MEROMORPHIC CONTINUATION OF THE SPECTRAL SHIFT FUNCTION

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Abstract. We obtain a representation of the derivative of the spectral shift function $\xi(\lambda, h)$ in the framework of semi-classical "black box" perturbations. Our representation implies a meromorphic continuation of $\xi(\lambda, h)$ involving the semi-classical resonances. Moreover, we obtain a Weyl type asymptotics of the spectral shift function as well as a Breit-Wigner approximation in an interval $(\lambda - \delta, \lambda + \delta),\ 0 < \delta < \epsilon h$.

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1. Introduction

The purpose of this paper is to obtain a meromorphic continuation of the derivative of the spectral shift function $\xi(\lambda, h)$. This problem is closely related to the trace formulae (see [13], [34], [35] [21], [23], [30], [28], [29]) and to resonances expansions ([7], [32]). For compact perturbations the function $\xi(\lambda, h)$ coincides with the *scattering phase*

$$\sigma(\lambda, h) = \frac{1}{2\pi i} \log \det S(\lambda, h),\ \ \lambda \in \mathbb{R},$$

where $S(\lambda, h) = I + A(\lambda, h) :\ L^2(S^{n-1}) \longrightarrow L^2(S^{n-1})$ is the scattering operator and for more information about the spectral shift function we refer to [33]. In the classical case ($h = 1$) the first result proving a representation of $\sigma(\lambda) = \sigma(\lambda, 1)$ containing the resonances $z_j \in \mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ was established by Melrose [16] for obstacle scattering in odd dimensions $n \geq 3$. More precisely, given a function $\chi(t) \in C^\infty(\mathbb{R})$ such that $0 \leq \chi(t) \leq 1,\ \chi(t) = 1$ for $t \leq 2,\ \chi(t) = 0$ for $t \geq 3$, Melrose showed that

$$\sigma(\lambda) = \sigma_{\text{sing}}(\lambda) + \sigma_{\text{reg}}(\lambda),$$

with

$$\frac{d}{d\lambda}\sigma_{\text{sing}}(\lambda) = -\frac{1}{\pi} \sum_j \chi\Big(\frac{|z_j|}{\lambda}\Big) \frac{\operatorname{Im} z_j}{|\lambda - z_j|^2},\ \ \sigma_{\text{sing}}(0) = 0,\ \ \lambda \in \mathbb{R},$$

$$\sigma_{\text{reg}}(\lambda) \in S^n(\mathbb{R})\,.$$

Since $\sigma(\lambda, h)$ is the logarithmic derivative of the *scattering determinant*

$$s(\lambda, h) = \det(I + A(\lambda, h))\,,$$

it is natural to examine the behavior of $s(z, h)$ for z in the "physical half plane", where we have no resonances. This idea was developed by Guillopé and Zworski [13] for the analysis of the scattering resonances for certain Riemann surfaces and in the classical case $h = 1$, Zworski [34], [35] gave an elegant proof of the trace formula for "black box" compact perturbations based on the meromorphic continuation of $s(z)$ (see [34] for other works on trace formulae).

In [21], [23] the Breit-Wigner approximation for the scattering phase has been justified for "black box" scattering with compact perturbations in the classical and the semi-classical cases.

Among the ideas introduced in [21], [23], one of the main point in [23] was the estimate of the holomorphic function $g(z,h)$,

$$|g(z,h)| \leq C(\Omega)h^{-n^{\#}},\ n^{\#} \geq n \tag{1.1}$$

in the local factorization

$$s(z,h) = e^{g(z,h)} \frac{\overline{P(\overline{z},h)}}{P(z,h)},\ z \in \Omega\,,$$

where

$$P(z,h) = \prod_{\substack{w \in \mathrm{Res\ L(h)} \cap \Omega_\epsilon,\\ \mathrm{Im}\, w \neq 0}} (z-w)\,,$$

$$\Omega = (a,b) + i(-c,c),\ 0 < a < b,\ c > 0,\ \Omega_\epsilon = \{z \in \mathbb{C} : d(\Omega, z) < \epsilon\},\ \epsilon > 0\,.$$

Here $L(h)$ is a compactly supported perturbation of the operator $-h^2\Delta,\ 0 < h \leq h_0$, and $n^{\#}$ depends on the estimates of the number of the eigenvalues of the reference operator. The local factorization implies immediately

$$\partial_z \sigma(z,h) = \frac{1}{2\pi i}\partial_z g(z,h) + \frac{1}{2\pi i} \sum_{\substack{w \in \mathrm{Res}\ L(h)\, \cap\, \Omega_\epsilon,\\ \mathrm{Im}\, w \neq 0}} \Big(\frac{1}{z-\overline{w}} - \frac{1}{z-w}\Big),\ z \in \Omega \tag{1.2}$$

and for $\lambda \in (a,b)$ we obtain an analogue of the formula of Melrose mentioned above. Combining (1.2) with the Birman-Krein formula one obtains easily the trace formula of [28] exploiting the meromorphic continuation of $\partial_z \sigma(z,h)$ in $\{z \in \mathbb{C} : \mathrm{Im} \leq 0\}$ (see Theorem 1 in [23]). Moreover, a similar factorization has been established in [23] in domains $\lambda + h\Omega$ with an improved estimate for the holomorphic function $g(z,h)$.

In the case of "black box" long-range perturbations the existence of the scattering operator and that of the scattering determinant are far from apparent. In this direction Sjöstrand [28], [29] proposed powerful techniques based on the complex scaling operators, introduced in [30], and complex analysis. The scattering determinant is replaced by $D(z,h) = \det(I + \tilde{K}(z))$, where $\tilde{K}(z)$ is trace class operator which is not uniquely determined and the resonances are the zeros of $D(z,h)$. Applying the approach of Sjöstrand, J.-F. Bony [1], [2], established upper and lower bounds on the number of the semi-classical resonances in small domains and the Breit-Wigner approximation has been extended to long-range perturbations in [3]. For a pair of self-adjoint operators $L_j(h),\ j = 1,2$, satisfying some assumptions (see Section 2) the spectral shift function $\xi(\lambda,h)$ is a distribution in $\mathcal{D}'(\mathbb{R})$ such that

$$< \xi'(\lambda,h), f(\lambda) >_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})} = \mathrm{tr_{bb}}\Big(f(L_2(h)) - f(L_1(h))\Big), f(\lambda) \in C_0^\infty(\mathbb{R})\,,$$

where $\mathrm{tr_{bb}}$ is a generalized trace defined in Section 2. We denote by Res $L_j(h), j = 1,2$ the set of the resonances $w \in \overline{\mathbb{C}}_-$ of $L_j(h)$.

In this work we are strongly inspired by the approach in [23] and our main goal is to obtain an analogue of (1.2) in the cases when a scattering determinant is not available. We show that the representation (1.2) remains true in the general case of semi-classical "black box" scattering, replacing $\sigma'(\lambda,h)$ by the "regular part"

$$\xi'(\lambda,h) - \Big[\sum_{w \in \mathrm{Res}\ L_j(h)\, \cap (a,b)} \delta(\lambda - w)\Big]_{j=1}^{2},$$

where here and throughout the paper we use the notation $[a_j]_{j=1}^2 = a_2 - a_1$. Our principal result is the following.

Theorem 1. *Assume that $L_j(h)$, $j = 1, 2$, satisfy the assumptions of Section 2. Let $\Omega \subset\subset e^{i]-2\theta, 2\theta[}]0, +\infty[$, $0 < \theta < \pi/2$, be an open simply connected set and let $W \subset\subset \Omega$ be an open simply connected and relatively compact set which is symmetric with respect to $\mathbb{R}$. Assume that $J = \Omega \cap \mathbb{R}^+$, $I = W \cap \mathbb{R}^+$ are intervals. Then for $\lambda \in I$ we have the representation*

$$\xi'(\lambda, h) = \frac{1}{\pi} \operatorname{Im} r(\lambda, h) + \Big[\sum_{\substack{w \in \operatorname{Res} L_j \cap \Omega, \\ \operatorname{Im} w \neq 0}} \frac{-\operatorname{Im} w}{\pi |\lambda - w|^2} + \sum_{w \in \operatorname{Res} L_j \cap J} \delta(\lambda - w) \Big]_{j=1}^2, \tag{1.3}$$

where $r(z, h) = g_+(z, h) - \overline{g_+(\overline{z}, h)}$, $g_+(z, h)$ is a function holomorphic in Ω and $g_+(z, h)$ satisfies the estimate

$$|g_+(z, h)| \leq C(W) h^{-n^{\#}}, \ z \in W \tag{1.4}$$

with $C(W) > 0$ independent on $h \in]0, h_0]$.

Remarks.

- The terms related to the resonances are measures. In fact, the resonances w, $\operatorname{Im} w < 0$, are related to harmonic measures
$$\omega_{\mathbb{C}_-}(w, E) = -\frac{1}{\pi} \int_E \frac{\operatorname{Im} w}{|t - w|^2} dt, \ E \subset \mathbb{R} = \partial \mathbb{C}_- ,$$
while the resonances $w \in \mathbb{R}^+$ coincide with the embedded eigenvalues of $L_j(h), j = 1, 2$. Moreover, in a small neighborhood $U_\lambda(h)$ of every $\lambda \in I \setminus \cup_{j=1}^2 \{\lambda \in \mathbb{R} : \lambda \in \sigma_{pp}(L_j(h))\}$ the derivative $\xi'(\lambda, h)$ coincides with a *real analytic function* on $U_\lambda(h)$. In particular, if we have no embedded positive eigenvalues of $L_j(h)$ in I, then $\xi'(\lambda, h)$ is real analytic in I.

- The representations of $\xi'(\lambda, h)$ obtained in [25], [6] involve the traces of the cut-off resolvents $\chi(L_j - \lambda \mp i0)^{-1}\chi$, $\chi \in C_0^\infty(\mathbb{R}^n)$, and some regular terms whose meromorphic continuation is far from apparent. The form of $\xi'(\lambda, h)$ in [25], [6] has been used for the investigation of the Weyl type asymptotics of $\xi(\lambda, h)$ (see also [17], [5] for semi-classical asymptotics in the trapping case).

The proof of (1.3) relies heavily on the work of Sjöstrand [29], while the arguments in [23] were self-contained and based on the semi-classical estimates of the scattering determinant. Having in mind (1.3), we obtain in the general case of "black box" semi-classical scattering several results:

I) We establish a Weyl type asymptotics of the spectral shift function in the general framework of semi-classical "black box" perturbations improving our previous result [6] and working without any assumption on the behavior of the resonances close to the real axis. We generalize the results of Christiansen [8] for compact perturbations and those of Robert [25] for long-range perturbations. Theorem 1 allows to consider the sum of the harmonic measures related to the resonances w, $\operatorname{Im} w \neq 0$, as a monotonic function and to apply a Tauberian argument as in [16].

II) We present a new direct and short proof of the recent result of J.-F. Bony and Sjöstrand [3] on the Breit-Wigner approximation in the long-range case (see Theorem 3). For this purpose the Weyl asymptotics obtained in Theorem 2 plays an essential role. Moreover, Theorem 2 and Theorem 3 are established under the "black box" assumptions in Section 2 and the condition (5.1). Thus we have an unified approach to these problems. Next, assuming the existence of free resonances domain, we obtain a Breit-Wigner approximation involving only the resonances w lying in small "boxes"

$$\{w \in \mathbb{C} : |\operatorname{Re} w - \lambda| \leq R(h), \ |\operatorname{Im} w| \leq R_1(h)\}$$

with $R(h) = \sqrt{hR_1(h)} = \mathcal{O}(h^\infty)$.

III) In the same way as in [23], we obtain the local trace formula of Sjöstrand [28], [29] in a slightly stronger version (see Section 7). Moreover, we prove a trace formula involving the unitary groups $e^{-i\frac{t}{h}L_j(h)}$, $j = 1, 2$ (see Theorem 5) which is a semi-classical version of the classical trace formulae.

We expect that the approach of our work could be useful in other situations as in the analysis of periodic potentials [10] or the study of matrix Schrödinger operators [18] if a representation like (1.3) is established.

The plan of the paper is the following. In Section 2 we introduce the "black box" scattering assumptions and in Section 3 we obtain a formula for $\xi'(\lambda, h)$ involving the limits of the functions $\sigma_\pm(z)$ as $\operatorname{Im} z \to 0$. Theorem 1 is proved in Section 4 and in Section 5 we establish a Weyl type asymptotics for the spectral shift function $\xi(\lambda, h)$. The semi-classical Breit-Wigner approximation is established in Section 6 together with a stronger approximation based on some recent results of Stefanov [31]. In Section 7 we prove some trace formulae combining (1.3) with the arguments of [23]. In particular, we obtain a trace formula involving the unitary groups $e^{-ih^{-1}L_j}$. Finally, in Section 8 the Breit-Wigner approximation is applied to establish the existence of clusters of resonances close to the real axis.

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2. Preliminaries

We start by the abstract "black box" scattering assumptions introduced in [30], [28] and [29]. The operators $L_j(h) = L_j, j = 1, 2, \ 0 < h \leq h_0$, are defined in domains $\mathcal{D}_j \subset \mathcal{H}_j$ of a complex Hilbert space $\mathcal{H}_j$ with an orthogonal decomposition

$$\mathcal{H}_j = \mathcal{H}_{R_0,j} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)),\ B(0, R_0) = \{x \in \mathbb{R}^n : |x| \leq R_0\},\ \ R_0 > 0,\ \ n \geq 2.$$

Below $h > 0$ is a small parameter and we suppose the assumptions satisfied for $j = 1, 2$. We suppose that $\mathcal{D}_j$ satisfies

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D}_j = H^2(\mathbb{R}^n \setminus B(0, R_0)), \tag{2.1}$$

uniformly with respect to h in the sense of [28]. More precisely, equip $H^2(\mathbb{R}^n \setminus B(0, R_0))$ with the norm $\| < hD >^2 u\|_{L^2}$, $< hD >^2 = 1 + (hD)^2$, and equip $\mathcal{D}_j$ with the norm $\|(L_j + i)u\|_{\mathcal{H}_j}$. Then we require that $\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} : \mathcal{D}_j \longrightarrow H^2(\mathbb{R}^n \setminus B(0, R_0))$ is uniformly bounded with respect to h and this map has a uniformly bounded right inverse.

Assume that

$$\mathbb{1}_{B(0,R_0)}(L_j+i)^{-1} \text{is compact} \tag{2.2}$$

and

$$(L_j u)|_{\mathbb{R}^n\setminus\overline{B(0,R_0)}} = Q_j\Big(u|_{\mathbb{R}^n\setminus\overline{B(0,R_0)}}\Big), \tag{2.3}$$

where Q_j is a formally self-adjoint differential operator

$$Q_j u = \sum_{|\nu|\le 2} a_{j,\nu}(x;h)(hD_x)^\nu u, \tag{2.4}$$

with $a_{j,\nu}(x;h) = a_{j,\nu}(x)$ independent of h for $|\nu| = 2$ and $a_{j,\nu} \in C_b^\infty(\mathbb{R}^n)$ uniformly bounded with respect to h.

We assume also the following properties:

There exists $C > 0$ such that

$$l_{j,0}(x,\xi) = \sum_{|\nu|=2} a_{j,\nu}(x)\xi^\nu \ge C|\xi|^2, \tag{2.5}$$

$$\sum_{|\nu|\le 2} a_{j,\nu}(x;h)\xi^\nu \longrightarrow |\xi|^2,\ \ |x| \longrightarrow \infty \tag{2.6}$$

uniformly with respect to h.

There exists $\overline{n} > n$ such that we have

$$\Big|a_{1,\nu}(x;h) - a_{2,\nu}(x;h)\Big| \le \mathcal{O}(1)\langle x\rangle^{-\overline{n}} \tag{2.7}$$

uniformly with respect to h. This assumption will guarantee that for every $f \in C_0^\infty(\mathbb{R})$ the operator $f(L_1) - f(L_2)$ is "trace class near infinity".

There exist $\theta_0 \in]0, \frac{\pi}{2}[$, $\epsilon > 0$ and $R_1 > R_0$ so that the coefficients $a_{j,\nu}(x;h)$ of Q_j can be extended holomorphically in x to

$$\Gamma = \{r\omega;\ \omega \in \mathbb{C}^n,\ \text{dist}\,(\omega, S^{n-1}) < \epsilon,\ r \in \mathbb{C}, r \in e^{i[0,\theta_0]}]R_1, +\infty[\} \tag{2.8}$$

and (2.6), (2.7) extend to Γ.

