SEMI-CLASSICAL BEHAVIOUR OF THE SCATTERING PHASE FOR TRAPPING PERTURBATIONS OF THE LAPLACIAN

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Abstract– We study the influence of trapping trajectories on the semi-classical asymptotic of the scattering phase (spectral shift function), $s_h(\lambda)$ associated to Schrödinger operators, if λ is a non critical energy level. Some results are known when the set of closed trajectories has Liouville measure 0 and in the case of a potential well. In this paper, for smooth and sufficiently decreasing potentials, we describe the behaviour of $s_h(\lambda + rh)$ in terms of the continuity properties of a certain oscillating function $Q(h,r)$. This function, introduced by V. Petkov and G. Popov to study clustering of eigenvalues, is related to the periodic trajectories. If $Q(h,r)$ is uniformly continuous in r for any $h \in (0, h_0]$, we obtain a Weyl type asymptotic of $s_h(\lambda + rh)$, with a second term containing $Q(h,r)$. On the other hand, the point of discontinuity of $Q(h,r)$ in r may give rise to a "clustering" phenomena for the scattering phase. Hence, in contrast to non-trapping case, the derivative $\frac{d}{d\lambda}s_h(\lambda)$ can have no complete asymptotic expansion.

1 Introduction

Let us denote by H_0 the self-adjoint extension in $L^2(\mathbb{R}^n)$ of $(-h^2\Delta)$, $h>0$, Δ being the usual Laplace operator. We want to discuss the semi-classical asymptotic (i.e. as $h \searrow 0$) of the scattering phase $s_h(\lambda)$ for a class of perturbations H_1 of H_0 of the following form: $H_1 = -h^2\Delta_g + B(x,hD)$, Δ_g being the symmetric Laplace-Beltrami operator, acting on $L^2(\mathbb{R}^n)$, associated with a metric g and

$$B(x,hD) := \frac{1}{2}\Big(a(x).hD + hD.a(x)\Big) + V(x), \qquad D = (-i\partial/\partial_{x_j})_{1\le j\le n}$$

is a symmetric first order differential operator, where $a = (a_j)_{1\le j\le n}$, V, and $g = \{g_{ij}\}_{1\le i,j\le n}$ are C^∞ on $\mathbb{R}^n$. Moreover, we assume the following condition: (H_δ): $-$ For any $\alpha \in \mathbb{N}^n$, there exist $C_\alpha > 0$, $\delta > n$ such that:

$$\mid \partial_x^\alpha\Big(g-g_0\Big)(x)\mid + \mid \partial_x^\alpha a(x)\mid + \mid \partial_x^\alpha V(x)\mid \le C_\alpha \langle x\rangle^{-\delta-|\alpha|},$$

where for any $x \in \mathbb{R}^n$, $g_0(x) = \mathbb{1} = \{\delta_{ij}\}$ is the flat Euclidean metric and $\langle x\rangle = (1+\mid x\mid^2)^{\frac{1}{2}}$.

Hamiltonians of this type appear in quantum mechanics under the form:

$$L(g,A,V) \stackrel{\text{def}}{=} -G^{-1/4}\sum_{j,k=1}^{n}\Big(h\partial_j + iA_j\Big)G^{1/2}g^{jk}\Big(h\partial_k + iA_k\Big)G^{-1/4} + V\ ,$$

where $\partial_j = \partial/\partial_{x_j}$, $\{g^{jk}\} = g^{-1}$, $G = det(g)$ and $A = (A_1, ..., A_n)$ represents a magnetic potential.

Let us introduce the classical Hamiltonian associated with H_1:

$$a_0(x,\xi) =< g(x)^{-1}\xi, \xi > + a(x).\xi + V(x).$$

For the Hamiltonian vector field $H_{a_0} = (\partial_\xi a_0, -\partial_x a_0)$, we will denote by $\Phi_{a_0}(t) = \exp(tH_{a_0})$, the Hamiltonian flow.

In this paper, we consider a non critical energy level λ, i.e. $\nabla a_0(x,\xi) \neq 0$ on the set $\Sigma_\lambda = a_0^{-1}(\lambda)$. Our goal is to study the influence of trapped trajectories on the semi-classical asymptotic of the scattering phase. We will see that, near a trapping energy level, an oscillatory term could appear in a Weyl type asymptotic formula.

Let us now recall some results about scattering theory and the definition of the scattering phase. We consider the wave operators:

$$W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} \exp(-itH_1)\exp(itH_0)$$

which exist and are complete under the assumption (H_δ). The scattering operator defined by $S = W_-^* W_+$ can be decomposed on the spectral representation of H_0 as follows:

$$(Sf)(\lambda,\omega) = (S(\lambda)f(\lambda,.))(\omega),$$

where $\lambda > 0$, $\omega \in S^{n-1} = \{x \in \mathbb{R}^n; \mid x \mid = 1\}$. The scattering matrix $S(\lambda)$ is an unitary operator on $L^2(S^{n-1})$. Moreover, using the stationary representation formula of $S(\lambda)$ (see [31]), we can prove that $Id - S(\lambda)$ is a trace class operator. Then it is possible to define $\det S(\lambda)$ and the scattering phase $s_h(\lambda)$, modulo $\mathbb{Z}$, as a smooth function satisfying:

$$\det S_h(\lambda) = e^{2i\pi s_h(\lambda)}.$$

According to the Birman-Krein theory ([1],[31]), $s_h(\lambda)$ can be identified with the spectral shift function and the Krein formula :

$$Tr\Big(f(H_1) - f(H_0)\Big) = -\int_{\mathbb{R}} s_h(\lambda) f'(\lambda) d\lambda, \qquad \forall f \in C_0^\infty(\mathbb{R}), \tag{1}$$

holds.

It is well known that for $\lambda > 0$ we have

$$s_h(\lambda) = h^{-n}\gamma_0(\lambda) + \mathcal{O}(h^{1-n}), \qquad h \searrow 0 \tag{2}$$

uniformly with respect to λ in compact subset of $]0,+\infty[$, and $\gamma_0 \in C^\infty(]0,+\infty[)$ with

$$\gamma_0(\lambda) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{a_0(x,\xi)\le\lambda} d\xi - \int_{|\xi|^2\le\lambda} d\xi \right) dx. \tag{3}$$

In the case $g = \mathbb{1}$ and $a = 0$, we have:

$$\gamma_0'(\lambda) = (2\pi)^{-n}\frac{\omega_n}{2} \int_{\mathbb{R}^n} \left((\lambda - V(x))_+^{\frac{n}{2}-1} - (\lambda)^{\frac{n}{2}-1} \right) dx.$$

The Weyl formula (2) can be proved by the classical method of Hörmander-Levitan ([12], [6], [9], [26]...), splitting $s_h(\lambda)$ into two monotone pieces (see section 4 of [22]).

Moreover, if the set of closed trajectories has Liouville measure 0 on Σ_λ, then ([20], [24], [26]) :

$$s_h(\lambda) = h^{-n}\gamma_0(\lambda) + o(h^{1-n}), \qquad h \searrow 0. \tag{4}$$

At last, if J is a compact non-trapping interval in $]0,+\infty[$, that is if for any $\lambda \in J$, the classical flow associated to the Hamiltonian $a_0(x,\xi)$ has not trapped trajectories in Σ_λ, then ([27], [24], [26]):

$$\frac{d}{d\lambda} s_h(\lambda) \asymp \sum_{j\ge 0} h^{-n+2j}\alpha_{2j}(\lambda), \qquad h \searrow 0. \tag{5}$$

Let us mention that similar results are valid in high energy asymptotics for obstacle perturbation problems ([4], [14], [15], [18], [16], [25]) and for metric perturbation problems ([15], [22]). These forms of asymptotics are also known for the Dirac operator ([2], [3]).

The assumptions of null measure or of non-trapping condition are physically restrictives. It is important to understand the influence of the trapping trajectories on the asymptotics of scattering phase.

For trapping trajectories due to the presence of a potential well, it is known that some resonances occur [11] and the resonances close to the real axis produce big variations in the scattering phase (see Breit-Wigner formula in [7] and [24]).

The scattering phases associated to the Dirichlet and Neumann problems for the elasticity operator is studied by F. Cardoso and G. Vodev [5]. The asymptotic behaviour of the scattering phase is connected with the counting function of the Rayleigh resonances converging rapidly to the real axis. This yields a Weyl asymptotic with a second term.

The case of metric perturbation of the Laplacian was treated by V. Petkov [17]. He proves a Weyl type formula (in high energy regime) for scattering phase involving an oscillatory term which is related to the periodic trajectories of the Hamiltonian vector field H_{a_0}.

We will prove similar estimates concerning the semi-classical asymptotics. We consider $s_h(\lambda)$ at the energy level $\lambda(h) = \lambda + rh$, where the energy shift parameter r is bounded.

In this work, our aim is to describe the semi-classical behaviour of $s_h(\lambda+rh)$ in terms of the continuity properties of a certain oscillating function $Q(h,r)$. This term, defined below, involves dynamic and topological characteristics of the corresponding closed trajectories in Σ_λ. We shall obtain either semi-classical asymptotics of $s_h(\lambda + rh)$ or energy levels $\lambda(h) = \lambda + r(\lambda, h)h$ where s_h varies rapidely for small h.

Let us introduce some notation to define $Q(h,r)$. A point $\nu \in \Sigma_\lambda$ is called periodic, if there exists $T > 0$ such that

$$\Phi_{a_0}^T(\nu) = \nu. \tag{6}$$

For a periodic point ν, by $\gamma(\nu) = \{\Phi_{a_0}^t(\nu) : 0 \leq t \leq T(\nu)\} = \{(x(t), \xi(t)) : 0 \leq t \leq T(\nu)\}$ we denote the primitive periodic trajectory issued from ν, where $T(\nu) = T_\gamma$ is the period of $\gamma(\nu)$ (i.e. the smallest positive number T for which (6) holds).

