SEMICLASSICAL RESOLVENT ESTIMATES FOR TRAPPING PERTURBATIONS

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Abstract. We study the semiclassical estimates of the resolvent $R(\lambda + i\tau)$, $\lambda \in J \subset\subset \mathbb{R}^+$, $\tau \in]0,1]$ of a self-adjoint operator $L(h)$ in the space of bounded operators $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$, $s > 1/2$. In the general case of long-range trapping "black-box" perturbations we prove that the estimate of the cut-off resolvent $\|\chi(x)R(\lambda+i0)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \leq C\exp(Ch^{-p})$, $\chi(x) \in C_0^\infty(\mathbb{R}^n)$, $p \geq 1$ implies the estimate $\|R(\lambda + i\tau)\|_{s,-s} \leq C_1 \exp(C_1 h^{-p})$.

1. Introduction

The purpose of this paper is to obtain semiclassical estimates for the resolvent

$$R(z) = (L(h) - z)^{-1},\ z \in \mathbb{C} \setminus \mathbb{R}$$

of a self-adjoint operator $L = L(h)$ depending on $h \in]0, h_0]$ for

$$z \in \mathcal{B}_\pm = \{z = \lambda + i\tau \in \mathbb{C} : \lambda \in J,\ \pm\tau \in]0,1]\},$$

$J =]\mu_0, \mu_1[\subset\subset \mathbb{R}^+$. The operator L is defined in a domain $\mathcal{D} \subset \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0)),\ B(0,R_0) = \{x \in \mathbb{R}^n : |x| \leq R_0\},\ n \geq 2$$

and L satisfies the long-range "black box" assumptions (1.7)-(1.15) described below. Introduce the spaces

$$\mathcal{H}^{0,s} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0,R_0), \langle x\rangle^s dx),\ \langle x\rangle = (1+|x|^2)^{1/2}.$$

First we show that for $s > 1/2$ and fixed $h > 0$ the limits

$$R(\lambda \pm i0) = \lim_{\epsilon\to 0,\ \epsilon>0} R(\lambda \pm i\epsilon),\ \ \lambda \in J \tag{1.1}$$

exist in the space of bounded operators $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$, where here and below we omit in $R(z)$ the dependence of h. Next we obtain semiclassical resolvent estimates

$$\|R(\lambda \pm i\tau)\|_{s,-s} \leq Cr(h),\ \lambda \in J,\ \ \tau \in]0,1],\ \ h \in]0,h_0], \tag{1.2}$$

where $\|.\|_{s,-s}$ denotes the norm in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$. Such estimates have been obtained by many authors in great generality in the case when every $\lambda \in J$ is a *non-trapping energy level* for the principal symbol $l_0(x,\xi)$ of a differential operator L (see [19], [6], [20], [9], [26], [16], [17], [25] and the survey article [18] for other references). The proofs of these estimates are essentially based on Mourre method [13], [4], [14] and the main point is to construct a conjugate operator $A(h)$ for which the Mourre inequality (5.2) holds (see Appendix 5.1). Roughly speaking, we call these operators *non-trapping* perturbations and in this case we have $r(h) = h^{-1}$ in (1.2). Moreover, for such perturbations there are no resonances converging sufficiently fast to the real axis as $h \to 0$.

The case of *trapping energy levels* is more complicated. First, without the non-trapping assumption, in general, a Mourre type inequality is not true uniformly with respect to $h \in]0, h_0]$. Secondly, resonances converging exponentially fast to the real axis could exist and, consequently, we must have $r(h) = e^{Ch^{-p}}$ with some $p \geq 1$. In the special case of a Schrödinger operator $-h^2\Delta + V(x)$, with potential $V(x)$ having the form of a "well in an island", the results in [7], [8] imply (1.2) with $r(h) = e^{Ch^{-1}}$. In our principal result we assume the following estimate

$$\|\chi(x)R(\lambda + i0)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \leq C\exp(Ch^{-p}),\ \lambda \in J,\ 0 < h \leq h_0, \tag{1.3}$$

with $p \geq 1$ and some constants $C > 0,\ h_0 > 0$, provided $\chi(x) \in C_0^\infty(\{x : |x| \leq \rho_1\})$ is such that $\chi(x) = 1$ for $|x| \leq \rho_0,\ R_0 < \rho_0 < \rho_1$. Here the constant $\rho_0 = \rho_0(J)$ depends on the interval J and the choice of ρ_0 is related to the properties of Hamiltonian trajectories examined in Section 2. Recently, for long-range trapping perturbations in the exterior of a connected obstacle Burq [3] established the estimate

$$\|(L_\theta(h) - \lambda)^{-1}\|_{L^2(\mathcal{H}_\theta)} \leq Ce^{C_1 h^{-p}},\ \lambda \in J, \tag{1.4}$$

with $p = 1$ for the complex scaling operator $L_\theta(h)$ defined in a dense domain in $\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$, where Γ_θ is obtained by modifying $\mathbb{R}^n$ outside the ball $\{x : |x| \geq \rho_1\}$ containing the support of χ (see [23], [21] for a precise definition and Section 3). On the other hand, the estimate (1.4) implies immediately (1.3) since we have

$$\chi(x)R(\lambda + i0)\chi(x) = \chi(x)(L_\theta(h) - \lambda)^{-1}\chi(x)$$

(see Lemma 3.5 in [23]). To treat another case when (1.3) holds, denote by Res $L(h)$ the set of resonances of $L(h)$ and assume that there exist $\epsilon > 0,\ c > 0$ and $q \geq 1$ so that

$$\mathrm{dist}\,\{\mathrm{Res}\, L(h),\ [\mu_0, \mu_1]\} \geq \epsilon \exp(-ch^{-q}),\ 0 < h \leq h_0.$$

Then an application of Lemma 1 in [24] with $g(h) = \exp(-\frac{c}{2}h^{-q})$ yields (1.4) with $p \geq q$. Thus we have several sufficient conditions leading to (1.3).

In this work we show that for a fixed interval $J \subset\subset \mathbb{R}^+$, taking $\chi \in C_0^\infty(\{x : |x| \leq \rho_1\})$ as above, the estimate (1.3) of the cut-off resolvent $\|\chi(x)R(\lambda + i0)\chi(x)\|_{\mathcal{H}\to\mathcal{H}},\ \lambda \in J$, implies an estimate of $\|R(\lambda + i\tau)\|_{s,-s}$ with the same order p in the bound $\exp(Ch^{-p})$. Thus, for long-range perturbations in $\mathbb{R}^n$, studied in [3], we prove the conjecture of Robert [17] who claimed that (1.2) with $r(h) = C\exp(Ch^{-p})$ holds for general trapping perturbations. Under this assumption, Robert [17] obtained a Weyl type asymptotic for the scattering phase related to two h-admissible pseudodifferential operators $L_1(h),\ L_2(h)$. Our main result is the following.

Theorem 1. *Let $L(h)$ be a self-adjoint operator satisfying the "black box" assumptions* $(1.7) - (1.15)$. *Then for fixed $s > 1/2$ and for fixed sufficiently small h the limits*

$$R(\lambda \pm i0) = \lim_{\epsilon\to 0,\ \pm\epsilon>0} R(\lambda \pm i\epsilon)$$

exist in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$ uniformly with respect to $\lambda \in J =]\mu_0, \mu_1[$. Let $\rho_0 > R_0$ be a constant, depending on J, such that all bounded trajectories of the Hamilton field H_{l_0} are included in $\{x : |x| \leq \rho_0/2\}$. Assume that the cut-off resolvent $\chi(x)R(\lambda \pm i0)\chi(x)$ with $\chi(x) \in C_0^\infty(\{x : |x| \leq \rho_1\})$, $\chi = 1$ for $|x| \leq \rho_0 < \rho_1$, satisfies the estimate

$$\|\chi(x)R(\lambda \pm i0)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \leq C\exp(Ch^{-p}),\ \lambda \in]\mu_0 - \delta, \mu_1 + \delta[,\ 0 < h \leq h_0 \tag{1.5}$$

with $\delta > 0$ and $p \geq 1$. Then for each $s > \frac{1}{2}$ there exist $C_s > 0,\ h_1 > 0$ such that for $(\lambda, \tau) \in J\times]0, 1]$ we have

$$\|R(\lambda \pm i\tau)\|_{s,-s} \leq C_s \exp(C_s h^{-p}),\ \ 0 < h \leq h_1. \tag{1.6}$$

Remark 1. *It is important to note that under our assumptions for every fixed interval $J \subset\subset \mathbb{R}^+$ there exists a constant $\rho_0 > R_0$ with the properties mentioned in Theorem* 1 *(see Section* 2*). The precise choice of ρ_0 and ρ_1 will be given in Proposition* 3.