Let $R > R_0$, $T = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$, $\tilde{R} > 2R$. Set

$$\mathcal{H}_j^{\#} = \mathcal{H}_{R_0,j} \oplus L^2(T\setminus B(0,R_0))$$

and consider a differential operator

$$Q_j^{\#} = \sum_{|\nu|\le 2} a_{j,\nu}^{\#}(x;h)(hD)^\nu$$

on T with $a_{j,\nu}^{\#}(x;h) = a_{j,\nu}(x;h)$ for $|x| \le R$ satisfying (2.3), (2.4), (2.5) with $\mathbb{R}^n$ replaced by T. Consider a self-adjoint operator $L_j^{\#} : \mathcal{H}_j^{\#} \longrightarrow \mathcal{H}_j^{\#}$ defined by

$$L_j^{\#} u = L_j\varphi u + Q_j^{\#}(1-\varphi)u,\ u \in \mathcal{D}_j^{\#},$$

with domain

$$\mathcal{D}_j^{\#} = \{u \in \mathcal{H}_j^{\#} :\ \varphi u \in \mathcal{D}_j,\ (1-\varphi)u \in H^2\},$$

where $\varphi \in C_0^\infty(B(0,R);[0,1])$ is equal to 1 near $\overline{B(0,R_0)}$.

Denote by $N(L_j^{\#}, [-\lambda, \lambda])$ the number of eigenvalues of $L_j^{\#}$ in the interval $[-\lambda, \lambda]$. Then we assume that

$$N(L_j^{\#}, [-\lambda, \lambda]) = \mathcal{O}(\Big(\frac{\lambda}{h^2}\Big)^{n_j^{\#}/2}),\ n_j^{\#} \geq n,\ \lambda \geq 1. \tag{2.9}$$

Finally, we suppose that with some constant $C \geq 0$ independent on h we have

$$\text{sp } L_j(h) \subset [-C, \infty[,\ \ j = 1, 2, \tag{2.10}$$

where sp (L) denotes the spectrum of L. This condition is a technical one and we expect that by a more fine version of Proposition 1 we could cover the general case.

Given $f \in C_0^\infty(\mathbb{R})$ independent on h and $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $\overline{B(0, R_0)}$ we can define $\mathrm{tr_{bb}}[f(L_j)]_{j=1}^2$, as in [28], [29], by the equality

$$\mathrm{tr_{bb}}\Big(f(L_2) - f(L_1)\Big) = [\mathrm{tr}(\chi f(L_j)\chi + \chi f(L_j)(1-\chi) + (1-\chi) f(L_j)\chi)]_{j=1}^2$$
$$+\mathrm{tr}[(1-\chi) f(L_j)(1-\chi)]_{j=1}^2 .$$

Following [28], [29], we can define the resonances $w \in \overline{\mathbb{C}}_-$ by the complex scaling method as the eigenvalues of the complex scaling operators $L_{j,\theta},\ j = 1, 2$. We denote by Res $L_j(h),\ j = 1, 2$, the set of resonances and set $n^{\#} = \max\{n_1^{\#}, n_2^{\#}\}$.

3. Representation of the derivative of the spectral shift function

Consider the resolvents

$$R_j(\lambda \pm i\epsilon) = i \int_0^{\pm\infty} e^{it\lambda} e^{-it(L_j \mp i\epsilon)} dt,\ \ \lambda \in \mathbb{R},\ \ \epsilon > 0.$$

$$R_j(\lambda - i\epsilon) = -i \int_{-\infty}^0 e^{it\lambda} e^{-it(L_j + i\epsilon)} dt.$$

Given a function $f(\lambda) \in C_0^\infty(\mathbb{R})$, we have

$$\frac{1}{2\pi i} \int R_j(\lambda + i\epsilon) f(\lambda) d\lambda = \frac{1}{2\pi} \int_0^\infty \hat{f}(-t) e^{-itL_j - t\epsilon} dt,$$

$$-\frac{1}{2\pi i} \int R_j(\lambda - i\epsilon) f(\lambda) d\lambda = \frac{1}{2\pi} \int_{-\infty}^0 \hat{f}(-t) e^{-itL_j + t\epsilon} dt,$$

where $\hat{f}$ denotes the Fourier transform of f. Choose $z_0 \in \mathbb{R}^-$ which is away from sp $(L_j),\ \ j = 1, 2$, and set $g(\lambda) = (\lambda - z_0)^m f(\lambda)$, where the integer $m > n/2$ will be taken sufficiently large and independent on h. Applying the above formula, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\, \mathrm{tr_{bb}} \int \Big[(L_j - z_0)^{-m}\Big((\lambda + i\epsilon - z_0)^m R_j(\lambda + i\epsilon) - (\lambda - i\epsilon - z_0)^m R_j(\lambda - i\epsilon)\Big)\Big]_{j=1}^2 f(\lambda) d\lambda \\ = \frac{1}{2\pi}\, \mathrm{tr_{bb}} \Big[(L_j - z_0)^{-m}\Big(\int_0^\infty e^{-\epsilon t - itL_j}(\hat{g}(-t) + i\epsilon G_{+,\epsilon}(t)) dt \\ + \int_{-\infty}^0 e^{\epsilon t - tL_j}(\hat{g}(-t) + i\epsilon G_{-,\epsilon}(t)) dt\Big)\Big]_{j=1}^2 .\end{aligned} \tag{3.1}$$

Here $G_{\pm,\epsilon}(t)$ are some functions in $\mathcal{S}(\mathbb{R})$ related to the Fourier transform of $\lambda^k f(\lambda),\ 0 \leq k \leq m-1$, which are uniformly bounded with respect to $0 < \epsilon < 1$. To justify the limit $\epsilon \downarrow 0$ in (3.1), we need to establish the estimates of the trace uniformly with respect to $\epsilon > 0$. To do this we will prove the following.

Lemma 1. *For any $t \in \mathbb{R}$, the trace $\mathrm{tr}_{\mathrm{bb}}\Big[(L_j - z_0)^{-m} e^{-itL_j}\Big]_{j=1}^2$ is well defined, and*

$$\mathrm{tr}_{\mathrm{bb}}\Big[(L_j - z_0)^{-m} e^{-itL_j}\Big]_{j=1}^2 = \mathcal{O}(h^{-n^{\#}}(1+|t|)).$$

Proof. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be equal to 1 near $\overline{B(0,R_1)}$, $R_1 > R_0$. Since the operators $\chi(L_j - z_0)^{-m}$ and $(L_j - z_0)^{-m}\chi$ are trace class (see [28]) and e^{-itL_j} is uniformly bounded with respect to t, it is clear that $\chi(L_j - z_0)^{-m} e^{-itL_j}$ and $(L_j - z_0)^{-m} e^{-itL_j}\chi$ are trace class ones with trace bounded by $\mathcal{O}(h^{-n^{\#}})$. To be more precise let us note that in [29] the condition (2.10) is not assumed and we can formally apply the results of [29] for $z_0 \in \mathbb{C} \setminus \mathbb{R}$. In our case $z_0 \in \mathbb{R}^-$ and according to the resolvent equation we have

$$(L_j - z_0)^{-m} = (L_j - z_1)^{-m}\Big(I + (z_0 - z_1)(L_j - z_0)^{-1}\Big)^m .$$

So taking $z_1 \in \mathbb{C} \setminus \mathbb{R}$, we obtain the trace class properties mentioned above.

Now consider the operator

$$\Big[(1-\chi)(L_j - z_0)^{-m} e^{-itL_j}(1-\chi)\Big]_{j=1}^2 .$$

By Duhamel formula we obtain

$$(1-\chi)(L_j - z_0)^{-m} e^{-itL_j}(1-\chi) = e^{-itQ_j}(1-\chi)(L_j - z_0)^{-m}(1-\chi)$$
$$+ i\int_0^t e^{-i(t-s)Q_j}[\chi, L_j](L_j - z_0)^{-m} e^{-isL_j}\, ds .$$

The integrand is a trace class operator with trace bounded by $\mathcal{O}(h^{-n^{\#}})$ and it remains to study the operator

$$\Big[e^{-itQ_j}(1-\chi)(L_j - z_0)^{-m}(1-\chi)\Big]_{j=1}^2 .$$

For $R_1 > R_0$, $\chi_0 \in C_0^\infty(\mathbb{R}^n)$ equal to 1 near $\overline{B(0,R_1)}$ and $\chi_0 \prec \chi$ we have

$$(L_j - z_0)^{-1}(1-\chi) = (1-\chi_0)(Q_j - z_0)^{-1}(1-\chi) + (L_j - z_0)^{-1}[Q_j, \chi_0](Q_j - z_0)^{-1}(1-\chi) .$$

Here and below the notation $\varphi \prec \psi$ means that $\psi = 1$ on supp φ. Choose cut-off functions $\theta_N \prec \ldots \prec \theta_1 \prec \chi$ so that $\theta_N = 1$ on $\overline{B(0,R_0)}$ and apply the telescopic formula

$$(L_j - z_0)^{-1}[Q_j, \chi_0](Q_j - z_0)^{-1}(1-\chi)$$
$$= (L_j - z_0)^{-1}[Q_j, \chi_0](Q_j - z_0)^{-1}[Q_j, \theta_N](Q_j - z_0)^{-1}[Q_j, \theta_{N-1}]\ldots[Q_j, \theta_1](Q_j - z_0)^{-1}(1-\chi) .$$

For $N > n/2$ this operator is trace class. In fact, for $\tilde{\chi} \in C_0^\infty$ equal to 1 on supp θ_N the operator

$$\tilde{\chi}(Q_j - i)^{-N/2}(Q_j - i)^{N/2}[Q_j, \theta_N](Q_j - z_0)^{-1}\ldots[Q_j, \theta_1](Q_j - z_0)^{-1}(1-\chi)$$

is trace class, while $(L_j - z_0)^{-1}[Q_j, \chi_0](Q_j - z_0)^{-1}$ is bounded. Here we have used the fact that Q_j are elliptic operators and

$$(Q_j - z_0)^{-1} = \mathcal{O}(1) : H^N(\mathbb{R}^n) \longrightarrow H^{N+2}(\mathbb{R}^n), \ \forall N \in \mathbb{N}.$$

Repeating this procedure, we obtain modulo trace class operators

$$e^{-itQ_j}(L_j - z_0)^{-m}(1-\chi)$$

$$= e^{-itQ_j}(1-\theta_m)(Q_j-z_0)^{-1}...(1-\theta_1)(Q_j-z_0)^{-1}(1-\chi)\,.$$

In the same way, since $\theta_k \prec \theta_{k-1}$, each term $\theta_k(Q_j-z_0)^{-1}(1-\theta_{k-1})$ in the above product is trace class operator and modulo a trace class operator we are going to study

$$\Big[e^{-itQ_j}(Q_j-z_0)^{-m}(1-\chi)\Big]_{j=1}^2.$$

Consider the difference

$$(Q_2-z_0)^{-m}e^{-itQ_2}-(Q_1-z_0)^{-m}e^{-itQ_1}$$
$$= e^{-itQ_2}\Big((Q_2-z_0)^{-m}-(Q_1-z_0)^{-m}\Big)+\Big(e^{-itQ_2}-e^{-itQ_1}\Big)(Q_1-z_0)^{-m}.$$

For the first term at the right hand side observe that the operator $(Q_2-z_0)^{-m}-(Q_1-z_0)^{-m}$ for $m>n/2$ is a trace class one (see [9], [24], [28]). To handle the second term, notice that

$$\Big(e^{-itQ_2}-e^{-itQ_1}\Big)(Q_1-z_0)^{-m}=i\int_0^t e^{-i(t-s)Q_2}(Q_1-Q_2)(Q_1-z_0)^{-m}e^{-isQ_1}ds$$

and use the fact that $(Q_1-Q_2)(Q_1-z_0)^{-m}$ is trace class for $m>\frac{n}{2}+1$. □

According to Lemma 1, in the equation (3.1) we can take the limit $\epsilon \downarrow 0$ with respect to the norm in the space of trace class operators and taking into account the definition of $\mathrm{tr}_{\mathrm{bb}}(.)$, we get

$$\lim_{\epsilon\downarrow 0}\frac{1}{2\pi}\mathrm{tr}_{\mathrm{bb}}\Big[(L_j-z_0)^{-m}\Big(\int_0^\infty e^{-\epsilon t-itL_j}(\hat{g}(-t)+i\epsilon G_{+,\epsilon}(t))dt$$
$$+\int_{-\infty}^0 e^{\epsilon t-itL_j}(\hat{g}(-t)+i\epsilon G_{-,\epsilon}(t))dt\Big)\Big]_{j=1}^2$$
$$=\frac{1}{2\pi}\mathrm{tr}_{\mathrm{bb}}\Big[(L_j-z_0)^{-m}\int_{-\infty}^\infty e^{-itL_j}\hat{g}(-t)dt\Big]_{j=1}^2$$
$$=\mathrm{tr}_{\mathrm{bb}}\Big[(L_j-z_0)^{-m}g(L_j)\Big]_{j=1}^2=\mathrm{tr}_{\mathrm{bb}}\Big(f(L_1)-f(L_2)\Big)=<\xi'(\lambda,h),f(\lambda)>_{\mathcal{D}'(\mathbb{R}),\mathcal{D}(\mathbb{R})}\,.$$

Thus we have proved the following.

Proposition 1. *We have*

$$\xi'(\lambda,h)=\frac{1}{2\pi i}\lim_{\epsilon\downarrow 0}\mathrm{tr}_{\mathrm{bb}}\Big[\Big((\lambda+i\epsilon-z_0)^m(L_j-\lambda-i\epsilon)^{-1} \tag{3.2}$$
$$-(\lambda-i\epsilon-z_0)^m(L_j-\lambda+i\epsilon)^{-1}\Big)(L_j-z_0)^{-m}\Big]_{j=1}^2,$$

where the limit is taken in the sense of distributions $\mathcal{D}'(\mathbb{R})$.

Introduce the functions

$$\sigma_\pm(z)=(z-z_0)^m\mathrm{tr}_{\mathrm{bb}}\Big[(L_j-z)^{-1}(L_j-z_0)^{-m}\Big]_{j=1}^2,\ \ \pm\mathrm{Im}\,z>0. \tag{3.3}$$

which are well defined (see [29] and Proposition 2 below). The relation

$$\mathrm{tr}_{\mathrm{bb}}\Big[(L_j-(\lambda-i\epsilon))^{-1}(L_j-z_0)^{-m}\Big]_{j=1}^2=\overline{\mathrm{tr}_{\mathrm{bb}}\Big[(L_j-(\lambda+i\epsilon))^{-1}(L_j-z_0)^{-m}\Big]_{j=1}^2},$$

implies immediately

$$\sigma_-(z)=\overline{\sigma_+(\overline{z})},\ \ \mathrm{Im}\,z<0. \tag{3.4}$$

The equality (3.4) plays a crucial role in the proof of (1.3) and our choice of real z_0 is related to the above relation.

4. Meromorphic continuation of the spectral shift function

In this section we prove our principal result given in Theorem 1. Taking $0 < \theta \leq \theta_0 < \pi/2$, consider the complex scaling operators $L_{j,\theta}$ related to L_j, $j = 1, 2$, introduced by Sjöstrand and Zworski (see [30], [28] and Section 2 in [29]). More precisely, given $\epsilon_0 > 0$, $R_1 > R_0$, consider a function

$$f_\theta(t) :]0, \frac{\pi}{2}[\times[0, \infty[\ni (\theta, t) \mapsto \mathbb{C}$$

which is injection for every θ and has the properties:

$$f_\theta(t) = t \text{ for } 0 \leq t \leq R_1\,,$$

$$0 \leq \arg f_\theta(t) \leq \theta,\ \partial_t f_\theta \neq 0\,,$$

$$\arg f_\theta(t) \leq \arg \partial_t f_\theta(t) \leq \arg f_\theta + \epsilon_0\,,$$

$$f_\theta(t) = e^{i\theta} t, \text{ for } t \geq T_0\,,$$

where T_0 depends on ϵ_0 and R_1. Next consider the map

$$\kappa_\theta : \mathbb{R}^n \ni x = t\omega \mapsto f_\theta(t)\omega \in \mathbb{C}^n\,,\ t = |x|$$

and introduce $\Gamma_\theta = \kappa_\theta(\mathbb{R}^n)$ which coincides with $\mathbb{R}^n$ along $B(0, R_1)$. We define

$$\mathcal{H}_{j,\theta} = \mathcal{H}_{R_0,j} \oplus L^2(\Gamma_\theta \setminus B(0, R_0))$$

and $L_{j,\theta} : \mathcal{H}_{j,\theta} \longrightarrow \mathcal{H}_{j,\theta}$ with domain $\mathcal{D}_j$ as the operator

$$L_{j,\theta} u = L_j(\chi_1 u) + Q_j|_{\Gamma_\theta}(1 - \chi_1)u\,,$$

$\chi_1 \in C_0^\infty(B(0, R_1))$ being a function equal to 1 near $\overline{B(0, R_0)}$.