For any periodic point ν, we define the action along the corresponding primitive periodic trajectory $\gamma(\nu)$ by:

$$S(\nu) := \int_{\gamma(\nu)} \xi dx = \int_0^{T(\nu)} \xi(t)x'(t)dt.$$

We denote by $m(\gamma) \in \mathbb{Z}_4$ the Maslov index related to $\gamma(\nu)$ and set

$$q(\nu) := -\frac{\pi}{2}m(\gamma).$$

Let Π be the set of all periodic points on Σ_λ, and let us introduce the oscillating function

$$Q(h,r) := (2\pi)^{-n} \int_\Pi \left[\pi - h^{-1}S(\nu) + q(\nu) - rT(\nu)\right]_{2\pi} T(\nu)^{-1} d\nu, \qquad (7)$$

where $d\nu = \mid \nabla a_0 \mid^{-1} dS$ stands for the Liouville measure on Σ_λ, dS being the induced Lebesgue measure. The residium $[z]_{2\pi}$ is defined by:

$$z = [z]_{2\pi} + 2\pi k, \qquad -\pi < [z]_{2\pi} \leq \pi, \qquad k \in \mathbb{Z}.$$

The function $Q(h,r)$ was introduced by V. Petkov and G. Popov [19] to study the semi-classical behaviour of the number of eigenvalues situated in the interval $[E + rh - ch, E + rh + ch]$. This function could be considered as a semi-classical analogue of the oscillating function defined by T. Guriev and Yu. Safarov [8], and Yu. Safarov ([28], [29]), who investigated the distribution of the eigenvalues at high energies for elliptic operators in compact manifolds.

It is obtained in the proposition 6.1 of [19] (see proposition 5.4 below), assuming there exist $r_0 > 0$, $h_0 > 0$ and $\omega_0 > 0$ such that

$$h^{-1}S(\nu) + rT(\nu) - q(\nu) \geq \omega_0, \quad \forall \nu \in \Pi_+, \ \ h \in]0, h_0], \ \ |r| \leq r_0, \qquad (8)$$

where Π_+ is the set of points $\nu \in \Pi$ of a positive Lebesgue density in Π. Let us note that the condition (8) holds if Σ_λ is of contact type, or under an assumption of *strictly* ξ*-convexity* for $a_0(x,\xi)$ (see [19] for more details).

Let us recall some properties of the function $Q(h,r)$. The limits

$$Q(h, r \pm 0) = \lim_{\epsilon \searrow 0} Q(h, r \pm \epsilon)$$

exist for any $0 < h \leq h_0$ and any r. Moreover,

$$Q(h, r+0) - Q(h, r-0) = (2\pi)^{1-n} \int_{\Omega_{h,r}} \frac{d\nu}{T(\nu)},$$

where $\Omega_{h,r} := \{\nu \in \Pi : h^{-1}S(\nu) - q(\nu) + rT(\nu) \equiv 0(2\pi)\}$.

The main difficulty, in the study of s_h, is to take into account all periodic trajectories of H_{a_0}. To overcome this difficulty, we are inspired by results of V. Petkov and G. Popov [19] used for counting function, where the introduction of absolutely periodic points (adapted to the semi-classical case) allow the construction of suitable phase functions in special symplectic coordinates.

The difference with counting function, is that the scattering phase is not monotone. Then to apply a Tauberian theorem (see section 8) and results of [19], a first step is to split s_h into two terms : $\sigma_1(h,\lambda)$ a monotone function which involve the contribution of all periodic trajectories and $\sigma_b(h,\lambda)$ which does not involve the contribution of the periodic trajectories.

This splitting of s_h is the object of the section 3, where we prove a perturbation trace formula.

In section 4, we study the asymptotic behaviour of $\sigma_b(h,\lambda)$, splitting it into two monotone terms $\sigma_{b^\pm}(h,\lambda)$ and using standard arguments of asymptotic analysis. It is related with the singularities at $t = 0$ of the propagator $U_h(t) = e^{ith^{-1}H_1}$.

In section 5 the results of [19] are adapted for $\sigma_1(h,\lambda)$.

The results stated in section 2 are proved in section 6 while in section 7, we give an example of Schrödinger operator for which "clustering" phenomena occurs for the scattering phase.

2 Main results

Theorem 2.1 *Let us assume (8) and (H_δ), $\delta > n$. Let $\lambda > 0$ be a non critical energy level for $a_0(x,\xi)$.*

For $r_0 > 0$ there exist $h_0 > 0$, $\epsilon_0 > 0$ such that for all $h \in (0,h_0]$, $\mid r \mid < r_0$ and $0 < \epsilon < \epsilon_0$ we have :

$$\begin{aligned} h^{1-n}Q(h,r-\epsilon) - C_0\epsilon h^{1-n} + o_\epsilon(h^{1-n}) \\ \leq s_h(\lambda + rh) - h^{-n}\gamma_0(\lambda) - h^{1-n}r\gamma_0'(\lambda) \leq \\ h^{1-n}Q(h,r+\epsilon) + C_0\epsilon h^{1-n} + o_\epsilon(h^{1-n}), \end{aligned} \tag{9}$$

with $C_0 > 0$ independent of h, r, ϵ.

Corollary 2.2 *Suppose $Q(h,r)$ uniformly continuous with respect to $r \in [r_1,r_2]$, $r_1 < r_2$, for $h \in (0,h_0]$.*

a) Then for any $r \in [r_1, r_2]$ and $h \in (0, h_0]$, we have a second term Weyl type asymptotic :

$$s_h(\lambda + rh) = h^{-n}\gamma_0(\lambda) + h^{1-n}\Big(Q(h,r) + r\gamma_0'(\lambda)\Big) + o(h^{1-n}). \tag{10}$$

b) Let $r_1 < R_1 < R_2 < r_2$. Then there exists $c_0 > 0$ such that for any $r \in [R_1, R_2]$, $\mid c \mid \leq c_0$ and $h \in (0, h_0]$, we have :

$$s_h(\lambda+rh+ch) - s_h(\lambda+rh) = ch^{1-n}\gamma_0'(\lambda) + h^{1-n}\Big(Q(h,r+c) - Q(h,r)\Big) + o_{c_0}(h^{1-n}).$$

Remark 2.3 *In particular, if the set of closed trajectories has Liouville measure 0 on Σ_λ, then $Q(h,r)$ vanishes, and for $r = 0$ the corollary 2.2 yields the formula (4).*

Corollary 2.4 *Suppose that there exist $\lambda_0 > 0$, an integer p and a subset $\Pi^1 \subset \Pi \subset \Sigma_{\lambda_0}$ of a positive Lebesgue measure such that,*

$$r(h) := \Big([q(\nu) - h^{-1}S(\nu)]_{2\pi} + 2\pi p\Big)T(\nu)^{-1}\,, \qquad h \in (0, h_0]$$

does not depend on $\nu \in \Pi^1$. Then we have:

a) There exists $c_0 > 0$ such that for any $c \in (0, c_0]$, we have :

$$\liminf_{h \searrow 0} h^{n-1}\Big(\pm s_h(\lambda_0 + r(h)h \pm ch) \mp s_h(\lambda_0 + r(h)h)\Big) \geq (2\pi)^{1-n}\int_{\Pi^1} \frac{d\nu}{T(\nu)}. \tag{11}$$

b) There exists a compact set $K \subset]0, +\infty[$ containing λ_0 so that $h^n \frac{d}{d\lambda}s_h(\lambda)$ is not uniformly bounded with respect to $(h, \lambda) \in (0, h_0] \times K$.

In particular, $\frac{d}{d\lambda}s_h(\lambda)$ does not satisfy a complete asymptotic expansion of the form (5).

3 A Perturbation Trace formula

We start with a representation formula of s_h, adapted to the application of a Tauberian theorem and of results of [19]. We introduce the generator of dilation group of $\mathbb{R}^n$: $\mathcal{A}_h := \frac{1}{2}(x.hD + hD.x)$ and a function $\zeta \in C_0^\infty(\mathbb{R}^n)$ which will be equal to 1 on a compact set containing the x-components of the periodic trajectories (as proved in the Appendix, under the assumption (H_δ), this compact set exists).

Proposition 3.1 *For any $f \in C_0^\infty(\mathbb{R}\setminus 0)$, and $\zeta \in C_0^\infty(\mathbb{R}^n)$ we have the trace formula :*

$$-\int f'(\lambda)s_h(\lambda)d\lambda = Tr\Big(\zeta^2 f(H_1)\Big) + Tr\left(\tilde{W}(h)H_1^{-1}f(H_1)\right), \qquad (12)$$

where

$$\tilde{W}(h) := \Big(\frac{[\zeta^2, H_1]}{2ih}\mathcal{A}_h + (1-\zeta^2)W(h)\Big),$$

is a 1-order h-admissible pseudodifferential operator vanishing on the interior of $\zeta^{-1}(\{1\})$ and $W(h) := (H_1 - H_0) + (2ih)^{-1}[(H_1 - H_0), \mathcal{A}_h]$.

Let us mention that an analogous formula was proved and exploited in [25] for obstacle perturbations. We can also prove similar formula for a more general class of functions and operators (see [30]), but $f \in C_0^\infty(\mathbb{R}\setminus 0)$ is sufficient to study the scattering phase.

