Spectral Asymptotic in the Large Coupling Limit

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Abstract : in this paper, we study a singular perturbation of an eigenvalues problem related to supra-conductor wave guides. Using boundary layer tools we perform a complete asymptotic expansion of the eigenvalues as the conductivity tends to $+\infty$.

1 Introduction

Let Ω and $\mathcal{O}$ two bounded connected smooth domains of $I\!R^2$ such that $\Omega \subset\subset \mathcal{O}$. We denote $\mathcal{U} = \mathcal{O} \setminus \Omega$.
We consider the self adjoint operator $\mathcal{H}^\sigma$ on $L^2(\mathcal{O})$ with domain $H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ defined by:

$$\mathcal{H}^\sigma = -\Delta + \sigma \mathbf{1}_\Omega, \tag{1.1}$$

where σ is a positive real number. In this paper we study the asymptotic behavior of the eigenvalues of $\mathcal{H}^\sigma$ as σ tends to $+\infty$. The limit operator is $\mathcal{H}^\infty = -\Delta$ defined on $H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})$, that is $u = 0$ on $\partial\mathcal{U} = \partial\mathcal{O} \cup \partial\Omega$.

Remark 1.1 *Such a problem appears in the study of an electromagnetic wave guide $\mathcal{O} \times I\!R$ with section $\mathcal{O}$, where $\mathcal{U} \times I\!R$ is a dielectric body and $\Omega \times I\!R$ is a supra-conductor material with very large conductivity. For mathematical studies concerning electromagnetic wave guides, see [3] and [4].*

According to general monotone convergence theorems (see [15]), it is well known that $\mathcal{H}^\sigma$ tends to $\mathcal{H}^\infty$ in strong resolvent sense, as $\sigma \to +\infty$. In our case, where the operators are self-adjoint, non negative with compact resolvents, this implies that the eigenvalues of $\mathcal{H}^\sigma$ tend to ones of $\mathcal{H}^\infty$ (see also [5]). To our knowledge, there is no result concerning the rate of the convergence (see below for more references).
The goal of this paper is to build an asymptotic expansion of the eigenvalues of $\mathcal{H}^\sigma$ when σ tends to $+\infty$.
We denote $R^\sigma(\xi) = (\mathcal{H}^\sigma - \xi I)^{-1}$ the resolvent of $\mathcal{H}^\sigma$, defined on $L^2(\mathcal{O})$ with values in $H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ for $\xi \notin \text{spec } \mathcal{H}^\sigma$.
We first prove that the resolvent of $\mathcal{H}^\sigma$ admits a first order asymptotic expansion :

Theorem 1.1 *We define the operator $R^\infty(\xi)$ by : if $f \in H^1(\mathcal{O})$, $R^\infty(\xi)(f)$ is defined by $R^\infty(\xi)(f)(x) = U_0(x)$ if $x \in \mathcal{U}$ and $R^\infty(\xi)(f)(x) = 0$ if $x \in \Omega$, where*

$$\begin{cases} U^0 = 0 \ \text{ on } \partial\Omega \cup \partial\mathcal{O}, \\ \\ (-\Delta - \xi) U^0 = f \ \text{ on } \mathcal{U}. \end{cases}$$

On the other hand we define $R_1^\sigma(\xi)$ by : if $f \in H^1(\mathcal{O})$,

$$R_1^\sigma(\xi)(f)(x) = \begin{cases} \psi(x)\alpha_1(x) e^{-\sqrt{\sigma}\varphi(x)} \ \text{ if } x \in \Omega, \\ \\ U^1(x) \ \text{ if } x \in \mathcal{U}, \end{cases}$$

where

$$\begin{cases} U^1 = 0 \ \text{on} \ \partial\mathcal{O}, \\ \\ U^1 = \dfrac{\partial U^0}{\partial \nu} \ \text{on} \ \partial\Omega, \\ \\ (-\Delta - \xi)U^1 = 0 \ \text{on} \ \mathcal{U}, \end{cases}$$

and

- α_1 *is an extension of* $U^1|_{\partial\Omega}$ *on* Ω,
- *for* $x \in \Omega$, $\varphi(x) = dist(x, \partial\Omega)$,
- ψ *is a cut-off function equal to 1 in a neighborhood of* $\partial\Omega$.

Then when σ *tends to* $+\infty$, $R^\sigma(\xi)$ *admits an asymptotic expansion of the type :*

$$R^\sigma(\xi) = R^\infty(\xi) + \frac{1}{\sqrt{\sigma}} R_1^\sigma(\xi) + \frac{1}{\sqrt{\sigma}} K_2^\sigma(\xi),$$

and there exists C *independent on* σ *such that the remainder term* $K_2^\sigma(\xi)$ *satisfies :*

$$\forall\, f \in H^1(\mathcal{O}), \quad \begin{cases} \|K_2^s(\xi)(f)\|_{H^1(\mathcal{O})} \leq C\|f\|_{H^1(\mathcal{O})}, \\ \\ \|K_2^s(\xi)(f)\|_{L^2(\mathcal{O})} \leq \dfrac{C}{\sigma^{\frac{1}{4}}}\|f\|_{H^1(\mathcal{O})}. \end{cases}$$

Remark 1.2 *We denote* $\tilde{R}^\infty(\xi) = (\mathcal{H}^\infty - \xi I)^{-1}$ *the resolvent of* $\mathcal{H}^\infty$, *defined on* $L^2(\mathcal{U})$ *with values in* $H^2(\mathcal{U}) \cap H_0^1(\mathcal{U})$. *Then the first term of the asymptotic expansion of* $R^\sigma(\xi)$ *is* $R^\infty(\xi) = e\tilde{R}^\infty(\xi)r$, *where* $e : L^2(\mathcal{U}) \longrightarrow L^2(\mathcal{O})$ *is the extension by zero in* Ω *and* $r : L^2(\mathcal{O}) \longrightarrow L^2(\mathcal{U})$ *is the restriction operator.*

Theorem 1.1 prove the existence of a first order asymptotic expansion of the resolvent and we deduce the following theorem concerning eigenvalues :

Theorem 1.2 *Let* λ^∞ *be an eigenvalue of* $\mathcal{H}^\infty$ *with multiplicity* m. *We fix* $\eta > 0$ *such that* $B(\lambda^\infty, \eta) \cap spec\ \mathcal{H}^\infty = \{\lambda^\infty\}$, *where* $B(\lambda^\infty, \eta) = \{\xi \in \mathbb{C},\ |\lambda^\infty - \xi| \leq \eta\}$.
Then for sufficiently large σ, $\mathcal{H}^\sigma$ *has exactly* m *eigenvalues counted according to their multiplicities in* $B(\lambda^\infty, \eta)$, *and these eigenvalues* λ_i^σ *admit an asymptotic expansion of the type :*

$$\lambda_i^\sigma = \lambda^\infty + \frac{1}{\sqrt{\sigma}}\mu_i + o(\frac{1}{\sqrt{\sigma}}). \tag{1.2}$$

Furthermore if we denote $(f_1, \ldots, f_m)$ *an* $L^2(\mathcal{U})$ *orthonormal basis of the eigenspace* $ker(\mathcal{H}^\infty - \lambda^\infty I)$, *then* $(\mu_i)_{1 \leq i \leq m}$ *are the eigenvalues of the matrix* A *with coefficients :*

$$a_{ij} = \int_{\partial\Omega} \frac{\partial f_i}{\partial \nu}\frac{\partial f_j}{\partial \nu}.$$

Remark 1.3 *The computation of the eigenvalues and eigenvectors of the limit operator* $\mathcal{H}^\infty$ *give easily a first order approximation of the eigenvalues of* $\mathcal{H}^\sigma$, *since* A *is deduced from the eigenvectors of* $\mathcal{H}^\infty$.

Remark 1.4 *In the proof of Theorem 1.2, we show in fact that the remainder term in (1.2) is* $\mathcal{O}\left(\frac{1}{\sigma^{\frac{3}{4}}}\right)$. *This estimate is not optimal as we will see in Theorem 1.3.*

We can not build an asymptotic expansion of the resolvent at any order in $\mathcal{L}(H^1; H^1)$ as it is done at order 1 in Theorem 1.1. Nevertheless it is possible to prove that the eigenvalues of H^σ admit an asymptotic expansion at any order :

Theorem 1.3 *Let* λ^∞ *be an eigenvalue of* $\mathcal{H}^\infty$. *With the notations of Theorem 1.2, for any* i, $1 \leq i \leq m$, λ_i^σ *admits an asymptotic expansion at any order, that is there exists a sequence* $(\mu_i^j)_{j\in\mathbb{N}}$ *such that for all* N,

$$\lambda_i^\sigma = \lambda^\infty + \sum_{j=1}^{N} \frac{1}{\sigma^{\frac{j}{2}}}\mu_i^j + \mathcal{O}\left(\frac{1}{\sigma^{\frac{N+1}{2}}}\right).$$

Remark 1.5 *Theorem 1.1 remains valid in all dimension, but in Theorem 1.2, we use a Sobolev embedding which prevents the generalization of our proof at any dimension. Nevertheless the proof of Theorem 1.3 does not depend on the dimension, so the eigenvalues of* $\mathcal{H}^\sigma$ *admit an asymptotic expansion at any order in all dimension.*

Without any assumption on the dimension and on the multiplicity of the eigenvalues, it seems to be the first asymptotic result (at all order) in Large-coupling Limit. Large coupling limit are essentially discussed for operators with continuous spectrum. For example, for periodic problems let us quote works of Hempel-Lineau-Herbst (see [11] and their references). They use a Floquet decomposition (or direct fiber-integral decomposition [14]) and a monotonic convergence theorem in each fiber. Concerning perturbations of the Laplacian on $\mathbb{R}^n$ of the form $H_\lambda = -\Delta + V + \lambda \mathbf{1}_\Omega$, with $V \in L^2(\mathbb{R}^n)$, Demuth [7] use a Feynman-Kac formula to prove trace norm convergence of the resolvent $(H_\lambda - z)^{-r}$, $r \geq 1 + n/2$, to the resolvent of associated Dirichlet problem. More general perturbations of the type $H_\lambda = -\Delta + V + \lambda W$, with $V \in L^\infty(\mathbb{R}^n)$ and $0 \leq W \in L^\infty(\mathbb{R}^n)$ are studied by Gesztesy and al. [8]. Exploiting monotonic convergence theorems they prove that the discrete and essential spectrums of H_λ tend to the ones of the associated Dirichlet operator. Furthermore, in [8], using WKB machinery, 1-order asymptotic expansions are given in 1-dimension and for multiplicity $m \leq 2$. Further order asymptotic expansion is proved by Ashbaugh-Harrel [2] in 1-dimension (on $\mathbb{R}^+$) for $H_\lambda = -\Delta + V + \lambda \mathbf{1}_\Omega$, with V continuous supported in $[0,1]$ and $W = (x-1)^p \mathbf{1}_{[1,+\infty[}$. By an analytic implicit function theorem they prove that the eigenvalues of H_λ are analytic with respect to $\lambda^{-\frac{1}{p+2}}$.