Let $\Omega \subset e^{i]-2\theta,2\theta[}]0, +\infty[$ be a simply connected open relatively compact set such that $\Omega \cap \mathbb{R}^+ = J$ is an interval. The spectrum of $L_{j,\theta}$ outside of $e^{-2i\theta}[0, +\infty[$ consists of the negative eigenvalues of L_j and the eigenvalues in $e^{-i[0,2\theta[}]0, +\infty[$ (see [28]). Since the spectrum of L_j is bounded from below, we may choose $z_0 \in \mathbb{R}^-$, $z_0 \notin \overline{\Omega}$, so that z_0 is away from sp (L_j) and sp $(L_{j,\theta})$, $j = 1, 2$. Given a positive number $\delta > 0$, we can apply Proposition 4.1 of Sjöstrand [29], saying that for all $z \in \Omega \cap \{z : \operatorname{Im} z \geq \delta\}$ we have

$$\operatorname{tr}_{\mathrm{bb}}\Big[(L_j - z)^{-1}(L_j - z_0)^{-m}\Big]_{j=1}^2 = \operatorname{tr}_{\mathrm{bb}}\Big[(L_{j,\theta} - z)^{-1}(L_{j,\theta} - z_0)^{-m}\Big]_{j=1}^2, \tag{4.1}$$

where in the definition of the complex scaling operators $L_{j,\theta}$ the parameter ϵ_0 is chosen small enough. Notice that the choice of $z_0 \in e^{i[3\epsilon_0,\ \min(\pi,\ 2\pi - 2\theta - 3\epsilon_0)]}]0, +\infty[$ in [29] says that we may take $z_0 \in \mathbb{R}^-$, assuming $\theta < \frac{\pi}{2} - \frac{3}{2}\epsilon_0$.

Below we assume δ and θ fixed and we will drop in the notations L_j the index j writing $L_.$ when the properties are satisfied for both operators L_j, $j = 1, 2$. Following [29], Section 4, there exists an operator $\hat{L}_{.,\theta} : \mathcal{D}_. \longrightarrow \mathcal{H}_.$ so that

$$K_{.,\theta} = \hat{L}_{.,\theta} - L_{.,\theta} \text{ has rank } \mathcal{O}(h^{-n^\#})$$

and for all $N,\ M \in \mathbb{N}$ we have

$$K_{.,\theta} = \mathcal{O}(1) : \mathcal{D}(L_.^N) \longrightarrow \mathcal{D}(L_.^M).$$

Secondly, $K_{.,\theta}$ is compactly supported, that is if $\chi \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 on $B(0,R)$ for $R \geq R_0$ large enough, we have $K_{.,\theta} = \chi K_{.,\theta}\chi$ and, finally, for every $N \in \mathbb{N}$ we have

$$(\hat{L}_{.,\theta} - z)^{-1} = \mathcal{O}(1) : \ \mathcal{D}(L_.^N) \longrightarrow \mathcal{D}(L_.^{N+1}),$$

uniformly for $z \in \overline{\Omega}$. These properties imply for $z \in \Omega \cap \{\operatorname{Im} z > 0\}$ the representation

$$(L_{.,\theta} - z)^{-1} = (\hat{L}_{.,\theta} - z)^{-1} + (L_{.,\theta} - z)^{-1} K_{.,\theta} (\hat{L}_{.,\theta} - z)^{-1}. \tag{4.2}$$

The contributions related to the resolvent $(\hat{L}_{.,\theta} - z)^{-1}$ are examined in the following.

Proposition 2. *There exists a function $a_+(z,h)$ holomorphic in Ω such that for $z \in \Omega \cap \{\operatorname{Im} z > 0\}$ we have*

$$\sigma_+(z) = \operatorname{tr}\Big[(L_{j,\theta} - z)^{-1} K_{j,\theta} (\hat{L}_{j,\theta} - z)^{-1}\Big]_{j=1}^2 + a_+(z,h). \tag{4.3}$$

Moreover,

$$|a_+(z,h)| \leq C(\Omega) h^{-n^\#}, \ \ z \in \Omega \tag{4.4}$$

with a constant $C(\Omega)$ independent on $h \in]0, h_0]$.

Remark. The singularities of $\sigma_+(z)$ for $\operatorname{Im} z \downarrow 0$ are independent on $z_0 \in \mathbb{R}^-$ and $m \in \mathbb{N}$.

Proof. According to (4.2), for $z \in \Omega \cap \{\operatorname{Im} z \geq \delta\}$ we have

$$\sigma_+(z) = (z - z_0)^m \operatorname{tr}_{\mathrm{bb}}\Big[(\hat{L}_{j,\theta} - z)^{-1}(L_{j,\theta} - z_0)^{-m}\Big]_{j=1}^2 \tag{4.5}$$

$$+ (z - z_0)^m \Big[\operatorname{tr}\Big((L_{j,\theta} - z)^{-1} K_{j,\theta} (\hat{L}_{j,\theta} - z)^{-1} (L_{j,\theta} - z_0)^{-m}\Big)\Big]_{j=1}^2. \tag{4.6}$$

¿From the resolvent equation we obtain

$$(z - z_0)^m (L_{j,\theta} - z_0)^{-m} (L_{j,\theta} - z)^{-1} = (L_{j,\theta} - z)^{-1} - \sum_{k=1}^m (z - z_0)^{k-1} (L_{j,\theta} - z_0)^{-k}.$$

To treat (4.6) we use the cyclicity of the trace and the above equality and conclude that this term is equal to $\operatorname{tr}\Big[(L_{j,\theta} - z)^{-1} K_{j,\theta} (\hat{L}_{j,\theta} - z)^{-1}\Big]_{j=1}^2$ modulo a function holomorphic in Ω and bounded by $\mathcal{O}(h^{-n^\#})$.

Now we pass to the analysis of (4.5). Our purpose is to show that (4.5) is holomorphic in Ω and bounded by $\mathcal{O}(h^{-n^\#})$. By construction, $(\hat{L}_{j,\theta} - z)^{-1}$ is holomorphic on Ω and for any cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ on $\overline{B(0,R_0)}$ with $\operatorname{supp} \chi \subset B(0,R_1)$ the operators $\chi(L_{j,\theta} - z_0)^{-m}$, $(L_{j,\theta} - z_0)^{-m}\chi$ are trace class ones. Hence the function $\operatorname{tr}\Big((\hat{L}_{j,\theta} - z)^{-1}(L_{j,\theta} - z_0)^{-m}\chi\Big)$ is holomorphic in Ω. On the other hand,

$$(L_{j,\theta} - z_0)^{-m} (\hat{L}_{j,\theta} - z)^{-1} - (\hat{L}_{j,\theta} - z)^{-1} (L_{j,\theta} - z_0)^{-m} \tag{4.7}$$

$$= (L_{j,\theta} - z_0)^{-m} (L_{j,\theta} - z)^{-1} K_{j,\theta} (\hat{L}_{j,\theta} - z)^{-1} - (L_{j,\theta} - z)^{-1} K_{j,\theta} (\hat{L}_{j,\theta} - z)^{-1} (L_{j,\theta} - z_0)^{-m}.$$

Consequently, for $\operatorname{Im} z > 0$ if $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function and $\chi_1 \prec \chi$, applying the cyclicity of the trace once more, we get

$$\operatorname{tr}\Big(\chi_1(\hat{L}_{j,\theta} - z)^{-1}(L_{j,\theta} - z_0)^{-m}(1-\chi)\Big) = 0.$$

Thus it remains to examine

$$\tau_+(z) = \operatorname{tr}\Big[(1-\chi_1)(\hat{L}_{j,\theta} - z)^{-1}(1-\chi)(L_{j,\theta} - z_0)^{-m}(1-\chi)\Big]_{j=1}^2.$$

Consider the operator $Q_{\cdot,\theta} = Q_\cdot|_{\Gamma_\theta}$ and note that for $\psi \in C^\infty$ supported away from $B(0, R_1)$ we have $L_{\cdot,\theta}\psi = Q_{\cdot,\theta}\psi$. Repeating the construction of $\hat{L}_{\cdot,\theta}$ in Section 4, [29], we can find an operator $\hat{Q}_{\cdot,\theta} : H^2(\Gamma_\theta) \longrightarrow L^2(\Gamma_\theta)$ so that

$$\hat{Q}_{\cdot,\theta} - Q_{\cdot,\theta} \text{ has rank } \mathcal{O}(h^{-n}),$$

the operator $\hat{Q}_{\cdot,\theta} - Q_{\cdot,\theta}$ is compactly supported and for $z \in \overline{\Omega}$ we have

$$(\hat{Q}_{\cdot,\theta} - z)^{-1} = \mathcal{O}(1) : D(Q_\cdot^N) \longrightarrow D(Q_\cdot^{N+1}), \ \forall N \in \mathbb{N}.$$

Moreover, for $\psi \in C^\infty$ supported away from $B(0, R_1)$ we have $\hat{L}_{\cdot,\theta}\psi = \hat{Q}_{\cdot,\theta}\psi$ and for $\chi \in C_0^\infty(\Gamma_\theta)$ equal to 1 on a sufficiently large set, $z \in \Omega$ and $\chi_1 \prec \chi_0 \prec \chi$ we obtain

$$\begin{aligned}(\hat{L}_{\cdot,\theta} - z)^{-1}(1-\chi) &= (1-\chi_0)(\hat{Q}_{\cdot,\theta} - z)^{-1}(1-\chi)\\ &+(\hat{L}_{\cdot,\theta} - z)^{-1}[\hat{Q}_{\cdot,\theta}, \chi_0](\hat{Q}_{\cdot,\theta} - z)^{-1}(1-\chi).\end{aligned}$$

As above, we assume that $z_0 \in \mathbb{R}^-$ is chosen so that $z_0 \notin \operatorname{sp}(Q_j)$, $z_0 \notin \operatorname{sp}(Q_{j,\theta})$, $j = 1, 2$. For simplicity of the notations below we omit the index θ and we get

$$\begin{aligned}\tau_+(z) &= \operatorname{tr}\Big[(1-\chi_0)(\hat{Q}_j - z)^{-1}(1-\chi)(L_j - z_0)^{-m}(1-\chi)\Big]_{j=1}^2\\ &+\operatorname{tr}\Big[(1-\chi_1)(\hat{L}_j - z)^{-1}[\hat{Q}_j, \chi_0](\hat{Q}_j - z)^{-1}(1-\chi)(L_j - z_0)^{-m}(1-\chi)\Big]_{j=1}^2.\end{aligned}$$

Obviously, $[\hat{Q}_j, \chi_0] = [Q_j, \chi_0] + M_j$ with a trace class operator M_j. To show that the operator $[Q_j, \chi_0](\hat{Q}_j - z)^{-1}(1-\chi)$ is a trace class one, we apply the telescopic formula choosing cut-off functions $\theta_N \prec \theta_{N-1} \prec \ldots \prec \theta_1 \prec \chi$ and write

$$\begin{aligned}[Q_j, \chi_0](\hat{Q}_j - z)^{-1}(1-\chi) &= [Q_j, \chi_0](\hat{Q}_j - z)^{-1}\chi(Q_j - i)^{-m}\\ \times\Big[(Q_j - i)^m[\hat{Q}_j, \theta_N]&(\hat{Q}_j - z)^{-1}[\hat{Q}_j, \theta_{N-1}]\ldots[\hat{Q}_j, \theta_1](\hat{Q}_j - z)^{-1}(1-\chi)\Big]\end{aligned}$$

with $N \geq 2m > n$. The operator in the brackets $[\ldots]$ and $[Q_j, \chi_0](\hat{Q}_j - z)^{-1}$ are bounded, while $\chi(Q_j - i)^{-m}$ is trace class. Thus the term involving $[\hat{Q}_j, \chi_0]$ is holomorphic in Ω and bounded by $\mathcal{O}(h^{-n^\#})$.

As in the proof of Proposition 1, we have

$$\|(1-\chi)(L_j - z_0)^{-m}(1-\chi) - (1-\chi)(Q_j - z_0)^{-m}(1-\chi)\|_{\operatorname{tr}} = \mathcal{O}(h^{-n^\#}).$$

Moreover, $(Q_j - z_0)^{-m}\chi$ is trace class and, consequently, there exists a function $b(z, h)$ holomorphic in Ω and bounded by $\mathcal{O}(h^{-n^\#})$ so that

$$\tau_+(z) = b(z,h) + \operatorname{tr}\Big[(1-\chi)(\hat{Q}_j - z)^{-1}(Q_j - z_0)^{-m}(1-\chi)\Big]_{j=1}^2. \tag{4.8}$$

We write

$$(\hat{Q}_2 - z)^{-1}(Q_2 - z_0)^{-m} - (\hat{Q}_1 - z)^{-1}(Q_1 - z_0)^{-m}$$
$$= (\hat{Q}_2 - z)^{-1}\Big[(Q_2 - z_0)^{-m} - (Q_1 - z_0)^{-m}\Big] + \Big[(\hat{Q}_2 - z)^{-1} - (\hat{Q}_1 - z)^{-1}\Big](Q_1 - z_0)^{-m} = I + II\,.$$

According to [28], [29], the operator $(Q_2 - z_0)^{-m} - (Q_1 - z_0)^{-m}$ is trace class one and the contribution of I is holomorphic and bounded by $\mathcal{O}(h^{-n^{\#}})$. For II we obtain the representation

$$II = (\hat{Q}_2 - z)^{-1}(\hat{Q}_1 - \hat{Q}_2)(\hat{Q}_1 - z)^{-1}(Q_1 - z_0)^{-m}\,.$$

It is clear that $\hat{Q}_1 - \hat{Q}_2 = Q_1 - Q_2 + K_{1,2}$ with a finite rank operator $K_{1,2}$, and modulo a trace class operator we have

$$II = (\hat{Q}_2 - z)^{-1}\Big((Q_1 - Q_2)(Q_2 - z_0)^{-m}\Big)\Big((Q_2 - z_0)^{m}(\hat{Q}_1 - z)^{-1}(Q_1 - z_0)^{-m}\Big)\,.$$

The second factor is a trace class operator, while the first and the third ones are bounded operators. Consequently, II has the same property as I. Combining the above results, we conclude that $\tau_+(z)$ is holomorphic in Ω and bounded by $\mathcal{O}(h^{-n^{\#}})$.

To establish (4.3), notice that the right hand side of this equality is holomorphic for $z \in \Omega \cap \{\operatorname{Im} z > 0\}$. The left hand side is also holomorphic in this domain since we may apply (4.1) with different $\delta > 0$, $\epsilon_0 > 0$ and $0 < \theta < \frac{\pi}{2} - \frac{3}{2}\epsilon_0$. By analytic continuation we deduce (4.3) and the proof of Proposition 2 is complete. □

Proof of Theorem 1. To obtain a meromorphic continuation of $\sigma_+(z)$ through the real axis, it suffices to do this for the trace involving $K_{j,\theta}$. Next we will follow closely the argument of Sjöstrand [29] and since θ is fixed, we will omit it in the notations. Setting $\tilde{K}_.(z) = K_.(z - \hat{L}_.)^{-1}$, from (4.31) in [29] we get the representation

$$-\operatorname{tr}((L_. - z)^{-1}K_.(\hat{L}_. - z)^{-1}) = \operatorname{tr}\Big((1 + \tilde{K}_.(z))^{-1}\frac{\partial}{\partial z}\tilde{K}_.(z)\Big)$$
$$= \partial_z \log\det(1 + \tilde{K}_.(z))$$

and the resonances of $L_.$ are precisely the zeros of the function

$$D(z,h) = \det(1 + \tilde{K}_.(z)) = \mathcal{O}(1)\exp(Ch^{-n^{\#}}). \tag{4.9}$$

Notice that the multiplicities of the resonances and the zeros coincide. Below in the notations we omit the subscript . since the argument does not depend on $j = 1, 2$. Let Res (L) be the resonances of L and let

$$D(z,h) = G(z,h)\prod_{w \in \mathrm{Res}\,(L)\cap\Omega}(z - w)\,,$$

where $G(z,h)$ and $\frac{1}{G(z,h)}$ are holomorphic in Ω and the resonances in the product are repeated following their multiplicity. Obviously,

$$\partial_z \log D(z,h) = \partial_z \log G(z,h) + \sum_{w \in \mathrm{Res}\,(L)\cap\Omega}\frac{1}{z - w}$$

and according to the estimate (4.54) in [29], we get

$$\Big|\frac{\partial}{\partial z}\log G(z,h)\Big| \le C(\tilde{\Omega})h^{-n^{\#}},\ \ z \in \tilde{\Omega}, \tag{4.10}$$

where $\tilde{\Omega} \subset\subset \Omega$ is an arbitrary open simply connected domain and $C(\tilde{\Omega})$ is independent on $h \in]0, h_0]$.

