Semi-classical asymptotics of Riesz means [1]

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Abstract– The semi-classical asymptotic behaviour of the Riesz means of a distribution of eigenvalues is investigated, at a non-critical energy level. For Schrödinger type operators, we obtain the second term related to the periodic trajectories of the classical Hamiltonian. This oscillating term is explained for Riesz means of order one with the aim of discussing a Lieb-Thirring conjecture.

1 Introduction and result

Let us consider $P(h) := p^w(h; x, hD_x)$ an h-admissible self-adjoint operator on $L^2(\mathbb{R}^n)$, defined by the Weyl quantization, $h > 0$ being a small parameter (see [10], [24], [8])

$$P(h)\varphi(x) = \frac{1}{(2\pi h)^n} \int \exp(i\langle x - y, \xi\rangle) p\Big(h; \frac{x+y}{2}, h\xi\Big)\varphi(y)dyd\xi, \;\; \varphi(x) \in \mathcal{S}(\mathbf{R}^n).$$

The typical example is the Schrödinger operator, $P(h) = -h^2\Delta + V$, for which the Weyl symbol is $p(h; x, \xi) =\mid \xi \mid^2 + V(x)$.

In the general case, the symbol of $P(h)$ satisfies:

$$p(h, x, \xi) \sim \sum_{j \geq 0} h^j p_j(x, \xi),$$

with $p_0(x, \xi)$, the principal symbol.

Let

$$\lambda_0 := \liminf_{|x|+|\xi| \to \infty} p_0(x, \xi);$$

Then for $\lambda < \lambda_0$, the spectrum of $P(h)$ in $] - \infty, \lambda]$ is discrete (i.e. is formed by finitely many isolated eigenvalues of finite multiplicity) (see [10]).

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For $\lambda < \lambda_0$ define the Riesz mean of order $\gamma > 0$ by:

$$R_\gamma(h,\lambda) = \sum_{e_j(h)\le\lambda} \Big(\lambda - e_j(h)\Big)^\gamma = Tr\left(\lambda_+^\gamma\Big(\lambda - P(h)\Big)\right),$$

where $e_1(h) \le e_2(h) \le \ldots \le e_j(h) \le \ldots < \lambda_0$ are the eigenvalues counted according to their multiplicities and $\lambda_+^\gamma(\tau) := \tau^\gamma 1\!\!1_{[0,+\infty[}$, $1\!\!1_{[0,+\infty[}$ is the characteristic function of $[0,+\infty[$.

The interest in the quantity $R_\gamma(h,\lambda)$ is partly connected with its physical meaning (see the references of [11] and [5]). In particular, an essential ingredient of the proof of "stability" of matter by E.H. Lieb and W. Thirring [20] is a bound on the sum of the eigenvalues of a one-particle Hamiltonian.

In this paper, we consider a non critical energy level λ, i.e. $\nabla p_0(x,\xi) \neq 0$ on the set $\Sigma_\lambda = p_0^{-1}(\lambda)$ and we study the semi-classical asymptotics of $R_\gamma(h,\lambda)$.

From [11] we know that $R_\gamma(h,\lambda)$ satisfies a Weyl type formula (with remainder $O(h^{1+\gamma-n})$). For $\gamma = 0$ (counting function), V. Petkov and G. Popov [22], improve this asymptotic formula with an oscillating second term related to the behaviour of periodic trajectories of the Hamiltonian flow $\Phi_{p_0}^t := \exp(tH_{p_0})$ associated to the Hamiltonian vector field $H_{p_0} = (\partial_\xi p_0, -\partial_x p_0)$. We extend this result for $\gamma > 0$.

Let us introduce some notations. A point $\nu \in \Sigma_\lambda$ is called periodic, of primitive period $T(\nu)$, if there exists $T > 0$ such that $\Phi_{p_0}^T(\nu) = \nu$, $T(\nu)$ being the smallest such $T > 0$. We denote by Π the set of all periodic points on Σ_λ. For a periodic point ν, $\gamma(\nu) = \{\Phi_{p_0}^t(\nu) : 0 \le t \le T(\nu)\} = \{(x(t),\xi(t)) : 0 \le t \le T(\nu)\}$ is the primitive periodic trajectory issued from ν. Let $m(\gamma) \in \mathbb{Z}_4$ denote the Maslov index related to $\gamma(\nu)$ and $p_1(x,\xi)$ the sub-principal symbol of $P(h)$. Thus, we define :

$$z(h^{-1},\nu) := h^{-1}\int_{\gamma(\nu)} \xi dx - \int_0^{T(\nu)} p_1\Big(x(t),\xi(t)\Big)dt + \frac{\pi}{2}m(\gamma). \tag{1}$$

$$Q^\gamma(h,\lambda) := \frac{2\Gamma(\gamma+1)}{(2\pi)^n}\int_\Pi \sum_{k=1}^{\infty} \frac{1}{\Big(kT(\nu)\Big)^{\gamma+1}} sin\Big(kz(h^{-1},\nu) - \gamma\frac{\pi}{2}\Big)d\nu, \quad \gamma > 0 \tag{2}$$

where $d\nu =\mid \nabla p_0 \mid^{-1} dS$ stands for the Liouville measure on Σ_λ.