In the following, we fix an eigenvalue λ^∞ of the operator $\mathcal{H}^\infty$. The resolvent of $\mathcal{H}^\infty$ is compact, so there exists $\eta > 0$ such that spec $\mathcal{H}^\infty \cap B(\lambda^\infty, \eta) = \{\lambda^\infty\}$.
We will use the following notations :

- E^σ is the sum of the eigenspaces associated with the eigenvalues of $\mathcal{H}^\sigma$ contained in $B(\lambda^\infty, \eta)$.
- P^σ is the spectral projection onto E^σ. It is given by :

$$P^\sigma = \frac{-1}{2i\pi} \int_{\mathcal{C}(\lambda^\infty,\eta)} R^\sigma(\xi) d\xi, \tag{1.3}$$

  where $\mathcal{C}(\lambda^\infty, \eta)$ is the circle of center λ^∞ and of radius η.

- $\tilde{E}^\infty$ is the eigenspace of $\mathcal{H}^\infty$ associated to λ^∞. We denote $\tilde{P}^\infty$ the spectral projection onto $\tilde{E}^\infty$.
- $E^\infty = e(\tilde{E}^\infty)$ and $P^\infty = e\tilde{P}^\infty r$ which satisfies :

$$P^\infty = \frac{-1}{2i\pi}\int_{\mathcal{C}(\lambda^\infty,\eta)} R^\infty(\xi)d\xi. \tag{1.4}$$

In order to perform the asymptotic expansion of the resolvent we use boundary layers machinery inside Ω, that is we seek $R^\sigma(\xi)(f)$ on the form :

$$R^\sigma(\xi)(f)(x) = V^0(x,\sqrt{\sigma}\varphi(x)) + \frac{1}{\sqrt{\sigma}}V^1(x,\sqrt{\sigma}\varphi(x)) + \ldots,$$

where $\varphi(x)$ denotes the distance from x to $\partial\Omega$. The term $\psi(x)\alpha_1(x)e^{-\sqrt{\sigma}\varphi(x)}$ represents the formation of the thin layer near the boundary of Ω.

Remark 1.6 *Boundary layers should appear in the case of viscous perturbation of hyperbolic and parabolic systems (see [9], [10] and [6]).*

In the proof of Theorem 1.2, following Kato [12], we introduce the invertible operator $\mathcal{A}^\sigma = 1 - P^\infty + P^\sigma P^\infty$. This operator maps E^∞ onto E^σ. The first order asymptotic expansion of the resolvent gives a first order asymptotic expansion of $\mathcal{A}^\sigma$ in the space $\mathcal{L}(H^1;H^1)$.

We denote $Q^\sigma = P^\infty[\mathcal{A}^\sigma]^{-1}\mathcal{H}^\sigma\mathcal{A}^\sigma P^\infty$. This operator belongs to $\mathcal{L}(E^\infty)$ and has the same eigenvalues than $\mathcal{H}^\sigma P^\sigma$. Using asymptotic expansions of the resolvent we obtain a first order asymptotic expansion of Q^σ. Applying classical finite dimensional results due to Kato (see [12]) we deduce Theorem 1.2.

In order to prove Theorem 1.3, we introduce the unitary operator $\mathcal{B}^\sigma$ defined by :

$$\mathcal{B}^\sigma = (1 - W^\sigma)^{-\frac{1}{2}}\Big((1-P^\sigma)(1-P^\infty) + P^\sigma P^\infty\Big),$$

with $W^\sigma = (P^\sigma - P^\infty)^2$, and we remark that $\tilde{Q}^\sigma = P^\infty[\mathcal{B}^\sigma]^{-1}\mathcal{H}^\sigma\mathcal{B}^\sigma P^\infty$ has the same eigenvalues than $\mathcal{H}^\sigma$. On the other hand, with algebraic arguments, we see that it suffices to perform the asymptotic expansion of the resolvent for $f \in E^\infty$ that is $f = 0$ on Ω and $f \in \mathcal{C}^\infty(\mathcal{U}) \cap H_0^1(\mathcal{U})$. This is possible at any order in an algebra and we can compose the asymptotic expansions to obtain that $\tilde{Q}^\sigma$ admit an asymptotic expansion at any order. Since $\tilde{Q}^\sigma$ is self-adjoint, the eigenvalues of $\tilde{Q}^\sigma$ admit then a complete asymptotic expansion.

The article is organized as follows. In Section 2, we prove technical estimates and we introduce the boundary layers tools. The asymptotic expansions of the resolvent are given in Section 3. We conclude the proof of Theorem 1.2 in Section 4. The last section is devoted to the proof of Theorem 1.3.

2 Preliminaries

2.1 Boundary layers tools

We denote $\varphi : \Omega \longrightarrow \mathbb{R}^+$ the distance from x to $\partial\Omega$. The open set Ω being smooth, there exists $\Omega_1 \subset \Omega$ a neighborhood of the boundary $\partial\Omega$ such that φ is smooth on Ω_1. Then we have $|\nabla\varphi| = 1$ on Ω_1 and $\nabla\varphi = -\nu$ on $\partial\Omega$, where ν is the outward unitary normal on $\partial\Omega$.
We have the following proposition :

Proposition 2.1 *Let $\gamma \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in H^{\frac{1}{2}}(\Omega_1)$. Then there exists an unique $\alpha \in H^{\frac{1}{2}}(\Omega_1)$ such that*

$$\begin{cases} 2(\nabla\varphi, \nabla\alpha) + \Delta\varphi\alpha = f \text{ in } \Omega_1 \\ \\ \alpha = \gamma \text{ on } \partial\Omega. \end{cases} \tag{2.5}$$

Furthermore there exists a constant C such that :

$$\|\alpha\|_{H^{\frac{1}{2}}(\Omega_1)} \leq C\left(\|\gamma\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|f\|_{H^{\frac{1}{2}}(\Omega_1)}\right).$$

Proof. Even if it means reducing Ω_1, we give a parameterization of Ω_1 of the form :

$$\begin{array}{rccl} \Lambda: & \partial\Omega \times [0,\delta] & \longrightarrow & \Omega_1 \\ & (x,s) & \longmapsto & x - s\nu(x) \end{array}$$

We denote $\tilde{\alpha} = \alpha \circ \Lambda$. Equation (2.5) is equivalent to :

$$\begin{cases} 2\dfrac{\partial\tilde{\alpha}}{\partial s} + \Delta\varphi(\Lambda(x,s))\tilde{\alpha} = (f \circ \Lambda)(x,s), \\ \\ \tilde{\alpha}(x,0) = \gamma(x). \end{cases} \tag{2.6}$$

This is a linear differential equation with regular coefficients. Cauchy-Lipschitz Theorem shows that there exists a unique $\tilde{\alpha} \in \mathcal{C}^\infty(0,\delta; H^{\frac{1}{2}}(\partial\Omega))$ satisfying (2.6) and there exists a constant C independent on γ such that :

$$\|\tilde{\alpha}\|_{\mathcal{C}^1(0,\delta;H^{\frac{1}{2}})} \leq C\left(\|\gamma\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|f\|_{H^{\frac{1}{2}}(\Omega_1)}\right).$$

Since $\alpha = \tilde{\alpha} \circ \Lambda^{-1}$, we obtain that $\alpha \in H^{\frac{1}{2}}(\Omega_1)$ and that there exists a constant C such that

$$\|\alpha\|_{H^{\frac{1}{2}}(\Omega_1)} \leq C\left(\|\gamma\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|f\|_{H^{\frac{1}{2}}(\Omega_1)}\right).$$

2.2 Estimates

First we recall in Lemma 2.1 and Lemma 2.2 two classical *a priori* estimates.

Lemma 2.1 *There exists a constant C such that for any $\xi \in \mathcal{C}(\lambda^\infty, \eta)$, for any $u \in H^1(\mathcal{U})$ such that $(-\Delta - \xi)u \in L^2(\mathcal{U})$, we have :*

$$\|u\|_{L^2(\mathcal{U})} \leq C\left(\|u\|_{L^2(\partial\mathcal{U})} + \|(-\Delta - \xi)u\|_{L^2(\mathcal{U})}\right).$$

Proof. Let $f \in L^2(\mathcal{U})$. There exists $w \in H^2(\mathcal{U}) \cap H^1_0(\mathcal{U})$ such that $-\Delta w - \xi w = f$ in $\mathcal{U}$. Moreover, $\|w\|_{H^2(\mathcal{U})} \leq C\|f\|_{L^2(\mathcal{U})}$, where C does not depend on $\xi \in \mathcal{C}(\lambda^\infty, \eta)$ and f.
Then we have :

$$\begin{aligned} \int_U uf &= \int_U u(-\Delta w - \xi w) \\ &= -\int_{\mathcal{U}} \Delta u w - \xi \int_{\mathcal{U}} uw + \int_{\partial\mathcal{U}} u \frac{\partial w}{\partial \nu}. \end{aligned}$$

Hence,

$$\left|\int_{\mathcal{U}} uf\right| \leq \|-\Delta u - \xi u\|_{L^2(\mathcal{U})}\|w\|_{L^2(\mathcal{U})} + \|u\|_{L^2(\partial\mathcal{U})}\|\frac{\partial w}{\partial \nu}\|_{L^2(\mathcal{U})}$$

$$\leq C\left(\|-\Delta u - \xi u\|_{L^2(\mathcal{U})} + \|u\|_{L^2(\partial\mathcal{U})}\right)\|f\|_{L^2(\mathcal{U})}.$$

These estimates are true for all $f \in L^2(\mathcal{U})$, so by a duality argument, we obtain :

$$\|u\|_{L^2(\mathcal{U})} \leq C\left(\|-\Delta u - \xi u\|_{L^2(\mathcal{U})} + \|u\|_{L^2(\partial\mathcal{U})}\right).$$

Remark 2.1 *More general estimates of this type are proved in [13].*

Lemma 2.2 *There exists a constant C such that for all $v \in H^1(\Omega)$,*

$$\|v\|^2_{L^2(\partial\Omega)} \leq C\left(\|v\|^2_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}\right).$$

Sketch of the Proof. We prove the estimate for regular maps and we conclude the proof by a density argument (see [1]).