Going back to the representation (3.2) and taking into account (3.4), we observe that for $\lambda \in I \subset \mathbb{R}^+$, $\operatorname{Im} w \neq 0$, we have

$$-\frac{1}{2\pi i}\lim_{\epsilon \downarrow 0}\Big(\frac{1}{\lambda + i\epsilon - w} - \frac{1}{\lambda - i\epsilon - \overline{w}}\Big) = \frac{-\operatorname{Im} w}{\pi|\lambda - w|^2}\,,$$

while for $w \in \mathbb{R}$ we get

$$-\frac{1}{2\pi i}\lim_{\epsilon \downarrow 0}\Big(\frac{1}{\lambda + i\epsilon - w} - \frac{1}{\lambda - i\epsilon - w}\Big) = \delta(\lambda - w)\,,$$

where both limits are taken in the sense of distributions. Combining Propositions 1, 2 and the above arguments we complete the proof of Theorem 1. $\square$

The representation (1.3) shows that modulo a constant the spectral shift function $\xi(\lambda, h)$ coincides with the distribution

$$\xi(\lambda, h) = \frac{1}{\pi}\Big[\sum_{\substack{w \in \operatorname{Res} L_j(h) \\ \operatorname{Im} w \neq 0}} \int_{\lambda_0}^{\lambda} \frac{|\operatorname{Im} w|}{|\mu - w|^2} d\mu\Big]_{j=1}^2$$

$$+\Big[\#\{\mu \in [\lambda_0, \lambda] :\, \mu \in \sigma_{pp}(L_j(h))\}\Big]_{j=1}^2 + \frac{1}{\pi}\int_{\lambda_0}^{\lambda} \operatorname{Im} r(\mu, h) d\mu,\;\; \lambda_0 > 0,\; \lambda_0 \notin I\,.$$

In particular, for $\lambda \in I \setminus \cup_{j=1}^2\{\lambda \in \mathbb{R} : \lambda \in \operatorname{sp}_{pp}(L_j(h))\}$ the distribution $\xi(\lambda, h)$ is continuous and the function

$$\eta(\lambda, h) = \xi(\lambda, h) - \Big[\#\{\mu \in [\lambda_0, \lambda] :\, \mu \in \operatorname{sp}_{pp}(L_j(h))\}\Big]_{j=1}^2$$

is real analytic in I.

5. Weyl asymptotics

In this section we obtain a Weyl type asymptotics for the spectral shift function. We generalize the results of Christiansen [8] and Robert [25] covering the "black box" long-range perturbations of the Laplacian and we improve our previous result (see Theorem 2 in [6]) working without any condition on the behavior of the resonances close to the real axis.

We will say that $\lambda \in \mathbb{R}$ is a *non-critical energy level* for Q if for all $(x, \xi) \in \Sigma_\lambda = \{(x, \xi) \in \mathbb{R}^{2n} : l(x, \xi) = \lambda\}$ we have $\nabla_{x,\xi} l(x, \xi) \neq 0$, $l(x, \xi)$ being the principal symbol of Q. Given a Hamiltonian $l(x, \xi)$, denote by

$$\exp(tH_l)(x_0, \xi_0) = (x(t, x_0, \xi_0),\; \xi(t, x_0, \xi_0))$$

the trajectory of the Hamilton flow $\exp(tH_l)$ passing through $(x_0, \xi_0) \in \Sigma_\lambda$. Recall that $\lambda \in J$ is a *non-trapping energy level* for $l(x, \xi)$ if for every $R > 0$ there exists $T(R) > 0$ such that for $(x_0, \xi_0) \in \Sigma_\lambda$, $|x_0| < R$, the x-component of the trajectory of $\exp(tH_l)$ passing through (x_0, ξ_0) satisfies

$$|x(t, x_0, \xi_0)| > R,\;\; \forall |t| > T(R).$$

Denote by $N(L_j^\#, I)$ the number of eigenvalues of $L_j^\#$ in the interval I. From the assumptions (2.5) and (2.10) we deduce easily that there exists a constant $C^\#$ such that the spectrums of $L_j^\#$, $j = 1, 2$, do not intersect the interval $]-\infty, -C^\#]$ and consequently $N(L_j^\#,]-\infty, -C^\#]) = 0$. In fact, let $\chi_0,\ \chi,\ \chi_1 \in C_0^\infty(B(0,R);[0,1])$ be equal to 1 on $\overline{B(0,R_0)}$ and let $\chi_1 \succ \chi \succ \chi_0$. Using the resolvent equality we get

$$\begin{aligned}(L_j^\# - z)^{-1} &= (L_j^\# - z)^{-1}\chi + (L_j^\# - z)^{-1}(1-\chi)\\ &= \chi_1(L_j - z)^{-1}\chi - (L_j^\# - z)^{-1}[Q_j^\#, \chi_1](L_j - z)^{-1}\chi\\ &\quad +(1-\chi_0)(Q_j^\# - z)^{-1}(1-\chi) + (L_j^\# - z)^{-1}[Q_j^\#, \chi_0](Q_j^\# - z)^{-1}(1-\chi).\end{aligned}$$

Then

$$\begin{aligned}&(L_j^\# - z)^{-1}\Big(1 + [Q_j^\#, \chi_1](L_j - z)^{-1}\chi - [Q_j^\#, \chi_0](Q_j^\# - z)^{-1}(1-\chi)\Big)\\ &\qquad = \chi_1(L_j - z)^{-1}\chi + (1-\chi_0)(Q_j^\# - z)^{-1}(1-\chi).\end{aligned}$$

According to the assumptions (2.5) and (2.10) there exists $C^\#$ such that spectrums of L_j, $Q_j^\#$, $j = 1, 2$, do not intersect the interval $]-\infty, -C^\#]$, hence for $z \in]-\infty, -C^\#]$, the resolvents $(L_j - z)^{-1}$, $(Q_j^\# - z)^{-1}$ are bounded and we obtain immediately

$$[Q_j^\#, \chi_1](L_j - z)^{-1}\chi - [Q_j^\#, \chi_0](Q_j^\# - z)^{-1}(1-\chi) = \mathcal{O}(h).$$

Consequently, for h small enough and $z \in]-\infty, -C^\#]$, the resolvent $(L_j^\# - z)^{-1}$ is bounded and $z \notin \mathrm{sp}(L_j^\#)$. In the following we will use the notation

$$N(L_j^\#, \lambda) = N(L_j^\#,]-C^\#, \lambda]),\ \ j = 1, 2\,.$$

The spectral shift function $\xi(\lambda, h)$ is determined modulo a constant and from (2.10) we deduce that $\xi(\lambda, h)$ is constant on $]-\infty, -C_1]$ for C_1 sufficiently large. In the following, without loss of the generality, we may choose $\xi(\lambda, h)$ so that $\xi(\lambda, h) = 0$ on $]-\infty, -C^\#]$. Moreover, in this section we consider $\xi(\lambda, h) = \lim_{\epsilon\downarrow 0} \xi(\lambda + \epsilon, h)$ as a function continuous from the right. The main result in this section is a Weyl type asymptotics for the spectral shift function.

Theorem 2. *Assume that L_j, $j = 1, 2$ satisfy the assumptions of Section 2. Let $0 < E_0 < E_1$ and suppose that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for Q_j, $Q_j^\#$, $j = 1, 2$. Assume that there exist positive constants B, ϵ_1, C_1, h_1 such that for any $\lambda \in [E_0 - \epsilon_1, E_1 + \epsilon_1]$, $h/B \le \delta \le B$ and $h \in]0, h_1]$ we have*

$$N(L^\#, [\lambda - \delta, \lambda + \delta]) \le C_1 \delta h^{-n^\#},\ \ j = 1, 2. \tag{5.1}$$

Then there exist $\omega(\lambda) \in C^0(\mathbb{R})$, $h_0 > 0$ such that

$$\xi(\lambda, h) = \Big[N(L_j^\#, \lambda])\Big]_{j=1}^{2} + \omega(\lambda)h^{-n} + \mathcal{O}(h^{1-n^\#}) \tag{5.2}$$

uniformly with respect to $\lambda \in [E_0, E_1]$ and $h \in]0, h_0]$.

Remark. Notice that if λ is a non-critical energy level, then for $\epsilon > 0$ small enough each $\mu \in]\lambda - \epsilon, \lambda + \epsilon[$ is also non-critical one. Consequently, (5.2) remains valid on some interval $[E_0 - \alpha, E_1 + \alpha]$, $\alpha > 0$. Recall that the operators $L_j^\#$, $j = 1, 2$, have been defined in Section 2 by using the operators $Q_j^\#$, $j = 1, 2$, whose coefficients satisfy $a_{j,\nu}^\#(x; h) = a_{j,\nu}(x; h)$ for $|x| \le R$, $R > R_0$. If the principal symbol $l_j(x.\xi)$ of Q_j is non-critical for $\lambda \in [E_0, E_1]$, we can extend $a_{j,\nu}^\#(x; h)$ for $|x| > R$ in a such way that $\lambda \in [E_0, E_1]$ become non-critical for $Q_j^\#$. This continuation changes the operator $L_j^\#$ but as it has been proved by J.-F. Bony [1], the assumption (5.1) does not depend on the continuation of $a^{\#}_{j,\nu}(x;h)$.

To prove Theorem 2, we will introduce intermediate operators exploiting the following result of J.-F. Bony (see also [27]).

Proposition 3. *([2]) Assume that L satisfy the assumptions of Section 2 and suppose that each $\lambda \in [E_0, E_1]$ is a non-critical energy level for Q. Given a fixed $\lambda \in [E_0, E_1]$, there exists a differential operator $\tilde{L}$, such that*

(a) The pair $(L, \tilde{L})$ satisfies the assumptions of Section 2, with $\overline{n} = n + 1$,

(b) There exists an interval $I_0 \ni \lambda$, such that each $\mu \in I_0$ is non-trapping and non-critical energy level for $\tilde{L}$,

(c) The operator $\tilde{L}$ has no resonances in a complex neighborhood Ω_0 of I_0 and Ω_0 is independent on h.

Now denote by $\xi(\lambda; A, B)$ the spectral shift function related to the operators A and B. Using the above proposition for each operator L_j, $j = 1, 2$, we can construct operators $\tilde{L}_j$, $j = 1, 2$, and decompose the spectral shift function $\xi(\lambda; L_1, L_2)$ as follows

$$\xi(\lambda; L_1, L_2) = \xi(\lambda; L_1, \tilde{L}_1) + \xi(\lambda; \tilde{L}_1, \tilde{L}_2) - \xi(\lambda; L_2, \tilde{L}_2).$$

Moreover, if every $\lambda \in [\alpha, \beta]$ is non-trapping and non-critical energy level for $\tilde{L}_j$, $j = 1, 2$, we have a complete asymptotic expansion of $\xi(\lambda; \tilde{L}_1, \tilde{L}_2)$ and its derivatives (see [25]) uniformly with respect to $\lambda \in [\alpha, \beta]$. Here we may estimate the difference $\tilde{L}_1 - \tilde{L}_2 = (\tilde{L}_1 - L_1) + (L_1 - L_2) + (L_2 - \tilde{L}_2)$ by applying our assumptions on $Q_1 - Q_2$. Thus it is sufficient to prove the theorem for $\lambda \in I_2 \subset I_0$ and the pair (L_1, L_2) with $L_2 = Q_2$ being a differential operator having no resonances in a complex neighborhood Ω_0 of I_0 and such that every $\lambda \in I_0$ is non-trapping and non-critical energy level for L_2. Then the assertion follows by applying the local result and covering the compact interval $[E_0, E_1]$ by small intervals.

We denote $\xi(\lambda, h)$ the spectral shift function for the operators (L_1, L_2). Applying Theorem 1 in the domain Ω_0, we deduce that there exists a function $g_+(z, h)$ holomorphic in Ω_0 such that for $\lambda \in I_0 = W_0 \cap \mathbb{R}$, $W_0 \subset\subset \Omega_0$ we have

$$\xi'(\lambda, h) = \frac{1}{\pi} \operatorname{Im} g_+(\lambda, h) + \sum_{\substack{w \in \operatorname{Res} L_1 \cap \Omega_0, \\ \operatorname{Im} w \neq 0}} \frac{-\operatorname{Im} w}{\pi |\lambda - w|^2} + \sum_{w \in \operatorname{Res} L_1 \cap I_0} \delta(\lambda - w), \tag{5.3}$$

where $g_+(z, h)$ satisfies the estimate

$$|g_+(z, h)| \leq C(W_0) h^{-n^{\#}}, \ \ z \in W_0 \tag{5.4}$$

with $C(W_0) > 0$ independent on $h \in]0, h_0]$.

In the following, we fix an open interval $I_0 \subset \mathbb{R}^+$ so that each $\mu \in I_0$ is a non-critical energy level for Q_j, $j = 1, 2$, and we introduce open intervals $I_2 \subset\subset I_1 \subset\subset I_0$. It is convenient to decompose $\xi(\lambda, h)$ for $\lambda \in I_2$ into a sum of a term independent on λ and a second one localized in I_0 where (5.3) holds.

Lemma 2. *Let $C^{\#} > 0$ be such that the spectrums of L_j and $L_j^{\#}$, $j = 1, 2$, do not intersect the interval $[-\infty, -C^{\#}]$. Let φ_1, $\varphi_2 \in C_0^\infty(\mathbb{R}; \mathbb{R}^+)$ be such that supp $\varphi_1 \subset (-\infty, \gamma_1)$, supp $\varphi_2 \subset I_1$, $\varphi_2 = 1$ on $I_2 = (\gamma_1, \gamma_2)$ and $\varphi_1 + \varphi_2 = 1$ on $[-C^{\#} - \eta_0, \gamma_2]$, $\eta_0 > 0$. Then for $\lambda \in I_2$ we have*

$$\xi(\lambda, h) = \mathrm{tr}_{\mathrm{bb}} \Big[\varphi_1(L_j)\Big]_{j=1}^2 + G_{\varphi_2}(\lambda) + M_{\varphi_2}(\lambda), \tag{5.5}$$

where

$$G_{\varphi_2}(\lambda) = \frac{1}{\pi} \int_{]-\infty,\lambda]} \mathrm{Im}\, g_+(\mu, h)\varphi_2(\mu) d\mu,$$

$$M_{\varphi_2}(\lambda) = \sum_{\substack{w \in \mathrm{Res}\ L_1 \cap \Omega_0, \\ \mathrm{Im}\, w \neq 0}} \int_{]-\infty,\lambda]} \frac{-\,\mathrm{Im}\, w}{\pi |\mu - w|^2} \varphi_2(\mu) d\mu + \sum_{w \in \mathrm{Res}\ L_1 \cap]-C^{\#}, \lambda]} \varphi_2(w) \tag{5.6}$$

and we omit in M_{φ_2} and G_{φ_2} the dependence of h.

Proof. Roughly speaking, for $\lambda \in I_2$, if we express the action of the distributions as integrals, we must have

$$\xi(\lambda, h) = \int_{-\infty}^{\lambda} \varphi_1(\mu)\xi'(\mu, h) d\mu + \int_{-\infty}^{\lambda} \varphi_2(\mu)\xi'(\mu, h) d\mu.$$

Since φ_1 vanishes on I_2, the first term is independent on $\lambda \in I_2$ and equal to $\mathrm{tr}_{\mathrm{bb}} \Big[\varphi_1(L_j)\Big]_{j=1}^2$. For the second one we may apply (5.3) since φ_2 is supported in $I_1 \subset I_0$.

For a rigorous proof of the above representation, take $f \in C_0^\infty(I_2)$ and introduce

$$F(\lambda) = (\varphi_1 + \varphi_2)(\lambda) \int_{\lambda}^{+\infty} f(\mu) d\mu$$

which is compactly supported. Since supp $f \subset I_2$ and $\varphi_1 + \varphi_2 = 1$ on I_2, we have

$$F'(\lambda) = -f(\lambda) + (\varphi_1' + {\varphi'}_2)(\lambda) \int_{\lambda}^{+\infty} f(\mu) d\mu,$$

where the second term vanishes on $[-C^{\#} - \eta_0, +\infty[$. Our choice of $\xi(\lambda, h) = 0$ on $]-\infty, -C^{\#}]$ makes possible to write

$$\langle \xi, f \rangle_{\mathcal{D}', \mathcal{D}} = -\langle \xi, F' \rangle_{\mathcal{D}', \mathcal{D}} = \langle \xi', F \rangle_{\mathcal{D}', \mathcal{D}}.$$

Next the equality $\varphi_1 \int_\lambda^{+\infty} f = \varphi_1 \int_{\mathbb{R}} f$ yields

$$\langle \xi', \varphi_1 \int_{\lambda}^{+\infty} f \rangle_{\mathcal{D}', \mathcal{D}} = \Big(\int_{\mathbb{R}} f\Big) \langle \xi', \varphi_1 \rangle_{\mathcal{D}', \mathcal{D}} = \Big(\int_{\mathbb{R}} f\Big)\, \mathrm{tr}_{\mathrm{bb}} \Big[\varphi_1(L_j)\Big]_{j=1}^2.$$

For the term involving φ_2, we apply (5.3) and we get

$$\langle \xi', \varphi_2 \int_{\lambda}^{+\infty} f \rangle_{\mathcal{D}', \mathcal{D}} = \langle G'_{\varphi_2}, \psi \int_{\lambda}^{+\infty} f \rangle_{\mathcal{D}', \mathcal{D}} + \langle M'_{\varphi_2}, \psi \int_{\lambda}^{+\infty} f \rangle_{\mathcal{D}', \mathcal{D}}$$

for $\psi \in C^\infty(\mathbb{R})$ equal to 1 on $\mathbb{R}^+$ and vanishing on $]-\infty, -1]$. The above relations imply (5.5) in the sense of distributions since $G_{\varphi_2}\psi' = M_{\varphi_2}\psi' = 0$ and $\psi f = f$. □

To prove Theorem 2, we will apply a Tauberian argument for the increasing function $M_{\varphi_2}(\lambda)$. Consider a function $\theta(t) \in C_0^\infty(]-\delta_1, \delta_1[)$, $\theta(0) = 1$, $\theta(-t) = \theta(t)$, such that the Fourier transform

$\hat{\theta}$ of θ satisfies $\hat{\theta}(\lambda) \geq 0$ on $\mathbb{R}$ and assume that there exist $0 < \epsilon_0 < 1$, $\delta_0 > 0$ so that $\hat{\theta}(\lambda) \geq \delta_0 > 0$ for $|\lambda| \leq \epsilon_0$. Next introduce

$$\left(\mathcal{F}_h^{-1}\theta\right)(\lambda) = (2\pi h)^{-1} \int e^{it\lambda/h}\theta(t)dt = (2\pi h)^{-1}\hat{\theta}(-h^{-1}\lambda).$$

Remark. It is obvious that the Lemma 2 holds if we take a partition of unity $\varphi_1^2 + \varphi_2^2$ over $[-C^{\#} - \eta_0, \gamma_2]$ with cut-off functions φ_j, $j = 1, 2$.