The function $Q^\gamma(h,\lambda)$ could be considered as a semi-classical analog of the oscillating function defined by Yu. Safarov [29], who investigated Riesz Means of the distribution of the eigenvalues at high energies for elliptic operators on compact manifolds. For $\gamma = 0$ (counting function), an analogous function was introduced by V. Petkov and G. Popov [22], as the semi-classical version of the oscillating function defined by T. Guriev and Yu. Safarov [9], and Yu. Safarov ([27], [28]), in the investigation of the distribution of the eigenvalues at high energies for elliptic operators on compact manifolds.

Theorem 1.1 *Let $\lambda < \lambda_0$ be a non critical energy level of p_0. For $\gamma > 0$, as $h \searrow 0$, we have*

$$R_\gamma(h,\lambda) = \sum_{j=0}^{[\gamma]+} h^{j-n} C_{j,\gamma}(\lambda) + h^{1+\gamma-n} Q^\gamma(h,\lambda) + o(h^{1+\gamma-n}),$$

where $[\gamma]_+$ *is the smallest integer* $> \gamma$, *and* $C_{j,\gamma}(\lambda)$ *are computable smooth functions. In particular for* $\gamma = l \in \mathbb{N}^*$,

$$Q^l(h,\lambda) = \frac{-(2\pi)^{l+1}}{(l+1)(2\pi)^n} \int_{\Pi} B_{l+1}\Big(\frac{[z(h^{-1},\nu)-\pi]_{2\pi}+\pi}{2\pi}\Big)\frac{d\nu}{T(\nu)^{l+1}}. \tag{3}$$

$(B_l(x))_{l\in\mathbb{N}}$ *being the Bernoulli Polynomials and the residium* $[z]_{2\pi}$ *being defined:*

$$z = [z]_{2\pi} + 2\pi k, \qquad -\pi < [z]_{2\pi} \leq \pi, \qquad k \in \mathbb{Z}.$$

For the harmonic oscillator ($P(h) = -h^2\Delta + x^2$), this asymptotic was stated by B. Helffer and D. Robert when $\gamma \leq 1$ (see relation 2.5 of [12]). They compute $R_\gamma(h,\lambda)$ using study of the De Hass-Van Alphen effect [13].

The proof of theorem 1.1 is based on a standard method in asymptotic analysis of spectral functions, which was initiated by L. Hörmander ([15], [14]...) in the high-energy case for elliptic operators on a compact manifold. Classicaly, the semi-classical asymptotic expansions are connected with the singularities of the trace of the propagator at $t = 0$ and at $t = T(\nu)$ the periods of the primitive periodic trajectories. The singularity $t = 0$ gives the term $\sum h^{j-n}C_{j,\gamma}(\lambda)$ and the oscillating term $Q^\gamma(h,\lambda)$ comes from the non-zero singularities. The main difficulty, in comparaison with [11], is to take into account all periodic trajectories of H_{p_0}. To overcome this difficulty, we are inspired by results of V. Petkov and G. Popov [22], where the introduction of absolutely periodic points (adapted from [27, 28, 9] to the semi-classical case) allow the construction of suitable phase functions in special symplectic coordinates.

As in [11], the proof mainly consists of three steps:

First by the h-admissible C^∞ functionnal calculus ([10], [24], [8]) we are reduced to studying:

$$\tilde{R}_\gamma(h,\lambda) = \sum_{e_j(h)\leq\lambda} \Big(\lambda - e_j(h)\Big)^\gamma f\Big(e_j(h)\Big),$$

with $f \in C_0^\infty(]-\infty,\lambda_0[)$ supported in a non critical neighbourhood of λ. More precisely, we have

$$R(h,\lambda) = \tilde{R}_\gamma(h,\lambda) + Tr\Big(\psi_{\lambda,\gamma}(P)\Big),$$

with $\psi_{\lambda,\gamma} : \mu \mapsto (\lambda-\mu)^\gamma\Big(1-f(\mu)\Big)$ which is C^∞ and consequently, $Tr\Big(\psi_{\lambda,\gamma}(P)\Big) \asymp \sum h^{j-n}\alpha_{j,\gamma}(\lambda)$.

Then we establish, in section 2, a Tauberian theorem with a mollifier function, $\rho_{\epsilon h}$, for which the support of the Fourier transform is large when ϵ is small.

Finally, in section 3, we give the semi-classical asymptotic expansion of a regularized spectral function. We treat the singularities at $t = 0$ using standard arguments of asymptotic analysis (construction of a parametrix of the propagator) and for the singularities at $t = T(\nu)$ we exploit results of V. Petkov and G. Popov [22].

In section 4, we explain the oscillating term for Riesz means of order one when $P(h) = -h^2\Delta + V(x)$. In connection with a Lieb-Thirring conjecture, a motivation of this work was to prove that this term could be sufficiently large in comparaison to the Weyl term $C_{2,1}(\lambda)$. Unfortunately we are not able to make any conclusion in this direction and the problem remains open.

The results presented in this paper were annonced in the note [4].