Remark 3.2 *Of course H_1^{-1} is not well defined, however for $f \in C_0^\infty(\mathbb{R}\setminus 0)$, we can define $H_1^{-1}f(H_1)$ as the self-adjoint operator $\varphi(H_1)$ where $\varphi \in C_0^\infty(\mathbb{R})$ satisfies:*

$$\varphi(\lambda) = \begin{cases} \lambda^{-1}f(\lambda) & \textit{for} \quad \lambda \neq 0 \\ 0 & \textit{for} \quad \lambda = 0 \end{cases}$$

Remark 3.3 *To obtain the representation formula (12), we will use that H_0 satisfies:*

$$H_0 = -\frac{[H_0, \mathcal{A}_h]}{2ih}.$$

Then, by the cyclicity of traces, formally we have:

$$\text{"}Tr\Big(f(H_0)\Big) = Tr\Big(H_0H_0^{-1}f(H_0)\Big) = -\frac{1}{2ih}Tr\Big([H_0, \mathcal{A}_h]H_0^{-1}f(H_0)\Big) = 0\text{"}$$

Consequently, we will be able to express $s_h(\lambda)$ as a local spectral density of H_1 and to split $s_h(\lambda)$ into a sum of monotone functions.

If we want to compare H_1 with another Hamiltonian of the same form, we have not this kind of representation formula by spectral densities. In this case, we can use the asymptotic representation formula proved by D. Robert [24]. However some difficulties occur when we estimate resolvants without non-trapping assumptions.

Proof of the proposition 3.1. – Let $f \in C_0^\infty(\mathbb{R}\setminus 0)$. According to the Krein formula (1), we want to prove:

$$Tr\Big(f(H_1) - f(H_0)\Big) = Tr\Big(\zeta^2 f(H_1)\Big) + Tr\left(\tilde{W}(h)H_1^{-1}f(H_1)\right).$$

For $\zeta \in C_0^\infty(\mathbb{R}^n)$, $(\zeta^2, 1-\zeta^2)$ is a partition of the unity and $\zeta^2 f(H_1)$, $\zeta^2 f(H_0)$ are of trace class. Consequently,

$$Tr\Big(\left[f(H.)\right]_0^1\Big) = Tr\Big(\zeta^2 f(H_1)\Big) - Tr\Big(\zeta^2 f(H_0)\Big) + Tr\Big((1-\zeta^2)\left[f(H.)\right]_0^1\Big), \quad (13)$$

where $[f(H.)]_0^1 = f(H_1) - f(H_0)$. Next we shall prove that

$$Tr\Big((1-\zeta^2)\left[f(H.)\right]_0^1\Big) = Tr\Big(\zeta^2 f(H_0)\Big) + Tr\left(\tilde{W}(h) H_1^{-1} f(H_1)\right). \quad (14)$$

By using:

$$H_0 = -\frac{[H_0, \mathcal{A}_h]}{2ih}, \qquad H_1 = -\frac{[H_1, \mathcal{A}_h]}{2ih} + W(h) \quad (15)$$

we deduce that $\Big((1-\zeta^2)\left[f(H.)\right]_0^1\Big)$ has the form:

$$\Big((1-\zeta^2) W(h) H_1^{-1} f(H_1)\Big) - \left((1-\zeta^2)\left[\frac{[H., \mathcal{A}_h]}{2ih} H.^{-1} f(H.)\right]_0^1\right). \quad (16)$$

Moreover, $W(h) := (H_1 - H_0) + (2ih)^{-1}[(H_1 - H_0), \mathcal{A}_h]$ is a 1-order h-admissible pseudodifferential operator of weight $\langle x\rangle^{-\delta}\langle\xi\rangle$, with $\delta > n$. Consequently the first term in (16) is a trace class operator.

To compute the trace of the second term in (16), we apply the cyclicity of traces (see Appendix A of [22]) and the following relation:

$$(1-\zeta^2)[H., \mathcal{A}_h] = [H., (1-\zeta^2)\mathcal{A}_h] - [H., (1-\zeta^2)]\mathcal{A}_h$$

The term containing $[H., (1-\zeta^2)\mathcal{A}_h]$ vanishes, and we obtain:

$$Tr\left((1-\zeta^2)\left[\frac{[H., \mathcal{A}_h]}{2ih} H.^{-1} f(H.)\right]_0^1\right) = Tr\left(\left[\frac{[H., (1-\zeta^2)]}{2ih} \mathcal{A}_h H.^{-1} f(H.)\right]_0^1\right).$$

On the other hand, the operator $[H., (1-\zeta^2)] = -[H., \zeta^2]$ is a h-admissible pseudodifferential operator, compactly supported in suppζ, hence $[H., (1-\zeta^2)]\mathcal{A}_h H.^{-1} f(H.)$ is a trace class operator and the second term of (16) is equal to

$$Tr\left(\frac{[H_1, \zeta^2]}{2ih} \mathcal{A}_h H_1^{-1} f(H)\right) - Tr\left(\frac{[H_0, \zeta^2]}{2ih} \mathcal{A}_h H_0^{-1} f(H_0)\right). \quad (17)$$

Now, using $[H_0, \zeta^2]\mathcal{A}_h = [H_0, \zeta^2 \mathcal{A}_h] - \zeta^2 [H_0, \mathcal{A}_h]$, the cyclicity of traces again and (15), we prove:

$$Tr\left(\frac{[H_0, \zeta^2]}{2ih} \mathcal{A}_h H_0^{-1} f(H_0)\right) = Tr\Big(\zeta^2 f(H_0)\Big). \quad (18)$$

Thus (16), (17), (18) prove the relation (14) and we obtain the formula (12) applying the relations (13), (14). ■

We deduce now an useful pointwise representation formula for $s_h(\lambda)$. Let us introduce $E_{H_1}(\lambda)$, $\lambda > 0$, the spectral projector of H_1, on $]0, \lambda[$.

Corollary 3.4 *Let $0 < \bar{\lambda}_1 < \lambda_1 < \lambda_2 < \bar{\lambda}_2$ and $g \in C_0^\infty(]\bar{\lambda}_1, \bar{\lambda}_2[)$ be such that $g \equiv 1$ on $[\lambda_1, \lambda_2]$. Then, for all $\lambda \in]\lambda_1, \lambda_2[$, $s_h(\lambda)$ can be decomposed in the following form:*

$$s_h(\lambda) = \sigma_1(h, \lambda) + \sigma_b(h, \lambda), \tag{19}$$

where σ_1, σ_b are the spectral densities:

$$\sigma_1(h, .) \ : \ \lambda \longmapsto Tr\Big(\zeta g(H_1)^2 E_{H_1}(\lambda)\zeta\Big),$$

$$\sigma_b(h, .) \ : \ \lambda \longmapsto Tr\Big(b(x, hD, h) g(H_1)^2 E_{H_1}(\lambda)\Big),$$

with

$$b(x, hD, h) g(H_1)^2 := \left(\frac{[\zeta^2, H_1]}{2ih}\mathcal{A}_h + (1 - \zeta^2)W(h)\right) H_1^{-1} g(H_1)^2$$

a classical h-admissible pseudodifferential operator of order 0*. Moreover b belongs to the symbol class*

$$S(\langle x\rangle^{-\delta}) = \{s \in C^\infty(\mathbb{R}^{2n}) :\mid \partial_x^\alpha \partial_\xi^\beta s(x, \xi) \mid = O(\langle x\rangle^{-\delta - |\alpha|} \langle \xi\rangle^{-|\beta|})\} \tag{20}$$

with $\delta > n$.

Proof. $-$Using functional calculus for h-admissible pseudo-differential operators (see [21], [10]...) and condition (H_δ), we obtain the properties of b claimed in the proposition, and we prove easily that $\zeta g(H_1)^2$ and $b(x, hD, h)g(H_1)^2$ are trace class operators. Then the quantities $\sigma_1(h, \lambda)$, $\sigma_b(h, \lambda)$ are well defined. Moreover, $\sigma_1(h, .)$, $\sigma_b(h, .)$ and s_h are smooth functions on $]\lambda_1, \lambda_2[$ (see [23], [24]).

On the other hand, according to the fact that $s_h(\lambda)$ is defined modulo a constant term, it is sufficient to check derivative's relation of (19) in the distribution sense. This means that for any $f \in C_0^\infty(]\lambda_1, \lambda_2[)$, we want to prove:

$$\int f'(\lambda) s_h(\lambda) d\lambda = \int f'(\lambda)\sigma_1(h, \lambda) d\lambda + \int f'(\lambda)\sigma_b(h, \lambda) d\lambda.$$

By Fubini theorem, this is a consequence of proposition 3.1, because the spectral theorem yields:

$$-\int f'(\lambda)\sigma_1(h,\lambda)d\lambda = Tr\Big(\zeta g(H_1)^2 f(H_1)\zeta\Big) = Tr\Big(\zeta f(H_1)\zeta\Big).$$

$$-\int f'(\lambda)\sigma_b(h,\lambda)d\lambda = Tr\Big(b(x,hD,h)g(H_1)^2 f(H_1)\Big) = Tr\left(b(x,hD,h)f(H_1)\right).$$

■

Now, we are going to study $\sigma_1(h,\lambda)$ and $\sigma_b(h,\lambda)$. We will choose $\zeta \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on a compact set containing the x-components of periodic trajectories. Thus, the contribution of all periodic trajectories appears only in σ_1. It is the more difficult term to deal with. The asymptotic behaviour of σ_b will be obtained splitting it into two monotone terms and by using standard arguments of asymptotic analysis.

4 Asymptotic behaviour of σ_b

In this section, we prove the following proposition 4.1 using a Tauberian theorem (see Appendix). Let $b(x,hD,h)$ be a classical h-admissible pseudodifferential operator of order 0 with symbol in $S(\langle x\rangle^{-\delta})$ (see (20)), and $\delta > n$. Then, for $g \in C_0^\infty(]0,+\infty[)$ equal to 1 in a ¿neigbourhood of λ we define the local spectral density

$$\sigma_b(h,\lambda) = Tr\Big(b(x,hD,h)g(H_1)E_{H_1}(\lambda)g(H_1)\Big).$$

Proposition 4.1 *Assume that $\lambda > 0$ is a non-critical energy level and that the symbol of b vanishes on the set of periodic points Π.*

Then, there exists $h_0 > 0$ such that for any $h \in (0,h_0]$ and $0 < \epsilon < 1$ we have :

$$\mid \sigma_b(h,\lambda) - h^{-n}\alpha_0(\lambda,b) - h^{1-n}\alpha_1(\lambda,b) \mid \leq C_0\epsilon h^{1-n} + O_\epsilon(h^{2-n}), \qquad (21)$$

with $C_0 > 0$ independent of h,ϵ.