The next lemma permits to establish a connection between the asymptotics of the functions M_{φ_2} and $N^{\#}_{\varphi_2}$.

Lemma 3. *Let $\varphi_2 \in C_0^\infty(I_1; \mathbb{R}^+)$ and let $N^{\#}_{\varphi_2}(\lambda) = \operatorname{tr}\left(\varphi_2(L_1^{\#})\mathbf{1}_{]-C^{\#},\lambda]}(L_1^{\#})\right)$. Then there exists $\omega_{\varphi_2}(\lambda) \in C_0^0(I_0)$ such that for any $\lambda \in \mathbb{R}$ we have*

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M_{\varphi_2})(\lambda) = \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N^{\#}_{\varphi_2})(\lambda) - G'_{\varphi_2}(\lambda) + \omega_{\varphi_2}(\lambda)h^{-n} + \mathcal{O}(h^{1-n^{\#}}), \tag{5.7}$$

where $\mathcal{O}(h^{1-n^{\#}})$ is uniform with respect to $\lambda \in \mathbb{R}$. Moreover, we have

$$M_{\varphi_2}(\lambda) = (\mathcal{F}_h^{-1}\theta * M_{\varphi_2})(\lambda) + \mathcal{O}(h^{1-n^{\#}}) \tag{5.8}$$

$$= (\mathcal{F}_h^{-1}\theta * N^{\#}_{\varphi_2})(\lambda) - G_{\varphi_2}(\lambda) + \int_{-\infty}^{\lambda} \omega_{\varphi_2}(\mu)d\mu h^{-n} + \mathcal{O}(h^{1-n^{\#}})$$

uniformly with respect to $\lambda \in I_0$.

Proof. For simplicity of the notations we omit the subscript φ_2 and denote by M, G, $N^{\#}$, ω the functions M_{φ_2}, G_{φ_2}, $N^{\#}_{\varphi_2}$, ω_{φ_2}. According to (5.6) and (5.3), for any $\lambda \in \mathbb{R}$ we have

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M)(\lambda) = (\mathcal{F}_h^{-1}\theta * M')(\lambda) = (\mathcal{F}_h^{-1}\theta * \varphi_2\xi')(\lambda) - (\mathcal{F}_h^{-1}\theta * G')(\lambda).$$

Using the Cauchy inequalities, it follows easily that $G'(\lambda) = \mathcal{O}(h^{-n^{\#}})$ and $G''(\lambda) = \mathcal{O}(h^{-n^{\#}})$ and we obtain immediately

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * G)(\lambda) = G'(\lambda) + \mathcal{O}(h^{1-n^{\#}})$$

uniformly with respect to $\lambda \in \mathbb{R}$.

It remains to examine

$$(\mathcal{F}_h^{-1}\theta * \varphi_2\xi')(\lambda) = \operatorname{tr}_{\mathrm{bb}}\Big[(\mathcal{F}_h^{-1}\theta)\Big(\lambda - L_j\Big)\varphi_2(L_j)\Big]_{j=1}^2$$

$$= \frac{1}{2\pi h}\int e^{it\lambda h^{-1}}\theta(t)\operatorname{tr}_{\mathrm{bb}}\Big[e^{-ith^{-1}L_j}\varphi_2(L_j)\Big]_{j=1}^2 dt.$$

We will prove that

$$(\mathcal{F}_h^{-1}\theta * \varphi_2\xi')(\lambda) = \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N^{\#})(\lambda) + \omega(\lambda)h^{-n} + \mathcal{O}(h^{1-n}),\ \ \lambda \in \mathbb{R}, \tag{5.9}$$

where $\omega(\lambda) \in C_0^0(I_0)$ has compact support and $\mathcal{O}(h^{1-n})$ is uniform with respect to $\lambda \in \mathbb{R}$. As in Section 2, define the operator $L_1^{\#}$ on the torus $T_{\tilde{R}} = (\mathbb{R}/\tilde{R}\mathbb{Z})^n$ with $\tilde{R} > 2R > 2R_0$ and introduce $\chi \in C_0^\infty(\{x : |x| \leq \tilde{R}\})$ equal to 1 for $|x| \leq 2R > 2R_0$. We have

$$\operatorname{tr}_{\mathrm{bb}}\Big[e^{-ith^{-1}L_j}\varphi_2(L_j)\Big]_{j=1}^2 = \Big[\operatorname{tr}\Big(\chi e^{-ith^{-1}L_j}\varphi_2(L_j)\chi\Big)\Big]_{j=1}^2 + \operatorname{tr}_{\mathrm{bb}}\Big[e^{-ith^{-1}L_j}\varphi_2(L_j)(1-\chi^2)\Big]_{j=1}^2.$$

Applying the Duhamel formula and the semi-classical Egorov theorem (see Section 6 of [6] for more details), for $|t|$ sufficiently small we obtain

$$\mathrm{tr}_{\mathrm{bb}}\Big[e^{-ith^{-1}L_j}\varphi_2(L_j)(1-\chi^2)\Big]_{j=1}^2 = \mathrm{tr}\Big[e^{-ith^{-1}Q_j}\varphi_2(Q_j)(1-\chi^2)\Big]_{j=1}^2 + \mathcal{O}(h^\infty),$$

$$\begin{aligned}\mathrm{tr}\Big(\chi e^{-ith^{-1}L_1}\varphi_2(L_1)\chi\Big) &= \mathrm{tr}\Big(\chi e^{-ith^{-1}L_1^\#}\varphi_2(L_1^\#)\chi\Big) + \mathcal{O}(h^\infty)\\ &= \mathrm{tr}\Big(e^{-ith^{-1}L_1^\#}\varphi_2(L_1^\#)\Big) - \mathrm{tr}\Big(e^{-ith^{-1}Q_1^\#}\varphi_2(Q_1^\#)(1-\chi^2)\Big) + \mathcal{O}(h^\infty),\end{aligned}$$

where $Q_1^\#$ is a differential operator

$$Q_1^\# = \sum_{|\nu|\leq 2} a_{1,\nu}^\#(x;h)(hD)^\nu$$

on the torus $T_{\tilde{R}}$ introduced in Section 2 and $a_{1,\nu}^\#(x;h) = a_{1,\nu}(x;h)$ for $|x| < r_0$, $r_0 > 2R_0$. Using the classical constructions of a parametrix for small $|t|$ for the unitary groups $e^{-ith^{-1}Q_1^\#}$, $e^{-ith^{-1}L_2}$, combined with the fact that $\lambda \in I_0$ is non-critical for $Q_1^\#$, L_2 we deduce for $\lambda \in I_0$

$$\mathrm{tr}\Big((\mathcal{F}_h^{-1}\theta)\Big(\lambda - Q_1^\#\Big)\varphi_2(Q_1^\#)(1-\chi^2)\Big) = \omega_1(\lambda)h^{-n} + \mathcal{O}(h^{1-n}),$$

$$\mathrm{tr}\Big(\chi(\mathcal{F}_h^{-1}\theta)\Big(\lambda - L_2\Big)\varphi_2(L_2)\chi\Big) = \omega_2(\lambda)h^{-n} + \mathcal{O}(h^{1-n}),$$

with functions ω_1, $\omega_2 \in C_0^0(I_1)$ and $\mathcal{O}(h^{1-n})$ uniform with respect to $\lambda \in I_0$. The problem can be reduced to the application of the stationary phase method to some integrals where the integration is over a compact set. We refer to Chapter 10, [9], for more details. Since $\hat{\theta} \in \mathcal{S}(\mathbb{R})$, we can extend the above relations to all $\lambda \in \mathbb{R}$ with $\mathcal{O}(h^{1-n})$ uniform with respect to $\lambda \in \mathbb{R}$.

For the trace involving Q_j, $j = 1, 2$, we have for $\lambda \in I_0$

$$\mathrm{tr}\Big[(\mathcal{F}_h^{-1}\theta)\Big(\lambda - Q_j\Big)\varphi_2(Q_j)(1-\chi^2)\Big]_{j=1}^2 = \omega_{ext}(\lambda)h^{-n} + \mathcal{O}(h^{1-n}) \tag{5.10}$$

with $\omega_{ext} \in C_0^0(I_0)$ and $\mathcal{O}(h^{1-n})$ uniform with respect to $\lambda \in I_0$. The proof of (5.10) is more technical since we must integrate over a non-compact domain. In fact, it is similar to the calculation of the traces in Section 4 in [2] and for the sake of completeness we present a proof in Appendix. Moreover, we show in the Appendix that we can extend (5.10) to all $\lambda \in \mathbb{R}$ with $\mathcal{O}(h^{1-n})$ uniform with respect to $\lambda \in \mathbb{R}$. Taking together the asymptotics of the traces and the above relations, we obtain (5.9) and (5.7).

Now we will apply a Tauberian theorem (see for example, Theorem V-13 of [24]) for the increasing function $M_{\varphi_2}(\lambda)$. For this purpose we need the estimates

$$M_{\varphi_2}(\lambda) = \mathcal{O}(h^{-n^\#}),\ \frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * M_{\varphi_2})(\lambda) = \mathcal{O}(h^{-n^\#}),\ \forall \lambda \in \mathbb{R}. \tag{5.11}$$

The first one follows easily from (5.6). To establish the second one, we apply the equality (5.7). Thus it suffices to prove the estimate

$$\frac{d}{d\lambda}(\mathcal{F}_h^{-1}\theta * N_{\varphi_2}^\#)(\lambda) = (2\pi h)^{-1}\mathrm{tr}\Big(\hat{\theta}(\frac{L_1^\# - \lambda}{h})\varphi_2(L_1^\#)\Big) = \mathcal{O}(h^{-n^\#}),\ \forall \lambda \in \mathbb{R}. \tag{5.12}$$

To do this, assume first that $\lambda \in [E_0 - \epsilon_1, E_1 + \epsilon_1]$. Taking into account (5.1), we obtain

$$\Big|\mathrm{tr}\Big(\hat\theta(\frac{L_1^{\#}-\lambda}{h})\varphi_2(L_1^{\#})\Big)\Big| = \Big|\sum_{\mu \in \mathrm{sp}\,(L_1^{\#})\cap \mathrm{supp}\,\varphi_2} \hat\theta(\frac{\mu-\lambda}{h})\varphi_2(\mu)\Big|$$

$$\leq \sum_{k=0}^{C/h} \sum_{\frac{kh}{B} \leq |\mu-\lambda| \leq \frac{(k+1)h}{B}} \Big|\hat\theta(\frac{\mu-\lambda}{h})\varphi_2(\mu)\Big| \leq C\Big(h^{1-n^{\#}} + \sum_{k=1}^{C/h} \frac{(k+1)h^{1-n^{\#}}}{k^3}\Big) \leq Ch^{1-n^{\#}},$$

where we have used the inequality $|\hat\theta(\mu)| \leq C(1+|\mu|)^{-3}$. On the other hand, for $\lambda \notin [E_0-\epsilon_1, E_1+\epsilon_1]$ and $\mu \in \mathrm{supp}\ \varphi_2$, we have $|\mu - \lambda| \geq \delta_2 > 0$ and the term (5.11) is estimated by $\mathcal{O}(h^{\infty})$. Now a Tauberian argument implies the first assertion in (5.8). The second one is obtained by integration of (5.7) over $[\inf I_0, \lambda]$ combined with the equalities

$$M_{\varphi_2}(\mu) = G_{\varphi_2}(\mu) = N^{\#}_{\varphi_2}(\mu) = 0,\ \ \mu \leq \inf I_1$$

and the fact that $\hat\theta(t) \in \mathcal{S}(\mathbb{R})$. $\square$

Proof of Theorem 2. As mentioned above, it remains to show that

$$\xi(\lambda, h) = \xi(\lambda; L_1, L_2) = N(L_1^{\#}, \lambda) + \omega(\lambda)h^{-n} + \mathcal{O}(h^{1-n^{\#}}) \tag{5.13}$$

for a differential operator $L_2 = Q_2$ having no resonances in Ω_0 and such that each $\lambda \in I_0$ is non-trapping and non-critical energy level for L_2. According to Lemma 2 and Lemma 3, for $\lambda \in I_2$ we have

$$\xi(\lambda, h) = \mathrm{tr_{bb}}\Big[\varphi_1(L_j)\Big]_{j=1}^2 + (\mathcal{F}_h^{-1}\theta * N^{\#}_{\varphi_2})(\lambda) + \int_{-\infty}^{\lambda} \omega_{\varphi_2}(\mu)d\mu h^{-n} + \mathcal{O}(h^{1-n^{\#}}).$$

Given a function $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ on $\overline{B(0, R_0)}$, exploiting the functional calculus for smooth functions and the estimates for the trace (see [29]), we obtain

$$\begin{aligned}
\mathrm{tr_{bb}}\Big[\varphi_1(L_j)\Big]_{j=1}^2 &= \Big[\mathrm{tr}\Big(\chi\varphi_1(L_j)\chi\Big)\Big]_{j=1}^2 + \mathrm{tr_{bb}}\Big[\varphi_1(L_j)(1-\chi^2)\Big]_{j=1}^2 \\
&= \mathrm{tr}\Big(\chi\varphi_1(L_1^{\#})\chi\Big) - \mathrm{tr}\Big(\chi\varphi_1(L_2)\chi\Big) + \mathrm{tr}\Big[\varphi_1(Q_j)(1-\chi^2)\Big]_{j=1}^2 + \mathcal{O}(h^{\infty}) \\
&= \mathrm{tr}\Big(\varphi_1(L_1^{\#})\Big) + C(\varphi_1)h^{-n} + \mathcal{O}(h^{1-n}),
\end{aligned}$$

where $C(\varphi_1)$ is a constant depending on φ_1.

On the other hand, applying a Tauberian theorem for $N^{\#}_{\varphi_2}(\lambda) = \mathcal{O}(h^{-n^{\#}})$, we deduce

$$N^{\#}_{\varphi_2}(\lambda) = (\mathcal{F}_h^{-1}\theta * N^{\#}_{\varphi_2})(\lambda) + \mathcal{O}(h^{1-n^{\#}}),\ \ \forall \lambda \in \mathbb{R}.$$

Consequently, for $\lambda \in I_2$ we get

$$\xi(\lambda, h) = \mathrm{tr}\Big(\varphi_1(L_1^{\#})\Big) + \mathrm{tr}\Big(\varphi_2(L_1^{\#})\mathbf{1}_{]-C^{\#},\lambda]}(L_1^{\#})\Big) + \Big(C(\varphi_1) + \int_{-\infty}^{\lambda} \omega_{\varphi_2}(\mu)d\mu\Big)h^{-n} + \mathcal{O}(h^{1-n^{\#}}).$$

By construction we have

$$\varphi_1(L_j^{\#}) + \varphi_2(L_j^{\#})\mathbf{1}_{]-C^{\#},\lambda]}(L_j^{\#}) = \mathbf{1}_{]-C^{\#},\lambda]}(L_j^{\#}),\ \ \forall \lambda \in I_2$$

and this implies (5.13).

To obtain (5.2), we construct a covering of the interval $[E_0, E_1] \subset \cup_{\nu=1}^M J_\nu$ by small open intervals J_ν so that for every J_ν we can find an operator Q_ν with the properties of Proposition 3, where I_0 is replaced by J_ν. Next we introduce a partition of unity

$$\sum_{\nu=1}^{M} \varphi_\nu(x) = 1 \text{ on } [E_0, E_1],\ \ \varphi_\nu \in C_0^\infty(J_\nu; \mathbb{R}^+)$$

and we apply the above argument. This completes the proof of Theorem 2.