In the asymptotic expansion of the resolvent, the following *a priori* estimate will be used to estimates remainder terms.

Proposition 2.2 *Let $g \in L^2(\Omega)$, $\gamma \in L^2(\partial\Omega)$ and $\xi \in \mathcal{C}(\lambda^\infty, \eta)$. There exist $\sigma_0 > 0$ and a constant C, independent on g, γ and ξ, such that for all $\sigma \geq \sigma_0$, if a and b are the solution of the following system,*

$$\begin{cases} (-\Delta - \xi)b + \sigma b = g \ \text{ on } \Omega & (i) \\ (-\Delta - \xi)a = 0 \ \text{ on } \mathcal{U} & (ii) \\ a = b \ \text{ and } \dfrac{\partial b}{\partial \nu} = \dfrac{\partial a}{\partial \nu} + \gamma \ \text{ on } \partial\Omega & (iii) \\ a = 0 \ \text{ on } \partial\mathcal{O} & (iv) \end{cases}$$

then we have the following estimate :

$$\int_\Omega |\nabla b|^2 + \int_{\mathcal{U}} |\nabla a|^2 + \sigma\int_\Omega |b|^2 \leq \frac{C}{\sigma}\|g\|^2_{L^2(\Omega)} + \frac{C}{\sqrt{\sigma}}\|\gamma\|^2_{L^2(\partial\Omega)}. \tag{2.7}$$

Proof. We multiply the first equation by $\bar{b}$ and the second by $\bar{a}$. We add up the two equalities and we obtain :

$$\int_\Omega |\nabla b|^2 + \int_{\mathcal{U}} |\nabla a|^2 + \sigma\int_\Omega |b|^2 = \xi\int_\Omega |b|^2 + \xi\int_{\mathcal{U}} |a|^2 + \int_\Omega g\bar{b} + \int_{\partial\Omega}\frac{\partial b}{\partial \nu}\bar{b} - \int_{\partial\Omega}\frac{\partial a}{\partial \nu}\bar{a}.$$

Using (iii), we have :

$$\int_\Omega |\nabla b|^2 + \int_{\mathcal{U}} |\nabla a|^2 + \sigma\int_\Omega |b|^2 \leq |\xi|\int_\Omega |b|^2 + |\xi|\int_{\mathcal{U}} |a|^2 + \|g\|_{L^2(\Omega)}\|b\|_{L^2(\Omega)} + \int_{\partial\Omega}|\gamma||b|.$$

Now, we have the following estimates :

- For σ sufficiently large, $|\xi| \le \frac{\sigma}{8}$.
- $\|g\|_{L^2(\Omega)}\|b\|_{L^2(\Omega)} \le \frac{\sigma}{8}\|b\|^2_{L^2(\Omega)} + \frac{2}{\sigma}\|g\|^2_{L^2(\Omega)}$.
- According to Lemma 2.1,
$$\int_{\mathcal{U}} |a^2| \le C \int_{\partial\Omega} |a^2|,$$
and since $a = b$ on $\partial\Omega$, using Lemma 2.2, since $|\xi| \le \lambda^\infty + \eta$,
$$|\xi| \int_{\mathcal{U}} |a|^2 \le K\|b\|^2_{L^2(\Omega)} + K\|b\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)},$$
and for σ sufficiently large,
$$|\xi| \int_{\mathcal{U}} |a|^2 \le \frac{\sigma}{8}\|b\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla b\|^2_{L^2(\Omega)}.$$
- Finally, for $\sigma \ge \sigma_0$, with σ_0 sufficiently large,
$$\begin{aligned}
\left|\int_{\partial\Omega} \gamma\bar{b}\right| &\le \|\gamma\|_{L^2(\partial\Omega)}\|b\|_{L^2(\partial\Omega)} \\
&\le C\|\gamma\|_{L^2(\partial\Omega)}\|b\|_{L^2(\Omega)} + C\|\gamma\|_{L^2(\partial\Omega)}\|b\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla b\|^{\frac{1}{2}}_{L^2(\Omega)} \\
&\le \frac{\sigma}{8}\|b\|^2_{L^2(\Omega)} + \frac{1}{4}\|\nabla b\|^2_{L^2(\Omega)} + \frac{K}{\sqrt{\sigma}}\|\gamma\|^2_{L^2(\partial\Omega)}.
\end{aligned}$$

Using these estimates, we obtain the claimed result.

3 Asymptotic expansions of the resolvent

We fix $f \in L^2(\mathcal{O})$ and we denote u^σ (resp v^σ) the restriction of $R^\sigma(\xi)(f)$ on $\mathcal{U}$ (resp Ω). The couple (u^σ, v^σ) satisfies the following system :

$$\left\{\begin{array}{lr}
-\Delta u^\sigma - \xi u^\sigma = f \quad \text{on } \mathcal{U} & (1) \\
-\Delta v^\sigma + (\sigma - \xi)v^\sigma = f \quad \text{on } \Omega & (2) \\
u^\sigma = v^\sigma \quad \text{on } \partial\Omega & (3) \\
\dfrac{\partial u^\sigma}{\partial \nu} = \dfrac{\partial v^\sigma}{\partial \nu} \quad \text{on } \partial\Omega & (4) \\
u^\sigma = 0 \quad \text{on } \partial\mathcal{O} & (5)
\end{array}\right. \tag{3.8}$$

We will seek asymptotic expansion of u^σ and v^σ of the form :

$$u^\sigma(x) = U^0(x) + \frac{1}{\sqrt{\sigma}}U^1(x) + \frac{1}{\sigma}U^2(x) + \dots$$

$$v^\sigma(x) = V^0(x, \sqrt{\sigma}\varphi(x)) + \frac{1}{\sqrt{\sigma}}V^1(x, \sqrt{\sigma}\varphi(x)) + \frac{1}{\sigma}V^2(x, \sqrt{\sigma}\varphi(x)) + \dots$$

where the profiles $V^i(x,z)$ can be decomposed on the following way :

$$V^i(x,z) = \widetilde{V}^i(x,z) + \overline{V}^i(x),$$

where $\tilde{V}^i$ and their derivatives tend to zero when $z \longrightarrow +\infty$.
We formally replace u^σ and v^σ in (3.8) by their profile and we say that each order of the asymptotic expansion is zero.

3.1 Order 0

Proposition 3.1 *Let λ^∞ and η as in Theorem 1.2. Then for all $\xi \in \mathcal{C}(\lambda^\infty, \eta)$, $R^\sigma(\xi)$ satisfies the asymptotic expansion :*

$$R^\sigma(\xi) = R^\infty(\xi) + K_1^\sigma(\xi),$$

where $K_1^\sigma(\xi)$ is an operator defined on $L^2(\mathcal{O})$. Furthermore, there exists a constant C such that :

$$\forall\, \xi \in \mathcal{C}(\lambda^\infty,\eta),\ \forall\, \sigma > \sigma_0,\ \forall\, f \in L^2(\mathcal{O}),\ \left\{ \begin{array}{l} \|K_1^\sigma(\xi)(f)\|_{H^1(\mathcal{O})} \le \dfrac{C}{\sigma^{\frac{1}{4}}}\|f\|_{L^2(\mathcal{O})} \\ \\ \|K_1^\sigma(\xi)(f)\|_{L^2(\mathcal{O})} \le \dfrac{C}{\sqrt{\sigma}}\|f\|_{L^2(\mathcal{O})}. \end{array} \right.$$

Proof. Let $f \in L^2(\mathcal{O})$. We denote u^σ (resp. v^σ) the restriction of $R^\sigma(\xi)(f)$ on $\mathcal{U}$ (resp. Ω). We know that u^σ and v^σ satisfy (3.8). We write

$$\left\{ \begin{array}{l} u^\sigma(x) = U^0(x) + a^\sigma(x), \\ \\ v^\sigma(x) = b^\sigma(x), \end{array} \right.$$

where U_0 satisfies :

$$\left\{ \begin{array}{l} U^0 = 0 \text{ on } \partial\Omega, \\ \\ U^0 = 0 \text{ on } \partial\mathcal{O}, \\ \\ (-\Delta - \xi)U^0 = f \text{ on } \mathcal{U}, \end{array} \right.$$

that is

$$U^0 = (\widetilde{R}^\infty(\xi) \circ r)(f).$$

The remainder terms satisfy the following system :

$$\left\{ \begin{array}{l} (-\Delta - \xi)a^\sigma = 0 \text{ in } \mathcal{U}, \\ \\ (-\Delta - \xi)b^\sigma + \sigma b^\sigma = f \text{ on } \Omega, \\ \\ a^\sigma = b^\sigma \text{ on } \partial\Omega, \\ \\ \dfrac{\partial U^0}{\partial \nu} + \dfrac{\partial a^\sigma}{\partial \nu} = \dfrac{\partial b^\sigma}{\partial \nu} \text{ on } \partial\Omega. \end{array} \right.$$

We remark that there exist a constant C independent on ξ and f such that :

$$\|U^0\|_{H^2(\mathcal{U})} \leq C\|f\|_{L^2(\mathcal{U})}.$$

We apply proposition 2.2 and we obtain that there exists a constant C which does not depend on ξ and f such that :

$$\|\nabla a^\sigma\|^2_{L^2(\mathcal{U})} + \|\nabla b^\sigma\|^2_{L^2(\Omega)} + \sigma\|b^\sigma\|^2_{L^2(\Omega)} \leq C\frac{1}{\sigma}\|f\|^2_{L^2(\Omega)} + C\frac{1}{\sqrt{\sigma}}\|\frac{\partial U^0}{\partial \nu}\|^2_{L^2(\partial\Omega)},$$

thus

$$\|\nabla a^\sigma\|^2_{L^2(\mathcal{U})} + \|\nabla b^\sigma\|^2_{L^2(\Omega)} + \sigma\|b^\sigma\|^2_{L^2(\Omega)} \leq C\frac{1}{\sqrt{\sigma}}\|f\|^2_{L^2(\mathcal{O})}. \tag{3.9}$$

Now we have :

$$K_1^\sigma(\xi)(f)(x) = \begin{cases} a^\sigma(x) \text{ if } x \in \mathcal{U}, \\ b^\sigma(x) \text{ if } x \in \Omega, \end{cases}$$

and Estimate (3.9) give that :

$$\|\nabla K_1^\sigma(\xi)(f)\|_{L^2(\mathcal{O})} \leq \frac{C}{\sigma^{\frac{1}{4}}}\|f\|_{L^2(\mathcal{O})}.$$

With estimate (3.9) we have that $\|b^\sigma\|_{L^2(\Omega)} \leq \dfrac{C}{\sigma^{\frac{3}{4}}}\|f\|_{L^2(\mathcal{O})}$ and using Lemma 2.1 and Lemma 2.2, we can estimate $\|a^\sigma\|_{L^2(\mathcal{U})}$ and we obtain :

$$\begin{aligned}
\|a^\sigma\|_{L^2(\mathcal{U})} &\leq C\|a^\sigma\|_{L^2(\partial\Omega)} \\
&\leq C\|b^\sigma\|_{L^2(\partial\Omega)} \\
&\leq C\left(\|b^\sigma\|_{L^2(\Omega)} + \|b^\sigma\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla b^\sigma\|^{\frac{1}{2}}_{L^2(]]\Omega)}\right) \\
&\leq \frac{C}{\sqrt{\sigma}}\|f\|_{L^2(\mathcal{O})}.
\end{aligned}$$

Hence we have :

$$\|K_1^\sigma(\xi)(f)\|_{L^2(\mathcal{O})} \leq \frac{C}{\sqrt{\sigma}}\|f\|_{L^2(\mathcal{O})}.$$

This completes the proof of Proposition 3.1.