2 Tauberian Theorem

In this section we prove a Tauberian theorem (proposition 2.1) for $\tilde{R}_\gamma(h,\lambda)$, taking into account the non-zero singularities of the trace of the propagator. Towards this end we introduce a mollifier function for which the support of the Fourier transform is large. Let $\rho(\lambda) \in \mathcal{S}(\mathbb{R})$ denote an even non-negative function such that $\hat{\rho}(t) \in C_0^\infty(\mathbb{R})$, $\hat{\rho}(t) = 1$ on $[-\epsilon_1, \epsilon_1]$, and introduce, for $0 < \epsilon \leq 1$, the function $\rho_{\epsilon h}(\lambda) = \frac{1}{\epsilon h}\rho(\frac{\lambda}{\epsilon h})$.

Proposition 2.1 *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in]0, \epsilon_0]$*

$$\tilde{R}_\gamma(h,\lambda) = \Big(\tilde{R}_\gamma(h,.) * \rho_{\epsilon h}\Big)(\lambda) + O(\epsilon^\gamma h^{1+\gamma-n}). \tag{4}$$

The proof of this relation uses the following estimate stated by V. Petkov and D. Robert [23].

Lemma 2.2 *Let $\lambda < \lambda_0$ be a non critical energy level of p_0. Then*

$$\mid \tilde{R}_0(h,\lambda) - \tilde{R}_0(h,\lambda - \tau h) \mid \leq C(\lambda)(1+ \mid \tau \mid^2)h^{1-n},$$

for any $\tau \in \mathbb{R}$, locally uniformly with respect to λ, and where the $C(\lambda)$, which depends only on λ, is locally bounded.

Proof of the proposition 2.1. This proposition is essentially obtained by inserting the parameter ϵ in the proof of the Tauberian theorem of [11].

Denote by $\hat{R}_\gamma(h,.)$ the Fourier transform of $\lambda \mapsto \tilde{R}_\gamma(h,\lambda)$. We have, in $\mathcal{S}'$, the space of temperate distributions:

$$\hat{R}_\gamma(h,t) = \hat{\lambda}_+^\gamma Tr\Big(f(P)e^{-itP}\Big) = \Gamma(\gamma+1)e^{-i(\gamma+1)\pi/2}(t-i0)^{-\gamma-1}Tr\Big(f(P)e^{-itP}\Big).$$

Consequently, the Fourier transform of $\tilde{R}_\gamma(h,\lambda) - \Big(\tilde{R}_\gamma(h,.) * \rho_{\epsilon h}\Big)(\lambda)$ is given by

$$\Gamma(\gamma+1)e^{-i(\gamma+1)\pi/2}(t-i0)^{-\gamma-1}\Big(1-\hat{\rho}(\epsilon th)\Big)Tr\Big(f(P)e^{-itP}\Big).$$

However, as $\hat{\rho}(0) = 1$, then

$$(t-i0)^{-\gamma-1}\Big(1-\hat{\rho}(\epsilon th)\Big) = t^{-\gamma-1}\Big(1-\hat{\rho}(\epsilon th)\Big) = (\epsilon h)^{\gamma+1}\varphi^\gamma(\epsilon ht)$$

where $\varphi^\gamma(t) := t^{-\gamma-1}\Big(1-\hat{\rho}(t)\Big)$, $\gamma > 0$, is an integrable smooth function. Moreover, for any $k \in \mathbb{N}$, $t^k \frac{d^k}{dt^k}\varphi^\gamma$ is also integrable. Thus we have

$$\tilde{R}_\gamma(h,\lambda) - \Big(\tilde{R}_\gamma(h,.) * \rho_{\epsilon h}\Big)(\lambda) = K(\gamma)(\epsilon h)^{\gamma+1}(2\pi)^{-1}\int e^{i\lambda t}\varphi^\gamma(\epsilon ht)Tr\Big(f(P)e^{-itP}\Big)dt,$$

with K depending only on γ, $K(\gamma) = \Gamma(\gamma+1)e^{-i(\gamma+1)\pi/2}$. The change of variables $u = \epsilon ht$ yields:

$$\tilde{R}_\gamma(h,\lambda) - \Big(\tilde{R}_\gamma(h,.) * \rho_{\epsilon h}\Big)(\lambda) =$$

$$= K(\gamma)(\epsilon h)^{\gamma+1}(2\pi\epsilon h)^{-1}\int e^{iu\epsilon^{-1}h^{-1}\lambda}\varphi^{\gamma}(u)Tr\Big(f(P)e^{-iu\epsilon^{-1}h^{-1}P}\Big)du.$$

For $\gamma > 0$, we can apply Fubini's Theorem, and we obtain

$$\tilde{R}_{\gamma}(h,\lambda) - \Big(\tilde{R}_{\gamma}(h,.) * \rho_{\epsilon h}\Big)(\lambda) =$$

$$= K(\gamma)(\epsilon h)^{\gamma}(2\pi)^{-1}Tr\left(f(P)\int e^{-iu\epsilon^{-1}h^{-1}(P-\lambda)}\varphi^{\gamma}(u)du\right).$$

$$= K(\gamma)(\epsilon h)^{\gamma}(2\pi)^{-1}Tr\left(f(P)\hat{\varphi^{\gamma}}\left(\frac{P-\lambda}{\epsilon h}\right)\right),$$

with $\hat{\varphi^{\gamma}} \in C^{\infty}(\mathbb{R}\setminus 0)$, continuous at 0 and $\hat{\varphi^{\gamma}}(t) = O(|\, t\,|^{-k})$ for any $k \in \mathbb{N}$.