□

6. Breit-Wigner approximation

In this section we consider small domains of width h and we prove a semi-classical analogue of the Breit-Wigner approximation for $\xi(\lambda, h)$ (see [21], [23], [3] for similar results, [12] for the case of a potential having the form of an "well in the island" and [11] for the one dimensional critical case). In the following $\eta(\lambda, h)$ denotes the real analytic function defined by

$$\eta(\lambda, h) = \xi(\lambda, h) - \Big[\#\{\mu \in [E_0, \lambda] :\ \mu \in \mathrm{sp}_{pp}(L_j(h))\}\Big]_{j=1}^2.$$

Theorem 3. *Assume that $L_j(h)$, $j = 1, 2$ satisfy the assumptions of Theorem 2. Then for any $\lambda \in [E_0, E_1]$, any $0 < \delta < h/B$, $0 < B_1 < B$, and h sufficiently small we have*

$$\eta(\lambda + \delta, h) - \eta(\lambda - \delta, h) = \Big[\sum_{\substack{w \in \mathrm{Res}\, L_j(h), \\ \mathrm{Im}\, w \neq 0,\ |w-\lambda| < h/B_1}} \omega_{\mathbb{C}_-}(w, [\lambda - \delta, \lambda + \delta]) \Big]_{j=1}^2 + \mathcal{O}(\delta) h^{-n^\#}, \tag{6.1}$$

where $B > 0$ is the constant introduced in Theorem 2.

Remark. Following the recent result of J.-F. Bony [1] the assumption (5.1) implies the existence of positive constants D, ϵ_2, C_2, $h_2 > 0$ such that for any $\lambda \in [E_0 - \epsilon_2, E_1 + \epsilon_2]$, $h/D \leq \delta \leq D$ and $h \in]0, h_2]$ we have

$$\#\{z \in \mathbb{C} : z \in \mathrm{Res}\, L_j(h),\ |z - \lambda| \leq \delta\} \leq C_2 \delta h^{-n^\#},\ \ j = 1, 2. \tag{6.2}$$

Proof. We apply Theorem 1 in the interval $I_0 \supset (\lambda - \delta, \lambda + \delta)$, $0 < \delta \leq h/B_1$, and introduce the function

$$F(z, h) = \Big[\sum_{\substack{w \in \mathrm{Res}\, L_j,\ \mathrm{Im}\, w \neq 0, \\ h/B_1 \leq |w-\lambda| \leq C_4}} \Big(\frac{1}{z - w} - \frac{1}{z - \overline{w}}\Big)\Big]_{j=1}^2,\ \ z \in D(\lambda, h/B).$$

It is sufficient to show that

$$|F(z, h)| \leq C h^{-n^\#},\ \ |z - \lambda| \leq h/B. \tag{6.3}$$

We have

$$\partial_z F(z, h) = \Big[\sum_{\substack{w \in \mathrm{Res}\, L_j,\ \mathrm{Im}\, w \neq 0, \\ h/B_1 \leq |w-\lambda| \leq C_4}} \frac{1}{(z - \overline{w})^2} - \frac{1}{(z - w)^2}\Big]_{j=1}^2.$$

Let $l_0 \in \mathbb{N}$ be an integer such that $D \leq 2^{l_0-1}B$. Following the argument in [23] and applying (6.2), for any $z \in D(\lambda, h/B)$ we obtain

$$\sum_{\substack{w\in \mathrm{Res}\, L_j,\, \mathrm{Im}\, w\neq 0,\\ h/B_1 \leq |w-\lambda| < C_4}} \frac{1}{|z-w|^2} \leq \sum_{\substack{w\in \mathrm{Res}\, L_j,\, \mathrm{Im}\, w\neq 0,\\ h/B_1 \leq |w-\lambda| \leq \frac{2^{l_0}h}{D}}} \frac{1}{|z-w|^2}$$

$$+ \sum_{k=l_0}^{C\log(1/h)} \sum_{\frac{2^k h}{D} \leq |w-\lambda| \leq \frac{2^{k+1}h}{D}} \frac{1}{|z-w|^2}$$

$$\leq C 2^{l_0} D^{-1} h^{-1-n^{\#}} + C \sum_{k=l_0}^{C\log(1/h)} \frac{(2^{k+1}h)h^{-n^{\#}}}{(2^k h)^2} \leq C h^{-1-n^{\#}}\,.$$

Here and below we denote by $C > 0$ different constants which may change from line to line and which are independent on h and the choice of λ in the interval $[E_0, E_1]$. Thus we get the estimate

$$|\partial_z F(z,h)| \leq C h^{-n^{\#}-1},\ \ z \in D(\lambda, h/B)\,.$$

It remains to find an estimate of $|F(\mu_0, h)| = |\,\mathrm{Im}\, F(\mu_0, h)|$ at a suitable point $\mu_0 = \mu_0(h)$. [1] Set $\nu = \frac{h}{B} < \frac{h}{B_1}$ and suppose that for all $\mu \in \mathbb{R}$, $|\mu - \lambda| \leq \nu$, we have $|\,\mathrm{Im}\, F(\mu, h)| \geq M h^{-n^{\#}}$, $M > 0$. The continuity of the function $\mathrm{Im}\, F(\mu, h)$ implies that $\mathrm{Im}\, F(\mu, h)$ is either positive or negative in $[\lambda - \nu, \lambda + \nu]$. Assuming $\mathrm{Im}\, F(\mu, h)$ positive, we get

$$\frac{M h^{-n^{\#}+1}}{B\pi} \leq \frac{1}{2\pi}\int_{\lambda-\nu}^{\lambda+\nu} \mathrm{Im}\, F(\mu,h)d\mu \leq \frac{1}{\pi}\int_{\lambda-\nu}^{\lambda+\nu} \Big[\sum_{\substack{w\in \mathrm{Res}\, L_j,\, \mathrm{Im}\, w\neq 0\\ |w-\lambda|\leq C}} \frac{|\,\mathrm{Im}\, w|}{|\mu-w|^2}\Big]_{j=1}^{2} d\mu$$

$$+\frac{1}{\pi}\sum_{j=1}^{2}\int_{\lambda-\nu}^{\lambda+\nu} \sum_{\substack{w\in \mathrm{Res}\, L_j,\, \mathrm{Im}\, w\neq 0,\\ |w-\lambda|<h/B_1}} \frac{|\,\mathrm{Im}\, w|}{|\mu-w|^2} d\mu$$

$$\leq |\eta(\lambda+\nu, h) - \eta(\lambda-\nu, h)| + C h^{1-n^{\#}}\,.$$

Here we have used the inequality

$$\int_{\lambda-\nu}^{\lambda+\nu} \frac{|\,\mathrm{Im}\, w|}{|\mu-w|^2} d\mu \leq \int_{-\infty}^{\infty} \frac{|\,\mathrm{Im}\, w|}{|\mu-w|^2} d\mu \leq \pi$$

and (6.2) to estimate the number of resonances in $\{w : |w-\lambda| < h/B_1\}$. Notice that if $D \leq B_1$, we have $\{w : |w-\lambda| < h/B_1\} \subset \{w : |w-\lambda| < h/D\}$. Next the assumption (5.1) combined with Theorem 2 yield the estimate

$$|\xi(\lambda+\nu, h) - \xi(\lambda-\nu, h)| \leq C h^{1-n^{\#}}.$$

Thus,

$$|\eta(\lambda+\nu, h) - \eta(\lambda-\nu, h)| \leq |\xi(\lambda+\nu, h) - \xi(\lambda-\nu, h)|$$

$$+\sum_{j=1}^{2} \sharp\{\mu \in \mathrm{sp}_{pp}(L_j) :\ |\mu-\lambda| \leq \nu\} \leq C h^{1-n^{\#}}\,,$$

[1] There is some similarity between the proof of the existence of $\mu_0(h)$ and that of the existence of a suitable point $z_0(h)$, $\mathrm{Im}\, z_0(h) \geq \delta > 0$ in Section 4 in [23] so that $\log|\det S(z_0(h), h)| \geq -C h^{-n^{\#}}$.

where for the second inequality we have used once more (6.2), observing that the positive eigenvalues of L_j coincide with the resonances on $\mathbb{R}^+$. Consequently, we obtain a bound for M. Hence there exists a constant $C > 0$ and $\mu_0 \in [\lambda - \nu, \lambda + \nu]$ so that

$$|F(\mu_0, h)| \leq C h^{-n^\#}. \tag{6.4}$$

Writing

$$F(z, h) = F(\mu_0, h) + \int_{\mu_0}^{z} \partial_z F(z, h) dz, \ \ |z - \lambda| \leq h/B\,,$$

we obtain (6.3). The case $\operatorname{Im} F(\mu, h) < 0$ can be treated by the same argument exploiting the inequality $-\operatorname{Im} F(\mu, h) \geq M h^{-n^\#}$, $|\mu - \lambda| \leq \nu$. By an integration over the interval $(\lambda - \delta, \lambda + \delta)$, we complete the proof of (6.1). □

Remark. Our proof goes without a factorization in small domains $\{z \in \mathbb{C} : |z - \lambda| \leq Ch\}$ and a suitable trace formula (see Lemma 6.2 in [23] and Theorem 1.3 in [3]). The above argument can be applied to simplify the proof of Lemma 6.2 in [23].

Next, the estimate (6.3) of $F(z, h)$ yields immediately the following.

Corollary 1. *Under the assumptions of Theorem 3 for $\mu \in \mathbb{R}$, $|\mu - \lambda| < h/B$ we have the representation*

$$\xi'(\mu, h) = \frac{1}{\pi} \operatorname{Im} q(\mu, h) + \Big[\sum_{\substack{w \in \mathrm{Res}\, L_j,\, |w-\lambda| < h/B_1 \\ \mathrm{Im}\, w \neq 0}} \frac{-\operatorname{Im} w}{\pi |\mu - w|^2} + \sum_{\substack{w \in (\mathrm{Res}\, L_j \cap \mathbb{R}), \\ |w - \lambda| < h/B}} \delta(\mu - w) \Big]_{j=1}^{2}, \tag{6.5}$$

where $q(z, h) = p(z, h) - \overline{p(\overline{z}, h)}$, $p(z, h)$ is holomorphic in $D(\lambda, h/B)$ and $p(z, h)$ satisfies the estimate

$$|p(z, h)| \leq C h^{-n^\#}, \ \ z \in D(\lambda, h/B)$$

with $C > 0$ independent on $h \in]0, h_0]$ and $\lambda \in [E_0, E_1]$.

We may slightly improve Theorem 3, noting that for every $0 < \epsilon < 1$ and $|\mu - \lambda| \leq \frac{\epsilon h}{B}$ we have

$$\sum_{\substack{w \in \mathrm{Res}\, L(h), \\ \epsilon h/B_1 \leq |w - \lambda| \leq h/B_1}} \frac{|\operatorname{Im} w|}{|\mu - w|^2} \leq \frac{h}{\epsilon^2 h^2} C h^{1 - n^\#} = \mathcal{O}_\epsilon(h^{-n^\#})\,.$$

Thus for $0 < \delta \leq \frac{\epsilon h}{B}$ the right hand part in (6.1) can be replaced by

$$\Big[\sum_{\substack{w \in \mathrm{Res} L_j(h),\ \mathrm{Im}\, w \neq 0, \\ |w - \lambda| \leq \epsilon h/B_1}} \omega_{\mathbb{C}_-}(w, [\lambda - \delta, \lambda + \delta]) \Big]_{j=1}^{2} + \mathcal{O}_\epsilon(\delta) h^{-n^\#}\,.$$

To obtain a stronger version involving the resonances in smaller "boxes", we need some additional information for the distribution of the resonances in $\{w \in \mathbb{C} : |w - \lambda| \leq \epsilon h\}$. In the case of the Schrödinger operator $L(h) = -h^2 \Delta + V(x)$ with $V(x) \in C_0^\infty(\mathbb{R}^n)$ real valued this is possible applying the recent result of Stefanov [31]. Set $a_0(x, \xi) = |\xi|^2 + V(x)$ and let $0 < E_0 < E_1$ be non-critical values of $a_0(x, \xi)$. Let

$$a_0^{-1}[E_0, E_1] = W_{\mathrm{int}} \cup W_{\mathrm{ext}}\,,$$

where W_{ext} is the *unbounded* connected component, while W_{int} is the union of bounded ones if there are such connected components. Assume that all points in W_{ext} are non-trapping (see [31] for a precise definition). Then, according to Theorem 6.1 in [31], there exists a function $0 < R_1(h) = \mathcal{O}(h^\infty)$ such that for any $M \in \mathbb{N}$ the operator $L(h)$ has no resonances in the set

$$\Omega_M(\lambda, h) = [E_0, E_1] + i[-Mh, -R_1(h)],\ \ 0 < h \leq h(M). \tag{6.6}$$

Setting $0 < R(h) = \sqrt{hR_1(h)} = \mathcal{O}(h^\infty)$, an elementary argument shows that for $\lambda \in [E_0, E_1]$ and $|\mu - \lambda| \leq R(h)/2$ we have

$$\sum_{\substack{w \in \mathrm{Res}\ L(h),\ |\operatorname{Im} w| \leq R_1(h) \\ R(h) \leq |Re w - \lambda| \leq h}} \frac{|\operatorname{Im} w|}{|\mu - w|^2} \leq C h^{-n^{\#}} .$$

In the next result we treat a formally symmetric differential operator

$$L_1(h) = \sum_{|\alpha| \leq 2} a_\alpha(x, h)(hD_x)^\alpha$$

on $L^2(\mathbb{R}^n)$ satisfying the assumptions of Section 2. Given a fixed $\lambda \in]E_0, E_1[$, as in the previous section, we may construct an operator $L_2(h)$ having the properties (a) - (c) of Proposition 3. Applying Theorem 3 for $L_j(h)$, $j = 1, 2$, and $\{z \in \mathbb{C} : |z - \lambda| \leq h/B_1\} \subset W$, and assuming that we have a free resonances domain, we obtain the following improvement of Corollary 1.

Corollary 2. *Let $E_0 < \lambda < E_1$ be fixed. Let $L_2(h)$ be chosen so that $L_j(h)$, $j = 1, 2$, satisfy the assumptions of Theorem 3 and $L_2(h)$ has no resonances in the disk $\{z \in \mathbb{C} : |z - \lambda| \leq h/B_1\}$. Suppose that there exists a function $0 < R_1(h) = \mathcal{O}(h^\infty)$ such that $L_1(h)$ has no resonances in the set*

$$[E_0, E_1] + i[-\epsilon h, -R_1(h)],\ \epsilon > 0,\ \ 0 < h \leq h(\epsilon) .$$

Then for $|\mu - \lambda| < \frac{R(h)}{2}$ and h sufficiently small we have

$$\xi'(\mu, h) = \frac{1}{\pi} \operatorname{Im} q(\mu, h) + \sum_{\substack{w \in \mathrm{Res}\ L_1,\ |\operatorname{Re} w - \lambda| < R(h) \\ 0 < |\operatorname{Im} w| \leq R_1(h)}} \frac{-\operatorname{Im} w}{\pi |\mu - w|^2} + \sum_{\substack{w \in \mathrm{Res}\ L_1 \cap \mathbb{R}, \\ |w - \lambda| < R(h)/2}} \delta(\mu - w) \tag{6.7}$$

with $R(h) = \sqrt{hR_1(h)} = \mathcal{O}(h^\infty)$ and $q(\mu, h)$ as in Corollary 1.

7. Local trace formula

In this section we prove a local trace formula which is a slightly stronger version of that in [28], [29] (see [23] for compactly supported perturbations). Exploiting Theorem 1, we repeat with trivial modifications the argument of Section 5, [23], to get the following.

