1} F'(z)(I - F(z))^{k-1} \right) dz = \text{Ind}_{\partial \Omega} F,
\]

because the sum in the above formula is holomorphic in \(\Omega\).

Moreover, we have a Rouché Theorem:

**Theorem 6.4** ([13, Theorem 2.2], [12, Theorem 4.4.3]). For \(D \subset \mathbb{C}\) a bounded open set with piecewise \(C^1\)-boundary and \(Z \subset D\) a finite set, let \(F : \overline{D} \setminus Z \rightarrow GL(E)\) be a holomorphic function which is finite meromorphic and Fredholm at each points of \(Z\) and let \(G : \overline{D} \setminus Z \rightarrow L(E)\) be a holomorphic function which is finite meromorphic at each points of \(Z\) such that

\[\|F(z)^{-1}G(z)\| < 1\]

for \(z \in \partial D\).

Then \(F + G\) is finite meromorphic and Fredholm at each points of \(Z\) and

\[\text{Ind}_{\partial D}(F + G) = \text{Ind}_{\partial D}(F)\]

**References**


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