The idea of our proof is to construct a self-adjoint operator $\tilde{L} = \tilde{L}(h)$ on $L^2(\mathbb{R}^n)$ such that each $\lambda \in J$ is a non-trapping energy level for $\tilde{L}$ and $L(h)\psi = \tilde{L}(h)\psi$ for any $\psi \in C_0^\infty(\mathbb{R}^n)$ supported away from a ball $B(0, \rho_1)$. For this purpose we show that the bounded trajectories related to the symbol of $L(h)$ and to the energy levels $\lambda \in J$ are included in a compact set $K(J) \subset \mathbb{R}^{2n}$ depending on J. Next we represent $R(z)$ by a sum of terms involving the resolvent $\tilde{R}(z)$ of $\tilde{L}$ (see Section 3) and an application of the semiclassical estimates for non-trapping energy levels reduces the problem to the estimation of $\|\chi(x)R(z)\chi(x)\|_{\mathcal{H}\mapsto\mathcal{H}}$ (see Proposition 3). In Section 2 our approach is motivated by the proof of Lemma 1.2 in [8] concerning the operator $L(h) = -h^2\Delta + V(x)$, where the potential $V(x)$ involves a "well in an island". Having an "island" around the trapped trajectories, Gérard, Martinez and Robert have introduced a perturbation $W_0(x) \in C_0^\infty(\mathbb{R}^n)$ such that the perturbed symbol $|\xi|^2 + V(x) + W_0(x)$ becomes a non-trapping Hamiltonian for energy levels $\lambda \in J$. In the general case we prove a similar result by constructing a perturbation $\tilde{l}(x,\xi)$ of $l_0(x,\xi)$ (see Proposition 2) so that all $\lambda \in J$ are non-trapping energy levels for $\tilde{l}(x,\xi)$. This implies immediately the existence of an operator $\tilde{L}(h)$ with the properties mentioned above. Finally, let us mention that the estimate (1.6) combined with the argument of Robert [17] lead to a Weyl asymptotic of the scattering phase for general long-range perturbations in $\mathbb{R}^n$ when $\mathcal{H} = L^2(\mathbb{R}^n)$.

Now we will recall the abstract "black box" assumptions given in [23], [21] and [22]. Suppose that $\mathcal{D}$ satisfies

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} = H^2(\mathbb{R}^n \setminus B(0,R_0)), \tag{1.7}$$

uniformly with respect to h in the sense of [21]. More precisely, equip $H^2(\mathbb{R}^n \setminus B(0,R_0))$ with the norm $\| < hD >^2 u\|_{L^2}$, $< hD >^2 = 1 + (hD)^2$, and equip $\mathcal{D}$ with the norm $\|(L+i)u\|_{\mathcal{H}}$. Then we require that $\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} : \mathcal{D} \longrightarrow H^2(\mathbb{R}^n \setminus B(0,R_0))$ is uniformly bounded with respect to h and this map has a uniformly bounded right inverse.

Assume that

$$\mathbb{1}_{B(0,R_0)}(L+i)^{-1} \text{is compact} \tag{1.8}$$

and

$$(Lu)|_{\mathbb{R}^n \setminus \overline{B(0,R_0)}} = Q\Big(u|_{\mathbb{R}^n \setminus \overline{B(0,R_0)}}\Big), \tag{1.9}$$

where Q is a formally self-adjoint differential operator

$$Qu = \sum_{|\alpha| \leq 2} a_\alpha(x;h)(hD_x)^\alpha u, \tag{1.10}$$

with $a_\alpha(x;h) = a_\alpha(x)$ independent of h for $|\alpha| = 2$ and $a_\alpha(x;h) \in C_b^\infty(\mathbb{R}^n)$ uniformly bounded with respect to h.

Next we assume the ellipticity and a long-range decay of the symbol $\sum_{|\alpha|\leq 2} a_\alpha(x;h)\xi^\alpha$. There exists $C > 0$ such that

$$l_0(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq C\langle\xi\rangle^2,\ \ \langle\xi\rangle = (1+|\xi|^2)^{1/2}. \tag{1.11}$$

There exists $\gamma > 0$ such that for every $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, we have

$$|\partial_x^\alpha \partial_\xi^\beta \Big(\sum_{|\alpha|\leq 2} a_\alpha(x;h)\xi^\alpha - |\xi|^2 \Big)| \leq C_{\alpha,\beta} \langle x\rangle^{-\gamma-|\alpha|}\langle\xi\rangle^2 \tag{1.12}$$

uniformly with respect to h.

To study the analytic properties of the cut-off resolvent related to the resonances near J, we need some technical assumptions which we will use only in Proposition 3 and Lemma 3.

There exist $\theta_0 \in]0,\pi[$, $\epsilon > 0$ and $R_1 > R_0$ so that the coefficients $a_\alpha(x;h)$ of Q can be extended holomorphically in x to

$$\{r\omega;\ \omega \in \mathbb{C}^n,\ \text{dist}\,(\omega, S^{n-1}) < \epsilon,\ r \in \mathbb{C},\ |r| > R_1,\ \arg r \in [-\epsilon, \theta_0 + \epsilon)\} \tag{1.13}$$

and (1.12) extend to this larger set. Let $R > R_0$, $T = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$, $\tilde{R} > 2R$. Set

$$\mathcal{H}^\# = \mathcal{H}_{R_0} \oplus L^2(T \setminus B(0,R_0))$$

and consider a differential operator

$$Q^\# u = \sum_{|\alpha|\leq 2} a_\alpha^\#(x;h)(hD)^\alpha u$$

on T with $a^{\#}_{\alpha}(x;h) = a_{\alpha}(x;h)$ for $|x| < R$ satisfying (1.9), (1.10), (1.11) with $\mathbb{R}^n$ replaced by T. Consider a self-adjoint operator

$$L^{\#}u = L(\varphi u) + Q^{\#}((1-\varphi)u),\ u \in \mathcal{D}^{\#}$$

with domain

$$\mathcal{D}^{\#} = \{u \in \mathcal{H}^{\#} :\ \varphi u \in \mathcal{D},\ (1-\varphi)u \in H^2\},$$

where $\varphi \in C_0^{\infty}(B(0,R);[0,1])$ is equal to 1 near $\overline{B(0,R_0)}$. Denote by $N(L^{\#},[-\lambda,\lambda])$ the number of eigenvalues of $L^{\#}$ in the interval $[-\lambda,\lambda]$ and assume that

$$N(L^{\#},[-\lambda,\lambda]) = \mathcal{O}(\Big(\frac{\lambda}{h^2}\Big)^{n^{\#}/2}),\ n^{\#} \geq n,\ \lambda \geq 1. \tag{1.14}$$

Finally, we suppose that

$$\sigma_{pp}(L(h)) \cap [\mu_0 - \delta, \mu_1 + \delta] = \emptyset,\ h \in]0, h_0], \tag{1.15}$$

where $\delta > 0$ is the constant in (1.5). A typical example for an operator satisfying the conditions (1.7)-(1.15) is an elliptic self-adjoint Schrödinger operator

$$P = \sum_{|\alpha|\leq 2} a_{\alpha}(x;h)(hD_x)^{\alpha}$$

with C^{∞} coefficients in the exterior $\Omega = \mathbb{R}^n \setminus \mathcal{O}$ of a bounded obstacle $\mathcal{O}$ with C^{∞} boundary $\partial\Omega$ with Dirichlet condition on $\partial\Omega$ (see [3]). In this case $\mathcal{D} = H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H_0^1(\mathbb{R}^n \setminus \mathcal{O})$ and P is a positive operator.

Under the above assumptions the resonances close to the real axis can be defined by the method of complex scaling [23], [21] and they coincide with the poles of the meromorphic continuation of the resolvent $(L(h) - z)^{-1} : \mathcal{H}_{\rm comp} \longrightarrow \mathcal{D}_{\rm loc}$ from $\operatorname{Im} z > 0$ to a conic neighborhood of the positive real axis in the lower half plane. The set of resonances will be denoted by Res $L(h)$. Notice that the assumption (1.13) is necessary for the introduction of the complex scaling operator $L_{\theta}(h)$, while the condition (1.14) is exploited for the estimation of the function counting the resonances in a compact set (see [21], [23]). These assumptions play an important role in the proof of Lemma 3 in Section 3. Finally, (1.15) guarantees that we have no resonances lying in the interval $]\mu_0 - \delta, \mu_1 + \delta[$.

The paper is organized as follows. Section 2 is devoted to the analysis of the non-trapping energy levels and to the construction of the symbol $\tilde{l}(x,\xi)$. In Section 3 we prove Theorem 1 introducing the operator $\tilde{L}(h)$ and exploiting the semiclassical estimates for non-trapping energy levels. In Section 4 we obtain some microlocal estimates closely related to the bounds of $\|R(z)\|_{s,-s}$. In another work [2] we apply the estimates given in Proposition 5 for the analysis of the semiclassical asymptotic of the spectral shift function related to trapping "black box" perturbations.

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2. Classical trajectories and non-trapping energy levels

Let $l(x,\xi) \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R})$ be a classical Hamiltonian and let $H_l = (\partial_{\xi} l, -\partial_x l)$ be the associated Hamilton vector field

$$H_l(f)(x,\xi) = \{l,f\}(x,\xi) = (\partial_{\xi} l \partial_x f - \partial_x l \partial_{\xi} f)(x,\xi),\ \ f \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n).$$

Denote by Φ_l^t the Hamilton flow of l given by

$$\Phi_l^t = \exp(tH_l) \ : \ (x_0, \xi_0) \mapsto \Big(x(t, x_0, \xi_0), \xi(t, x_0, \xi_0)\Big).$$

Therefore for each $\lambda \in \mathbb{R}$, the energy surface $\Sigma_\lambda = l^{-1}(\lambda)$ is stable by Φ_l^t. Throughout the exposition below we set $J =]\mu_0, \mu_1[\subset\subset \mathbb{R}^+$. We will say that (x, ξ) is a non-critical point for $l(x, \xi)$ if $\nabla_{x,\xi} l(x, \xi) \neq 0$ and λ is a non-critical energy level for $l(x, \xi)$, if for all $(x, \xi) \in \Sigma_\lambda = l^{-1}(\lambda)$ we have $\nabla_{x,\xi} l(x, \xi) \neq 0$. We start by an analysis of the trapped trajectories.