3.2 First order asymptotic expansion

First step. Formal asymptotic expansion
With the usual notations we seek formal first order asymptotic expansion on the form :

$$\begin{cases} u^\sigma(x) = U^0(x) + \dfrac{1}{\sqrt{\sigma}}U^1(x) + \ldots \\ v^\sigma(x) = V^0(x, \sqrt{\sigma}\varphi(x)) + \dfrac{1}{\sqrt{\sigma}}V^1(x, \sqrt{\sigma}\varphi(x)) + \ldots \end{cases}$$

We recall that u^σ and v^σ satisfy (3.8). We will replace u^σ and v^σ by their asymptotic expansion in (3.8) and we will identify the different powers of σ.

We will use the following notations :

- $x = (x_1, \ldots, x_n)$ are the coordinates in $I\!R^n$,
- $V_z = \dfrac{\partial V}{\partial z}$ and $V_{zz} = \dfrac{\partial^2 V}{\partial z^2}$,
- $\nabla V = \begin{pmatrix} \dfrac{\partial V}{\partial x_1} \\ \cdot \\ \cdot \\ \dfrac{\partial V}{\partial x_n} \end{pmatrix}$ and $\nabla V_z = \begin{pmatrix} \dfrac{\partial^2 V}{\partial x_1 \partial z} \\ \cdot \\ \cdot \\ \dfrac{\partial^2 V}{\partial x_n \partial z} \end{pmatrix}$,
- $\Delta V = \dfrac{\partial^2 V}{\partial x_1^2} + \ldots + \dfrac{\partial^2 V}{\partial x_n^2}$.

We remark that if $w(x) = V(x, \sqrt{\sigma}\varphi(x))$,

$$\nabla w(x) = \nabla V(x, \sqrt{\sigma}\varphi(x)) + \sqrt{\sigma}\nabla\varphi(x) V_z(x, \sqrt{\sigma}\varphi(x)),$$

and

$$\begin{aligned} \Delta w(x) = \; & \sigma |\nabla\varphi|^2 V_{zz}(x, \sqrt{\sigma}\varphi(x)) \\ & + \sqrt{\sigma}\Big(2(\nabla\varphi(x), \nabla V_z(x, \sqrt{\sigma}\varphi(x))) + \; \Delta\varphi(x) V_z(x, \sqrt{\sigma}\varphi(x)) \Big) \\ & + \Delta V(x, \sqrt{\sigma}\varphi(x)). \end{aligned}$$

Using that $|\nabla\varphi| = 1$ we formally obtain the following equations :

$$\begin{array}{ll}
\text{from (3.8.1)}, & \left\{ \begin{array}{ll} \text{(i)} & (-\Delta - \xi) U^0 = f \text{ in } \mathcal{U}, \\ \text{(ii)} & (-\Delta - \xi) U^1 = 0 \text{ in } \mathcal{U}, \end{array} \right. \\
\text{from (3.8.2)}, & \left\{ \begin{array}{ll} \text{(iii)} & V^0 - V^0_{zz} = 0 \text{ in } \Omega, \\ \text{(iv)} & V^1 - V^1_{zz} - \Big[(2\nabla\varphi, \nabla V^0_z) + \Delta\varphi V^0_z \Big] = 0 \text{ in } \Omega, \end{array} \right. \\
\text{from (3.8.3)}, & \left\{ \begin{array}{ll} \text{(v)} & U^0 = V^0 \text{ in } \partial\Omega, \\ \text{(vi)} & U^1 = V^1 \text{ in } \partial\Omega, \end{array} \right. \\
\text{from (3.8.4)}, & \left\{ \begin{array}{ll} \text{(vii)} & V^0_z = 0 \text{ in } \partial\Omega, \\ \text{(viii)} & \dfrac{\partial U^0}{\partial \nu} = -V^1_z \text{ in } \partial\Omega, \end{array} \right. \\
\text{from (3.8.5)}, & \left\{ \begin{array}{ll} \text{(ix)} & U^0 = 0 \text{ in } \partial\mathcal{O}, \\ \text{(x)} & U^1 = 0 \text{ in } \partial\mathcal{O}. \end{array} \right.
\end{array} \tag{3.10}$$

Using the equations (3.10), we obtain :

- from (iii), we have $V^0(x,z) = \alpha_0(x)e^{-z}$.
- from (vii), $\alpha_0(x) = 0$ on $\partial\Omega$.
- in (iv) we arbitrary cut the equation in two terms and we say that $V^1 - V^1_{zz} = 0$, i.e. $V^1(x,z) = \alpha_1(x)e^{-z}$, and we have $(2\nabla\varphi, \nabla\alpha_0) + \Delta\varphi\alpha_0 = 0$ in Ω.

  Using Proposition 2.1 we obtain that :

$$\alpha_0 = 0 \text{ on } \Omega.$$

- Using (i) and (ix), since (v) implies that $U^0 = 0$ on $\partial\Omega$, we have :

$$\begin{cases} (-\Delta - \xi)U^0 = f \text{ on } \mathcal{U}, \\ U^0 = 0 \text{ on } \partial\mathcal{U}, \end{cases} \tag{3.11}$$

  that is

$$U^0 = (\widetilde{R}^\infty(\xi) \circ r)(f).$$

- Using (viii) and (vi) we have :

$$\alpha_1 = U^1 = \frac{\partial U^0}{\partial\nu} \text{ on } \partial\Omega,$$

  thus U^1 is uniquely determined by :

$$\begin{cases} (-\Delta - \xi)U^1 = 0 \text{ on } \mathcal{U}, \\ U^1 = 0 \text{ on } \partial\mathcal{O}, \\ U^1 = \dfrac{\partial U^0}{\partial\nu} \text{ on } \partial\Omega. \end{cases} \tag{3.12}$$

- Finally, in order to define α_1, we will use a classical trace relevement from $\partial\Omega$ to Ω denoted by $\mathcal{E}_\Omega$ defined on $H^{\frac{3}{2}}(\partial\Omega)$ with values on $H^2(\Omega)$, and we set

$$\alpha_1 = \mathcal{E}_\Omega(\frac{\partial U^0}{\partial\nu}).$$

In short, if we fix $f \in H^1(\mathcal{O})$, we define $R^\infty(\xi)(f) = e(U^0)$ by (3.11) and we define $R_1^\sigma(\xi)(f)$ by :

$$R_1^\sigma(\xi)(f)(x) = \begin{cases} \psi(x)\alpha_1(x)e^{-\sqrt{\sigma}\varphi(x)} \text{ if } x \in \Omega, \\ U^1(x) \text{ if } x \in \mathcal{U}, \end{cases} \tag{3.13}$$

where U_1 satisfies (3.12), α_1 is defined by

$$\alpha_1 = \mathcal{E}_\Omega(U^1) \text{ on } \Omega, \tag{3.14}$$

and where ψ is a cut-off function with support in $\overline{\Omega_1}$ and equal to 1 in a neighborhood of $\partial\Omega$. The cut-off function ψ is used to avoid the problems of non regularity of φ far from $\partial\Omega$.

The main properties of the operator $R_1^\sigma(\xi)$ are listed in the following lemma.

Lemma 3.1 *The operator $R_1^\sigma(\xi)$ satisfies :*

1. *$R_1^\sigma(\xi)(f)$ only depends on rf, the restriction of f to $\mathcal{U}$.*
2. *$rR_1^\sigma(\xi)$ does not depend on σ, and there exists a constant C such that for all σ,*
$$\forall\ f \in H^1(\mathcal{O}),\quad \|(rR_1^\sigma(\xi))(f)\|_{H^2(\mathcal{U})} \leq C\|rf\|_{H^1(\mathcal{O})}. \tag{3.15}$$
3. *There exists a constant C such that :*
$$\forall\ f \in H^1(\mathcal{O}),\quad \|R_1^\sigma(\xi)(f)\|_{L^2(\Omega)} \leq \frac{C}{\sigma^{\frac{1}{4}}}\|rf\|_{H^1(\mathcal{U})},$$
$$\forall\ f \in H^1(\mathcal{O}),\quad \|R_1^\sigma(\xi)(f)\|_{H^1(\Omega)} \leq C\sigma^{\frac{1}{4}}\|rf\|_{H^1(\mathcal{U})},$$
4. *$P^\infty R_1^\sigma(\xi)P^\infty = \dfrac{1}{(\lambda^\infty - \xi)^2}AP^\infty$, where A is a linear map in E^∞. The coefficients of the matrix A in an orthonormal basis $(f_1, \ldots, f_m)$ of E^∞ are given by the formula :*
$$a_{ij} = \int_{\partial\Omega} \frac{\partial f_i}{\partial \nu}\frac{\partial f_j}{\partial \nu}.$$