Fix $\tau_0 > 0$. We have

$$Tr\left(f(P)\hat{\varphi^{\gamma}}\left(\frac{P-\lambda}{\epsilon h}\right)\right) = \sum_{m\in\mathbb{Z}} Tr\left(1\!\!1_{I(\lambda+m\epsilon\tau_0 h,\epsilon\tau_0 h)}(P)f(P)\hat{\varphi^{\gamma}}\left(\frac{P-\lambda}{\epsilon h}\right)\right),$$

where $I(a,r) := [a, a+r]$. However,

$$Tr\left(1\!\!1_{I(\lambda+m\epsilon\tau_0 h,\epsilon\tau_0 h)}(P)f(P)\hat{\varphi^{\gamma}}\left(\frac{P-\lambda}{\epsilon h}\right)\right) = \int_{\lambda+m\epsilon\tau_0 h}^{\lambda+(m+1)\epsilon\tau_0 h}\hat{\varphi^{\gamma}}\left(\frac{P-\lambda}{\epsilon h}\right)d\tilde{R}_0(h,\tau),$$

which is less than

$$\left(\sup_{t\in[m\tau_0,(m+1)\tau_0]} |\,\hat{\varphi^{\gamma}}(t)\,|\right) |\,\tilde{R}_0(h,\lambda+m\epsilon\tau_0 h) - \tilde{R}_0(h,\lambda+(m+1)\epsilon\tau_0 h)\,|\,.$$

Consequently, from lemma 2.2 and because $\hat{\varphi^{\gamma}}(t) = O(|\, t\,|^{-k})$, $k \in \mathbb{N}$ we deduce

$$|\, Tr\left(f(P)\hat{\varphi^{\gamma}}\left(\frac{P-\lambda}{\epsilon h}\right)\right) |\leq C(\lambda)\sum_{m\in\mathbb{Z}}(1+|\, m\,|)^{-k}(1+m^2\epsilon^2)h^{1-n}.$$

Thus, taking k large enough, we obtain

$$|\,\tilde{R}_{\gamma}(h,\lambda) - \Big(\tilde{R}_{\gamma}(h,.) * \rho_{\epsilon h}\Big)(\lambda)\,|\leq C_0(\lambda,\gamma)\epsilon^{\gamma}h^{1+\gamma-n} \qquad h \searrow 0,$$

locally uniformly in λ. ■

3 Semi-classical asymptotics

Now we have to study the behaviour of the regularized spectral function, $\Big(\tilde{R}_{\gamma}(h,.) * \rho_{\epsilon h}\Big)(\lambda)$. We prove proposition 3.1. Towards that end, we isolate the singularity of the propagator at $t = 0$ from the non-zero singularities which are connected with the periodic trajectories of the classical hamiltonian H_{p_0}.

The singularity at $t = 0$ is treated in a standard way by constructing a parametrix of the propagator and using a stationnary phase theorem (as in [10]). For the singularities at the periods $T > 0$, we use a recent result of V. Petkov and G. Popov (theorem 5.1 of [22]).

Proposition 3.1 *There exists $h_0 > 0$ such that for any $h \in]0, h_0]$ and for $\epsilon > 0$,*

$$\Big(\tilde{R}_\gamma(h,.) * \rho_{\epsilon h}\Big)(\lambda) = \sum_{j=0}^{[\gamma]_+} h^{j-n} C_{j,\gamma}(\lambda) + h^{1+\gamma-n} \tilde{Q}_\epsilon^\gamma(h,\lambda) + o_\epsilon(h^{1+\gamma-n}) \tag{5}$$

where

$$\tilde{Q}_\epsilon^\gamma(h,\lambda) := \frac{2\Gamma(\gamma+1)}{(2\pi)^n} \int_\Pi \sum_{k=1}^{\infty} \frac{\hat{\rho}(\epsilon k T(\nu))}{\Big(kT(\nu)\Big)^{\gamma+1}} sin\Big(kz(h^{-1},\nu) - \gamma\frac{\pi}{2}\Big) d\nu.$$

To split the singularities, let us introduce :

$$\Phi_h(\lambda,\epsilon) = (2\pi h)^{-1} \int \exp(ith^{-1}\lambda)\Big(1-\hat{\theta}(t)\Big)\hat{\rho}(\epsilon t)dt,$$

$$\Psi_h(\lambda,\epsilon) = (2\pi h)^{-1} \int \exp(ith^{-1}\lambda)\hat{\theta}(t)\hat{\rho}(\epsilon t)dt,$$

where $\theta \in \mathcal{S}(\mathbb{R})$ and $\hat{\theta} \in C_0^\infty(\mathbb{R})$ is equal to 1 in a small neighbourhood of 0.

We have:

$$\rho_{\epsilon h}(\lambda) = \Phi_h(\lambda,\epsilon) + \Psi_h(\lambda,\epsilon),$$

and proposition 3.1 is a consequence of the two following lemmas.