Lemma 1. *Let $\mu_1 > \mu_0 > 0$ be fixed. Assume there exist $\mu \in]0, \mu_0[$, $R > 0$ such that*

$$|x| \geq R, \ (x, \xi) \in l^{-1}(]\mu_0, \mu_1[) \ \Rightarrow \ |\frac{1}{4}\{l, x \cdot \partial_\xi l\}(x, \xi) - l(x, \xi)| \leq \mu. \tag{2.1}$$

Then for each $\lambda \in J$ and each $(x_0, \xi_0) \in \Sigma_\lambda$ with $|x_0| > R$ we have $\sup_{t \in \mathbb{R}} |x(t, x_0, \xi_0)| = +\infty$.

Proof. For the simplicity of the notations we will omit (x_0, ξ_0) in $x(t, x_0, \xi_0)$, $\xi(t, x_0, \xi_0)$ if the dependence of (x_0, ξ_0) is not important. It is clear that the function $|x(t)|^2$ satisfies the equations

$$\begin{cases} (|x|^2)' = \left\{l, |x|^2\right\}(x, \xi) = 2x \cdot \partial_\xi l(x, \xi), \\ \quad (x \cdot \partial_\xi l(x, \xi))' = \left\{l, x \cdot \partial_\xi l\right\}(x, \xi). \end{cases} \tag{2.2}$$

Obviously, for $(x_0, \xi_0) \in \Sigma_\lambda$ we have $(x(t), \xi(t)) \in \Sigma_\lambda$ and this yields

$$(x \cdot \partial_\xi l(x, \xi))' = 4\lambda + \left\{l, x \cdot \partial_\xi l\right\}(x, \xi) - 4l(x, \xi). \tag{2.3}$$

Choose $|x_0| > R$ with $(x_0, \xi_0) \in l^{-1}(]\mu_0, \mu_1[)$ and denote by $[T_m, T_M] \subset \mathbb{R}$ the maximal interval containing $t_0 = 0$ so that

$$|\Big(\{l, x \cdot \partial_\xi l\} - 4l\Big)(x(t), \xi(t))| \leq 4\mu, \ \ t \in [T_m, T_M]. \tag{2.4}$$

The continuity of the function $x(t)$ implies the existence of an interval $[T_m, T_M] \neq \{0\}$. Using the relation (2.3), we deduce

$$(x \cdot \partial_\xi l(x, \xi))'(t) \geq 4(\lambda - \mu) \geq 4(\mu_0 - \mu), \ \ \forall t \in [T_m, T_M],$$

that is

$$\begin{cases} (x \cdot \partial_\xi l(x, \xi))(t) \geq x_0 \cdot \partial_\xi l(x_0, \xi_0) + 4(\mu_0 - \mu)t, \ \ \forall t \in [0, T_M], \\ (x \cdot \partial_\xi l(x, \xi))(t) \leq x_0 \cdot \partial_\xi l(x_0, \xi_0) + 4(\mu_0 - \mu)t, \ \ \forall t \in [T_m, 0]. \end{cases}$$

Then applying (2.2), we get

$$|x(t)|^2 \geq |x_0|^2 + 2x_0 \cdot \partial_\xi l(x_0, \xi_0)t + 4(\mu_0 - \mu)t^2, \ \ \forall t \in [T_m, T_M].$$

Consequently, if $\pm x_0 \cdot \partial_\xi l(x_0, \xi_0) \geq 0$ for $\pm t \geq 0$, we have

$$|x(t)|^2 \geq |x_0|^2 + 4(\mu_0 - \mu)t^2 \tag{2.5}$$

which implies immediately $T_M = +\infty$ (if $x_0 \cdot \partial_\xi l(x_0, \xi_0) \geq 0$) or $T_m = -\infty$ (if $x_0 \cdot \partial_\xi l(x_0, \xi_0) \leq 0$) and (2.5) holds for any $t \geq 0$ (any $t \leq 0$). Thus, for $|x_0| > R$ and $l(x_0, \xi_0) \in]\mu_0, \mu_1[$ we obtain $\sup_{t \in \mathbb{R}} |x(t, x_0, \xi_0)| = +\infty$ and the proof is complete.

□

The argument of proof above shows that the trajectories of Φ_l^t passing through a point $(x_0, \xi_0) \in \Sigma_\lambda$ with $|x_0| > R$ go to infinity if the product $x_0 \cdot \partial_\xi l(x_0, \xi_0)$ has a suitable sign. This phenomenon is well known for symbols $|\xi|^2 + V(x)$, when $V(x)$ is a short-range perturbation (see for instance [12]). The assumption (2.1) is usually known as *virial condition*. In the particular case, when $l(x, \xi) = |\xi|^2 + V(x)$, this condition means that $|x.V_x(x) + 2V(x)| \leq 2\mu$.

We will say that a trajectory $(x(t, x_0, \xi_0),\ \xi(t, x_0, \xi_0))$ of the flow Φ_l^t is *bounded* if there exists $A > 0$ such that $|x(t, x_0, \xi_0)| \leq A$, $\forall t \in \mathbb{R}$. In the next proposition we show that under the long-range assumption all bounded trajectories on Σ_λ, $\lambda \in J$, as well as all critical points of $l(x,\xi)$ are included in a fixed compact set $K(J) \subset \mathbb{R}^{2n}$ depending on the interval J.

Proposition 1. *Let $\mu_1 > \mu_0 > 0$. Assume there exists $\gamma > 0$ such that for every $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, $|\alpha| \leq 1$, $|\beta| \leq 2$ we have*

$$|\partial_x^\alpha \partial_\xi^\beta \Big(l(x,\xi) - |\xi|^2\Big)| \leq C_{\alpha,\beta} \langle x \rangle^{-\gamma - |\alpha|} \langle \xi \rangle^2.$$

Then there exists a compact set $K(J) \subset \mathbb{R}^{2n}$ such that for each $\lambda \in J$, the set $K(J) \cap \Sigma_\lambda$ contains all bounded trajectories of the Hamilton flow Φ_l^t on Σ_λ and all critical points of $l(x,\xi)$ in Σ_λ.

Remark 2. *In a more restrictive setup, this result was stated and exploited by the first author* [1] *in the analysis of the scattering phase for short-range trapping perturbations of the Laplacian.*

Proof. First, observe that for $(x,\xi) \in \Sigma_\lambda$, $\lambda \in J$ we have

$$\langle \xi \rangle^2 \leq C_{0,0} \langle x \rangle^{-\gamma} \langle \xi \rangle^2 + \mu_1 + 1.$$

Thus for R_1 large enough and $|x| > R_1$ we get

$$\lambda \in J,\ (x,\xi) \in \Sigma_\lambda,\ |x| > R_1 \Rightarrow |\xi| \leq M. \tag{2.6}$$

Next we will show that outside a fixed compact set $K(J)$ the x-component $x(t, x_0, \xi_0)$ of the trajectories of Φ_l^t goes to infinity in the sense that if $x(t, x_0, \xi_0)$ passes through a point $y \in K(J)$, then $\sup_{t \in \mathbb{R}} |x(t, x_0, \xi_0)| = \infty$.

As we have seen, applying Lemma 1, it is sufficient to establish the existence of $R > 0$ such that (2.1) holds with $0 < \mu < \mu_0$. We write

$$\begin{aligned} \{l, x \cdot \partial_\xi l\} &= \partial_\xi l \cdot \partial_x (x \cdot \partial_\xi l) - \partial_x l \cdot \partial_\xi (x \cdot \partial_\xi l) \\ &= |\partial_\xi l|^2 + \textstyle\sum_{j=1}^n \partial_{\xi_j} l \; x \cdot \partial_\xi \partial_{x_j} l - \partial_x l \cdot \partial_\xi (x \cdot \partial_\xi l). \end{aligned} \tag{2.7}$$

Setting $r(x,\xi) = l(x,\xi) - |\xi|^2$, we observe that

$$|\partial_\xi l|^2 = 4|\xi|^2 + 4\xi \cdot \partial_\xi r + |\partial_\xi r|^2$$

which implies

$$\frac{1}{4}|\partial_\xi l|^2 - l(x,\xi) = \xi \cdot \partial_\xi r + \frac{1}{4}|\partial_\xi r|^2 - r(x,\xi). \tag{2.8}$$

Moreover, since $\partial_{x_j} l = \partial_{x_j} r$, we deduce the following representation

$$\frac{1}{4}\{l, x \cdot \partial_\xi l\} - l(x,\xi)$$

$$= \xi \cdot \partial_\xi r + \frac{1}{4}|\partial_\xi r|^2 - r(x,\xi) + \frac{1}{4}\sum_{j=1}^n \partial_{\xi_j} l \; x \cdot \partial_\xi \partial_{x_j} r - \partial_x r \cdot \partial_\xi (x \cdot \partial_\xi l).$$

Combining this with the assumption on the derivatives of l, we obtain

$$|\frac{1}{4}\{l, x \cdot \partial_\xi l\}(x,\xi) - l(x,\xi)| \leq C \langle x \rangle^{-\gamma} \langle \xi \rangle^4.$$