Proof. Let $f \in H^1(\mathcal{O})$. We define $U^0 \in H^3(\mathcal{U})$ by (3.11). So U_0 only depends on rf. Hence $\dfrac{\partial U^0}{\partial \nu}|_{\partial\Omega} \in H^{\frac{3}{2}}(\partial\Omega)$ only depends on rf and by classical results on elliptic equations we obtain inequality (3.15).
In addition, by property of $\mathcal{E}_\Omega$,

$$\begin{aligned}\|\alpha_1\|_{H^2(\Omega)} \leq&\ \ C\|\frac{\partial U^0}{\partial \nu}\|_{H^{\frac{3}{2}}(\partial\Omega)}\\ \leq&\ \ C\|rf\|_{H^1(\mathcal{U})},\end{aligned}$$

and we have :

$$\begin{aligned}\left\|R_1^\sigma(\xi)\ (f)\right\|_{L^2(\Omega)} \leq&\ \ \|\psi\|_{L^\infty}\|\alpha_1\|_{L^\infty}\|e^{-\sqrt{\sigma}\varphi}\|_{L^2}\\ \leq&\ \ \frac{C}{\sigma^{\frac{1}{4}}}\|rf\|_{H^1(\mathcal{U})}.\end{aligned}$$

Furthermore

$$\begin{aligned}\left\|\nabla(R_1^\sigma(\xi)\ (f))\right\|_{L^2(\Omega)} \leq&\ \ \|\psi\|_{L^\infty}\|\alpha_1\|_{L^\infty}\|\nabla(e^{-\sqrt{\sigma}\varphi})\|_{L^2} + \|\nabla\psi\|_{L^\infty}\|\alpha_1\|_{L^\infty}\|e^{-\sqrt{\sigma}\varphi}\|_{L^2}\\ &+\|\psi\|_{L^\infty}\|\nabla\alpha_1\|_{L^2}\|(e^{-\sqrt{\sigma}\varphi}\|_{L^\infty}\\ \leq&\ \ C\sigma^{\frac{1}{4}}\|rf\|_{H^1(\mathcal{U})}\end{aligned}$$

Remark 3.1 *We use here the Sobolev embedding $H^2(\Omega) \subset L^\infty(\Omega)$ which is valid in dimension two. This is only here that there is a restriction on the dimension.*

Let us prove now the last assertion.
Let $(f_1, \dots, f_m)$ be an orthonormal basis of E^∞. Let us compute $P^\infty R_1^\sigma(\xi) P^\infty(f_j)$.
We note $U^0 \in H_0^1(\mathcal{U})$ the solution of :

$$(-\Delta - \xi) U^0 = f_j \text{ in } \mathcal{U}.$$

We remark that $U^0 = \dfrac{1}{\lambda^\infty - \xi} f_j$.

We consider $U^1 \in H^1(\mathcal{U})$ the solution of :

$$\begin{cases} (-\Delta - \xi) U^1 = 0 \text{ in } \mathcal{U}, \\[2ex] U^1 = \dfrac{\partial U^0}{\partial \nu} \text{ on } \partial\Omega \\[2ex] U^1 = 0 \text{ on } \partial\mathcal{O}. \end{cases}$$

We have :

$$P^\infty R_1^\sigma(\xi) P^\infty(f_j) = \sum_{i=1}^m \left(\int_{\mathcal{U}} U^1 \cdot f_i \right) f_i.$$

Now we have :

$$\begin{aligned} \int_{\mathcal{U}} U^1 f_i = & -\frac{1}{\lambda^\infty} \int_{\mathcal{U}} U^1 \Delta f_i \\ = & -\frac{1}{\lambda^\infty} \int_{\mathcal{U}} \Delta U^1 f_i + \frac{1}{\lambda^\infty} \int_{\partial\Omega} U^1 \frac{\partial f_i}{\partial \nu} \\ = & \frac{\xi}{\lambda^\infty} \int_{\mathcal{U}} U^1 f_i + \frac{1}{\lambda^\infty} \int_{\partial\Omega} \frac{\partial U^0}{\partial \nu} \frac{\partial f_i}{\partial \nu} \end{aligned}$$

Hence we obtain that :

$$\int_{\mathcal{U}} U^1 f_i = \frac{1}{(\lambda^\infty - \xi)^2} \int_{\partial\Omega} \frac{\partial f_j}{\partial \nu} \cdot \frac{\partial f_i}{\partial \nu},$$

thus

$$P^\infty R_1^\sigma(\xi) P^\infty(f_j) = \frac{1}{(\lambda^\infty - \xi)^2} \sum_{i=1}^m \left(\int_{\partial\Omega} \frac{\partial f_j}{\partial \nu} \cdot \frac{\partial f_i}{\partial \nu} \right) f_i.$$

We are now able to prove the following proposition which contains the results of Theorem 1.1 :

Proposition 3.2 *The resolvent $R^\sigma(\xi)$ admits an asymptotic expansion of the form :*

$$R^\sigma(\xi) = R^\infty(\xi) + \frac{1}{\sqrt{\sigma}} R_1^\sigma(\xi) + \frac{1}{\sqrt{\sigma}} K_2^\sigma(\xi), \tag{3.16}$$

where $R_1^\sigma(\xi)$ is defined by (3.11), (3.12) and (3.14).
Furthermore there exists $\sigma_0 > 0$ and there exists a constant C independent on $\xi \in \mathcal{C}(\lambda^\infty, \eta)$, such that :

$$\forall\, \sigma > \sigma_0, \quad \forall\, \xi \in \mathcal{C}(\lambda^\infty, \eta),\ \forall\, f \in H^1(\mathcal{O}), \quad \begin{cases} \|K_2^\sigma(\xi)(f)\|_{H^1(\mathcal{O})} \leq C \|f\|_{H^1(\mathcal{O})}, \\[2ex] \|K_2^\sigma(\xi)(f)\|_{L^2(\mathcal{O})} \leq \dfrac{C}{\sigma^{\frac{1}{4}}} \|f\|_{H^1(\mathcal{O})}. \end{cases}$$

Proof. We fix $f \in H^1(\mathcal{O})$, we define U^0, U^1 and α_1 like in (3.11), (3.12), (3.14), and we set :

$$\left\{\begin{array}{l} u^\sigma(x) = U^0(x) + \dfrac{1}{\sqrt{\sigma}}U^1(x) + \dfrac{1}{\sqrt{\sigma}}a^\sigma(x) \text{ if } x \in \mathcal{U}, \\ \\ v^\sigma(x) = \dfrac{1}{\sqrt{\sigma}}\psi(x)\alpha_1(x)e^{-\sqrt{\sigma}\varphi(x)} + \dfrac{1}{\sqrt{\sigma}}b^\sigma(x) \text{ if } x \in \Omega. \end{array}\right.$$

The remainder terms a^σ and b^σ satisfy the following system :

$$\left\{\begin{array}{ll} \text{(i)} & (-\Delta - \xi)a^\sigma = 0 \text{ in } \mathcal{U}, \\ \\ \text{(ii)} & a^\sigma = b^\sigma \text{ in } \partial\Omega, \\ \\ \text{(iii)} & \dfrac{\partial a^\sigma}{\partial \nu} + \dfrac{\partial U^1}{\partial \nu} = \dfrac{\partial b^\sigma}{\partial \nu} + \dfrac{\partial \alpha_1}{\partial \nu} \text{ in } \partial\Omega, \\ \\ \text{(iv)} & (-\Delta - \xi)b^\sigma + \sigma b^\sigma = g^\sigma \text{ in } \Omega, \\ \\ \text{(v)} & a^\sigma = 0 \text{ on } \partial\mathcal{O}, \end{array}\right. \tag{3.17}$$

where

$$\begin{aligned} g^\sigma = \; & \sqrt{\sigma}f + \sqrt{\sigma}\psi\left(2(\nabla\varphi, \nabla\alpha^1) + \Delta\varphi\alpha_1\right)e^{-\sqrt{\sigma}\varphi} \\ & + \left(\psi\Delta\alpha_1 + \xi\psi\alpha_1 + \Delta\psi\alpha_1 + 2(\nabla\psi, \nabla\alpha_1) + 2\alpha_1(\nabla\psi, \nabla\varphi)\right)e^{-\sqrt{\sigma}\varphi}. \end{aligned}$$

We have the following estimates :

- $\|\dfrac{\partial \alpha_1}{\partial \nu}\|_{L^2(\partial\Omega)} \leq C\|\alpha_1\|_{H^2(\Omega)} \leq K\|\alpha_1\|_{H^{\frac{3}{2}}(\partial\Omega)} \leq K\|U_0\|_{H^3(\mathcal{U})}$, and we obtain :
$$\|\frac{\partial \alpha_1}{\partial \nu}\|_{L^2(\partial\Omega)} \leq K\|f\|_{H^1(\mathcal{U})}.$$
- $\|\dfrac{\partial U^1}{\partial \nu}\|_{L^2(\partial\Omega)} \leq C\|U^1\|_{H^2(\mathcal{U})} \leq C\|U^1\|_{H^{\frac{3}{2}}(\partial\Omega)} \leq C\|\dfrac{\partial U^0}{\partial \nu}\|_{H^{\frac{3}{2}}(\partial\Omega)}$ hence
$$\|\frac{\partial U^1}{\partial \nu}\|_{L^2(\partial\Omega)} \leq K\|f\|_{H^1(\mathcal{U})}.$$
- $\|g^\sigma\|_{L^2(\Omega)} \leq \sqrt{\sigma}\|f\|_{L^2(\Omega)} + K\sqrt{\sigma}\|2(\nabla\varphi, \nabla\alpha_1) + \Delta\varphi\alpha_1)\|_{L^2(\Omega)}\|e^{-\sqrt{\sigma}\varphi}\|_{L^\infty(\Omega)} + K\|\alpha_1\|_{H^2(\Omega)}$ so
$$\|g^\sigma\|_{L^2(\Omega)} \leq K\sqrt{\sigma}\|f\|_{L^2(\Omega)} + K\sqrt{\sigma}\|f\|_{H^1(\mathcal{U})}.$$

We apply Proposition 2.2 and we obtain that :

$$\|\nabla a^\sigma\|^2_{L^2(\mathcal{U})} + \|\nabla b^\sigma\|^2_{L^2(\Omega)} + \sigma\|b^\sigma\|^2_{L^2(\Omega)} \leq K\left(\|f\|^2_{L^2(\Omega)} + \|f\|^2_{H^1(\mathcal{U})}\right).$$

Furthermore, applying Lemma 2.1 and Lemma 2.2, since $a^\sigma = b^\sigma$ on $\partial\Omega$, we obtain that :

$$\|a^\sigma\|_{L^2(\mathcal{U})} \leq \frac{K}{\sigma^{\frac{1}{4}}}\left(\|f\|^2_{L^2(\Omega)} + \|f\|^2_{H^1(\mathcal{U})}\right).$$

So the proof of Proposition 3.2 is complete.