Lemma 3.2 *There exists $T_0 > 0$ such that for any $\hat{\theta} \in C_0^\infty([-T_0,T_0])$, $\hat{\theta}(0) = 1$ we have:*

$$\Big(\tilde{R}_\gamma(h,.) * \Psi_h(.,\epsilon)\Big)(\lambda) = h^{-n} \sum_{j\geq 0} h^j C_{j,\gamma}(\lambda),$$

where $C_{j,\gamma}(\lambda)$ are computable smooth functions (see [11]).

Proof. By definition of $\tilde{R}_\gamma$, we have:

$$\Big(\tilde{R}_\gamma(h,.) * \Psi_h(.,\epsilon)\Big) = \Big(\lambda_+^\gamma * \mathrm{d}\tilde{R}_0(h,.)\Big) * \Psi_h(.,\epsilon) = \lambda_+^\gamma * \Big(\mathrm{d}\tilde{R}_0(h,.) * \Psi_h(.,\epsilon)\Big). \tag{6}$$

Thus we want to apply convolution by λ_+^γ to an asymptotic expansion of $\Big(\mathrm{d}\tilde{R}_0(h,.) * \Psi_h(.,\epsilon)\Big)$.

First let us remark that for ϵ small enough, $\hat{\theta}(t)\hat{\rho}(\epsilon t) = \hat{\theta}(t)$, then

$$\Big(\mathrm{d}\tilde{R}_0(h,.) * \Psi_h(.,\epsilon)\Big)(\tau) = \frac{1}{2\pi h} \sum_{e_j} f(e_j)\theta\Big(\frac{e_j - \tau}{h}\Big).$$

Moreover, for $\tau \in \Omega_\delta := \{\tau \in \mathbb{R};\ \mathrm{dist}(\tau, \mathrm{supp} f) > \delta\}$, $\delta > 0$, using that $\theta \in \mathcal{S}(\mathbb{R})$ we have

$$\mid \Big(\mathrm{d}\tilde{R}_0(h,.) * \Psi_h(.,\epsilon)\Big)(\tau) \mid \leq \frac{C}{2\pi h} \sum_{e_j} \mid f(e_j) \mid (|e_j - \tau| + \delta)^{-N}(2h)^N, \qquad \forall N \in \mathbb{N}. \tag{7}$$

Now, let $\delta > 0$ and $g \in C_0^\infty(]-\infty,\lambda_0[)$ be such that g is supported in a non critical neighbourhood of λ and $g = 1$ on $\mathbb{R} \setminus \Omega_\delta$ (shrinking the support of f if necessary, such g exists).

According to (6) and (7), modulo $O(h^\infty)$, we deduce:

$$\Big(\tilde{R}_\gamma(h,.) * \Psi_h(.,\epsilon)\Big)(\lambda) = \int_{-\infty}^{\lambda} (\lambda-\tau)^\gamma g(\tau)\Big(\mathrm{d}\tilde{R}_0(h,.) * \Psi_h(.,\epsilon)\Big)(\tau)\mathrm{d}\tau$$

with $g \in C_0^\infty(\mathbb{R})$ supported in a non critical neighbourhood of λ.

However, in a standard way by constructing a parametrix of the propagator and using a stationnary phase theorem ([15], [10], [11]...) we prove that

$$g(\tau)\Big(\mathrm{d}\tilde{R}_0(h,.) * \Psi_h(.,\epsilon)\Big)(\tau) = h^{-n}\sum_{j\geq 0} h^j C_j(\tau)$$

and obtain lemma 3.2, by an integration.

■

Lemma 3.3 *For any $\theta \in \mathcal{S}(\mathbb{R})$ such that $\hat{\theta} \in C_0^\infty(\mathbb{R})$ is equal to 1 in a small neighbourhood of 0, we have:*

$$\Big(\tilde{R}_\gamma(h,.) * \Phi_h(.,\epsilon)\Big)(\lambda) = h^{1+\gamma-n}\tilde{Q}_\epsilon^\gamma(h,\lambda) + o_\epsilon(h^{1+\gamma-n}).$$

Proof. As above, we have

$$\Big(\tilde{R}_\gamma(h,.) * \Phi_h(.,\epsilon)\Big) = \Big(\lambda_+^\gamma * \mathrm{d}\tilde{R}_0(h,.)\Big) * \Phi_h(.,\epsilon) = \Big(\lambda_+^\gamma * \Phi_h(.,\epsilon)\Big) * \mathrm{d}\tilde{R}_0(h,.).$$

However, the Fourier transform of $\lambda_+^\gamma * \Phi_h(.,\epsilon)$ is:

$$\Gamma(\gamma+1)e^{-i(\gamma+1)\pi/2}(t-i0)^{-\gamma-1}\Big(1-\hat{\theta}(-th)\Big)\hat{\rho}(\epsilon th) = K(\gamma)h^{\gamma+1}\Phi_\epsilon^\gamma(th),$$

with

$$\Phi_\epsilon^\gamma(u) := u^{-\gamma-1}\Big(1-\hat{\theta}(-u)\Big)\hat{\rho}(\epsilon u)$$

which is a smooth function compactly supported in $[-\epsilon_1\epsilon^{-1}, \epsilon_1\epsilon^{-1}] \setminus 0$.