In general, the function σ_b is not monotone. To apply a Tauberian theorem, first of all, we split σ_b into two monotone terms.

Lemme 4.2 *Let $b(x,hD,h)$ be as above with symbol supported in an open set $\Omega \subset \mathbb{R}^{2n}$ (Ω is not necessarily bounded). Then we have:*

a) There exist classical, non-negative, h-admissible pseudodifferential operators $b^{\pm}(x, hD, h)$ of order 0, with symbol in $S(\langle x\rangle^{-\delta})$, $\delta > n$, supported in Ω, such that:

$$b(x, hD, h) = b^{+}(x, hD, h) - b^{-}(x, hD, h) + R(h),$$

where $R(h)$ is a trace class operator satisfying:

$$\| R(h) \|_{tr} = O(h^{\infty}).$$

Moreover, there exist closed operators $B^{\pm}(h)$ such that $b^{\pm}(x, hD, h) = B^{\pm}(h)B^{\pm}(h)^{*}$.

b) The local spectral densities $\sigma_{b^{\pm}}(h, \lambda)$ associated with $b^{\pm}(x, hD, h)$ are non-decreasing and non-negative functions on $\mathbb{R}$ and we have:

$$\sigma_{b}(h, \lambda) = \sigma_{b^{+}}(h, \lambda) - \sigma_{b^{-}}(h, \lambda) + O(h^{\infty}).$$

Moreover, for $0 < \bar{\lambda}_1 < \bar{\lambda}_2$ and $g \in C_0^{\infty}(]\bar{\lambda}_1, \bar{\lambda}_2[)$, $\sigma_{b^{\pm}}(h, \lambda)$ satisfy:

$$\sigma_{\cdot}(h, \lambda) = \begin{cases} 0 & \text{for} \quad \lambda < \bar{\lambda}_1 \\ \sigma_{\cdot}(h, \bar{\lambda}_2) & \text{for} \quad \lambda \geq \bar{\lambda}_2 \end{cases} \tag{22}$$

and $\sigma_{b^{\pm}}(h, \lambda) = O(h^{-n})$ locally uniformly with respect to $(h, \lambda) \in (0, h_0] \times \mathbb{R}$.

Proof. $-$Without loss of generality, we can asume that $b(x, hD, h)$ is a self-adjoint operator. Using that the symbol of $b(x, hD, h)$ belongs to $S(\langle x\rangle^{-\delta})$, we conclude that

$$A(h) = \langle x\rangle^{\frac{\delta}{2}} b(x, hD, h) \langle x\rangle^{\frac{\delta}{2}}$$

is an h-admissible, self-adjoint and bounded operator on $\mathcal{L}(L^2)$. Here we have used a Calderon-Vaillancourt theorem. Moreover, there exist $h_0 > 0$ and $M > 0$ such that

$$\| A(h) \| \leq M \qquad \forall h \in (0, h_0].$$

The idea of the proof of this lemma is to decompose b in the form:

$$b = \Big(\langle x\rangle^{-\frac{\delta}{2}} 2M \langle x\rangle^{-\frac{\delta}{2}}\Big) - \Big(\langle x\rangle^{-\frac{\delta}{2}} (2M - A) \langle x\rangle^{-\frac{\delta}{2}}\Big).$$

However, the properties of support are not preserved in this decomposition. For this purpose we introduce a cut-off function $\chi \in C^{\infty}(\mathbb{R}^{2n})$, supported in Ω such that $0 \leq \chi \leq 1$ and $\chi(x, \xi) = 1$ on the support of b. Next we define $b^{\pm}(x, hD, h)$ by:

$$\begin{cases} b^{+}(x, hD, h) &= 2M \langle x\rangle^{-\frac{\delta}{2}} \chi(x, hD)^2 \langle x\rangle^{-\frac{\delta}{2}} \\ b^{-}(x, hD, h) &= \langle x\rangle^{-\frac{\delta}{2}} \chi(x, hD) \Big(2M\mathrm{Id} - A(h)\Big) \chi(x, hD) \langle x\rangle^{-\frac{\delta}{2}} \end{cases}$$

These operators are clearly classical, non-negative, h-admissible pseudodifferential operators of order 0, with symbols in $S(\langle x\rangle^{-\delta})$, $\delta > n$, supported in Ω. Moreover we have $b^{\pm}(x, hD, h) = B^{\pm}(h)B^{\pm}(h)^*$ with

$$\left\{\begin{array}{lll} B^+(h) & = & \sqrt{2M}\langle x\rangle^{-\frac{\delta}{2}}\chi(x, hD) \\ B^-(h) & = & \langle x\rangle^{-\frac{\delta}{2}}\chi(x, hD)\sqrt{2M\mathrm{Id} - A(h)} \end{array}\right. ,$$

and:

$$(b^+ - b^-)(x, hD, h) = \langle x\rangle^{-\frac{\delta}{2}}\chi(x, hD)A(h)\chi(x, hD)\langle x\rangle^{-\frac{\delta}{2}}.$$

However, taking account that supports of $(1-\chi)$ and of the symbol of $A(h)$ are separates, we deduce that

$$b^+(x, hD, h) - b^-(x, hD, h) = b(x, hD, h) + R(h),$$

with a trace class operator $R(h)$ satisfying:

$$\| R(h) \|_{tr} = O(h^\infty).$$

Now we obtain:

$$\sigma_b(h, \lambda) = \sigma_{b^+}(h, \lambda) - \sigma_{b^-}(h, \lambda) + O(h^\infty).$$

Therefore, the cyclicity of traces yields:

$$\sigma_{b^\pm}(h, \lambda) = Tr\Big(B^\pm(h)^* g(H_1)E_{H_1}(\lambda)g(H_1)B^\pm(h)\Big).$$

In conclusion the non-negative and non-decreasing properties of $\sigma_{b^\pm}(h, .)$ are obtained from the same properties for $E_{H_1}(.)$.

The relations (22) are consequences of spectral theorem which yields:

$$g(H_1)E_{H_1}(\lambda) = \left\{\begin{array}{lll} 0 & \text{for} & \lambda < \bar{\lambda}_1 \\ g(H_1) & \text{for} & \lambda \geq \bar{\lambda}_2. \end{array}\right.$$

To check $\sigma_{b^\pm}(h, \lambda) = O(h^{-n})$, we introduce smooth function which regularize the characteristic function $\mathbb{1}_{[0,\lambda]}$ and follow the functional calculus in ([10], [21]). This complete the proof of lemma 4.2. ■

Our analyse shows that the conditions $i)$, $ii)$ and $iii)$ of the Tauberian theorem in the Appendix are satisfied for $\sigma_{b^\pm}(h, \lambda)$.

Finally, to check the asymptotic (40) in the Tauberian theorem, we can use the proposition 4.3 (stated below) for $W(h) = b^\pm(x, hD_x, h)$.

Let ρ be an even function, $\hat{\rho}(0) = 1$, and let the support of $\hat{\rho}$ be contained in an small interval $[-\epsilon_1, \epsilon_1]$, $\epsilon_1 > 0$. Set $\rho_h(\lambda) = \frac{1}{h}\rho(\frac{\lambda}{h})$.

Proposition 4.3 *Let $W(h)$ be a classical, non negative, h-admissible pseudodifferential operator of order 0 with principal h-symbol, $w_0 \in S(\langle x\rangle^{-\delta})$. Then:*

a) for $\delta > n$ and $g \in C_0^\infty(]0,+\infty[)$ equal to 1 in a neigbourhood of λ we can define the local spectral density:

$$\sigma_{W,h}(\lambda) = Tr\Big(W(h)g(H_1)E_{H_1}(\lambda)g(H_1)\Big) \tag{23}$$

b) there exists $\epsilon_1 > 0$ sufficiently small such that for any $\lambda > 0$, we have:

$$\frac{d}{d\lambda}\Big(\sigma_{W,h} * \rho_h\Big)(\lambda) = h^{-n}\beta_0(\lambda, W) + h^{1-n}\beta_1(\lambda, W) + \mathcal{O}_\lambda(h^{2-n}),$$

where $\beta.(\lambda, W)$ are smooth with respect to λ, and for λ in the interior of $g^{-1}(\{1\})$

$$\beta_0(\lambda, W) = (2\pi)^{-n}\int_{a_0(x,\xi)=\lambda} \frac{w_0(x,\xi)}{\mid \nabla a_0(x,\xi)\mid} d\nu_\lambda$$

ν_λ being the induced Lebesgue measure on the surface $\{(x,\xi)\ :\ a_0(x,\xi) = \lambda\}$ and $\mathcal{O}_\lambda$ is locally uniform with respect to λ.

c) If the support of the symbol of $W(h)$ has no common points with the set of periodic points Π, then the asymptotic of b) holds for any $\epsilon_1 > 0$.

This proposition can be proved in a standard way by constructing a parametrix of the propagator and using a non-stationnary phase theorem (see [6], [13], [20], [27], [23]...).

Proof of the proposition 4.1. – Applying lemma 4.2 with $\Omega = \mathbb{R}^{2n}\setminus\Pi$, we have

$$\sigma_b(h,\lambda) = \sigma_{b^+}(h,\lambda) - \sigma_{b^-}(h,\lambda) + O(h^\infty)$$

where $\sigma_{b^\pm}(h,.)$ are non-negative, non-decreasing spectral densities which satisfy assumptions of theorem 8.1 and of proposition 4.3 *c)*.