4 Proof of Theorem 1.2

Proposition 4.1 *The projector P^σ admits an asymptotic expansion of the form :*

$$P^\sigma = P^\infty + \frac{1}{\sqrt{\sigma}} P^{1,\sigma} + \frac{1}{\sqrt{\sigma}} M^\sigma,$$

where $P^{1,\sigma}$ is an linear operator defined on $H^1(\mathcal{O})$. Furthermore, there exist a constant C and σ_0 such that for all $\sigma > \sigma_0$ and for all $f \in H^1(\mathcal{O})$,

$$\begin{cases} \|rP^{1,\sigma}(f)\|_{H^1(\mathcal{U})} \leq C\|rf\|_{H^1(\mathcal{U})} \\ \\ \|P^{1,\sigma}(f)\|_{H^1(\Omega)} \leq C\sigma^{\frac{1}{4}}\|rf\|_{H^1(\mathcal{U})} \\ \\ \|P^{1,\sigma}(f)\|_{L^2(\Omega)} \leq \dfrac{C}{\sigma^{\frac{1}{4}}}\|rf\|_{H^1(\mathcal{U})} \end{cases}$$

and

$$\begin{cases} \|M^\sigma(f)\|_{H^1(\mathcal{O})} \leq C\|f\|_{H^1(\mathcal{O})} \\ \\ \|M^\sigma(f)\|_{L^2(\mathcal{O})} \leq \dfrac{C}{\sigma^{\frac{1}{4}}}\|f\|_{H^1(\mathcal{U})}. \end{cases}$$

Furthermore, $rP^{1,\sigma}$ is independent on σ and $P^\infty P^{1,\sigma} P^\infty = 0$.

Proof. The operators $\mathcal{H}^\sigma$ and $\mathcal{H}^\infty$ are self-adjoint thus the projections P^σ and P^∞ are given by the relations (1.3) and (1.4).
Therefore, according to Lemma 3.1 and Proposition 3.2, if we set

$$P^{1,\sigma} = \frac{-1}{2i\pi}\int_{\mathcal{C}(\lambda^\infty,\eta)} R^{1,\sigma}(\xi)d\xi$$

and

$$M^\sigma = \frac{-1}{2i\pi}\int_{\mathcal{C}(\lambda^\infty,\eta)} K^{2,\sigma}(\xi)d\xi,$$

the first assertion and the estimates are straightforward.
The second assertion is a consequence of Lemma 3.1 since

$$\int_{\mathcal{C}(\lambda^\infty,\eta)} (\lambda^\infty - \xi)^{-2}d\xi = 0.$$

Proof of Theorem 1.2.
Using Kato's method (see [12]), we introduce the operator $\mathcal{A}^\sigma$:

$$\mathcal{A}^\sigma = I - P^\infty + P^\sigma P^\infty.$$

This operator maps E^∞ into E^σ and leaves the orthogonal of E^∞ invariant.
Using Proposition 4.1, we obtain an asymptotic expansion of $\mathcal{A}^\sigma$:

$$\mathcal{A}^\sigma = I + \frac{1}{\sqrt{\sigma}} P^{1,\sigma}P^\infty + \frac{1}{\sqrt{\sigma}} M^\sigma P^\infty, \tag{4.18}$$

We remark that there exists a constant C such that :

$$\forall\ \sigma > \sigma_0,\ \forall\ f \in H^1(\mathcal{O}),\ \left\{ \begin{array}{l} \left\| \dfrac{1}{\sqrt{\sigma}} P^{1,\sigma} P^\infty(f) \right\|_{H^1(\mathcal{O})} \leq \dfrac{C}{\sigma^{\frac{1}{4}}} \|f\|_{H^1(\mathcal{O})} \\ \left\| \dfrac{1}{\sqrt{\sigma}} M^\sigma P^\infty(f) \right\|_{H^1(\mathcal{O})} \leq \dfrac{C}{\sqrt{\sigma}} \|f\|_{H^1(\mathcal{O})}. \end{array} \right. \tag{4.19}$$

So $\mathcal{A}^\sigma$ is invertible in $\mathcal{L}(H^1(\Omega); H^1(\Omega))$ and

$$[\mathcal{A}^\sigma]^{-1} = Id - \frac{1}{\sqrt{\sigma}} P^{1,\sigma} P^\infty - \frac{1}{\sqrt{\sigma}} M^\sigma P^\infty + \sum_{n=2}^{+\infty} \frac{(-1)^n}{\sigma^{\frac{n}{2}}} \left[P^{1,\sigma} P^\infty +\ M^\sigma P^\infty \right]^n.$$

Using assertion (4.19) we remark that :

$$\left\| \sum_{n=3}^{+\infty} \frac{(-1)^n}{\sigma^{\frac{n}{2}}} \left[P^{1,\sigma} P^\infty +\ M^\sigma P^\infty \right]^n \right\|_{\mathcal{L}(H^1(\mathcal{O}))} \leq \frac{K}{\sigma^{\frac{3}{4}}}.$$

Furthermore, using that $P^\infty P^{1,\sigma} P^\infty = 0$, we obtain that

$$\begin{aligned} P^\infty [\mathcal{A}^\sigma]^{-1} = \quad & P^\infty - \tfrac{1}{\sqrt{\sigma}} P^\infty M^\sigma P^\infty + \tfrac{1}{\sigma} P^\infty M^\sigma P^\infty M^\sigma P^\infty \\ & + \sum_{n=3}^{+\infty} \frac{(-1)^n}{\sigma^{\frac{n}{2}}} \left[P^{1,\sigma} P^\infty +\ M^\sigma P^\infty \right]^n. \end{aligned}$$

Now since P^∞ is a regularizing operator,

$$\|P^\infty M^\sigma P^\infty\|_{\mathcal{L}(H^1; H^1)} \leq C \|M^\sigma P^\infty\|_{\mathcal{L}(H^1; L^2)} \leq \frac{K}{\sigma^{\frac{1}{4}}}.$$

So we have :

$$P^\infty (\mathcal{A}^\sigma)^{-1} = P^\infty + \frac{1}{\sqrt{\sigma}} N^\sigma, \tag{4.20}$$

with

$$\|N^\sigma\|_{\mathcal{L}(H^1)} \leq \frac{K}{\sigma^{\frac{1}{4}}}.$$

As a consequence, the eigenvalues of $\mathcal{H}^\sigma$ contained in $B(\lambda^\infty, \eta)$ are the eigenvalues of the operator Q^σ defined on E^∞ by :

$$Q^\sigma = P^\infty (\mathcal{A}^\sigma)^{-1} \mathcal{H}^\sigma P^\sigma \mathcal{A}^\sigma P^\infty.$$

Since we have the relation :

$$H^\sigma P^\sigma = -\frac{1}{2i\pi} \int_{\mathcal{C}(\lambda^\infty, \eta)} \xi R^\sigma(\xi) d\xi, \tag{4.21}$$

we are lead to study the asymptotic expansion of $P^\infty (\mathcal{A}^\sigma)^{-1} R^\sigma P^\sigma \mathcal{A}^\sigma P^\infty$.

Using Proposition 3.2, and equations (4.18) and (4.20) we have :

$$\begin{aligned} P^\infty (\mathcal{A}^\sigma)^{-1} R^\sigma P^\sigma \mathcal{A}^\sigma P^\infty = \quad & \left(P^\infty + \frac{1}{\sqrt{\sigma}} N^\sigma \right) \left(R^\infty(\xi) + \frac{1}{\sqrt{\sigma}} R_1^\sigma(\xi) + \frac{1}{\sqrt{\sigma}} K_2^\sigma \right) \\ & \times \left(P^\infty + \tfrac{1}{\sqrt{\sigma}} P^{1,\sigma} P^\infty + \tfrac{1}{\sqrt{\sigma}} M^\sigma P^\infty \right), \end{aligned}$$

i.e.

$$P^\infty(\mathcal{A}^\sigma)^{-1}R^\sigma P^\sigma \mathcal{A}^\sigma P^\infty = T_1 + \dots T_6$$

with

$$T_1 = P^\infty R^\infty(\xi)P^\infty + \frac{1}{\sqrt{\sigma}}P^\infty R^\infty(\xi)P^{1,\sigma}P^\infty + \frac{1}{\sqrt{\sigma}}P^\infty R^\infty(\xi)M^\sigma P^\infty,$$

$$T_2 = \frac{1}{\sqrt{\sigma}}P^\infty R_1^\sigma(\xi)P^\infty + \frac{1}{\sigma}P^\infty R_1^\sigma(\xi)P^{1,\sigma}P^\infty + \frac{1}{\sigma}P^\infty R_1^\sigma(\xi)M^\sigma P^\infty,$$

$$T_3 = \frac{1}{\sqrt{\sigma}}P^\infty K_2^\sigma(\xi)P^\infty + \frac{1}{\sigma}P^\infty K_2^\sigma(\xi)P^{1,\sigma}P^\infty + \frac{1}{\sigma}P^\infty K_2^\sigma(\xi)M^\sigma P^\infty,$$

$$T_4 = \frac{1}{\sqrt{\sigma}}N^\sigma R^\infty(\xi)P^\infty + \frac{1}{\sigma}N^\sigma R^\infty(\xi)P^{1,\sigma}P^\infty + \frac{1}{\sigma}N^\sigma R^\infty(\xi)M^\sigma P^\infty,$$

$$T_5 = \frac{1}{\sigma}N^\sigma R_1^\sigma(\xi)P^\infty + \frac{1}{\sigma^{\frac{3}{2}}}N^\sigma R_1^\sigma(\xi)P^{1,\sigma}P^\infty + \frac{1}{\sigma^{\frac{3}{2}}}N^\sigma R_1^\sigma(\xi)M^\sigma P^\infty,$$

$$T_6 = \frac{1}{\sigma}N^\sigma K_2^\sigma(\xi)P^\infty + \frac{1}{\sigma^{\frac{3}{2}}}N^\sigma K_2^\sigma(\xi)P^{1,\sigma}P^\infty + \frac{1}{\sigma^{\frac{3}{2}}}N^\sigma K_2^\sigma(\xi)M^\sigma P^\infty.$$

Since P^∞ is a regularizing operator, we remark that

$$\|P^\infty K_2^\sigma(\xi)\|_{\mathcal{L}(H^1;H^1)} \le \|P^\infty\|_{\mathcal{L}(L^2;H^1)}\|K_2^\sigma(\xi)\|_{\mathcal{L}(H^1;L^2)} \le \frac{C}{\sigma^{\frac{1}{4}}}.$$

Using Lemma 3.1, Proposition 3.2 and Proposition 4.1, we estimate the terms $T_1, \dots, T_6$. We recall that $P^\infty P^{1,\sigma}P^\infty = 0$.