Theorem 4. *Assume that $L_j(h)$ satisfy the assumptions of Section 2. Let $\Omega \subset e^{i]-2\theta_0, 2\theta_0[}]0, \infty[$ be an open, simply connected, relatively compact set such that $I = \Omega \cap \mathbb{R}$ is an interval. Suppose that f is holomorphic on a neighborhood of Ω and that $\psi \in C_0^\infty(\mathbb{R})$ satisfies*

$$\psi(\lambda) = \begin{cases} 0, & d(I, \lambda) > 2\epsilon, \\ 1, & d(I, \lambda) < \epsilon, \end{cases}$$

where $\epsilon > 0$ is sufficiently small. Then

$$\mathrm{tr}_{\mathrm{bb}} \Big[(\psi f)(L_j(h))\Big]_{j=1}^2 = \Big[\sum_{z \in \mathrm{Res}\ (L_j(h)) \cap \Omega} f(z) \Big]_{j=1}^2 + E_{\Omega, f, \psi}(h) \tag{7.1}$$

with

$$|E_{\Omega, f, \psi}(h)| \leq M(\psi, \Omega) \sup\{|f(z)|\ :\ 0 \leq d(\Omega, z) \leq 2\epsilon,\ \operatorname{Im} z \leq 0\} h^{-n^{\#}} .$$

Proof. Choose an almost analytic extension $\tilde{\psi}$ of ψ so that $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{C})$, $\tilde{\psi} = 1$ on Ω and

$$\text{supp } \overline{\partial}_z \tilde{\psi} \subset \{z \in \mathbb{C} : \epsilon \leq d(\Omega, z) \leq 2\epsilon\}.$$

Setting $\Omega_\epsilon = \{z \in \mathbb{C} : d(\Omega, z) \leq \epsilon\}$, we have

$$\text{tr}_{\text{bb}} \Big[(\psi f)(L_j(h)) \Big]_{j=1}^2 = < \xi'(\lambda, h), (\psi f)(\lambda) >$$

$$= \Big[\sum_{w \in \text{Res } (L_j(h)) \cap \text{ supp } \psi} (\psi f)(w) \Big]_{j=1}^2 + \frac{1}{2\pi i} \int (\psi f)(\lambda) r(\lambda, h) d\lambda$$

$$+ \frac{1}{2\pi i} \int (\psi f)(\lambda) \Big[\sum_{\substack{w \in \text{Res } (L_j(h)) \cap \Omega_{2\epsilon}, \\ \text{Im } w \neq 0}} \Big(\frac{1}{\lambda - \overline{w}} - \frac{1}{\lambda - w} \Big) \Big]_{j=1}^2 d\lambda \,.$$

The integral involving $r(\lambda, h)$ can be estimated using (1.4) with $W = \Omega_{2\epsilon}$. For the integral containing the resonances we apply Green formula and we get the term

$$\Big[\sum_{z \in \text{Res } L_j, \, \text{Im } z \neq 0} (\tilde{\psi} f)(z) \Big]_{j=1}^2$$

$$+ \frac{1}{\pi} \int_{\mathbb{C}_-} (\overline{\partial}_z \tilde{\psi})(z) f(z) \Big[\sum_{\substack{w \in \text{Res } (L_j(h)) \cap \Omega_{2\epsilon}, \\ \text{Im } w \neq 0}} \Big(\frac{1}{z - \overline{w}} - \frac{1}{z - w} \Big) \Big]_{j=1}^2 \mathcal{L}(dz) \,,$$

where $\mathcal{L}(dz)$ is the Lebesgue measure on $\mathbb{C}$. As in the proof of Theorem 1 in [23], we apply the inequality

$$\int_{\Omega_1} \frac{1}{|z - w|} \mathcal{L}(dz) \leq 2 \sqrt{2\pi |\Omega_1|}$$

and an upper bound for the number of the resonances in $\Omega_{2\epsilon}$ to obtain the result. □

Since we have no restrictions on the behavior of the holomorphic function $f(z)$ on $\Omega \cap \{\text{Im } z > 0\}$, we may apply the above argument choosing $f(z) = e^{-itz/h}$, $t \in \mathbb{R}$, to get the following.

Theorem 5. *Let Ω and ψ be as in Theorem 4 and let $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{C})$ be an almost analytic extension of ψ supported in $\Omega_{2\epsilon}$. Then for any $0 < \delta < 1$ and $t \geq h^\delta$ we have*

$$\text{tr}_{\text{bb}} \Big[\psi(L_j(h)) e^{-i \frac{t}{h} L_j(h)} \Big]_{j=1}^2 = \Big[\sum_{w \in \text{Res } (L_j(h)) \cap \Omega_{2\epsilon}} \tilde{\psi}(w) e^{-itw/h} \Big]_{j=1}^2 + \mathcal{O}_\delta(h^\infty). \tag{7.2}$$

Moreover, for $t \geq \epsilon > 0$ and $N \in \mathbb{N}$ there exists $h_N > 0$ such that for $0 < h \leq h_N$ we have

$$\text{tr}_{\text{bb}} \Big[\psi(L_j(h)) e^{-i \frac{t}{h} L_j(h)} \Big]_{j=1}^2 = \Big[\sum_{\substack{w \in \text{Res } (L_j(h)) \cap \Omega_{2\epsilon} \\ |\text{ Im } w| \leq -Nh \log h}} \tilde{\psi}(w) e^{-itw/h} \Big]_{j=1}^2 + \mathcal{O}_\epsilon(h^{N\epsilon - n^\#}). \tag{7.3}$$

Proof. Choose an almost analytic extension $\tilde{\psi}$ of ψ as in Theorem 4. Applying Green formula, we must examine the integrals

$$\int_{\mathbb{C}_-} \overline{\partial}_z \tilde{\psi}(z) e^{-itz/h} r(z, h) \mathcal{L}(dz) \,,$$

$$\int_{\mathbb{C}_-} \overline{\partial}_z \tilde{\psi}(z) e^{-itz/h} \Big[\sum_{w \in \text{Res } L_j(h) \cap \Omega_{2\epsilon}} \Big(\frac{1}{z - w} - \frac{1}{z - \overline{w}} \Big) \Big]_{j=1}^2 \mathcal{L}(dz) \,.$$

Choose $\mu > 0,\ 0 < \delta + \mu < 1$. For $-h^\mu \leq \operatorname{Im} z \leq 0$ we have

$$|\overline{\partial}_z \tilde{\psi}| \leq C_N |\operatorname{Im} z|^N \leq C_N h^{\mu N},\ \forall N \in \mathbb{N}$$

and the integration over $-h^\mu \leq \operatorname{Im} z \leq 0$ combined with the argument of the proof of Theorem 4 yield a term bounded by $\mathcal{O}(h^\infty)$. On the other hand, for $t \geq h^\delta,\ \operatorname{Im} z \leq -h^\mu$ we get

$$|e^{-itz/h}| \leq e^{-th^{\mu-1}} \leq e^{-h^{\delta+\mu-1}} = \mathcal{O}_\delta(h^\infty)$$

and this implies (7.2). For the second assertion we have

$$\left|e^{-itw/h}\right| \leq e^{tN \log h} \leq h^{N\epsilon}$$

for $|\operatorname{Im} w| \geq -Nh \log h$ and this completes the proof. $\square$

Remark. For non-trapping compactly supported perturbations $L(h)$ (see [32], [4]) and for non-trapping long-range perturbations $L(h) = -h^2\Delta + V(x)$ of the Laplacian (see [15]) there are no resonances of $L(h)$ in the domain

$$-Nh \log \frac{1}{h} \leq \operatorname{Im} z \leq 0,\ \ 0 < h \leq h_N\,.$$

For such perturbations the left hand side of (7.3) is equal to $\mathcal{O}_\epsilon(h^{N\epsilon - n^\#})$ and we obtain an analogue of the classical trace formula for non-trapping perturbations.

8. Existence of resonances close to the real axis

In this section we consider the operator $L(h) = -h^2\Delta_g + V(x)$, where Δ_g is symmetric Laplace-Beltrami operator on $L^2(\mathbb{R}^n)$ associated to a metric $g(x) = \{g_{i,j}(x)\}_{1\leq i,j\leq n}$ and $V(x) \in C^\infty(\mathbb{R}^n)$ is a real valued function. We assume that there exists $\rho > n$ so that

$$|\partial_x^\alpha (g_{i,j}(x) - \delta_{i,j})| + |\partial_x^\alpha V(x)| \leq C_\alpha < x >^{-\rho-|\alpha|},\ \ 1 \leq i,j \leq n,\ \ \forall\alpha. \tag{8.1}$$

Moreover, we assume that the coefficients $\{g_{i,j}(x)\}$ and $V(x)$ can be extended holomorphically in x to the domain given in (2.8) and the estimate (8.1) holds in this domain.

Consider the symbol

$$a_0(x,\xi) =< g(x)^{-1}\xi, \xi > +V(x)$$

and denote by H_{a_0} the Hamilton vector field associated to a_0 and by $\Phi^t = \exp(tH_{a_0})$ the Hamilton flow. Given $\lambda > 0$, let $\Sigma_\lambda = \{(x,\xi) \in \mathbb{R}^n : a_0(x,\xi) = \lambda\}$ be the energy surface and let $\nabla a_0(x,\xi) \neq 0$ on Σ_λ. A point $\nu \in \Sigma_\lambda$ is called *periodic*, if there exists $T > 0$ such that $\Phi^T(\nu) = \nu$ and the smallest $T > 0$ with this property is called period $T(\nu)$ of ν. Given a periodic point ν, consider the trajectory

$$\gamma(\nu) = \{\Phi^t(\nu) : 0 \leq t \leq T(\nu)\} = \{(x(t), \xi(t)) : 0 \leq t \leq T(\nu)\}$$

and define the *action* $S(\nu)$ along $\gamma(\nu)$ by

$$S(\nu) = \int_{\gamma(\nu)} \xi dx = \int_0^{T(\nu)} \xi(t) x'(t) dt\,.$$

Next we denote by $m(\nu) \in \mathbb{Z}_4$ the Maslov index related to $\gamma(\nu)$ and set $q(\nu) = -\frac{\pi}{2} m(\nu)$. Let Π be the set of all periodic points on Σ_λ and let

$$Q(h,r) = (2\pi)^{-n} \int_\Pi \Big[\pi - h^{-1}S(\nu) + q(\nu) - rT(\nu)\Big]_{2\pi} T(\nu)^{-1} d\nu, \tag{8.2}$$

where $d\nu$ is the Liouville measure on Σ_λ and the residue $-\pi < [z]_{2\pi} \leq \pi$ is defined so that $z = [z]_{2\pi} + 2\pi k,\ k \in \mathbb{Z}$. The set Π is bounded, the integrand in (8.2) is a measurable function and $T(\nu) \geq T_0 > 0,\ \forall \nu \in \Pi$. The oscillatory function $Q(h, r)$ has been introduced in [19] for the analysis of the semi-classical behavior of the eigenvalues and it is a semi-classical analogue of the oscillating function defined by Guriev and Safarov [14] and Safarov [26]. Notice that the limits $Q(h, r \pm 0) = \lim_{\epsilon \downarrow 0} Q(h, r \pm \epsilon)$ exist for each r and $0 < h \leq h_0$ and, moreover,

$$Q(h, r+0) - Q(h, r-0) = (2\pi)^{1-n} \int_{\Omega_{h,r}} \frac{d\nu}{T(\nu)}\,,$$

where $\Omega_{h,r} = \{\nu \in \Pi : h^{-1}S(\nu) - q(\nu) + rT(\nu) \equiv 0(2\pi)\}$. Following the arguments in Section 6, [21], we will prove the following.

Theorem 6. *Let $L(h) = -h^2\Delta_g + V(x)$, where the metric $g(x)$ and $V(x)$ satisfy the estimates (8.1) and let $\nabla a_0(x,\xi) \neq 0$ on $\Sigma_\lambda,\ \lambda > 0$. Assume that there exist an integer $p \in \mathbb{Z}$ and a subset $\Pi_0 \subset \Pi$ with positive Liouville measure $\mu(\Pi_0) > 0$ so that*

$$\Big(\Big[q(\nu) - h^{-1}S(\nu)\Big]_{2\pi} + 2\pi p\Big) T(\nu)^{-1} = r(h),\ 0 < h \leq h_0$$

does not depend on $\nu \in \Pi_0$. Then for $0 < h \leq h_1$ we have

$$\#\{w \in \operatorname{Res} L(h) : |w - \lambda - r(h)h| \leq h\} \geq \frac{(2\pi)^{1-n}}{2} h^{1-n} \int_{\Pi_0} \frac{d\nu}{T(\nu)}. \tag{8.3}$$

Remark. Clearly, $|r(h)| \leq \max\{|2p-1|,\ |2p+1|\}\pi(T_0)^{-1}$ and applying the Remark after Theorem 3, we conclude that

$$\#\{w \in \operatorname{Res} L(h) :\ |w - \lambda - r(h)h| \leq h\} \leq A h^{1-n}\,.$$

This shows that the order h^{1-n} in our estimate is optimal. On the other hand, it is clear that we may apply Theorem 3 with $B > 1$ and $B_1 = 1$.

Proof. Consider the scattering phase $\sigma(\lambda, h) = \frac{1}{2\pi i} \det S(\lambda, h)$, where the scattering operator $S(\lambda, h)$ is related to $L(h)$ and $L_0(h) = -h^2\Delta$. According to Birman-Krein theory (see for instance [33]), the scattering phase can be identified with the spectral shift function and, under our assumptions, we have not embedded positive eigenvalues. Following Theorem 2.1 in [5], and taking $|r(h)| \leq r_0$, $0 < \epsilon \leq \epsilon_0$, $0 < h \leq h_0$ and $\lambda > 0$ we have

$$\sigma\Big(\lambda + (r(h)+\epsilon)h, h\Big) - \sigma\Big(\lambda + (r(h)-\epsilon)h, h\Big)$$

$$\geq h^{1-n}\Big[Q\Big(h, r(h)+\epsilon/2\Big) - Q\Big(h, r(h)-\epsilon/2\Big)\Big] + 2\epsilon\gamma_0'(\lambda)h^{1-n} - C_0\epsilon h^{1-n} - o_\epsilon(h^{1-n})\,,$$

where

$$\gamma_0(\lambda) = (2\pi)^{-n} \int_{\mathbb{R}^n} \Big(\int_{a_0(x,\xi) \leq \lambda} d\xi - \int_{|\xi|^2 \leq \lambda} d\xi\Big) dx\,,$$

$C_0 > 0$ is independent on $r(h)$, ϵ and h and $o_\epsilon(h^{1-n})$ means that for any fixed $\epsilon > 0$ we have

$$\lim_{h \downarrow 0} \frac{o_\epsilon(h^{1-n})}{h^{1-n}} = 0.$$

On the other hand, for small $\epsilon > 0$ an application of Theorem 3 with $\delta = \epsilon h$ yields the estimate

$$\sigma\Big(\lambda + (r(h)+\epsilon)h, h\Big) - \sigma\Big(\lambda + (r(h)-\epsilon)h, h\Big)$$

$$\leq \#\{w \in \operatorname{Res} L(h) : |w - \lambda - r(h)h| \leq h\} + C_1 \epsilon h^{1-n},\ 0 < h \leq h_2 \leq h_0$$

with $C_1 > 0$ independent on $\epsilon, r(h)$ and h. We claim that

$$Q(h, r(h) + \epsilon/2) - Q(h, r(h) - \epsilon/2) \geq -(2\pi)^{-n} \epsilon \mu(\Pi) + (2\pi)^{1-n} \int_{\Pi_0} \frac{d\nu}{T(\nu)}. \tag{8.4}$$

In fact, according to the representation of the oscillatory function $Q(h, r)$ (see for instance, Proposition 1, [26]), we have

$$Q(h, r(h) + \epsilon/2) - Q(h, r(h) - \epsilon/2) = -\epsilon(2\pi)^{-n} \mu(\Pi)$$
$$+ (2\pi)^{1-n} \int_{\Pi} T^{-1}(\nu) \sum_{k \in \mathbb{Z}} \chi_{h,k}^{\epsilon}(\nu) d\nu\,,$$

where $\chi_{h,k}^{\epsilon}$ is the characteristic function of the set

$$\Omega_{h,k}^{\epsilon} = \{\nu \in \Pi : -\epsilon T(\nu) \leq h^{-1} S(\nu) - q(\nu) + r(h) T(\nu) - 2k\pi < \epsilon T(\nu)\}\,.$$

Obviously, for any $\nu \in \Pi_0$ we get

$$h^{-1} S(\nu) - q(\nu) + r(h) T(\nu) + 2M(\nu, h)\pi - 2p\pi = 0$$

with some $M(\nu, h) \in \mathbb{Z}$. Consequently,

$$\nu \in \Pi_0 \Longrightarrow \sum_{k \in \mathbb{Z}} \chi_{h,k}^{\epsilon}(\nu) \geq 1$$

and we obtain (8.4). Choosing $\epsilon > 0$ small enough, we arrange the inequality

$$-\epsilon(2\pi)^{-n} \mu(\Pi) - \epsilon(C_0 + C_1) + 2\epsilon \gamma_0'(\lambda) \geq -\frac{\alpha_0}{4}\,,$$

with $\alpha_0 = (2\pi)^{1-n} \int_{\Pi_0} \frac{d\nu}{T(\nu)}$. Next we fix $\epsilon > 0$ and choose $0 < h_1 \leq h_2$ so that for $0 < h \leq h_2$ we have

$$|o_{\epsilon}(h^{1-n})| \leq \frac{\alpha_0}{4} h^{1-n}\,.$$

Combining the above estimates for the difference $\sigma\Big(\lambda + (r(h) + \epsilon)h, h\Big) - \sigma\Big(\lambda + (r(h) - \epsilon)h, h\Big)$, we complete the proof.