The theorem 8.1 *a)* implies that there exists $h_0 > 0$ such that for any $h \in (0, h_0]$ and $\epsilon > 0$ we have :

$$\mid \sigma_{b^\pm}(h,\lambda) - \Big(\sigma_{b^\pm}(h,.) * \rho_{\epsilon h}\Big)(\lambda) \mid\leq \epsilon C_0\beta_0(\lambda, b^\pm)h^{1-n} + C_1(\epsilon,\lambda)h^{2-n}, \tag{24}$$

with $C_0 > 0$ and $C_1(\epsilon,\lambda)$ locally uniformly bounded with respect to λ.

On the other hand, from Proposition 4.3 *c)* we get that, for any $1 > \epsilon > 0$:

$$\Big(\sigma_{b^\pm}(h,.) * \rho_{\epsilon h}\Big)(\lambda) = h^{-n}\alpha_0(\lambda, b^\pm) + h^{1-n}\alpha_1(\lambda, b^\pm) + \mathcal{O}_{\lambda,\epsilon}(h^{2-n})$$

locally uniformly with respect to λ.

Then, there exists $h_0 > 0$ such that for any $h \in (0, h_0]$ and $0 < \epsilon < 1$ we have :

$$| \sigma_b(\lambda) - h^{-n}\alpha_0(\lambda, b) - h^{1-n}\alpha_1(\lambda, b) | \leq \epsilon C_0 \beta_0(\lambda, b) h^{1-n} + C_1(\epsilon, \lambda) h^{2-n},$$

with $C_0 > 0$ and $C_1(\epsilon, \lambda)$ locally uniformly bounded with respect to λ. This proves the proposition 4.1 ■

Remark 4.4 *In the proof of the Theorem 2.1 (sect. 6) the expression of* $\alpha_0(\lambda, b)$ *and of* $\alpha_1(\lambda, b)$ *will be not necessary. Thus in the proposition 4.1, we need only the existence of these coefficients.*

5 Asymptotic behaviour of σ_1

In this section, we prove the following proposition 5.1 using a Tauberian theorem (see Appendix) and some recent results of V. Petkov, G. Popov [19]. Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on a compact set containing the x-components of periodic trajectories. Then, for $g \in C_0^\infty(]0, +\infty[)$ equal to 1 in a neigbourhood of λ we define the local spectral density

$$\sigma_1(h, \lambda) = Tr\Big(\zeta^2 g(H_1) E_{H_1}(\lambda) g(H_1)\Big)$$

Proposition 5.1 *Let us assume (8) and let* $\lambda > 0$ *be a non critical energy level.*

For $r_0 > 0$ *there exist* $h_0 > 0$, $\epsilon_0 > 0$ *such that for all* $h \in (0, h_0]$, $| r |< r_0$ *and* $0 < \epsilon < \epsilon_0$ *we have :*

$$\begin{aligned} & h^{1-n} Q(h, r - \epsilon) - C_0 \epsilon h^{1-n} + o_\epsilon(h^{1-n}) \\ & \quad \leq \sigma_1(h, \lambda + rh) - h^{-n}\alpha_0(\lambda + rh, \zeta^2) - h^{1-n}\alpha_1(\lambda + rh, \zeta^2) \leq \\ & \qquad\qquad h^{1-n} Q(h, r + \epsilon) + C_0 \epsilon h^{1-n} + o_\epsilon(h^{1-n}), \end{aligned} \tag{25}$$

with $C_0 > 0$ *independent of* h, r, ϵ.

Using the cyclicity of traces, we have:

$$\sigma_1(h, \lambda) = Tr\Big(\zeta g(H_1) E_{H_1}(\lambda) g(H_1) \zeta\Big)$$

hence, as for $\sigma_{b^\pm}(h, \lambda)$, σ_1 is a non decreasing, non-negative function on $\mathbb{R}$ and for $0 < \bar{\lambda}_1 < \bar{\lambda}_2$, $g \in C_0^\infty(]\bar{\lambda}_1, \bar{\lambda}_2[)$, we check easily that $\sigma_1(h, \lambda)$ satisfy the conditions $i)$, $ii)$ and $iii)$ of the Tauberian theorem in the Appendix. The asymptotic (40) of $(\sigma_1' * \rho_h)(\lambda)$ is also fulfilled applying proposition 4.3 b) with $W = \zeta^2$.

Now, we are going to study the asymptotic of $(\sigma_1 * \rho_{\epsilon h})(\lambda + rh)$. First of all, we split $(\sigma_1 * \rho_{\epsilon h})(\lambda + rh)$ into two terms. We isolate the singularity of the propagator at $t = 0$ from the others, which are connected with the periodic trajectories of the classical hamiltonian H_{a_0}. More precisely, let us put :

$$\Phi_h(\lambda, \epsilon) = (2\pi h)^{-1} \int \exp(ith^{-1}\lambda)\Big(1 - \theta(t)\Big)\hat{\rho}(\epsilon t)dt,$$

$$\Psi_h(\lambda, \epsilon) = (2\pi h)^{-1} \int \exp(ith^{-1}\lambda)\theta(t)\hat{\rho}(\epsilon t)dt,$$

where $\theta \in C_0^\infty(\mathbb{R})$ is equal to 1 in a small neighbourhood of 0.

We have:

$$\rho_{\epsilon h}(\lambda) = \Phi_h(\lambda, \epsilon) + \Psi_h(\lambda, \epsilon).$$

Introduce the function :

$$\chi_h(\lambda, \epsilon) = (2\pi h)^{-1} \int \exp(ith^{-1}\lambda)(it)^{-1}\Big(1 - \theta(t)\Big)\hat{\rho}(\epsilon t)dt \qquad (26)$$

Then, $(\sigma_1 * \Phi_h(., \epsilon))(\lambda) = h\frac{d}{d\lambda}(\sigma_1 * \chi_h(., \epsilon))(\lambda)$ and modulo (h^∞) we have:

$$(\sigma_1 * \rho_{\epsilon h})(\lambda + rh) = \int_{-\infty}^{\lambda + rh} \frac{d}{d\lambda}(\sigma_1 * \Psi_h)(\mu)d\mu + h\frac{d}{d\lambda}(\sigma_1 * \chi_h)(\lambda + rh). \quad (27)$$

The first term in the right hand side of (27) can be treated by using b) of proposition 4.3

The analysis of the second term in the right hand side of (27) may be difficult, because it could involve analysis of oscillatory integrals with degenerate phase functions. To overcome this obstacle, we exploit some recent results of V. Petkov [17] and V. Petkov, G. Popov [19], who make use of the so called *absolutely periodic points* and *absolutely periodic trajectories*. These results are influenced by works from T. Guriev and Yu. Safarov ([8] , [28], [29]) for counting function on compact manifolds. The main tool of this paper is the following theorem due to V. Petkov, G. Popov (theorem 5.1 of [19]). The difference with [19] is that Σ_λ is not compact. But by using that ζ^2 is compactly supported, and equal to 1 on the periodic trajectories, the proof of theorem 5.1 in [19] can be easily adapted for our case. We have:

Theorem 5.2 *([19]) Let $\hat{\chi}(t) \in C_0^\infty(\mathbb{R})$ be such that $\hat{\chi}(t)$ vanishes in a neighbourhood of the origin. Then for any $\mid r \mid < r_0$ and $h \in (0, h_0]$, we have*

$$\Big(\sigma_1' * \chi_h\Big)(\lambda + rh) =$$

$$(2\pi h)^{-1}Tr\int exp(ith^{-1}(\lambda+rh))\hat{\chi}(t)\zeta^2 g(H_1)exp(ith^{-1}H_1)g(H_1)dt$$
$$=(2\pi h)^{-n}\sum_{k\in\mathbb{Z}\backslash 0}\int_{\Pi}\hat{\chi}(kT(\nu))exp\Big(ik(h^{-1}S(\nu)+rT(\nu)-q(\nu))\Big)d\nu+o_{\chi}(h^{-n}),$$

where $\chi_h(\lambda)=\frac{1}{h}\chi(\frac{\lambda}{h})$ *and* $o_{\chi}(h^{-n})$ *does not depend on* r.

Consequently, applying theorem 5.2 with χ_h defined by (26), and proposition 4.3 with $\rho_h=\Psi_h$, $W=\zeta^2$, we obtain:

Corollary 5.3 *For any* $\mid r\mid< r_0$, $\epsilon>0$ *and* $h\in(0,h_0]$, *we have*

$$\Big(\sigma_1(h,.)*\rho_{\epsilon h}\Big)(\lambda+rh)=$$
$$h^{-n}\alpha_0(\lambda+rh,\zeta^2)+h^{1-n}\alpha_1(\lambda+rh,\zeta^2)+h^{1-n}\tilde{Q}_{\epsilon}(h,r)+o_{\epsilon}(h^{1-n})$$

and

$$\tilde{Q}_{\epsilon}(h,r):=\frac{2}{(2\pi)^n}\int_{\Pi}\sum_{k=1}^{\infty}\frac{\hat{\rho}(\epsilon kT(\nu))}{kT(\nu)}sin\Big(k(h^{-1}S(\nu)+rT(\nu)-q(\nu))\Big)d\nu. \quad (28)$$

Proof of the proposition 5.1. – Let $\lambda>0$ and $r_0>0$, then there exists $h_0>0$, such that for any $h\in(0,h_0]$, $\mid r\mid<r_0$, we have $\lambda+rh\in[\bar{\lambda}_1,\bar{\lambda}_2]$. Consequently, applying the theorem 8.1 *b*) for σ_1 we deduce, that for each ϵ', $0<\epsilon'\leq 1$, each ϵ, $0<\epsilon\leq 1$:

$$\begin{aligned}\Big(\sigma_1*\rho_{\epsilon h}\Big)(\lambda+(r-\epsilon')h)-C_0\beta_0(\lambda+rh,\zeta^2){\epsilon'}^{-1}\epsilon h^{1-n}-C_1h^{2-n}\\ \leq\sigma_1(h,\lambda+rh)\leq\\ \Big(\sigma_1*\rho_{\epsilon h}\Big)(\lambda+(r+\epsilon')h)+C_0\beta_0(\lambda+rh,\zeta^2){\epsilon'}^{-1}\epsilon h^{1-n}+C_1h^{2-n}\end{aligned} \quad (29)$$

where for a h-admissible pseudo-differential operator $W(h)$, the smooth function $\beta_0(\lambda,W)$ is defined in proposition 4.3.