From this it is clear that we can find $R = R(\mu_0, \mu_1, \gamma) \geq R_1$ such that (2.1) holds for $|x| > R$. On the other hand, taking R so that the modulus of the RHS of (2.8) is bounded by μ, $0 < \mu < \mu_0$, we obtain $|\partial_\xi l| \neq 0$ on $l^{-1}(J) \cap \{(x,\xi) : |x| > R\}$. Consequently, each $(x,\xi) \in l^{-1}(J) \cap \{(x,\xi) : |x| > R\}$ is non-critical for l and for each $\lambda \in J$ every trajectory on Σ_λ with initial data $|\, x_0 \,| > R$ becomes non-bounded in the sense mentioned above. This completes the proof. $\square$

Now we pass to the analysis of the non-trapping energy levels in the sense given in [19]. Recall that $\lambda \in \mathbb{R}$ is a *non-trapping energy level* for $l(x,\xi)$ if for every $R > 0$ there exists $T_R > 0$ such that for $(x_0, \xi_0) \in \Sigma_\lambda$, $|x_0| < R$, we have

$$|x(t, x_0, \xi_0)| > R,\ \ \forall |t| > T_R.$$

Lemma 2. *Let $\mu_1 > \mu_0 > 0$ and let $l(x,\xi) \geq c_0|\xi|^2 + c_1$, $c_0 > 0$, $\forall (x,\xi) \in \mathbb{R}^n$. If there exists $\mu \in]0, \mu_0[$ such that*

$$(x,\xi) \in l^{-1}(]\mu_0, \mu_1[) \Rightarrow |\frac{1}{4}\{l, x\cdot\partial_\xi l\}(x,\xi) - l(x,\xi)| \leq \mu, \tag{2.9}$$

then each $\lambda \in J$ is a non-trapping energy level for $l(x,\xi)$.

Proof. Our purpose is to prove that for any $(x_0, \xi_0) \in \Sigma_\lambda$ we have $|x(t, x_0, \xi_0)| \longrightarrow +\infty$ in both directions $t \to \pm\infty$. Following the proof of Lemma 1, we have the estimate

$$|\Big(\{l, x\cdot\partial_\xi l\} - 4l\Big)(x(t), \xi(t))| \leq 4\mu, \ \ \forall t \in \mathbb{R}. \tag{2.10}$$

Notice that here there are no restrictions on the sign on t. Then it follows easily that

$$|x(t)|^2 \geq |x_0|^2 + 2x_0 \cdot \partial_\xi l(x_0, \xi_0)t + 4(\mu_0 - \mu)t^2, \ \ \forall t \in \mathbb{R}.$$

On the other hand, it is obvious that the set $\{\xi_0 : \ (x_0, \xi_0) \in \Sigma_\lambda \ , \ \mu_0 \leq \lambda \leq \mu_1\}$ is bounded. Consequently, for $R > 0$ and $|x_0| < R$, we conclude that $x_0 \cdot \partial_\xi l(x_0, \xi_0)$ remains bounded, and there exists $C_R > 0$ such that

$$|x(t)|^2 \geq 4|t|\Big((\mu_0 - \mu)|t| - C_R\Big), \ \forall t \in \mathbb{R}.$$

Thus, for all $|t| > 2C_R + 2\sqrt{C_R^2 + (\mu_0 - \mu)R^2}$ we have $|x(t)| > R$, hence λ is a non-trapping energy level. ☐

Introduce a function $\chi \in C_0^\infty(\{x \in \mathbb{R}^n : \ |x| \leq 2\})$, $0 \leq \chi \leq 1$ such that $\chi(x) = 1$ for $|x| \leq 1$ and set $\chi_R(x) = \chi(\frac{x}{R})$. The next proposition will play a crucial role in our analysis in Section 3.

Proposition 2. *Let $\mu_1 > \mu_0 > 0$. Assume there exists $\gamma > 0$ such that for every $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$, $|\alpha| \leq 1$, $|\beta| \leq 2$ we have*

$$|\partial_x^\alpha \partial_\xi^\beta \Big(l(x,\xi) - |\xi|^2\Big)| \leq C_{\alpha,\beta}\langle x\rangle^{-\gamma-|\alpha|}\langle\xi\rangle^2.$$

Then there exists $\rho > 0$ such that for $R \geq \rho$, each $\lambda \in J$ is a non-trapping and non-critical energy level for the Hamiltonian

$$\tilde{l}(x,\xi) = l(x,\xi) - \chi_R(x)\Big(l(x,\xi) - |\xi|^2\Big).$$

Proof. We will prove that it is possible to choose $R > 0$ large enough in order to satisfy the conditions of Lemma 2 for the symbol $\tilde{l}(x,\xi)$. We first observe that

$$\tilde{l}(x,\xi) = |\xi|^2 + \Big(1 - \chi_R(x)\Big)\Big(l(x,\xi) - |\xi|^2\Big)$$

combined with the assumptions on l and the fact that $(1 - \chi_R)$ has support in $\{x : \ |x| > R\}$ imply easily the estimate

$$\tilde{l}(x,\xi) \geq \langle\xi\rangle^2\Big(1 - C_{0,0}\langle R\rangle^{-\gamma}\Big) - 1.$$

Hence there exists R_1 such that if we take $R \geq R_1$ in the definition of $\tilde{l}(x,\xi)$ we have

$$\tilde{l}(x,\xi) \geq c_0|\xi|^2 + c_1, \ c_0 > 0, \ \forall (x,\xi) \in \mathbb{R}^n.$$

Now, as in the proof of Proposition 1, we have

$$\{\tilde{l}, x\cdot\partial_\xi\tilde{l}\} = |\partial_\xi\tilde{l}|^2 + \sum_{j=1}^n \partial_{\xi_j}\tilde{l}\; x\cdot\partial_\xi\partial_{x_j}\tilde{l} - \partial_x\tilde{l}\cdot\partial_\xi(x\cdot\partial_\xi\tilde{l}). \tag{2.11}$$

Next, setting $r(x,\xi) = \tilde{l}(x,\xi) - |\xi|^2$, the derivative of $\tilde{l} = |\xi|^2 + (1-\chi_R)r$ satisfies

$$|\partial_\xi\tilde{l}|^2 = 4|\xi|^2 + 4(1-\chi_R)\xi\cdot\partial_\xi r + (1-\chi_R)^2|\partial_\xi r|^2$$

and from this we see that

$$\frac{1}{4}|\partial_\xi\tilde{l}|^2 - \tilde{l}(x,\xi) = (1-\chi_R)\xi\cdot\partial_\xi r + \frac{1}{4}(1-\chi_R)^2|\partial_\xi r|^2 - (1-\chi_R)r(x,\xi). \tag{2.12}$$

On the other hand, it is clear that

$$\partial_{x_j}\tilde{l} = \partial_{x_j}(1-\chi_R)r = (1-\chi_R)\partial_{x_j}r - \frac{1}{R}(\partial_{x_j}\chi)\Big(\frac{x}{R}\Big)r$$

and, consequently,

$$\begin{aligned}&\sum_{j=1}^{n}\partial_{\xi_j}\tilde{l}\, x\cdot\partial_\xi\partial_{x_j}\tilde{l} - \partial_x\tilde{l}\cdot\partial_\xi(x\cdot\partial_\xi\tilde{l})\\ &= (1-\chi_R)\left(\sum_{j=1}^{n}\partial_{\xi_j}\tilde{l}\, x\cdot\partial_\xi\partial_{x_j}r - \partial_x r\cdot\partial_\xi(x\cdot\partial_\xi\tilde{l})\right)\\ &-\sum_{j=1}^{n}\frac{1}{R}(\partial_{x_j}\chi)\Big(\frac{x}{R}\Big)\Big(\partial_{\xi_j}\tilde{l}\, x\cdot\partial_\xi r - r\cdot\partial_{\xi_j}(x\cdot\partial_\xi\tilde{l})\Big). \qquad (2.13)\end{aligned}$$

The function $(1-\chi_R)$ (resp. $\partial_x\chi_R$) is supported in $\{|x|\geq R\}$, (resp. $\{2R\geq|x|\geq R\}$) and the assumptions on $r(x,\xi)$ combined with the relations (2.11), (2.12), (2.13) lead to

$$|\frac{1}{4}\{\tilde{l}, x\cdot\partial_\xi\tilde{l}\}(x,\xi) - \tilde{l}(x,\xi)| \leq C\langle R\rangle^{-\gamma}\langle\xi\rangle^4.$$

Now recall that $|\xi|$ remains bounded on the energy level Σ_λ. Then we can find $\rho=\rho(\mu_0,\mu_1,\gamma)\geq R_1$ such that for any $R\geq\rho$ the function $\tilde{l}(x,\xi)$ satisfies (2.9) and each $\lambda\in J$ is a non-trapping energy level for $\tilde{l}(x,\xi)$. Moreover, taking ρ so that the right hand side of (2.12) has a modulus bounded by μ, $0<\mu<\mu_0$, we obtain $|\partial_\xi\tilde{l}|\neq 0$ on $\tilde{l}^{-1}(J)$ and this completes the proof. $\square$

Remark 3. *For a more complete analysis of the qualitative aspects of classical trajectories, we refer to the recent work of Knauf* [12]. *The constant ρ determined in Proposition 2 corresponds to so called virial radius. For the Hamiltonian $l(x,\xi)=\frac{1}{2}|\xi|^2+V(x)$, Knauf obtained a topological condition for the existence of trapped trajectories.*

3. Resolvent estimates

We start with the following proposition reducing the proof of the resolvent estimates to that for the cut-off resolvent.