□

Example (see Section 7 in [5]). Let $L(h) = -h^2 \Delta + V(x)$ with

$$V(x) = \Phi_a(x - y_0)\Big(|x - y_0|^2 + b\Big)\,,$$

where $a > 0,\ b > 0$ and $y_0 \in \mathbb{R}^n$ are fixed and $\Phi_a(x) \in C_0^{\infty}(\mathbb{R}^n),\ \Phi_a(x) = 1$ for $|x| \leq 2a$. Let $0 < \epsilon < a/2$, $|\eta_0| = \sqrt{\lambda - b}$ and let $\lambda \in]b, b + a^2[$ be a non-critical energy level for $a_0(x, \xi) = |\xi|^2 + V(x)$. Therefore the set

$$\Pi_0 = \{(x, \xi) \in \Sigma_{\lambda} : |\xi - \eta_0|^2 + |x - y_0|^2 \leq \epsilon^2\}$$

has a positive Liouville measure and $\Pi_0 \subset \Pi$. Moreover, for every $\nu \in \Pi_0$ we have

$$T(\nu) = \pi,\ \ S(\nu) = (\lambda - b)\pi,\ \ q(\nu) = \frac{\pi}{2} m$$

with $m \in \mathbb{Z}$ independent on ν. We may apply Theorem 6 with $r_p(h) = \frac{1}{\pi}\Big[\frac{\pi}{2} m - h^{-1}(\lambda - b)\pi\Big]_{2\pi} + 2p,\ \ p \in \mathbb{Z}$, to conclude that

$$\#\{w \in \operatorname{Res} L(h) : |w - \lambda - r_p(h)h| < h\} \geq (2\pi)^{-n} \mu(\Pi_0) h^{1-n}\,.$$

On the other hand, for $p \neq j$ and $0 < h \leq h_0$ we have

$$\{w : |\operatorname{Re} w - \lambda - r_p(h)h| < h\} \cap \{w : |\operatorname{Re} w - \lambda - r_j(h)h| < h\} = \emptyset$$

and the clusters related to $p \neq j$ produce different resonances. Choosing $\delta > 0$ so that $]\lambda - \delta, \lambda + \delta[\subset]b, b + a^2[$, one obtains easily

$$\#\{w \in \operatorname{Res} L(h) : |w - \lambda| \leq \delta\} \geq \alpha\delta(2\pi)^{-n}\mu(\Pi_0)h^{-n}$$

with $\alpha > 0$ independent on δ. A stronger asymptotic for the number of the resonances in

$$[b, b + a^2] + i[-R(h), 0]$$

has been obtained by Stefanov [31]. Notice that in the above result we count only the resonances lying in clusters.

9. Appendix

In this Appendix we present a proof of (5.10). Following the Remark after Lemma 2, we will assume that $\varphi_2 = \psi^2$, $\psi \in C_0^\infty(I_1; \mathbb{R}^+)$, $I_1 \subset I_0$. Recall that $\lambda \in I_0$, supp $\theta(t) \subset [-\delta_1, \delta_1]$ and $\chi(x) = 1$ for $|x| \leq 2R$, $R > R_0$. It is easy to see that

$$\begin{aligned} &\operatorname{tr}\Big[\frac{1}{2\pi h}\int e^{it(\lambda - Q_j)h^{-1}}\theta(t)\psi^2(Q_j)(1-\chi^2)dt\Big]_{j=1}^2 \\ &= \frac{1}{2\pi h}\int e^{it\lambda h^{-1}}\theta(t)\operatorname{tr}\Big(\Big[\psi^2(Q_j)\Big]_{j=1}^2 e^{-itQ_2/h}(1-\chi^2)\Big)dt \\ &+ \frac{1}{2\pi h}\int e^{it\lambda h^{-1}}\,\theta(t)\,\operatorname{tr}\Big(\psi^2(Q_1)\Big[e^{-itQ_j/h}\Big]_{j=1}^2(1-\chi^2)\Big)dt = \mathcal{A} + \mathcal{B}. \end{aligned}$$

This representation is justified by applying Lemma 4.1 in [2] saying that

$$\|\Big[\psi^2(Q_j)\Big]_{j=1}^2\|_{\operatorname{tr}} = \mathcal{O}(h^{-n}),\ \ \|\psi^2(Q_1)\Big[e^{-itQ_j/h}\Big]_{j=1}^2\|_{\operatorname{tr}} = \mathcal{O}(h^{-1-n}).$$

We treat below $\mathcal{A}$ following closely the analysis of J.-F. Bony in Section 4.2, [2]. Put $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, where

$$\begin{aligned} \mathcal{A}_1 &= \frac{1}{2\pi h}\int e^{it\lambda h^{-1}}\theta(t)\operatorname{tr}\Big((\psi(Q_1) - \psi(Q_2))e^{-itQ_2/h}\psi(Q_2)(1-\chi^2)\Big)dt, \\ \mathcal{A}_2 &= \frac{1}{2\pi h}\int e^{it\lambda h^{-1}}\theta(t)\operatorname{tr}\Big(\psi(Q_1)(\psi(Q_1) - \psi(Q_2))e^{-itQ_2/h}(1-\chi^2)\Big)dt. \end{aligned}$$

We deal with the analysis of $\mathcal{A}_1$ only, since that of $\mathcal{A}_2$ is similar (see also Section 4.2, [2]). First, we find a pseudodifferential operator Q with symbol in $S^0(1)$ so that

$$\mathcal{A}_1 = \frac{1}{2\pi h}\int e^{it\lambda h^{-1}}\theta(t)\operatorname{tr}\Big(e^{-itQ_2/h}\psi(Q_2)Q(Q_1 - Q_2)\tilde{\psi}(Q_2)\Big)dt,$$

where $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ is such that $\tilde{\psi} = 1$ on supp ψ. We use the notations of [9] for h-pseudodifferential operators and set $\langle x\rangle = (1 + |x|^2)^{1/2}$. Moreover, modulo a term in $S^N(1)$, the symbol of Q is supported in $\{(x,\xi) : |x| > 2R\}$. Secondly, we obtain the existence of a pseudodifferential operator S with symbol

$$s(x,y,\xi;h) \in S^0\Big(\langle x\rangle^{-n-1}\langle\xi\rangle^{-N}\Big),\ \ \forall N \in \mathbb{N},$$

having compact support in ξ and $(x-y)$ and support in $\{(x,\xi) : |x| > 2R,\ (x,\xi) \in l_2^{-1}(I_1)\}$ so that

$$\mathcal{A}_1 = \frac{1}{2\pi h}\operatorname{tr}\Big(\int e^{it\lambda h^{-1}}\theta(t)e^{-itQ_2/h}S\,dt\Big) + \mathcal{O}(h^\infty).$$

Applying Theorem 2 in [2], we obtain the existence of a Fourier integral operator $\mathcal{U}_t$ such that for $|t| \leq \delta_1$ and δ_1 sufficiently small we have

$$\|\mathcal{U}_t - e^{-itL_2/h}S\|_{\rm tr} = \mathcal{O}(h^\infty).$$

Next, we write the kernel of the operator $\int e^{it\lambda h^{-1}}\theta(t)\mathcal{U}_t dt$ in the form

$$K(x,y;h) = \frac{1}{(2\pi h)^n}\int\int e^{i\left(t\lambda+\Phi(t,x,\xi)-y.\xi\right)/h}\theta(t)A(t,x,y,\xi;h)dtd\xi$$

and deduce that

$$\mathcal{A}_1 = \frac{1}{(2\pi h)^{n+1}}\int\int\int e^{i\left(t\lambda+\Phi(t,x,\xi)-x.\xi\right)/h}\theta(t)A(t,x,x,\xi;h)dt\,dx\,d\xi + \mathcal{O}(h^\infty)\,.$$

Here $\Phi(t,x,\xi)$ is the solution of the eikonal equation

$$\begin{cases}\partial_t\Phi + l_2(x,\partial_x\Phi) = 0,\\ \Phi(0,x,\xi) = x.\xi,\end{cases}$$

$l_j(x,\xi)$ being the principal symbol of $Q_j,\ \ j=1,2$, and all derivatives $\partial_t^\alpha\partial_x^\beta\partial_\xi^\gamma\Big(\Phi(t,x,\xi)-x.\xi\Big)$ are uniformly bounded for $(t,x,\xi)\in[-\delta_1,\delta_1]\times\mathbb{R}^n\times B(0,C_1)$ and $(\alpha,\beta,\gamma)\neq(0,0,0)$. Moreover, the symbol $A(t,x,x,\xi)$ has support in $\{(x,\xi): |x|>2R,\ |\xi|\leq C_1,\ \ (x,\xi)\in l_2^{-1}(I_1)\}$ so that for all α and $|t|\leq\delta_1$ we have

$$|\partial^\alpha A(t,x,x,\xi)| \leq C_\alpha\langle x\rangle^{-n-1}. \tag{9.1}$$

The last estimate enables us to calculate $\mathcal{A}_1$ by using an infinite partition of unity

$$\sum_{\alpha\in\mathbb{N}^n}\Psi(x-\alpha) = 1,\ \ \forall x\in\mathbb{R}^n\ ,$$

$\Psi\in C_0^\infty(K)$, $\Psi(x)\geq 0$, K being a neighborhood of the unit cube. Consequently, for every fixed $h\in]0,h_0]$ we have

$$\mathcal{A}_1 = \frac{1}{(2\pi h)^{n+1}}\lim_{m\to\infty}\int\int\int e^{i\left(t\lambda+\Phi(t,x,\xi)-x.\xi\right)/h}\theta(t)$$
$$\times\sum_{|\alpha|\leq m}\Psi(x-\alpha)A(t,x,x,\xi;h)dt\,dx\,d\xi + \mathcal{O}(h^\infty) = \lim_{m\to\infty}I_m + \mathcal{O}(h^\infty)$$

and we reduce the problem to the analysis of the integrals I_m over a compact set in (t,x,ξ). Concerning the phase function, we observe that

$$t\lambda + \Phi(t,x,\xi) - x.\xi = t\Big(\lambda - l_2(x,\xi) + \mathcal{O}(t)\Big)\,,$$

where $\mathcal{O}(t)$ is uniformly bounded on the support of $\theta(t)A(t,x,x,\xi)$ since the derivatives of $\Big(\Phi(t,x,\xi) - x.\xi\Big)$ are bounded on this set. Finally, to have an uniform bound for the remainder with respect to $m\to\infty$, notice that

$$|\partial_{x,\xi}l_2(x,\xi)| \geq \delta_2 > 0 \tag{9.2}$$

for $|\xi| \le C_1,\ (x,\xi) \in l_2^{-1}(\lambda),\ \lambda \in I_0$. The last condition follows easily from the form of the principal symbol

$$l_2(x,\xi) = |\xi|^2 + \sum_{|\alpha|=2} b_{\alpha,R}(x)\xi^\alpha + \sum_{|\alpha|\le 1} b_{\alpha,R}(x)\xi^\alpha$$

of the operator Q_2, constructed in [2], and the fact that

$$|b_{\alpha,R}(x)| + |\partial_x b_{\alpha,R}(x)| \le \epsilon_1(R)$$

with $\epsilon_1(R) \longrightarrow 0$ as $R \longrightarrow +\infty$ (see Section 2.3 in [2] for more details). Taking $R \gg 1$ sufficiently large, we arrange (9.2) uniformly with respect to $|\xi| \le C_1$ and $(x,\xi) \in l_2^{-1}(\lambda)$. Now the critical points of the phase function $(t\lambda + \Phi(t,x,\xi) - x.\xi)$ become $t = 0$, $l_2(x,\xi) = \lambda$ and by the stationary phase method we obtain

$$I_m = \frac{1}{(2\pi h)^n}\psi(\lambda)\int_{l_2(x,\xi)=\lambda} \sum_{|\alpha|\le m} \Psi(x-\alpha)A_1(0,x,\xi,\lambda)(1-\chi^2)(x)L_\lambda(d\omega) + \mathcal{O}(h^{1-n}),$$

where $L_\lambda(d\omega)$ is the Liouville measure on $l_2(x,\xi) = \lambda$ and the remainder $\mathcal{O}(h^{1-n})$ is uniform with respect to $\lambda \in I_0$ and $m \in \mathbb{N}$. Taking the limit $m \to \infty$, we obtain an asymptotics of $\mathcal{A}_1$.

For the analysis of $\mathcal{B}$ we use the representation

$$\Big[e^{-itQ_j/h}\Big]_{j=1}^2 = \frac{t}{ih}\int_0^1 e^{-istQ_1/h}(Q_1 - Q_2)e^{-i(1-s)tQ_2/h}ds.$$

Following the argument in Section 4.3, [2], we find pseudodifferential operators

$$Q \in \mathrm{Op}_h\Big(S^0(\langle x\rangle^{-n-1}\langle\xi\rangle^{-N})\Big),\ \tilde{Q} \in \mathrm{Op}_h\Big(S^0(\langle\xi\rangle^{-N})\Big)$$

with symbols $q(x,y,\xi;h),\ \tilde{q}(x,y,\xi;h)$ having compact support in ξ and $(x-y)$ so that

$$\mathcal{B} = \frac{1}{2\pi h^2}\mathrm{tr}\Big(\int e^{it\lambda/h}t\theta(t)\int_0^1 e^{-istQ_1/h}Qe^{-i(1-s)tQ_2/h}\tilde{Q}dsdt\Big) + \mathcal{O}(h^\infty).$$

Moreover, modulo a term in $S^N(1)$, the symbol of $\tilde{Q}$ is supported in $\{(x,\xi) : |x| > 2R\}$. Applying an approximation of the unitary groups $e^{-istQ_1/h}$, $e^{-i(1-s)tQ_2/h}$ by Fourier integral operators, we are reduced to study the integral

$$J = \frac{1}{(2\pi h)^{2n+2}}\int\int_0^1\int e^{it\lambda/h}t\theta(t)e^{i\big(\Phi_1(st,x,\xi)-z.\xi\big)/h}e^{i\big(\Phi_2((1-s)t,z,\eta)-x.\eta\big)/h}$$
$$\times B(t,s,X)dtdsdX,$$

where $X = (x,z,\xi,\eta)$ and the phase functions $\Phi_1(t,x,\xi)$, $\Phi_2(s,z,\eta)$ are related to the eikonal equations with symbols $l_1(x,\xi)$ and $l_2(z,\eta)$, respectively. The amplitude $B(t,s,X)$ has a compact support with respect to (ξ,η) and its support with respect to x is included in the set $\{(x,\xi) : |x| \ge 2R\}$. Moreover, $\partial^\alpha B(t,s,X)$ satisfy decreasing estimates with respect to (x,z) like those in (9.1).

In the same way, as in [2], we check that the critical points of the phase in the integral J are related to the closed trajectories composed as union of a curve

$$\{\exp\Big(\tau H_{l_1}\Big)(\rho) :\ 0 \le \tau \le st\}$$

of the Hamilton field H_{l_1} starting at same point $\rho \in \{(x,\xi) \in \mathbb{R}^n : |x| > 2R\}$ and a curve

$$\{\exp\Big(\tau H_{l_2}\Big)(\sigma) : 0 \leq \tau \leq (1-s)t\}, \;\; \sigma = \exp(stH_{l_1})(\rho)$$

of the Hamilton field H_{l_2}. For $0 < t \leq \delta_1$, δ_1 sufficiently small and $R > 0$ large enough, there are no such closed trajectories and the critical points are obtained for $t = 0$, only. We write the phase function in the form

$$t\Big[\lambda - sl_1(x,\xi) - (1-s)l_2(z,\eta) + \mathcal{O}(t)\Big] + (x-z)(\xi-\eta)$$

and the critical points become

$$t = 0, \; sl_1(x,\xi) + (1-s)l_2(x,\xi) = \lambda, \; x = z, \; \xi = \eta \,.$$

For $|x| \geq 2R$ and $0 \leq s \leq 1$, according to (2.6), we deduce

$$m_s(x,\xi) = sl_1(x,\xi) + (1-s)l_2(x,\xi) = |\xi|^2 + \eta_0(R)|\xi|^2$$
$$= l_1(x,\xi) + \eta_1(R)|\xi|^2 = l_2(x,\xi) + \eta_2(R)|\xi|^2$$

with $\eta_i(R) \longrightarrow 0$ as $R \to +\infty, \;\; i = 0,1,2$. Thus for $\lambda \in I_0$ and R large enough the energy surface

$$\Sigma_s(\lambda) = \{(x,\xi) : m_s(x,\xi) = \lambda, \; |x| \geq 2R\}$$

is non-degenerate. Repeating the argument used for $\mathcal{A}_1$, and applying the stationary phase method, we get an asymptotics

$$J = \frac{1}{(2\pi h)^n} b(\lambda) \int_0^1 \int_{m_s(x,\xi)=\lambda} B_1(s,x,\xi,\lambda) L_{s,\lambda}(d\omega) ds + \mathcal{O}(h^{1-n}) \,,$$

where $L_{s,\lambda}(d\omega)$ is the Liouville measure on $\Sigma_s(\lambda)$. Notice that the first term with power h^{-1-n} vanishes because we have the factor $t\theta(t)$ and the term involving h^{-n} yields the contribution to the leading term in (5.10). Moreover, $b(\lambda)$ has support in a small neighborhood of I_1 and taking $R > 0$ large, we may assume that $b(\lambda) \in C_0^0(I_0)$. This completes the proof of (5.10).

The above argument shows that for $\lambda \notin I_0$ the phase functions in I_m and J have no critical points over the support of the integrand. Consequently, by an integration by parts, we obtain

$$\mathrm{tr}\Big[(\mathcal{F}_h^{-1}\theta)\Big(\lambda - Q_j\Big)\varphi_2(Q_j)(1-\chi^2)\Big]_{j=1}^2 = \mathcal{O}(h^\infty)$$

uniformly with respect to $\lambda \notin I_0$.

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