Using corollary 5.3, the smoothness of α_0, α_1 and fact that $\frac{d}{d\lambda}\alpha_.=\beta_.$ we obtain:

$$\begin{aligned}h^{1-n}\tilde{Q}_{\epsilon}(h,r-\epsilon')-h^{1-n}\beta_0(\lambda+rh,\zeta^2)\Big(C_0{\epsilon'}^{-1}\epsilon+\epsilon'\Big)+o_{\epsilon}(h^{1-n})\\ \leq\sigma_1(h,\lambda+rh)-h^{-n}\alpha_0(\lambda+rh,\zeta^2)-h^{1-n}\alpha_1(\lambda+rh,\zeta^2)\leq\\ h^{1-n}\tilde{Q}_{\epsilon}(h,r+\epsilon')+h^{1-n}\beta_0(\lambda+rh,\zeta^2)\Big(C_0{\epsilon'}^{-1}\epsilon+\epsilon'\Big)+o_{\epsilon}(h^{1-n}),\end{aligned} \quad (30)$$

Next, the proposition 5.1 follows from the proposition 6.1 of [19] (see proposition 5.4 below) and for ϵ' fixed, we take $\epsilon={\epsilon'}^2$. We use also that β_0 is a smooth function, thus it is uniformly bounded with respect to $(\lambda+rh)$ in a compact set. ∎

Proposition 5.4 *([19]) Suppose the condition (8) satisfied. Let $Q(h,r)$ be defined by (7) and let $\tilde{Q}_\epsilon(h,r)$ be defined in corollary 5.3. Then, for any $0 < \epsilon' \leq 1$, $0 < \epsilon \leq 1$ and $\mid r \mid \leq r_0$, we have:*

$$Q(h,r-\epsilon')-C_1\epsilon'-C_0{\epsilon'}^{-1}\epsilon-o_\epsilon(1) \leq \tilde{Q}_\epsilon(h,r) \leq Q(h,r+\epsilon')+C_1\epsilon'+C_0{\epsilon'}^{-1}\epsilon+o_\epsilon(1)$$

with constants $C_0 > 0$ and $C_1 > 0$ independent of ϵ', ϵ and h.

6 Proof of the main results

In this section we prove our main results. First of all, let us recall the representation formula obtained in the section 3. Let $\lambda_2 > \lambda_1 > 0$ and s_h be the same as in the introduction. Then for $\lambda \in [\lambda_1, \lambda_2]$, we have:

$$s_h(\lambda) = \sigma_1(h,\lambda) + \sigma_b(h,\lambda),$$

where

$$\begin{cases} \sigma_1(h,\lambda) &=& Tr\Big(\zeta g(H_1)^2 E_{H_1}(\lambda)\zeta\Big) \\ \sigma_b(h,\lambda) &=& Tr\Big(b(x,hD,h)g(H_1)^2 E_{H_1}(\lambda)\Big), \end{cases}$$

with $g \in C_0^\infty(]0,+\infty[,\mathbb{R}^+)$ such that $g = 1$ on $[\lambda_1,\lambda_2]$, $E_{H_1}(\lambda)$ being the spectral projector of H_1 on $]0,\lambda]$, and $b(x,hD,h)$ a classical h-admissible pseudodifferential operator of order 0. The symbol of b vanishes on the set of periodic points, Π and belongs to the symbol class $S(\langle x\rangle^{-\delta})$, $\delta > n$.

Asymptotic properties of σ_b (resp. σ_1) are studied in section 4 (resp. section 5).

Proof of the Theorem 2.1. – Let $r_0 > 0$ and $\lambda > 0$ be non critical energy level such that Π is a compact subset of Σ_λ.

Then, applying proposition 4.1 and proposition 5.1, there exist $h_0 > 0$, $\epsilon_0 > 0$ such that for all $h \in (0,h_0]$, $\mid r \mid < r_0$ ¿and $0 < \epsilon < \epsilon_0$ we have :

$$\begin{aligned} h^{1-n}Q(h,r-\epsilon) - C_0\epsilon h^{1-n} + o_\epsilon(h^{1-n}) & \\ \leq s_h(\lambda+rh) - h^{-n}\alpha_0(\lambda+rh) - h^{1-n}\alpha_1(\lambda+rh) \leq & \\ h^{1-n}Q(h,r+\epsilon) + C_0\epsilon h^{1-n} + o_\epsilon(h^{1-n}), & \end{aligned} \tag{31}$$

with $C_0 > 0$ independent of h, r, ϵ. Here $\alpha_.(\lambda) = \alpha_.(\lambda,\zeta^2) + \alpha_.(\lambda,b)$.

To compute $\alpha_0(\lambda)$ and $\alpha_1(\lambda)$, we don't need the expression of $\alpha_.(\lambda,\zeta^2)$ and of $\alpha_.(\lambda,b)$. By definition (see proposition 4.3), they are independants of the presence of periodic trajectories, and they are connected only with the singularities of the propagator at $t = 0$. Then, $\alpha_0(\lambda)$ (resp. $\alpha_1(\lambda)$) is the coefficient of h^{-n} (resp. h^{1-n}) in the asymptotic (4), where the set of closed trajectories is of measure 0. Thus, $\alpha_0 = \gamma_0$, $\alpha_1 = 0$, and we prove the relation (9) of the Theorem 2.1 applying the Taylor formula for α_0 in the above relation (31). ■

Proof of the Corollary 2.2. – We prove this corollary in the same way as in [19] (Corollary 1.2). For $\delta > 0$, we are going to find $h_1 > 0$ such that the inequality

$$\mid h^{n-1}s_h(\lambda + rh) - h^{-1}\gamma_0(\lambda) - Q(h, r) - r\gamma_0'(\lambda) \mid\leq \delta$$

holds for any $0 < h < h_1$ and any $r \in [r_1, r_2]$. First, using the uniform continuity of Q with respect to r, there exists $0 < \epsilon < \frac{\delta}{4C_0}$, where C_0 is the constant of Theorem 2.1, so that

$$\mid Q(h, r \pm \epsilon) - Q(h, r) \mid\leq \frac{\delta}{4}$$

for any $r \in [r_1, r_2]$. Then from the relation (9) of the Theorem 2.1 and taking h_1 (with ϵ fixed) such that

$$\mid o_\epsilon(h^{1-n}) \mid\leq \frac{\delta}{4}h^{1-n}, \qquad \forall h \in (0, h_1],$$

we deduce the assertion $a)$ of Corollary 2.2.

The assertion $b)$ is a consequence of $a)$. For $r_1 < R_1 < R_2 < r_2$ fixed, let

$$c_0 = \min\Big(\frac{r_2 - R_2}{2}, \frac{R_1 - r_1}{2}\Big).$$

Then, for $r \in [r_1, r_2]$ we can also apply $a)$ for $r + c$ when $\mid c \mid< c_0$ and we deduce the assertion $b)$. ■

Proof of the Corollary 2.4. – $a)$ We prove the estimate (11) only for $\Big(s_h(\lambda_0 + r(h)h + ch) - s_h(\lambda_0 + r(h)h)\Big)$, the proof works in the same way for the other quantity.

Let Π^1 be a subset of Π with positive Lebesgue measure such that,

$$r(h) = \Big([q(\nu) - h^{-1}S(\nu)]_{2\pi} + 2\pi p\Big)T(\nu)^{-1}, \qquad h \in (0, h_0]$$

does not depend on $\nu \in \Pi^1$, with an integer p. Applying theorem 2.1 with $r = r(h)$, and choosing $\epsilon = c/4$, we have:

$$\begin{gathered} s_h(\lambda_0 + r(h)h + ch) - s_h(\lambda_0 + r(h)h) \geq \\ h^{1-n}c\gamma_0'(\lambda_0) + h^{1-n}\Big(Q(h, r + \tfrac{3c}{4}) - Q(h, r + \tfrac{c}{4})\Big) - \tfrac{c}{4}C_0h^{1-n} + o_c(h^{1-n}), \end{gathered} \tag{32}$$

for $0 < c \leq c_0$, c_0 sufficiently small.

We fix $0 < \delta < \int_{\Pi^1} \frac{d\nu}{T(\nu)}$, and we are going to find $c_2 > 0$ such that for any $0 < c \leq c_2$, we can find $h_1(c) > 0$ such that for each $h \in (0, h_1]$,

$$h^{n-1}\Big(s_h(\lambda_0 + r(h)h + ch) - s_h(\lambda_0 + r(h)h)\Big) \geq (2\pi)^{1-n}\left(\int_{\Pi^1} \frac{d\nu}{T(\nu)} - \delta\right). \quad (33)$$

First of all let us find $c_1 > 0$ such that for $0 < c \leq c_1$, $h \in (0, h_0]$

$$Q\Big(h, r(h) + \frac{3c}{4}\Big) - Q\Big(h, r(h) + \frac{c}{4}\Big) \geq (2\pi)^{1-n}\left(\int_{\Pi^1} \frac{d\nu}{T(\nu)} - \frac{\delta}{2}\right). \quad (34)$$

Let $T > 0$ be such that

$$\begin{cases} \mu(\{\nu \in \Pi; T(\nu) \geq T\}) \leq \frac{\delta T_0}{4}, \\ T(\nu) < T; \quad \forall \nu \in \Pi_1, \end{cases} \quad (35)$$

where μ is the Liouville measure on Σ_{λ_0}, and $T_0 = \min_{\nu \in \Pi} T(\nu)$.