Proposition 3. *Let $L(h)$ satisfy the "black box" assumptions* (1.7) − (1.12) *and let $J=]\mu_0,\mu_1[\subset\subset\mathbb{R}^+$. Let $\rho_0>2\rho>R_0$, where ρ is the constant of Proposition 2. Then there exists a self-adjoint differential operator $\tilde{L}=\tilde{L}(h)$ on $L^2(\mathbb{R}^n)$, satisfying the assumptions* (1.7) − (1.12) *such that each $\lambda\in J$ is a non-trapping and non-critical energy level for $\tilde{L}(h)$ and*

$$L(h)\psi = \tilde{L}(h)\psi$$

for any $\psi\in C^\infty(\mathbb{R}^n)$ supported away from $B(0,\rho_0)$. Let $\chi(x)\in C_0^\infty(\{x:|x|\leq\rho_1\})$, $\chi(x)=1$ for $|x|\leq\rho_0<\rho_1$. Then for every $s>\frac{1}{2}$ there exist $C>0$, $h_0>0$ such that we have

$$\|R(z)\|_{s,-s}\leq Ch^{-2}\Big(1+\|\chi R(z)\chi\|_{\mathcal{H}\to\mathcal{H}}\Big),$$

uniformly with respect to $z\in\mathcal{B}_\pm=\{z\in\mathbb{C}:\ (\mathrm{Re}\,z,\pm\mathrm{Im}\,z)\in J\times]0,1]\}$ and $h\in]0,h_0]$.

Proof. Choose a function $\Phi\in C_0^\infty(\{x\in\mathbb{R}^n:|x|\leq 1\};\mathbb{R})$, $0\leq\Phi\leq 1$ such that $\Phi(x)=1$ for $|x|\leq 1/2$ and set $\Phi_R(x)=\Phi(\frac{x}{R})$ for $R>R_0$, where R_0 is the constant in the "black box" assumptions. Introduce the self-adjoint operator on $L^2(\mathbb{R}^n)$

$$\begin{aligned}\tilde{L} &= (1-\Phi_R)L\Phi_R + \Phi_R L(1-\Phi_R) + (1-\Phi_R)L(1-\Phi_R) + \Phi_R(-h^2\Delta)\Phi_R \qquad (3.1)\\ &= Q + \Phi_R(-h^2\Delta - Q)\Phi_R, \qquad (3.2)\end{aligned}$$

where Q is the formally self-adjoint differential operator (1.10).

According to Proposition 2, taking $\rho_0 > 2\rho > R_0$, we know that every $\lambda \in J$ is a non-trapping and non-critical energy level in the sense mentioned in Section 2 for the principal symbol $\tilde{l}_0(x,\xi)$ of $\tilde{L}(h)$ having the form

$$\tilde{l}_0(x,\xi) = l_0(x,\xi) - \Phi^2_{\rho_0}(x)\Big(l_0(x,\xi) - |\xi|^2\Big)$$

with $l_0(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha$. Clearly, if $\psi = 0$ on $B(0,\rho_0)$, we have $\tilde{L}\psi = Q\psi = L\psi$.

Let $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on $B(0,\rho_0)$ and let $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of χ_1. The above property yields $\tilde{L}(1-\chi_1) = L(1-\chi_1)$ and we obtain

$$R(z) = R(z)\chi_2 + (1-\chi_1)\tilde{R}(z)(1-\chi_2) + R(z)[\tilde{L},\chi_1]\tilde{R}(z)(1-\chi_2), \tag{3.3}$$

where

$$\tilde{R}(z) = (\tilde{L}(h) - z)^{-1},\ \ z \in \mathbb{C},\ \mathrm{Im}\, z \neq 0.$$

This implies immediately the estimate

$$\|R(z)\|_{s,-s} \leq C\Big(\|R(z)\chi_2\|_{s,-s} + \|\tilde{R}(z)\|_{s,-s} + \|R(z)\chi_2\|_{s,-s}\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,s}\Big), \tag{3.4}$$

where here and below we denote by $C > 0$ different constants independent on h and z which may change from line to line. Next, $[\tilde{L},\chi_1]$ is a first order h-admissible differential operator with compactly supported coefficients, so there exists $C > 0$ such that

$$\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,s} \leq C\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,-s}.$$

Since the interval J contains non-trapping energy levels for $\tilde{L}(h)$, we can estimate $\|\tilde{R}(z)\|_{s,-s}$ and $\|[\tilde{L},\chi_1]\tilde{R}(z)\|_{s,-s}$ by Lemma 4 and Corollary 1 (see Appendix 5.1). Consequently, there exist $C > 0$ and $h_0 > 0$ such that for any $h \in]0,h_0]$ we have

$$\|R(z)\|_{s,-s} \leq Ch^{-1}\Big(1 + \|R(z)\chi_2\|_{s,-s}\Big). \tag{3.5}$$

On the other hand, using the equality

$$\|R(z)\chi_2\|_{s,-s} = \|\chi_2 R(\overline{z})\|_{s,-s}$$

and introducing χ_2 in the estimates (3.4), (3.5), we obtain

$$\|R(z)\chi_2\|_{s,-s} \leq Ch^{-1}\Big(1 + \|\chi_2 R(z)\chi_2\|_{\mathcal{H}\to\mathcal{H}}\Big). \tag{3.6}$$

Finally, (3.5) and (3.6) yield

$$\|R(z)\|_{s,-s} \leq Ch^{-2}\Big(1 + \|\chi_2 R(z)\chi_2\|_{\mathcal{H}\to\mathcal{H}}\Big)$$

and the proof is complete. □

In the following up to the end of our paper $\rho_1 > \rho_0 > R_0$ will denote the constants introduced in Proposition 3 and $\chi(x)$ is a cut-off function with the properties given in Proposition 3.

Proposition 4. *Assume that $L(h)$ satisfies the assumptions (1.7) – (1.15). Then for small $h_0 > 0$ and fixed $s > 1/2$ and $h \in]0,h_0]$ the limits*

$$R(\lambda \pm i0) = \lim_{\epsilon\to 0,\ \pm\epsilon>0} R(\lambda \pm i\epsilon)$$

exist in $\mathcal{L}(\mathcal{H}^{0,s},\mathcal{H}^{0,-s})$ uniformly with respect to $\lambda \in J$ and the function $J \ni \lambda \mapsto R(\lambda \pm i0) \in \mathcal{L}(\mathcal{H}^{0,s},\mathcal{H}^{0,-s})$ is continuous.

Proof. As in the proof of the Proposition 3, introduce an operator $\tilde{L} = \tilde{L}(h)$ with non-trapping energy levels in J. Let $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on $B(0, \rho_0)$ and let $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 on the support of χ_1. Set $R(z, z') = R(z) - R(z')$ and $\tilde{R}(z, z') = \tilde{R}(z) - \tilde{R}(z')$. According to (3.3), we have

$$R(z, z') = R(z, z')\chi_2 + (1 - \chi_1)\tilde{R}(z, z')(1 - \chi_2)$$
$$+R(z, z')[\tilde{L}, \chi_1]\tilde{R}(z)(1 - \chi_2) + R(z')[\tilde{L}, \chi_1]\tilde{R}(z, z')(1 - \chi_2)$$

and we deduce

$$\|R(z, z')\|_{s,-s} \leq C\|R(z, z')\chi_2\|_{s,-s}\Big(1 + \|[\tilde{L}, \chi_1]\tilde{R}(z)\|_{s,s}\Big)$$
$$+C\|\tilde{R}(z, z')\|_{s,-s} + C\|R(z')\chi_2\|_{s,-s}\, \|[\tilde{L}, \chi_1]\tilde{R}(z, z')\|_{s,s}.$$

For fixed $h \in]0, h_0]$, Lemma 6 in Appendix 5.2 implies that the cut-off resolvent $\chi_2(x)R(z)\chi_2(x)$ is analytic in $\overline{\mathcal{B}_\pm}$ and the norm $\|\chi_2 R(z)\chi_2\|_{\mathcal{H}\to\mathcal{H}}$ will be uniformly bounded with respect to $z \in \mathcal{B}_\pm$. Applying Proposition 3 for $s > 1/2$, we deduce that $\|R(z')\chi_2\|_{s,-s}$ is also uniformly bounded with respect to $z \in \mathcal{B}_\pm$. On the other hand, since J is non-trapping for $\tilde{L}$, using Corollary 1 we see that for any $s > 1/2$ there exists C_s such that

$$\|[\tilde{L}, \chi_1]\tilde{R}(z)\|_{s,s} \leq C_s h^{-1}$$

uniformly with respect to $z \in \mathcal{B}_\pm$. An application of Lemma 4 in Appendix 5.1 shows that for s and h fixed, there exists $C_{h,s}$ so that

$$\|\tilde{R}(z, z')\|_{s,-s} + \|[\tilde{L}, \chi_1]\tilde{R}(z, z')\|_{s,-s} \leq C_{h,s}\Big(|z - z'| + |z - z'|^{\delta_1}\Big),$$

with $\delta_1 = (s - 1/2)(2s - 1/2)$ uniformly with respect to $z, z' \in \mathcal{B}_\pm$. Thus with another constant $C_{h,s}$ we have

$$\|R(z, z')\|_{s,-s} \leq C_{h,s}\Big(|z - z'|^{\delta_1} + |z - z'| + \|R(z, z')\chi_2\|_{s,-s}\Big).$$

To deal with the term $\|R(z, z')\chi_2\|_{s,-s}$, we use the equality $\|R(z, z')\chi_2\|_{s,-s} = \|\chi_2 R(\overline{z}, \overline{z'})\|_{s,-s}$, and we introduce the cut-off function χ_2 in the above estimates in order to obtain

$$\|R(z, z')\chi_2\|_{s,-s} \leq C_{h,s}\Big(|z - z'|^{\delta_1} + |z - z'| + \|\chi_2 R(z, z')\chi_2\|_{s,-s}\Big).$$

This implies immediately

$$\|R(z, z')\|_{s,-s} \leq C_{h,s}\Big(|z - z'|^{\delta_1} + |z - z'| + \|\chi_2 R(z, z')\chi_2\|_{s,-s}\Big).$$

Applying Lemma 6 once more, we conclude that for fixed h and $s > 1/2$ the limits

$$R(\lambda \pm i0) = \lim_{\epsilon\to 0,\ \pm\epsilon>0} R(\lambda \pm i\epsilon), \quad \lambda \in J$$

exist in $\mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$ and $J \ni \lambda \mapsto R(\lambda \pm i0) \in \mathcal{L}(\mathcal{H}^{0,s}, \mathcal{H}^{0,-s})$ is continuous. □

For the proof of our principal result we need the following.