- $P^\infty R^\infty(\xi) = \dfrac{1}{\lambda^\infty - \xi}P^\infty$. Thus $T_1 = \dfrac{1}{\lambda^\infty - \xi}P^\infty + \dfrac{1}{\lambda^\infty - \xi}\tau_1^\sigma$ with $\tau_1^\sigma = \dfrac{1}{\sqrt{\sigma}}P^\infty M^\sigma P^\infty$, so

$$\|\tau_1^\sigma\|_{\mathcal{L}(H^1)} \le \frac{K}{\sigma^{\frac{3}{4}}}.$$

- $T_2 = \dfrac{1}{\sqrt{\sigma}}\dfrac{1}{(\lambda^\infty - \xi)^2}AP^\infty + \tau_2^\sigma$ where $\tau_2^\sigma = \dfrac{1}{\sigma}\Big(P^\infty R_1^\sigma(\xi)P^{1,\sigma}P^\infty + P^\infty R_1^\sigma(\xi)M^\sigma P^\infty\Big)$, and we have :

$$\|\tau_2^\sigma\|_{\mathcal{L}(H^1)} \le \frac{C}{\sigma}.$$

- $\|T_3\|_{\mathcal{L}(H^1)} \le \dfrac{1}{\sqrt{\sigma}}\|K_2^\sigma(\xi)\|_{\mathcal{L}(H^1;L^2)} + \dfrac{1}{\sigma}\|P^\infty K_2^\sigma(\xi)\|_{\mathcal{L}(H^1)}\Big(\|P^{1,\sigma}P^\infty\|_{\mathcal{L}(H^1)} + \|M^\sigma P^\infty\|_{\mathcal{L}(H^1)}\Big)$. Since P^∞ is a regularizing operator, and using Proposition 4.1 we obtain that

$$\|T_3\|_{\mathcal{L}(H^1)} \le \frac{K}{\sigma}.$$

In the same way we prove that T_4, T_5 and T_6 are bounded in $\mathcal{L}(H^1)$ by $\dfrac{K}{\sigma^{\frac{3}{4}}}$.
Therefore we obtain that

$$P^\infty(\mathcal{A}^\sigma)^{-1}R^\sigma(\xi)\mathcal{A}^\sigma P^\infty = (\lambda^\infty - \xi)^{-1}P^\infty + \frac{1}{\sqrt{\sigma}}(\lambda^\infty - \xi)^{-2}AP^\infty + o(\sigma^{-\frac{1}{2}}).$$

Hence integrating this asymptotic expansion on $\mathcal{C}(\lambda^\infty, \eta)$, we obtain that :

$$P^\infty(\mathcal{A}^\sigma)^{-1}\mathcal{H}^\sigma P^\sigma \mathcal{A}^\sigma P^\infty = \lambda^\infty P^\infty + \frac{1}{\sqrt{\sigma}} A P^\infty + o(\sigma^{-\frac{1}{2}}).$$

Theorem 1.2 is then a consequence of the well known results in finite dimensional spaces (see [12]).

5 Further order asymptotic

In this section we seek an asymptotic expansion of the eigenvalues of $\mathcal{H}^\sigma$ at any order. The main difficulty comes from the fact that the asymptotic at any order of the resolvent operator is not valid in an algebra. The first order approximation is only in $\mathcal{L}(H^1, L^2)$ and the further order ones are in $\mathcal{L}(H^k, L^2)$ with k sufficiently large. Then we can not compose the asymptotic expansions in the study of $Q^\sigma = P^\infty(\mathcal{A}^\sigma)^{-1} H^\sigma \mathcal{A}^\sigma P^\infty$.
The second difficulty is that Q^σ is not self-adjoint and the existence of an asymptotic expansion of the operator at any order does not imply the existence of the corresponding asymptotic expansion for the eigenvalues.
We consider $W^\sigma = (P^\sigma - P^\infty)^2$. This is a self-adjoint operator which tends to zero as σ tends to $+\infty$, according to Proposition 3.1. Consequently, for σ sufficiently large, we can define :

$$\mathcal{B}^\sigma := (1 - W^\sigma)^{-\frac{1}{2}}\Big((1 - P^\sigma)(1 - P^\infty) + P^\sigma P^\infty\Big). \tag{5.22}$$

Since W^σ commute with P^σ and P^∞, $\mathcal{B}^\sigma$ maps E^∞ in E^σ and $(E^\infty)^\perp$ in $(E^\sigma)^\perp$. Moreover using that $P^\infty P^\infty = P^\infty$ and that $P^\sigma P^\sigma = P^\sigma$, we remark that

$$\mathcal{B}^{\sigma\ T} \mathcal{B}^\sigma = (1 - W^\sigma)^{-\frac{1}{2}}(1 - W^\sigma)(1 - W^\sigma)^{-\frac{1}{2}} = I,$$

that is $\mathcal{B}^\sigma$ is unitary and then $\mathcal{H}^\sigma P^\sigma$ and $\tilde{Q}^\sigma = P^\infty(\mathcal{B}^\sigma)^{-1} H^\sigma \mathcal{B}^\sigma P^\infty$ have the same eigenvalues.
The crucial result to obtain the asymptotic expansion at any order is the following proposition :

Proposition 5.1 *For $\mathcal{B}^\sigma$ defined above, we have:*

$$P^\infty(\mathcal{B}^\sigma)^{-1} H^\sigma \mathcal{B}^\sigma P^\infty = (1 - W^\sigma P^\infty)^{-\frac{1}{2}} P^\infty H^\sigma P^\sigma P^\infty (1 - W^\sigma P^\infty)^{-\frac{1}{2}} P^\infty.$$

Proof. By definition of $\mathcal{B}^\sigma$ and by using that W^σ commute with P^σ and P^∞ we have :

$$\mathcal{B}^\sigma P^\infty = (1 - W^\sigma)^{-\frac{1}{2}} P^\sigma P^\infty = P^\sigma P^\infty (1 - W^\sigma)^{-\frac{1}{2}} P^\infty.$$

In the same way, $P^\infty(\mathcal{B}^\sigma)^{-1} = P^\infty(1 - W^\sigma)^{-\frac{1}{2}} P^\infty P^\sigma$. Furthermore since $W^\sigma P^\infty = P^\infty W^\sigma$ and $P^\infty P^\infty = P^\infty$, we have :

$$(1 - W^\sigma)^{-\frac{1}{2}} P^\infty = (1 - W^\sigma P^\infty)^{-\frac{1}{2}} P^\infty.$$

This complete the proof of the proposition.

Therefore using the relation (4.21) the asymptotic expansion of $P^\infty(\mathcal{B}^\sigma)^{-1} H^\sigma \mathcal{B}^\sigma P^\infty$ will be a consequence of the asymptotic expansion of $(1 - W^\sigma P^\infty)^{-\frac{1}{2}} P^\infty R^\sigma(\xi) P^\infty (1 - W^\sigma P^\infty)^{-\frac{1}{2}} P^\infty$. Moreover since $W^\sigma P^\infty = P^\infty W^\sigma P^\infty = P^\infty - P^\infty P^\sigma P^\infty$, according to relation (1.3), we are reduced to discuss the asymptotic expansion of $P^\infty R^\sigma(\xi) P^\infty$. We remark that if $g \in H^1(\mathcal{O})$ then $P^\infty(g) \in \mathcal{C}^\infty(\mathcal{U})$ and vanishes on Ω, thus it suffices to perform an asymptotic expansion of $R^\sigma(\xi)(f)$ with $f \in H^1_0(\mathcal{U}) \cap \mathcal{C}^\infty(\mathcal{U})$ and $f = 0$ in Ω.

Proposition 5.2 *Let $N \in \mathbb{N}$. Let $f \in H^1(\mathcal{O})$ such that $f = 0$ in Ω and such that the restriction of f in $\mathcal{U}$ is in $\mathcal{C}^\infty(\mathcal{U})$. Then $R^\sigma(\xi)(f)$ admits an asymptotic expansion of the form :*

$$R^\sigma(\xi)(f) = \begin{cases} \displaystyle\sum_{k=0}^{N} \frac{1}{\sigma^{\frac{k}{2}}} U^k(x) \quad + \frac{1}{\sigma^{\frac{N}{2}}} a^\sigma(x) \ \textit{if } x \in \mathcal{U}, \\[2ex] \displaystyle\sum_{k=1}^{N} \frac{1}{\sigma^{\frac{k}{2}}} \psi(x)\alpha_k(x) e^{-\sqrt{\sigma}\varphi(x)} + \frac{1}{\sigma^{\frac{N}{2}}} b^\sigma(x) \ \textit{if } x \in \Omega, \end{cases}$$

and there exists a constant C independent on f and on $\sigma > \sigma_0$ sufficiently large such that the remainder terms a^σ and b^σ satisfy :

$$\begin{cases} \|a^\sigma\|_{L^2(\mathcal{U})} \leq \dfrac{C}{\sigma} \|f\|_{H^{2N}(\mathcal{U})}, \\[2ex] \|\nabla a^\sigma\|_{L^2(\mathcal{U})} \leq \dfrac{C}{\sigma^{\frac{1}{4}}} \|f\|_{H^{2N}(\mathcal{U})}, \\[2ex] \|b^\sigma\|_{L^2(\Omega)} \leq \dfrac{C}{\sigma^{\frac{3}{4}}} \|f\|_{H^{2N}(\mathcal{U})}, \\[2ex] \|\nabla b^\sigma\|_{L^2(\Omega)} \leq \dfrac{C}{\sigma^{\frac{1}{4}}} \|f\|_{H^{2N}(\mathcal{U})}. \end{cases}$$

Proof.