Set $U = \{\nu \in \Pi : \ T(\nu) < T\}$ and $U_2^h = U \setminus U_1^h$, where U_1^h is the set:

$$\{\nu \in U : \ \exists k \in \mathbb{Z}\,,\ 2\pi k \leq -h^{-1}S(\nu) + q(\nu) - \Big(r(h) + \frac{c}{2} \pm \frac{c}{4}\Big)T(\nu) < 2\pi(k+1)\}.$$

For $\nu \in \Pi^1$ and $h \in (0, h_0]$, we have

$$-h^{-1}S(\nu) + q(\nu) - r(h)T(\nu) \equiv 0(2\pi)$$

hence, $\Pi^1 \subset U_2^h$ for $h \in (0, h_0]$.

Consider the function

$$\begin{array}{rcl} G(h,\nu) & = & [\pi - h^{-1}S(\nu) + q(\nu) - \Big(r(h) + \frac{3c}{4}\Big)T(\nu)]_{2\pi} \\ & - & [\pi - h^{-1}S(\nu) + q(\nu) - \Big(r(h) + \frac{c}{4}\Big)T(\nu)]_{2\pi} \end{array}$$

For any $h \in (0, h_0]$, we have :

$$\begin{cases} \mid G(h,\nu) \mid & = & \frac{c}{2}T(\nu) & \text{if } \nu \in U_1^h \\ G(h,\nu) & = & 2\pi - \frac{c}{2}T(\nu) & \text{if } \nu \in U_2^h \end{cases}$$

and we deduce

$$(2\pi)^{n-1}\Big(Q(h, r(h) + \frac{3c}{4}) - Q(h, r(h) + \frac{c}{4})\Big) \geq -T_0^{-1}\mu(\Pi \backslash U) - \frac{c}{4\pi}\mu(U) + \int_{\Pi^1} \frac{d\nu}{T(\nu)}.$$

We use $\mu(U) = \mu(U_h^1) + \mu(U_h^2)$ together with $\Pi^1 \subset U_h^2$. From (35) taking $c_1 = \min\{c_0, \frac{\delta\pi}{\mu(U)}\}$, we get (34).

Now, let $c_2 = \min\{c_1, \dfrac{\delta(2\pi)^{1-n}}{C_0+ \mid \gamma_0'(\lambda_0) \mid}\}$, and $0 < h_1(c) \leq h_0$ such that

$$\mid o_c(h^{1-n}) \mid \leq \frac{\delta}{4} h^{1-n}.$$

Then, from (32) and (34), we deduce (33) for any $\delta > 0$. This give the inequality (11) of corollary 2.4 for each $c \in (0, c_2]$.

To prove b), let us suppose that for any compact subset K of $]0, +\infty[$, there exist $C_K > 0$ and $h_0 > 0$ such that for any $\lambda \in K$, $h \in (0, h_0]$ we have:

$$\frac{d}{d\lambda} s_h(\lambda) < C_K h^{-n}. \tag{36}$$

Now applying a Taylor formula for s_h in (33) we obtain that for $0 < \delta < \int_{\Pi^1} \frac{d\nu}{T(\nu)}$ fixed, and $0 < c \leq c_2(\delta)$, there exists $\theta \in (0, 1)$ such that

$$h_1^{n-1}\Big(\frac{d}{d\lambda} s_{h_1}(\lambda_0 + r(h_1)h_1 + \theta c h_1)\Big) c h_1 \geq (2\pi)^{1-n} \left(\int_{\Pi^1} \frac{d\nu}{T(\nu)} - \delta \right),$$

where h_1 depends on c. Then owing to (36), we deduce that for $0 < \delta < \int_{\Pi^1} \frac{d\nu}{T(\nu)}$ fixed and for any $0 < c \leq c_2(\delta)$ we have:

$$c C_K \geq (2\pi)^{1-n} \left(\int_{\Pi^1} \frac{d\nu}{T(\nu)} - \delta \right).$$

This is false for c sufficiently small. Then $h^n \frac{d}{d\lambda} s_h(\lambda)$ is not uniformly bounded with respect to $(h, \lambda) \in (0, h_0] \times K$. ■

7 Example of Schrödinger operator with trapped trajectories

We give an example for which "clustering" phenomena occurs for the scattering phase, near a trapping energy level.

Consider the Schrödinger operator $H_1 = -h^2\Delta + V(x)$, with V a smooth real valued potential on $\mathbb{R}^n$. We assume that V is compactly supported and admits a well in the following form.

For $a > 0$, let $\Phi_a \in C_0^\infty(\mathbb{R}^n)$ such that $\Phi_a(x) = 1$ on $\{\mid x \mid \leq 2a\}$ and let

$$V(x) = \Phi_a(x - y_0)\Big(\mid x - y_0 \mid^2 + b\Big)$$

with $y_0 \in \mathbb{R}^n$ and $b \geq 0$ fixed.

In this case for fixed energy level λ, all trapped trajectories lie on compact subset of Σ_λ, its x-components are in the convex hull of supp$V$ and its ξ-components are in $\{|\ \xi\ | \leq (\lambda - \min V)_+^{\frac{1}{2}}\}$. Next we check conditions of corollary 2.4 and we prove the following proposition.

Proposition 7.1 *For any regular energy level $\lambda \in]b, b+a^2[$ for the symbol $a_0(x,\xi) = |\ \xi\ |^2 + V(x)$, there exist $C > 0$, $c_0 > 0$, $m \in \mathbb{Z}$ such that for any $c \in]0, c_0[$, $p \in \mathbb{Z}$:*

$$\liminf_{h \searrow 0} h^{n-1}\Big(\pm s_h(\lambda + r(h)h + 2ph \pm ch) \mp s_h(\lambda + r(h)h + 2ph)\Big) \geq C, \quad (37)$$

where

$$r(h) = \frac{1}{\pi}\left[-\frac{\pi}{2}m - h^{-1}\pi(\lambda - b)\right]_{2\pi}$$

In particular, $\frac{d}{d\lambda}s_h(\lambda)$ does not satisfy a complete asymptotic expansion in a neighbourhood of λ.

Proof. $-$Let $\lambda \in]b, b+a^2[$ be a regular energy level for $a_0(x,\xi)$ and set

$$\Sigma_\lambda = \{(x,\xi) \in \mathbb{R}^{2n};\ \ |\ \xi\ |^2 + V(x) = \lambda\}.$$

Then Σ_λ is a smooth non-compact hypersurface and the differential $\mathrm{d}a_0$ does not vanish on it.

Fix $\epsilon \in]0, a/2]$, and consider the following set of a positive Lebesgue measure:

$$\Pi^1 := \{(x,\xi) \in \Sigma_\lambda;\ \ |\ \xi - \eta_0\ |^2 + |\ x - y_0\ |^2 \leq \epsilon^2\},$$

where $|\eta_0| = \sqrt{\lambda - b}$ is fixed.

Let us remark that

$$\Pi^1 = \{(x,\xi) \in \mathbb{R}^{2n};\ \ |\xi|^2 + |x - y_0|^2 + b = \lambda \ \text{ and } \ |\xi - \eta_0|^2 + |x - y_0|^2 \leq \epsilon^2\},$$

which clearly contain (y_0, η_0) and satisfies:

Lemme 7.2 *We have $\Pi^1 \subset \Pi$, and for any $\nu \in \Pi^1$,*

$$T(\nu) = \pi, \qquad S(\nu) = \pi(\lambda - b).$$

Proof. –Let $\nu = (x_0, \xi_0) \in \Pi^1$ and

$$\exp(tH_{a_0})(\nu) = \big(x(t), \xi(t)\big).$$

It is sufficient to prove that for any $t \in \mathbb{R}$:

$$|x(t) - y_0| \leq 2a, \tag{38}$$

because in this case, $V(x) = |x - y_0|^2 + b$ in a neighbourhood of the curve $\{x(t);\ t \in \mathbb{R}\}$, and hence we have:

$$\left\{ \begin{array}{lcr} x(t) & = & y_0 + \xi_0 \sin 2t + (x_0 - y_0)\cos 2t \\ \xi(t) & = & \xi_0 \cos 2t - (x_0 - y_0)\sin 2t \end{array} \right. , \tag{39}$$

which is periodic with period $T(\nu) = \pi$ and with action:

$$S(\nu) = \int_0^\pi \xi(t) x'(t) dt = \pi(|\xi_0|^2 + |x_0 - y_0|^2) = \pi(\lambda - b).$$

Now it remains to check (38) for any $t \in \mathbb{R}$. First, for $(x_0, \xi_0) \in \Pi^1$ we have $|x_0 - y_0| \leq \epsilon \leq a/2$. Then the inequality (38) holds in an interval $]-t_0, t_0[$, $t_0 > 0$. Let $]-T, T[$ be the maximal interval where (38) is valid. Suppose that $T < +\infty$. Then for any $t \in [-T, T]$, $\big(x(t), \xi(t)\big)$ is determined by (39). Consequently, using that

$$|\xi_0| \leq |\xi_0 - \eta_0| + |\eta_0| \leq \epsilon + \sqrt{\lambda - b},$$

and that $0 < \lambda - b < a^2$ and $0 < \epsilon < a/2$ we obtain:

$$|x(t) - y_0|^2 \leq 2\epsilon^2 + (\lambda - b) + 3\epsilon\sqrt{\lambda - b} \leq 3a^2, \qquad \forall t \in [-T, T].$$