Lemma 3. *Let $L(h)$ satisfy the assumptions (1.7) − (1.15) and assume the estimate (1.5) fulfilled. Let $\chi(x)$ be as in Proposition 3. Then there exists $C > 0$ such that for $(\lambda, \tau) \in J\times]0, 1]$ we have*

$$\|\chi(x)R(\lambda \pm i\tau)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \leq Ce^{Ch^{-p}},\ 0 < h \leq h_0. \tag{3.7}$$

Proof. Set $S_\theta = \{z \in \mathbb{C} : -2\theta < \arg z \leq \pi/4\}$, where $0 < \theta \leq \theta_0$ is fixed and $\theta_0 \in]0, \pi[$ was introduced in (1.13) making possible a complex scaling of $L(h)$. More precisely, following [23], [21], we construct a smooth function $[0, \pi] \times [0, \infty] \ni (\theta, t) \mapsto f_\theta(t) \in \mathbb{C}$ and by using the map

$$\kappa_\theta : \mathbb{R}^n \ni x = t\omega \mapsto f_\theta(t)\omega \in \mathbb{C}^n,\ t = |x|,$$
$$f_\theta(t) = t \text{ for } 0 \leq t \leq \rho_1,$$

we define the manifold Γ_θ which coincides with $\mathbb{R}^n$ along $B(0, \rho_1)$. Here $\rho_1 > \rho_0$ is the constant given in Proposition 3. The complex scaling operator $L_\theta = L_\theta(h)$ is defined in a domain D_θ of the space

$\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$. We refer to [23], [21] for more details.

Choose $0 < \delta < \mu_0/2$ so that (1.5) holds for $\lambda \in [\mu_0 - \delta, \mu_1 + \delta]$ and consider the compact set

$$\Omega_\delta = \{z \in S_\theta : \mu_0 - \delta \leq \operatorname{Re} z \leq \mu_1 + \delta\}.$$

The construction of L_θ and an application of Lemma 1 in [24] show that for every $m \in \mathbb{N}$ we have

$$\|\chi(L(h) - z)^{-1}\chi\| \leq C(\Omega_\delta, m) \exp\left(C(\Omega_\delta, m)h^{-k}\right),\ 0 < h \leq h_0(\Omega_\delta, m) \tag{3.8}$$

for

$$z \in \Omega_\delta \setminus \bigcup_{z_j \in \operatorname{Res} L(h) \cap \Omega_\delta} D(z_j : h^m)$$

with $k > n$ and $D(z_j : h^m) = \{z \in \mathbb{C} : |z - z_j| \leq h^m\}$. For the number of the scattering resonances lying in Ω_δ we have the estimate (see [23], [21])

$$\sharp\{z_j : z_j \in \operatorname{Res} L(h) \cap \Omega_\delta\} \leq C_1(\Omega_\delta) h^{-n^\sharp}$$

with $n^\sharp \geq n$. Next we fix an integer m satisfying $m \geq n^\sharp + 1$ and take $0 < h_1(\Omega_\delta, m) \leq h_0(\Omega_\delta, m)$ small enough to arrange

$$4C_1(\Omega_\delta) h^{m - n^\sharp} < \delta$$

for $0 < h \leq h_1(\Omega_\delta, m)$. This implies immediately the existence of $\gamma_1(h)$, $\gamma_2(h)$ so that

$$\mu_0/2 \leq \mu_0 - \delta < \gamma_1(h) < \mu_0 - \frac{\delta}{2},\ \ \mu_1 + \frac{\delta}{2} < \gamma_2(h) < \mu_1 + \delta\,,$$

and

$$\gamma_j(h) + i\tau \notin \bigcup_{z_j \in \operatorname{Res} L(h) \cap \Omega_\delta} D(z_j : h^m),\ \ j = 1, 2,\ \tau \geq 0\,.$$

Thus for fixed $k > n$ the estimate (3.8) holds for $z = \gamma_j(h) + i\tau,\ j = 1, 2,\ \tau \geq 0$. Set $\epsilon = \frac{\delta}{10}$ and introduce the domain

$$\Pi_\delta = \{z = \lambda + i\tau : \mu_0 - \delta \leq \lambda \leq \mu_1 + \delta,\ 0 \leq \tau \leq h^M\} \subset \Omega_\delta\,,$$

where $2M > k$. By a modification of the argument in Lemma 2 in [24], we may find a function $f(z, h)$ holomorphic in Π_δ with the properties:

(i) $|f(z, h)| \leq e$ in Π_δ,

(ii) $|f(z, h)| \geq 1/2$ in $[\mu_0 - \epsilon, \mu_1 + \epsilon] + i[0, h^M]$,

(iii) for $z \in \Pi_\delta \cap \{z : \operatorname{Re} z \leq \mu_0 - 4\epsilon$, or $\operatorname{Re} z \geq \mu_1 + 4\epsilon\}$ we have

$$|f(z, h)| \leq C_\epsilon \exp\Big(-\frac{\epsilon^2}{2h^{2M}}\Big),\ 0 \leq h \leq h(\epsilon).$$

To construct $f(z, h)$, take a function $\psi(x) \in C_0^\infty(\mathbb{R}^n)$, $0 \leq \psi(x) \leq 1$, such that

$$\psi(x) = \begin{cases} 0, & x \notin [\mu_0 - 3\epsilon, \mu_1 + 3\epsilon], \\ 1, & x \in [\mu_0 - 2\epsilon, \mu_1 + 2\epsilon] \end{cases}$$

and put

$$f(z, h) = (\pi\alpha^2)^{-1/2} \int \exp\Big(-\frac{(x - z)^2}{\alpha^2}\Big)\psi(x)dx,\ \alpha = h^M.$$

The check of (i) is trivial. To prove (ii), write $z = u + iv$ and take $u \in [\mu_0 - \epsilon, \mu_1 + \epsilon]$, $0 \leq v \leq \alpha$. Then

$$|f(z, h) - 1| \leq (\pi\alpha^2)^{-1/2} \exp\Big(\frac{v^2}{\alpha^2}\Big) \int \exp\Big(-\frac{(x - u)^2}{\alpha^2}\Big)|\psi(x) - 1|dx$$

$$\leq \pi^{-1/2}e\int e^{-y^2}|\psi(\alpha y+u)-1|dy \leq 2\pi^{-1/2}e\int_{\mathbb{R}\setminus B(0,\frac{\epsilon}{\alpha})} e^{-y^2}dy \leq \frac{1}{2}$$

for $0 < h \leq h(\epsilon)$, since $\frac{\epsilon}{\alpha} \longrightarrow \infty$ as $h \to 0$. Finally, to deal with (iii), assume for example that $u \leq \mu_0 - 4\epsilon$. We have

$$|f(z,h)| \leq (\pi\alpha^2)^{-1/2}\exp\Big(\frac{v^2}{\alpha^2}\Big)\int \exp\Big(-\frac{(x-u)^2}{\alpha^2}\Big)\psi(x)dx$$

$$\leq A\pi^{-1/2}h^{-M}\exp\Big(-\frac{\epsilon^2}{\alpha^2}\Big) \leq C_\epsilon \exp\Big(-\frac{\epsilon^2}{2\alpha^2}\Big),\ 0 \leq h \leq h(\epsilon).$$

Our assumptions imply that the cut-off resolvent

$$F(z,h) = \chi(x)R(z)\chi(x)$$

has no poles on $]\mu_0-\delta, \mu_1+\delta[$ (see Lemma 6 in Appendix 5.2) and the function $G(z,h) = f(z,h)F(z,h)$ is holomorphic in Π_δ. For $\operatorname{Im} z = 0$ we have $\|G(z,h)\| \leq eCe^{\frac{C}{h^p}}$, while for $\operatorname{Im} z = h^M$ we get

$$\|G(z,h)\| \leq \frac{Ce}{h^M} \leq e^{\frac{C_1}{h}}.$$

For $z \in \Pi_\delta$, $\operatorname{Re} z = \gamma_j(h),\ j = 1,2$, combining the estimates for $|f(z,h)|$ and $\|F(z,h)\|$, we obtain

$$\|G(z,h)\| \leq CC_\epsilon e^{\frac{C}{h^k}}e^{-\frac{\epsilon^2}{h^{2M}}} \leq C_2,\ 0 < h \leq h(\epsilon).$$

As in [24], by applying the maximum principle for $G(z,h)$ in Π_δ, we deduce the estimate (3.7) for $\lambda \in J,\ 0 < \tau \leq h^M$. The result for $h^M \leq \tau \leq 1$ follows trivially from the inequality $\|\chi(x)R(\lambda+i\tau)\chi(x)\|_{\mathcal{H}\to\mathcal{H}} \leq \frac{C}{\tau}$ and the proof is complete. ☐

Combining Propositions 3, 4 and Lemma 3, we obtain Theorem 1.