Then, after the change of variables $u = th$, we have:

$$\Big(\tilde{R}_\gamma(h,.) * \Phi_h(.,\epsilon)\Big)(\lambda) = K(\gamma)h^{\gamma+1}(2\pi h)^{-1}\int e^{iuh^{-1}\lambda}\Phi_\epsilon^\gamma(u)Tr\Big(f(P)e^{-iuh^{-1}P}\Big)du.$$

From theorem 5.1 of [22] we deduce that

$$\Big(\tilde{R}_\gamma(h,.) * \Phi_h(.,\epsilon)\Big)(\lambda) =$$

$$K(\gamma)h^{\gamma+1}(2\pi h)^{-n}\sum_{k\in\mathbb{Z}\setminus 0}\int_\Pi \Phi_\epsilon^\gamma(kT(\nu))\exp\Big(ikz(h^{-1},\nu)\Big)\mathrm{d}\nu + o_\epsilon(h^{\gamma+1-n}),$$

where $z(h^{-1},\nu)$ is defined by (1).

However, by definition of $\hat{\theta}$ (in lemma 3.2), we have $\hat{\theta}(kT(\nu)) = 0$. Then, for $k \in \mathbb{Z}\setminus 0$

$$\Phi_\epsilon^\gamma(kT(\nu)) = \frac{1}{\Big(kT(\nu)\Big)^{\gamma+1}}\hat{\rho}(\epsilon kT(\nu)).$$

Moreover, $\hat{\rho}$ is an even function and

$$\frac{1}{\Big(-kT(\nu)\Big)^{\gamma+1}} = \frac{-e^{i\gamma\pi}}{\Big(kT(\nu)\Big)^{\gamma+1}}.$$

Consequently,

$$\sum_{k\in\mathbb{Z}\setminus 0} \Phi_\epsilon^\gamma(kT)\exp(ikz) = \sum_{k\geq 1} \frac{\hat{\rho}(\epsilon kT)}{(kT)^{\gamma+1}}\Big(e^{ikz} - e^{i\gamma\pi}e^{-ikz}\Big)$$

$$= e^{i\gamma\pi/2}\sum_{k\geq 1}\frac{\hat{\rho}(\epsilon kT)}{(kT)^{\gamma+1}} 2i\sin(kz-\gamma\pi/2).$$

Finally, using that $K(\gamma) = \Gamma(\gamma+1)e^{-i(\gamma+1)\pi/2}$, we deduce the lemma. ■

Thus we have proved relation (5), and the asymptotic result stated in theorem 1.1 follows from (4), (5) and the following relation (8):

Lemma 3.4

$$\forall \gamma > 0, \quad \tilde{Q}_\epsilon^\gamma(h,\lambda) = Q^\gamma(h,\lambda) + O(\epsilon^\gamma). \tag{8}$$

Proof. By definition, we have:

$$\tilde{Q}_\epsilon^\gamma(h,\lambda) - Q^\gamma(h,\lambda) = \frac{2\Gamma(\gamma+1)}{(2\pi)^n}\int_\Pi \sum_{k=1}^{\infty} \frac{\hat{\rho}(\epsilon kT)-1}{\Big(kT\Big)^{\gamma+1}} sin\Big(kz - \gamma\frac{\pi}{2}\Big)\mathrm{d}\nu,$$

with $z = z(h^{-1},\nu)$, $T = T(\nu)$.

However, $\hat{\rho}(\epsilon kT) - 1$ vanishes for $k \leq \epsilon_1(\epsilon T)^{-1}$ and $0 \leq 1-\hat{\rho}(\epsilon kT) \leq 1$. Then

$$\mid \tilde{Q}_\epsilon^\gamma(h,\lambda) - Q^\gamma(h,\lambda) \mid \leq \frac{2\Gamma(\gamma+1)}{(2\pi)^n}\int_\Pi \frac{1}{T^{\gamma+1}} \sum_{k\geq\epsilon_1(\epsilon T)^{-1}} \frac{1}{k^{\gamma+1}}\mathrm{d}\nu.$$

Now, using that

$$\sum_{k\geq N+1}\frac{1}{k^{\gamma+1}} \leq \frac{1}{\gamma N^\gamma},$$

we obtain

$$\mid \tilde{Q}_\epsilon^\gamma(h,\lambda) - Q^\gamma(h,\lambda) \mid \leq \frac{2\Gamma(\gamma+1)}{(2\pi)^n}\frac{\epsilon^\gamma C(\gamma)}{\gamma{\epsilon_1}^\gamma}\int_\Pi \frac{1}{T(\nu)}\mathrm{d}\nu.$$

■

Remark 3.5 *This lemma corresponds to proposition 6.1 of [22] where the term $\tilde{Q}_\epsilon^\gamma$ is estimated for $\gamma = 0$. In our case ($\gamma > 0$) the estimate is easier because we have an absolutely convergent series. We no longer need the assumption of contact type (see assumption (6.1) of [22]).*

Proof of the theorem 1.1. – From (4), (5) and (8), we deduce:

$$\tilde{R}_\gamma(h,\lambda) = \sum_{j=0}^{[\gamma]_+} h^{j-n} C_{j,\gamma}(\lambda) + h^{1+\gamma-n} Q^\gamma(h,\lambda) + O(\epsilon^\gamma h^{1+\gamma-n}) + o_\epsilon(h^{1+\gamma-n}),$$

for any $h \in]0,h_0]$ and $\epsilon \in]0,\epsilon_0]$. Now choosing ϵ and then h_0 sufficiently small, we deduce the asymptotic result stated in theorem 1.1.