Therefore, choosing $\epsilon' > 0$ sufficiently small, (38) holds for any t in an interval $[-T - \epsilon', T + \epsilon']$, and we conclude that $T = +\infty$. This prove the lemma 7.2. ■

End of the proof of the proposition 7.1 – The subprincipal symbol of H_1 vanishes and, on Π^1 the Maslov index m is constant, then $q(\nu) = -\frac{\pi}{2}m$ does not depend on $\nu \in \Pi^1$. Therefore from lemma 7.2 we deduce that the quantity

$$r(h) = \Big([q(\nu) - h^{-1}S(\nu)]_{2\pi} + 2\pi p\Big)T(\nu)^{-1}\ , \qquad h \in (0, h_0]$$

does not depend on $\nu \in \Pi^1$ for any $p \in \mathbb{Z}$ and it is given by

$$r(h) = \frac{1}{\pi}[-\frac{\pi}{2}m - h^{-1}\pi(\lambda - b)]_{2\pi} + 2p.$$

Hence we obtain the proposition 7.1 with $C = \dfrac{2}{(2\pi)^n}\mu(\Pi^1)$. ■

8 Appendix

8.1 Tauberian theorem

Let $\sigma_h(\lambda)$, $h \in (0, h_0]$, be a family of non-decreasing, non-negative functions in $\mathbb{R}$ and let $0 < \lambda_1 < \lambda < \lambda_2$. Suppose that $\sigma_h(\lambda)$ has the following properties:

(i) $\sigma_h(\lambda) = 0$ for $\lambda < \lambda_1$

(ii) $\sigma_h(\lambda) = \sigma_h(\lambda_2)$ for $\lambda \geq \lambda_2$

(iii) $\sigma_h(\lambda) \leq Kh^{-n}$, $\forall h \in (0, h_0]$, $\forall \lambda \in \mathbb{R}$.

Let $\rho(\lambda) \in \mathcal{S}(\mathbb{R})$, $\rho(-\lambda) = \rho(\lambda)$, and $\rho(\lambda) \geq 0$ on $\mathbb{R}$, while $\rho(\lambda) \geq \delta_0 > 0$ for $\mid \lambda \mid \leq \epsilon_0$. We assume that the Fourier transform of $\rho(\lambda)$ satisfies

$$\hat{\rho}(t) \in C_0^\infty(\mathbb{R}), \quad \text{supp } \hat{\rho} \subset [-\epsilon_1, \epsilon_1],\ \epsilon_1 > 0,\ \hat{\rho}(0) = 1.$$

Introduce for $0 < \epsilon \leq 1$ the function $\rho_{\epsilon h}(\lambda) = \frac{1}{\epsilon h}\rho(\frac{\lambda}{\epsilon h})$, which satisfies:

$$\int \rho_{\epsilon h}(\lambda) \exp(ith^{-1}\lambda) d\lambda = \hat{\rho}(\epsilon t).$$

Theorem 8.1 *([20], [19])*

a) Assume that there exist functions $\gamma_0(\lambda)$, $\gamma_1(\lambda) \in C^1(\mathbb{R})$ such that for $\epsilon > 0$,

$$\frac{d}{d\lambda}\Big(\sigma_h * \rho_{\epsilon h}\Big)(\lambda) = \gamma_0(\lambda)h^{-n} + \gamma_1(\lambda)h^{1-n} + \mathcal{O}_\lambda(h^{2-n}), \qquad (40)$$

where $\mathcal{O}_\lambda$ is locally uniform with respect to λ.

Then for each $\lambda \in [\lambda_1, \lambda_2]$, we have

$$\mid \sigma_h(\lambda) - \Big(\sigma_h * \rho_{\epsilon h}\Big)(\lambda) \mid \leq C_0\gamma_0(\lambda)\epsilon h^{1-n} + C_1(\epsilon, \lambda)h^{2-n}$$

with $C_0 > 0$ and $C_1(\epsilon, \lambda)$ locally uniformly bounded with respect to λ.

b) Assume (40) with $\epsilon = 1$, then for each ϵ', $0 < \epsilon' \leq 1$, each ϵ, $0 < \epsilon \leq 1$, and $\lambda \in [\lambda_1, \lambda_2]$, we have

$$\Big(\sigma_h * \rho_{\epsilon h}\Big)(\lambda - \epsilon' h) - C_0 \gamma_0(\lambda) {\epsilon'}^{-1} \epsilon h^{1-n} - C_1 h^{2-n} \leq \sigma_h(\lambda)$$
$$\leq \Big(\sigma_h * \rho_{\epsilon h}\Big)(\lambda + \epsilon' h) + C_0 \gamma_0(\lambda) {\epsilon'}^{-1} \epsilon h^{1-n} + C_1 h^{2-n}$$

with $C_0 > 0$, $C_1 > 0$ independent of ϵ', ϵ, λ and h.

8.2 Trapped trajectories

In the section 3, we have used that all the periodic trajectories of the classical Hamiltonian lie on a compact set. It is a direct consequence of the following proposition 8.3. Let $a_0(x, \xi)$ a classical Hamiltonian and $H_{a_0} = (\partial_\xi a_0, -\partial_x a_0)$ the associated vector field. Let us consider the Hamiltonian flow on $\Sigma_\lambda = a_0^{-1}(\lambda)$:

$$\Phi_{a_0}(t) = \exp(tH_{a_0}) \ : \ (x_0, \xi_0) \mapsto \Big(x(t, x_0, \xi_0), \xi(t, x_0, \xi_0)\Big).$$

Definition 8.2 *We say that $\{\Phi_{a_0}(t)(x_0, \xi_0) \ ; \ t \in \mathbb{R}\}$ is a trapped trajectory on Σ_λ if it is a bounded subset of Σ_λ.*

We consider the classical Hamiltonian

$$a_0(x, \xi) =< g(x)^{-1}\xi, \xi > +a(x).\xi + V(x)$$

with $a = (a_j)_{1 \leq j \leq n}$, V, and $g = \{g_{ij}\}_{1 \leq i,j \leq n}$ satisfying (H_δ) (see the introduction).

Proposition 8.3 *Under the assumption (H_δ) with $\delta > 1$, the trapped trajectories of the Hamiltonian flow H_{a_0} on Σ_λ, $\lambda > 0$, are included in a compact set.*

Sketch of Proof. – For $\lambda > 0$ fixed, we prove there exists $R_\lambda^0 > 0$ such that any trajectory $\Big(x(t), \xi(t)\Big) := \Big(x(t, x_0, \xi_0), \xi(t, x_0, \xi_0)\Big)$, with $\mid x_0 \mid > R_\lambda^0$ is a non trapped trajectory in Σ_λ. It is clear that the ξ-component of any element in Σ_λ is bounded, then we check that $x(t, x_0, \xi_0)$ goes to infinity as $\mid t \mid$ tends to infinity.

Let us denote $a_{0,1}(x, \xi) := a_0(x, \xi) - \mid \xi \mid^2$. Then from the Hamiltonian system:

$$\left\{ \begin{array}{lcl} x' & = & 2\xi + \nabla_\xi a_{0,1}(x, \xi) \\ \xi' & = & -\nabla_x a_{0,1}(x, \xi) \end{array} \right.$$

we deduce the following relations:

$$(x^2)' = 4x.\xi + 2x.\nabla_\xi a_{0,1}(x,\xi), \tag{41}$$

$$(x.\xi)' = \{a_0, x.\xi\} = 2 \mid \xi \mid^2 + \xi.\nabla_\xi a_{0,1}(x,\xi) - x.\nabla_x a_{0,1}(x,\xi).$$

Then, for $(x_0,\xi_0) \in \Sigma_\lambda$, consequently for $(x,\xi) \in \Sigma_\lambda$, we have:

$$(x.\xi)' = 2\lambda - 2a_{0,1}(x,\xi) + \xi.\nabla_\xi a_{0,1}(x,\xi) - x.\nabla_x a_{0,1}(x,\xi). \tag{42}$$

On the other hand let us remark that by the assumption (H_δ), the symbol $a_{0,1}$ satisfies:

For any α, β in $\mathbb{N}^n$ there exists $C = C(\lambda, \alpha, \beta) > 0$ such that:

$$\mid \partial_\xi^\beta \partial_x^\alpha a_{0,1}(x,\xi) \mid \leq C\langle x\rangle^{-\delta-|\alpha|} \qquad \forall (x,\xi) \in \Sigma_\lambda,$$

with $\delta > n \geq 1$.

Consequently there exists $R_\lambda > 0$ such that

$$\mid x \mid \geq R_\lambda \Rightarrow \mid -2a_{0,1}(x,\xi) + \xi.\nabla_\xi a_{0,1}(x,\xi) - x.\nabla_x a_{0,1}(x,\xi) \mid \leq \lambda. \tag{43}$$

Now, by continuity of $t \mapsto x(t)$ and by using the relations (41)-(43) we can find $R_\lambda^0 > R_\lambda$ such that

$$\mid x_0 \mid \geq R_\lambda^0 \Rightarrow \mid x(t, x_0, \xi_0) \mid^2 \geq R_\lambda^2 + 2\lambda(t - t_0)^2,$$

for any $t \geq t_0$ (when $x_0.\xi_0 \geq 0$) or for any $t \leq t_0$ (when $x_0.\xi_0 \leq 0$).

We deduce that $t \mapsto \big(x(t,x_0,\xi_0), \xi(t,x_0,\xi_0)\big)$ is not a trapped trajectory when $\mid x_0 \mid > R_\lambda^0$. This proves that the trapped trajectories in Σ_λ are included in the compact set $\{(x,\xi) \in \Sigma_\lambda \ ; \ \mid x \mid \leq R_\lambda^0\}$. ■

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