4. Microlocal resolvent estimates

In this section we obtain microlocal resolvent estimates in the trapping case. They are inspired by the microlocal estimates in Lemma 2.3 of [20] corresponding to the non-trapping case. In [20] (see also [17] for more general Hamiltonians), Robert and Tamura constructed an approximation of the propagator $\exp(ith^{-1}L)$ by Fourier integral operators which is uniform with respect to the time t and h. Following the argument in [20], it is convenient to localize in the outgoing or incoming regions of the phase space

$$\Gamma^\pm(R,d,\sigma_\pm) = \{(x,\xi) \in \mathbb{R}^{2n}\ :\ |x| > R,\ d^{-1} < |\xi| < d,\ \frac{\langle x.\xi\rangle}{|x||\xi|} \gtrless \sigma_\pm\}, \tag{4.1}$$

where $d > 1$, $-1 < \sigma_\pm < 1$ and $R \gg 1$ is large enough.

Consider a symbol $\omega_\pm(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ such that supp $\omega_\pm \subset \Gamma^\pm(R,d,\sigma_\pm)$ with $R > \rho_1 > \rho_0$, ρ_0, ρ_1 being the constants introduced in Proposition 3. Assume that $\omega_\pm(x,\xi)$ satisfy the estimates

$$|\partial_x^\alpha\partial_\xi^\beta\omega_\pm(x,\xi)| \leq C_{\alpha,\beta,L}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-L},\ \forall\alpha,\ \forall\beta \tag{4.2}$$

and any $L \gg 1$. Then we have the following.

Proposition 5. *Let $J \subset\subset \mathbb{R}^+$, let $\psi \in C^\infty(\mathbb{R}^n)$ be supported sufficiently away from $B(0,\rho_1)$, $\psi(x) = 1$ for $|x| \gg 1$ and let $\chi(x)$ be the function of Proposition 3. Then for any $(\lambda,\tau) \in J\times[0,1]$ and $h \in]0,h_0]$ the following assertions hold:*

i) For any $s > 1/2$, $\delta > 1$ and $N \in \mathbb{N}$ there exist $C_N > 0$ and $h_0 > 0$ such that for $h \in]0,h_0]$ we have

$$\|\psi R(\lambda\pm i\tau)\psi\omega_\pm(x,hD_x)\|_{-s+\delta,-s} \leq C_N h^{-1}(1+h^N\|\chi R(\lambda\pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}}). \tag{4.3}$$

ii) If $\sigma_+ > \sigma_-$, then for any $s \gg 1$ and $N \in \mathbb{N}$, there exist $C_N > 0$ and $h_0 > 0$ such that for $h \in]0,h_0]$ we have

$$\|\omega_\mp(x,hD_x)\psi R(\lambda\pm i\tau)\psi\omega_\pm(x,hD_x)\|_{-s,s} \leq C_N h^N(1+\|\chi R(\lambda\pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}}). \tag{4.4}$$

iii) Let $\varphi \in C_0^\infty(B(0,\rho_0))$, $\varphi = 1$ on $\overline{B(0,R_0)}$. Then for any $s \gg 1$ and $N \in \mathbb{N}$, there exist $C_N > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ we have

$$\|\varphi R(\lambda \pm i\tau)\psi\omega_\pm(x, hD_x)\|_{-s,0} \le C_N h^N (1 + \|\chi R(\lambda \pm i\tau)\chi\|_{\mathcal{H}\to\mathcal{H}}). \tag{4.5}$$

Proof. Introduce a function $\chi_1(x) \in C_0^\infty(B(0,\rho_1))$ such that $\chi_1(x) = 1$ on $\overline{B(0,\rho_0)}$ and $\chi\chi_1 = \chi_1$. As in the proof of Proposition 3, there exists a self-adjoint differential operator $\tilde{L} = \tilde{L}(h)$ on $L^2(\mathbb{R}^n)$, such that each $\lambda \in J$ is a non-trapping energy level for $\tilde{L}$ and the equality (3.3) implies

$$R(z)\psi\omega_\pm(x, hD_x) = (1 - \chi_1)\tilde{R}(z)\psi\omega_\pm(x, hD_x) + R(z)[\tilde{L}, \chi_1]\tilde{R}(z)\psi\omega_\pm(x, hD_x). \tag{4.6}$$

Then $i)$ is a consequence of Proposition 3 and Lemma 5, iii) (see Appendix 5.1). To treat (ii), observe that

$$\begin{aligned}\omega_\mp(x, hD_x)\psi R(z)\psi\omega_\pm(x, hD_x) &= \omega_\mp(x, hD_x)\psi\tilde{R}(z)\psi\omega_\pm(x, hD_x)\\ &+\omega_\mp(x, hD_x)\psi R(z)[\tilde{L}, \chi_1]\tilde{R}(z)\psi\omega_\pm(x, hD_x).\end{aligned}$$

Exploiting the adjoint operators, we get

$$\omega_\mp(x, hD_x)\psi R(z) = \Big(R(\overline{z})\psi\omega_\mp(x, hD_x)^*\Big)^*.$$

Thus using $i)$ and Lemma 5 once more, we obtain $ii)$. The analysis of $iii)$ follows a similar argument. □

5. Appendix

5.1. **Resolvent estimates in the non-trapping case.** In this section we recall some resolvent estimates in a non-trapping energy interval for a self-adjoint differential operator $\tilde{L} = \tilde{L}(h)$ defined in a dense domain in $L^2(\mathbb{R}^n)$. We say that $J \subset\subset \mathbb{R}^+$ is *non-trapping* for $\tilde{L}(h)$ if every $\lambda \in J$ is a *non-trapping energy level* for the principal symbol of $\tilde{L}(h)$ in the sense of Section 2. Denote by $\|.\|_{s,s'}$ the norm of $\mathcal{L}(L^{2,s}, L^{2,s'})$, where $L^{2,s}$ is the weight space $L^2(\mathbb{R}^n, \langle x\rangle^s dx)$.

Lemma 4. *Let $\tilde{L} = \tilde{L}(h)$ be a self-adjoint differential operator having the form* (1.10) *and satisfying* (1.11)*–*(1.12)*. Let $J \subset\subset \mathbb{R}^+$ be an open non-trapping interval for $\tilde{L}(h)$ and let $s > 1/2$. Then*

i) for $h \in]0, h_0]$ fixed, there exists $C_{h,s} > 0$ such that for any $z, z' \in \mathcal{B}_\pm$ we have

$$\|(\tilde{L} - z)^{-1} - (\tilde{L} - z')^{-1}\|_{s,-s} \le C_{h,s}|z - z'|^{\delta_1}, \quad \delta_1 = (s - 1/2)/(2s - 1/2).$$

ii) for $\lambda \in J$, the limits

$$(\tilde{L} - \lambda \mp i0)^{-1} := \lim_{\epsilon\to 0,\ \pm\epsilon>0}(\tilde{L} - \lambda \mp i\epsilon)^{-1}$$

exist in $\mathcal{L}(L^{2,s}, L^{2,-s})$, and there exist $C > 0$ and $h_0 > 0$ such that for $z \in \mathcal{B}_\pm$ and $h \in]0, h_0]$ we have

$$\|(\tilde{L} - z)^{-1}\|_{s,-s} \le Ch^{-1}. \tag{5.1}$$

Proof. This result can be obtained following the Mourre theory. First, there exists a *conjugate operator* $A = A(h)$ satisfying the following Mourre inequality: $\forall\Phi \in C_0^\infty(J)$, $\exists\gamma_0 > 0$ such that for all $h \in]0, h_0]$, we have

$$\Phi(\tilde{L})i^{-1}[\tilde{L}, A]\Phi(\tilde{L}) \ge \gamma_0 h\Phi(\tilde{L})^2. \tag{5.2}$$

For the construction of $A(h)$ we refer to [17], where a more general setup has been considered. The operator $A(h)$ has the form of a h-pseudodifferential operator with symbol

$$F(x, \xi) = K(x, \xi) + \frac{2x.\xi}{\langle\xi\rangle^2},$$

where $K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is compactly supported with respect to x and K is uniformly bounded with respect to (x, ξ).