First step. Formal asymptotic expansion.
As in the Section 3, with the usual notations, we seek (u^σ, v^σ) of the form:

$$\begin{cases} u^\sigma(x) = U^0(x) + \dfrac{1}{\sqrt{\sigma}} U^1(x) + \dfrac{1}{\sigma} U^2(x) + \ldots \\[2ex] v^\sigma(x) = V^0(x, \sqrt{\sigma}\varphi(x)) + \dfrac{1}{\sqrt{\sigma}} V^1(x, \sqrt{\sigma}\varphi(x)) + \dfrac{1}{\sigma} V^2(x, \sqrt{\sigma}\varphi(x)) + \ldots \end{cases}$$

We recall that u^σ and v^σ satisfy (3.8). Replacing u^σ and v^σ by their asymptotic expansion in (3.8) and identifying the different powers of $\sqrt{\sigma}$, we obtain :

- In $\mathcal{U}$, for $k = 0$:

  $$\begin{cases} (-\Delta - \xi)U^0 = f \text{ on } \mathcal{U} \\[1ex] U^0 = 0 \text{ on } \partial\mathcal{U} \end{cases}$$

  that is

  $$U^0 = (\widetilde{R}^\infty(\xi) \circ r)(f).$$

- In Ω, for $k \geq 0$:

  $$\begin{cases} (-\Delta - \xi)V^{k-1} - \Big[(2\nabla\varphi, \nabla V_z^k) + \Delta\varphi V_z^k\Big] + (V^{k+1} - V_{zz}^{k+1}) = 0 \text{ in } \Omega \\[2ex] -V_z^k = \dfrac{\partial U^{k-1}}{\partial\nu} - \dfrac{\partial V^{k-1}}{\partial\nu} \text{ on } \partial\Omega \end{cases}$$

  with $V^{-1} = V^0 = 0$ (see Section 3).

- In $\mathcal{U}$, for $k \geq 1$:

$$\begin{cases} (-\Delta - \xi)U^k = 0 \text{ on } \mathcal{U} \\ U^k = V^k \text{ on } \partial\Omega \\ U^k = 0 \text{ on } \partial\mathcal{O} \end{cases}$$

Then we construct functions V^k and U^k, $k \geq 1$, by induction. We set $V^k(x,z) = \alpha_k(x)e^{-z}$ with α_k solution of

$$\begin{cases} (2\nabla\varphi, \nabla\alpha_k) + \Delta\varphi\alpha_k = (\Delta + \xi)\alpha_{k-1} \text{ in } \Omega \\ \alpha_k = \dfrac{\partial U^{k-1}}{\partial\nu} - \dfrac{\partial\alpha_{k-1}}{\partial\nu} \text{ on } \partial\Omega \end{cases} \tag{5.23}$$

and U^k, solution of

$$\begin{cases} (-\Delta - \xi)U^k = 0 \text{ on } \mathcal{U} \\ U^k = \alpha_k \text{ on } \partial\Omega \\ U^k = 0 \text{ on } \partial\mathcal{O} \end{cases} \tag{5.24}$$

where $U^0 = rR^\infty(\xi)(f)$ and $\alpha_0 = 0$.

Second step. Properties of the profiles.
Using (5.23) we remark that if $\alpha_{k-1} \in H^p(\Omega)$ and $U^{k-1} \in H^p(\mathcal{U})$ then, $\alpha_k|_{\partial\Omega} \in H^{p-\frac{3}{2}}(\partial\Omega)$ and $(\Delta + \xi)\alpha_{k-1} \in H^{p-2}(\Omega)$. Hence $\alpha_k \in H^{p-2}(\Omega)$ with Proposition 2.1.
Using classical results on elliptic equations, we obtain now that $U^k \in H^{p-1}(\mathcal{U})$.
A straightforward induction allows us to claim that there exists a constant C independent on f and σ such that :

$$\forall \ k \leq N, \quad \|\alpha_k\|_{H^{2(N-k)+2}(\Omega)} + \|U^k\|_{H^{2(N-k)+2}(\mathcal{U})} \leq C\|f\|_{H^{2N}(\mathcal{U})}. \tag{5.25}$$

Third step. Estimation of the remainder terms.

For any $N \in \mathbb{N}$ we decompose u^σ and v^σ on the form :

$$\begin{cases} u^\sigma(x) = \displaystyle\sum_{k=0}^{N} \sigma^{-\frac{k}{2}} U^k(x) + \sigma^{-\frac{N}{2}} a^\sigma(x) \text{ if } x \in \mathcal{U}, \\ v^\sigma(x) = \displaystyle\sum_{k=0}^{N} \sigma^{-\frac{k}{2}} \psi(x)\alpha_k(x)e^{-\sqrt{\sigma}\varphi(x)} + \sigma^{-\frac{N}{2}} b^\sigma(x) \text{ if } x \in \Omega. \end{cases}$$

The remainder terms a^σ and b^σ satisfy the following system :

$$
\begin{cases}
\text{(i)} & (-\Delta - \xi)a^\sigma = 0 \text{ in } \mathcal{U}, \\
\text{(ii)} & a^\sigma = b^\sigma \text{ in } \partial\Omega, \\
\text{(iii)} & \dfrac{\partial a^\sigma}{\partial \nu} + \dfrac{\partial U^N}{\partial \nu} = \dfrac{\partial b^\sigma}{\partial \nu} + \dfrac{\partial \alpha^N}{\partial \nu} \text{ in } \partial\Omega, \\
\text{(iv)} & (-\Delta - \xi)b^\sigma + \sigma b^\sigma = g^\sigma \text{ in } \Omega, \\
\text{(v)} & a^\sigma = 0 \text{ on } \partial\Omega,
\end{cases}
\tag{5.26}
$$

where

$$
\begin{aligned}
g^\sigma = \; & \sum_{k=0}^{N-1} \sigma^{\frac{N-k}{2}} \left(2(\nabla\psi, \nabla\varphi)\alpha_{k+1} + \Delta\psi\alpha_k + 2(\nabla\psi, \nabla\alpha_k)\right) e^{-\sqrt{\sigma}\varphi} \\
& + \left(2(\nabla\psi, \alpha_N) + \Delta\psi\alpha_N\right) e^{-\sqrt{\sigma}\varphi} + \psi(\Delta + \xi)\alpha_N e^{-\sqrt{\sigma}\varphi},
\end{aligned}
$$

which is bounded in $L^2(\Omega)$ by $C\|f\|_{H^{2N}(\mathcal{U})}$ because $\varphi(x) > \varepsilon > 0$ on the support of the derivative of ψ, and using estimate (5.25).
We apply Proposition 2.2 and we obtain that :

$$
\|\nabla a^\sigma\|^2_{L^2(\mathcal{U})} + \|\nabla b^\sigma\|^2_{L^2(\Omega)} + \sigma\|b^\sigma\|^2_{L^2(\Omega)} \leq \frac{K}{\sqrt{\sigma}}\|f\|_{H^{2N}(\mathcal{U})}.
$$

Furthermore, applying Lemma 2.1, we obtain that :

$$
\|b^\sigma\|_{L^2(\partial\Omega)} \leq \frac{K}{\sigma}\|f\|_{H^{2N}}(\mathcal{U}).
$$

and applying Lemma 2.1 it concludes the proof of Proposition 5.2.

We consider $f \in E^\infty$. We remark that since $P^\infty f = f$ and since P^∞ is a regularizing operator, there exists a constant C such that :

$$
\|f\|_{H^{2N}(\mathcal{U})} = \|P^\infty f\|_{H^{2N}(\mathcal{U})} \leq C\|f\|_{L^2(\mathcal{U})}. \tag{5.27}
$$

We introduce now the profiles $U^0, \ldots, U^N$ defined by (5.24) and we set

$$
A^k(\xi)(f) = P^\infty(U^k).
$$

Using (5.25) and (5.27), since P^∞ is a regularizing operator, we obtain that there exists a constant C independent on f, ξ and $k \in \{1, \ldots, N\}$ such that :

$$
\|A^k(\xi)(f)\|_{L^2(\mathcal{U})} \leq C\|f\|_{L^2(\mathcal{U})}.
$$

Furthermore if we denote $K_N^\sigma(\xi)(f) = P^\infty(a^\sigma)$ where a^σ is defined by (5.26), we obtain the proposition :

Proposition 5.3 *For any* $N \in \mathbb{N}$

$$
P^\infty R^\sigma(\xi) P^\infty = (\lambda^\infty - \xi)^{-1} P^\infty + \sum_{k=1}^{N} \frac{1}{\sigma^{\frac{k}{2}}} A^k(\xi) P^\infty + \frac{1}{\sigma^{\frac{N}{2}}} K_N^\sigma,
$$

with $\|K_N^\sigma\|_{L^2(\mathcal{U})} \leq \frac{C}{\sqrt{\sigma}}$.

Corollary 5.1 *There exist $(P^k)_{k\geq 2} \in \mathcal{L}(E^\infty)^{\mathbb{N}}$ such that for any $N \geq 2$:*

$$P^\infty P^\sigma P^\infty = P^\infty + \sum_{k=2}^{N} \frac{1}{\sigma^{\frac{k}{2}}} P^k P^\infty + \frac{1}{\sigma^{\frac{N}{2}}} P_N^\sigma,$$

with $\|P_N^\sigma\|_{L^2(\mathcal{U})} \leq \frac{C}{\sqrt{\sigma}}$.

Proof. It is a direct consequence of the above Proposition using the relation (1.3). The coefficient of $\sigma^{-\frac{1}{2}}$ vanishes because $\int_C (\lambda^\infty - \xi)^{-2} d\xi = 0$. For $k \geq 2$, $P^k = \frac{-1}{2i\pi} \int_{\mathcal{C}(\lambda^\infty,\eta)} A^k(\xi) d\xi$.

Proof of Theorem 1.3. Let $\mathcal{B}^\sigma$ defined by (5.22). Since $\mathcal{B}^\sigma$ maps E^∞ on E^σ, the eigenvalues of $H^\sigma P^\sigma$ are the ones of $\tilde{Q}^\sigma = P^\infty (\mathcal{B}^\sigma)^{-1} H^\sigma \mathcal{B}^\sigma P^\infty$. This operator is self-adjoint then exploiting the perturbation theory in finite dimension we will deduce the asymptotic expansion of the eigenvalues of $\tilde{Q}^\sigma$ from the one of $\tilde{Q}^\sigma$ itself. According to the proposition 5.1,

$$\tilde{Q}^\sigma = (1 - W^\sigma P^\infty)^{-\frac{1}{2}} P^\infty H^\sigma P^\sigma P^\infty (1 - W^\sigma P^\infty)^{-\frac{1}{2}} P^\infty.$$

Using the relations $W^\sigma P^\infty = P^\infty - P^\infty P^\sigma P^\infty$ and (4.21), from Proposition 5.3 and its Corollary we deduce that $\tilde{Q}^\sigma$ admits an asymptotic expansion at any order with respect to $1/\sqrt{\sigma}$. Since $\tilde{Q}^\sigma$ is a self adjoint operator, using finite dimensional results (see [12]), we obtain that the eigenvalues of $\mathcal{H}^\sigma$ admit an asymptotic expansion at any order. This complete the proof of Theorem 1.3.

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