To compute $Q^l(h,\lambda)$ for $l \in \mathbb{N}^*$, we have to study the following series:

$$\sum_{k\geq 1} \frac{\sin(kz - l\pi/2)}{k^{l+1}} = \sum_{k\geq 1} \frac{\cos(kz-(l+1)\pi/2)}{k^{l+1}}.$$

This series is closely connected with the Fourier expansion of the Bernoulli polynomials $\Big(B_l(x)\Big)$. For $l > 1$ and $0 \leq x \leq 1$ we have:

$$B_l(x) = -2\frac{l!}{(2\pi)^l} \sum_{k\geq 1} \frac{\cos(2\pi kx - l\pi/2)}{k^l}. \tag{9}$$

Then we deduce that

$$\sum_{k\geq 1} \frac{\cos(kz-(l+1)\pi/2)}{k^{l+1}} = -\frac{(2\pi)^{l+1}}{2(l+1)!} B_{l+1}\Big(\frac{[z-\pi]_{2\pi}+\pi}{2\pi}\Big),$$

where the residium $[z]_{2\pi}$ is defined:

$$z = [z]_{2\pi} + 2\pi m, \qquad -\pi < [z]_{2\pi} \leq \pi, \qquad m \in \mathbb{Z}.$$

Consequently, for $\gamma = l \in \mathbb{N}^*$ we have

$$Q^l(h,\lambda) = \frac{-2\Gamma(l+1)}{(2\pi)^n} \times \frac{(2\pi)^{l+1}}{2(l+1)!} \int_\Pi B_{l+1}\Big(\frac{[z(h^{-1},\nu)-\pi]_{2\pi}+\pi}{2\pi}\Big) \frac{\mathrm{d}\nu}{T(\nu)^{l+1}}.$$

Thus we obtain the assertion in theorem 1.1. ■

Remark 3.6 *In the same way, we could improve the asymptotic formula of [25] for Riesz means of the average time-delay:*

$$\tau_\gamma^D(\lambda) := \int_{-\infty}^{\lambda} (\lambda-\mu)^\gamma \tau^D(\mu) d\mu.$$

Towards that end, we can use the representation formula of [3] for the scattering phase $s_h(\lambda)$, which is proportional to a primitive of the time-delay $\tau^D(\mu)$.

4 The case $\gamma = 1$ and Lieb-Thirring conjecture

Let us consider the Schrödinger operator, $P(h) = -h^2\Delta + V$, where V is a real-valued function.

In [20] E.H. Lieb and W. Thirring proved that if $\gamma > \max(0, 1 - n/2)$, then there exist universal best constants, $L_{\gamma,n}$, satisfying:

$$h^n R_\gamma(h,\lambda) \leq L_{\gamma,n} \int (\lambda - V)_+^{\gamma+n/2} dx. \tag{10}$$

In the case $n \geq 3$ and $\gamma = 0$ the bound (10) is known as Cwikel-Lieb-Rozenblum inequality (see [7], [19], [26], [6], [18]). For $n = 1$, $\gamma = 1/2$ the estimate (10) also holds ([31], [16]). On the other hand it is known that (10) fails for $\gamma = 0$ if $n = 2$ and for $0 \leq \gamma < 1/2$ if $n = 1$.

A natural problem is to compare $L_{\gamma,n}$ with the classical limit:

$$L_{\gamma,n}^{cl} := \lim_{h\to 0} \left[\int (\lambda - V)_+^{\gamma+n/2} dx\right]^{-1} h^n R_\gamma(h,\lambda).$$

By definition, we always have:

$$L_{\gamma,n}^{cl} \leq L_{\gamma,n}.$$

Moreover, in [1] it has been shown, that for fixed n, the ratio $L_{\gamma,n}/L_{\gamma,n}^{cl}$ is a monotone non-increasing function of γ.

Actually, one knows that

$$L_{\gamma,n}^{cl} < L_{\gamma,n}$$

if $n = 1$ and $1/2 \leq \gamma < 3/2$ ([20], [11]) or $n \geq 1$ and $\gamma < 1$ ([12]) and that

$$L_{\gamma,n}^{cl} = L_{\gamma,n}$$

for any $\gamma \geq 3/2$ in the general case ([17]) and for any $\gamma \geq 1$ when V is in a class of strictly convex quadratic potentials ([2]).