According to Mourre type results, we conclude that $\tilde{L}$ has no eigenvalues in J for h small enough (see [13], [4]) and for $s > 1/2$ the following assertions hold (see [14], [11] and [9],[16], [17] for operators depending on h):

i') for $h \in]0, h_0]$ fixed, there exists $C_h > 0$ such that for $z, z' \in \mathcal{B}_\pm = \{z \in \mathbb{C};\ (\operatorname{Re} z, \pm \operatorname{Im} z) \in J \times]0,1]\}$ we have

$$\|\langle A\rangle^{-s}\Big((\tilde{L}-z)^{-1} - (\tilde{L}-z')^{-1}\Big)\langle A\rangle^{-s}\|_{L^2\to L^2} \leq C_h|z-z'|^{\delta_1}.$$

ii') there exist $C > 0$ and $h_0 > 0$ such that for $z \in \mathcal{B}_\pm$ and $h \in]0, h_0]$ we have

$$\|\langle A\rangle^{-s}(\tilde{L}-z)^{-1}\langle A\rangle^{-s}\|_{L^2\to L^2} \leq Ch^{-1},\ <A> = (1+A^2)^{1/2}. \tag{5.3}$$

Let us note that the operators $\langle x\rangle^{-s}\langle A\rangle^{s}$, $\langle A\rangle^{s}\langle x\rangle^{-s}$, $s \geq 0$, are uniformly bounded with respect to h in $\mathcal{L}(L^2)$. This follows from the functional calculus developed by Dimassi-Sjöstrand (see Sections 7 and 8 in [5]) or by using a complex interpolation as in Lemma 8.2 of [14]. Thus, writing

$$\langle x\rangle^{-s}(\tilde{L}-z)^{-1}\langle x\rangle^{-s} = \langle x\rangle^{-s}\langle A\rangle^{s}\ \langle A\rangle^{-s}(\tilde{L}-z)^{-1}\langle A\rangle^{-s}\ \langle A\rangle^{s}\langle x\rangle^{-s},$$

we complete the proof. □

Corollary 1. *Let P be a p-order h-admissible differential operator, $p \leq 2$ and let $s > 1/2$. Then, under the assumptions of Lemma 4, the following assertions hold:*

i") for $h \in]0, h_0]$ fixed, there exists $C_{h,s} > 0$ such that for $z, z' \in \mathcal{B}_\pm$ we have

$$\|P\Big((\tilde{L}-z)^{-1} - (\tilde{L}-z')^{-1}\Big)\|_{s,-s} \leq C_{h,s}\Big(|z-z'| + |z-z'|^{\delta_1}\Big).$$

ii") there exists $C > 0$ and $h_0 > 0$ such that for $z \in \mathcal{B}_\pm$ and $h \in]0, h_0]$ we have

$$\|P(\tilde{L}-z)^{-1}\|_{s,-s} \leq Ch^{-1}. \tag{5.4}$$

Proof. The ellipticity of $\tilde{L}$ and the calculus of Dimassi-Sjöstrand (see Sections 7 and 8 in [5]) imply that for $s \geq 0$ the operator $\langle x\rangle^{-s}P(\tilde{L}-i)^{-1}\langle x\rangle^{s}$ is uniformly bounded in $\mathcal{L}(L^2)$ with respect to h. Then ii") is a direct consequence of ii) and the resolvent equation

$$(\tilde{L}-z)^{-1} = (\tilde{L}-i)^{-1} + (z-i)(\tilde{L}-i)^{-1}(\tilde{L}-z)^{-1}.$$

In the same way i") follows from i) and the relation

$$(\tilde{L}-z)^{-1} - (\tilde{L}-z')^{-1} = (z-z')(\tilde{L}-i)^{-1}(\tilde{L}-z)^{-1} + (z'-i)(\tilde{L}-i)^{-1}\Big((\tilde{L}-z)^{-1} - (\tilde{L}-z')^{-1}\Big).$$

□

Finally, we formulate some microlocal resolvent estimates which generalize Lemma 2.3 of [20].

Lemma 5. *Let $\tilde{L} = \tilde{L}(h)$ be a self-adjoint differential operator having the form (1.10) and satisfying (1.11)–(1.12). Let $J \subset\subset \mathbb{R}^+$ be an open non-trapping interval for $\tilde{L}(h)$ and let $\omega_\pm(x,\xi)$ be a symbol supported in an outgoing (incoming) region as in Section 4. Then for any $(\lambda, \tau) \in J \times]0,1]$ the following assertions hold:*

i) For any $s > 1/2$, $\delta > 1$ there exist $C > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ we have

$$\|(\tilde{L}-\lambda \mp i\tau)^{-1}\omega_\pm(x, hD_x)\|_{-s+\delta,-s} \leq Ch^{-1}. \tag{5.5}$$

ii) If $\sigma_+ > \sigma_-$, then for any $s \gg 1$ there exist $C > 0$ and $h_0 > 0$ such that for all $N \in \mathbb{N}$ and $h \in]0, h_0]$ we have

$$\|\omega_\mp(x, hD_x)(\tilde{L}-\lambda \mp i\tau)^{-1}\omega_\pm(x, hD_x)\|_{-s,s} \leq C_N h^N. \tag{5.6}$$

iii) Let $\rho_1 < R$ and let $\varphi \in C_0^\infty(B(0,\rho_1))$, $\varphi = 1$ on $\overline{B(0,R_0)}$. Let P be a p-order h-admissible differential operator, $p \leq 2$. Then for any $s \gg 1$ and $N \in \mathbb{N}$, there exist $C_N > 0$ and $h_0 > 0$ such that for $h \in]0, h_0]$ we have

$$\|\varphi P(\tilde{L}-\lambda \mp i\tau)^{-1}\omega_\pm(x, hD_x)\|_{-s,0} \leq C_N h^N. \tag{5.7}$$

The proof of this lemma is analogous to that of Lemma 2.3 of [20] exploiting the constructions of long time approximations in section 4 of [17] for a large class of long-range perturbations of the Laplacian. The essential point is the construction of the phases $\varphi_\pm$ (see Proposition 4.1 of [17]). To establish the estimates when we have a differential operator P, we apply the argument of the proof of Corollary 1 based on the resolvent equation and the fact that the operator $\langle x\rangle^{-s}P(\tilde{L}-i)^{-1}\langle x\rangle^{s}$ is uniformly bounded in $\mathcal{L}(L^2)$. We leave the details to the reader.

5.2. **Boundary values of the cut-off resolvent $\chi R(z)\chi$.** Our purpose is to show that for fixed $h\in]0,h_0]$ the limits

$$\chi(x)R(\lambda\pm i0)\chi(x)=\lim_{\epsilon\to 0,\ \pm\epsilon>0}\chi(x)R(\lambda\pm i\epsilon)\chi(x),\ \lambda\in J$$

exist in $\mathcal{L}(\mathcal{H},\ \mathcal{H})$, provided $\chi\in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on $B(0,\rho_0)$ and $\chi(x)=0$ for $|x|\geq\rho_0+a_0=\rho_1$. To do this, it is sufficient to show that there are no resonances $z_j\in \operatorname{Res} L(h)$ in the interval $]\mu_0-\delta,\mu_1+\delta[$.

Lemma 6. *Assume* (1.7) – (1.15) *fulfilled. Then for* $0<h\leq h_0$ *we have*

$$\operatorname{Res} L(h)\cap]\mu_0-\delta,\mu_1+\delta[=\emptyset.$$

Proof. The proof can be obtained from the results in [23], [21]. Here we will give only indications of how to extract it from there. Assume h fixed and let $z_0\in]\mu_0-\delta,\ \mu_1+\delta[$ be a resonance for $L(h)$. Denote by $\mathcal{H}_{\rho_0+a_0}$ the space of elements of $\mathcal{H}$ which vanish outside $B(0,\rho_0+a_0)$. Then by using a complex scaling argument and Lemma 3.5 in [23], we conclude that for $v\in\mathcal{H}_{\rho_0+a_0}$ and z in a neighborhood of z_0 we have

$$(L-z)^{-1}v=\sum_{j=1}^{N}(z_0-z)^{-j}v_{-j}+G(z) \tag{5.8}$$

with $G(z)$ holomorphic in a neighborhood of z_0 and N independent of v. Consider the spectral projection

$$\pi_{\theta,z_0}=\frac{1}{2\pi i}\int_{\gamma(z_0)}(z-L_\theta)^{-1}dz,$$

$L_\theta=L_\theta(h)$ being the complex scaling operator defined in the proof of Lemma 3, while $\gamma(z_0):\ [0,2\pi]\ni s\longrightarrow z_0+\epsilon_1 e^{is}$ with $\epsilon_1>0$ small enough. Let F_{θ,z_0} be the image $\pi_{\theta,z_0}(\mathcal{H}_\theta)$ and let

$$\pi_{0,z_0}=\frac{1}{2\pi i}\int_{\gamma(z_0)}(z-L)^{-1}dz:\ \mathcal{H}_{\text{comp}}\longrightarrow\mathcal{D}_{\text{loc}}.$$

As in Proposition 3.6 in [23], we conclude that there exists a bijection between the spaces F_{θ,z_0} and $\pi_{0,z_0}(\mathcal{H}_{\rho_0+a_0})$. Since z_0 is a resonance, the space F_{θ,z_0} is not trivial and we can find $v\in\mathcal{H}_{\rho_0+a_0}$ so that (5.8) holds with $\sum_{j=1}^{N}(z_0-z)^{-j}v_{-j}\neq 0$. For simplicity assume $v_{-j}=0$ for $j=2,...,N$ and let $v_{-1}\neq 0$. Take a cut-off function $\Phi(x)\in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $B(0,\rho_0)$ so that $(\Phi,v_{-1})_{\mathcal{H}}\neq 0$. Therefore,

$$\lim_{\epsilon\to 0}(\Phi,i\epsilon(L-z_0+i\epsilon)^{-1}v_{-1})_{\mathcal{H}}=(\Phi,P_{\{z_0\}}v_{-1})_{\mathcal{H}}\neq 0,$$

$P_{\{z_0\}}$ being the spectral projector of L related to z_0, and we get $z_0\in\sigma_{pp}(L)$ which contradicts (1.15). The case of a multiple pole can be treated in the same way. $\square$

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