A motivation of this paper was to contribute to the study in the case $\gamma = 1$. Applying theorem 1.1 for $P(h) = -h^2\Delta + V$ we obtain:

$$h^n R_\gamma(h,\lambda) = L_{\gamma,n}^{cl} \int (\lambda - V)_+^{\gamma+n/2} dx + h^2 C_{2,\gamma}(\lambda) + h^{1+\gamma} Q^\gamma(h,\lambda) + o(h^{\min(2,1+\gamma)}).$$

($C_{1,\gamma} = 0$ because the sub-principal symbol of P vanishes)

Thus we can state that $L_{\gamma,n}^{cl} < L_{\gamma,n}$ in the following situations:

1. for $\gamma > 1$ if there exists a potential V such that $C_{2,\gamma}(\lambda) > 0$
2. for $\gamma < 1$ if there exist a potential V and a sequence $h_k \searrow 0$ such that $Q^\gamma(h_k,\lambda) \geq \alpha > 0$
3. for $\gamma = 1$ if there exist a potential V and a sequence $h_k \searrow 0$ such that $C_{2,1}(\lambda) + Q^1(h_k,\lambda) \geq \alpha > 0$

The two first situations were discussed by Helffer-Robert ([11], [12]). They prove that situation 1 is satisfied for $n = 1$ and can not be satisfied for $n \geq 2$. Situation 2 is fulfilled with $V(x) = x^2$, $n \in \mathbb{N}$. On the other hand, for $V(x) = x^2$ we always have $C_{2,1}(\lambda) + Q^1(h, \lambda) < 0$ (see section 3 of [12]).

Our goal is now to exploit the general expression of the oscillating term to find a potential satisfaying the situation 3. For $\gamma = 1$, theorem 1.1 gives:

$$h^n R_1(h, \lambda) = L^{cl}_{1,n} \int (\lambda - V)_+^{1+n/2} dx + h^2 \Big(C_{2,1}(\lambda) + Q^1(h, \lambda) \Big) + o(h^2),$$

with (see [11])

$$C_{2,1}(\lambda) = -\frac{1}{24(2\pi)^n} \mathrm{vol}(S^{n-1}) \int (\lambda - V(x))_+^{n/2-1} \Delta V(x) dx,$$

and because $B_2(x) = x^2 - x + 1/6$, the oscillating term is

$$Q^1(h, \lambda) = \frac{1}{2(2\pi)^n} \int_\Pi \Big(\frac{\pi^2}{3} - [z(h^{-1}, \nu) - \pi]_{2\pi}{}^2 \Big) \frac{d\nu}{T(\nu)^2}.$$

Unfortunaly, we do not know if the sign of $\Big(C_{2,1}(\lambda) + Q^1(h, \lambda)\Big)$ is always negative or if there exists V such that it is positive. This remains an open question.

Actually, we can write down this term explicitely in particular cases. If the classical flow is totally periodic, that is Σ_λ is connected and there exists $T > 0$ such that $T(\nu) = T$, $\forall \nu \in \Sigma_\lambda$. Then the Maslov index and the action are constant. Thus, there exist $S > 0$ and $q = -m\pi/2$ such that $z(h^{-1}, \nu) = h^{-1}S - q$ for any $\nu \in \Sigma_\lambda$ and we have:

$$Q^1(h, \lambda) = \frac{1}{2(2\pi)^n} \frac{\mu(\Sigma_\lambda)}{T^2} \Big(\frac{\pi^2}{3} - [h^{-1}S - q - \pi]_{2\pi}{}^2 \Big),$$

with

$$\mu(\Sigma_\lambda) = \frac{1}{2} \mathrm{vol}(S^{n-1}) \int (\lambda - V(x))_+^{n/2-1} dx.$$

Then, for $\gamma = 1$, the coefficient $C_{2,1}(\lambda) + Q^1(h, \lambda)$ is equal to:

$$\frac{1}{12(2\pi)^n} \mathrm{vol}(S^{n-1}) \int (\lambda - V(x))_+^{n/2-1} \left(\frac{\pi^2}{T^2} - \frac{\Delta V(x)}{2} - \frac{3[h^{-1}S - q - \pi]_{2\pi}{}^2}{T^2} \right) dx,$$

which is optimal for $h_k = \frac{S}{q + (2k+1)\pi}$, $k \in \mathbb{N}$ and we have to study the sign of

$$C_{2,1}(\lambda) + Q^1(h_k, \lambda) = \frac{1}{12(2\pi)^n} \mathrm{vol}(S^{n-1}) \int (\lambda - V(x))_+^{n/2-1} \left(\frac{\pi^2}{T^2} - \frac{\Delta V(x)}{2} \right) dx.$$

For example, for strictly convex quadratic potentials, $V(x) = \sum_{i=1}^{n} k_i^2 x_i^2$, the classical flow is totally periodic if and only if for any $1 \leq i, j \leq n$, $k_i/k_j \in \mathbb{Q}$. In this case, there exists $l \in \mathbb{N}^*$ such that $T = l\pi/\sup|k_i|$ and $\Delta V = 2 \sum k_i^2$.

Thus $\left(C_{2,1}(\lambda)+Q^1(h_k,\lambda)\right)$ is always non positive, this agrees with de la Bretèche [2] who proved the Lieb-Thirring conjecture in the case of strictly convex quadratic potentials.

Moreover, let us note that this coefficient is improved when $k_i = k_j$, $\forall i,j$ (which implies $l=1$). This corresponds to the radial case studied by B. Helffer and D. Robert [12].

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