# Analyse spectrale pour la Physique Mathématique
## Rappels sur les opérateurs bornés

Vincent.Bruneau@u-bordeaux.fr

In this chapter, $\mathcal{H}$ will denote a separable Hilbert space on $\mathbb{C}$. We will denote by $\langle .,. \rangle$ and $\|.\|$ the scalar product on $\mathcal{H}$ and the associated norm.

### 1. Spectrum of bounded operators

**1.1. Definition and convergence.**

**Definition 1.1.** *A bounded operator $A$, on $\mathcal{H}$, is a linear map on $\mathcal{H}$ such that the set*

$$\{\|Ax\|;\ x \in \mathcal{H},\ \|x\| = 1\} = \left\{ \frac{\|Ax\|}{\|x\|};\ x \in \mathcal{H},\ x \neq 0 \right\}$$

*is bounded. The set of bounded operators on $\mathcal{H}$ is Banach space, denoted $\mathcal{L}(\mathcal{H})$, equipped with the norm*

$$\|A\| := \sup_{x \in \mathcal{H}, x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{x \in \mathcal{H},\, \|x\|=1} \|Ax\|.$$

For $A, B \in \mathcal{L}(\mathcal{H})$, we have immediately:

$$\|A\,B\| \leq \|A\|\,\|B\|, \quad \|A + B\| \leq \|A\| + \|B\|.$$

**Definition 1.2.** *Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of bounded operators on $\mathcal{H}$ and $A \in \mathcal{L}(\mathcal{H})$. We say:*

(i) *$(A_n)_n$ converge in norm to $A$ if $\|A_n - A\| \longrightarrow 0$ as $n \to +\infty$.*
(ii) *$(A_n)_n$ converge strongly to $A$ if for any $x \in \mathcal{H}$, $(A_n x)_n$ converge to $Ax$ in $\mathcal{H}$.*
(iii) *$(A_n)_n$ converge weakly to $A$ if for any $x, y \in \mathcal{H}$, $(\langle A_n x, y \rangle)_n$ converge to $\langle Ax, y \rangle$ in $\mathbb{C}$.*

**1.2. The adjoint operator.**

**Theorem 1.3.** *Let $A \in \mathcal{L}(\mathcal{H})$. There exists a unique operator $A^*$ on $\mathcal{H}$ such that for all $x, y \in \mathcal{H}$,*

$$\langle Ax, y \rangle = \langle x, A^* y \rangle.$$

*Moreover $A^* \in \mathcal{L}(\mathcal{H})$ and satisfies $\|A^*\| = \|A\|$.*

**Definition 1.4.** *The operator $A^*$ defined by the previous Theorem is called the* **adjoint operator** *of $A$.*

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**Proposition 1.5.** *For $A, B \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, we have:*

(i) $\|A^* A\| = \|A\|^2$
(ii) $(A + \lambda B)^* = A^* + \overline{\lambda} B^*$
(iii) $(AB)^* = B^* A^*$
(iv) $(A^*)^* = A$
(v) $(A^*)^{-1}(\{0\}) = (A(\mathcal{H}))^\perp$
(vi) *If $A^{-1}$ exists in $\mathcal{L}(\mathcal{H})$ then $(A^*)^{-1}$ exists in $\mathcal{L}(\mathcal{H})$ and $(A^*)^{-1} = (A^{-1})^*$.*

**Definition 1.6.** *We say that $A \in \mathcal{L}(\mathcal{H})$ is* **self-adjoint** *(auto-adjoint) if $A^* = A$.*

**Proposition 1.7.** *Let $A \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. Then*

$$\|A\| := \sup_{x \in \mathcal{H},\, \|x\|=1} |\langle Ax, y \rangle|.$$

1.3. **Spectrum of bounded operators.**

**Definition 1.8.** *Let $A \in \mathcal{L}(\mathcal{H})$.*

(i) *The* **spectrum** *(spectre) of $A$ is the set :*

$$\sigma(A) := \{\lambda \in \mathbb{C};\ (A - \lambda I) \text{ is not invertible } \}.$$

(ii) *The* **point spectrum** *(spectre ponctuel) is the set:*

$$\sigma_p(A) := \{\lambda \in \sigma(A);\ (A - \lambda I) \text{ is not injective } \}.$$

(iii) *The* **continuous spectrum** *(spectre continu) is the set:*

$$\sigma_c(A) := \{\lambda \in \sigma(A);\ (A - \lambda I) \text{ is injective, not surjective but } \overline{\mathrm{Ran}(A - \lambda I)} = \mathcal{H}\}.$$

(iv) *The* **residual spectrum** *(spectre residuel) is the set:*

$$\sigma_r(A) := \sigma(A) \backslash (\sigma_p(A) \cup \sigma_c(A)) = \{\lambda \in \sigma(A);\ (A - \lambda I) \text{ is injective, and } \big(\mathrm{Ran}(A - \lambda I)\big)^{\perp} \neq \{0\}\}.$$

By definition, the spectrum is the disjoint union of these three different spectrums:

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A).$$

**Definition 1.9.** *Let $A \in \mathcal{L}(\mathcal{H})$. The set $\rho(A) := \mathbb{C} \setminus \sigma(A)$ is called the* **resolvent set** *(ensemble résolvant). For $\lambda \in \rho(A)$, the bounded operator*

$$R_A(\lambda) := (A - \lambda I)^{-1}$$

*is called the* **resolvent operator** *(or the resolvent of $A$).*

**Proposition 1.10.** *Let $A \in \mathcal{L}(\mathcal{H})$. The resolvent set, $\rho(A)$, is an open subset of $\mathbb{C}$ and $\lambda \mapsto R_A(\lambda)$ is analytic on $\rho(A)$ with value in $\mathcal{L}(\mathcal{H})$. Moreover $\rho(A) \neq \mathbb{C}$ and consequently $\sigma(A)$ is a non empty closed subset of $\mathbb{C}$.*

**Definition 1.11.** *Let $A \in \mathcal{L}(\mathcal{H})$. The quantity*

$$r(A) := \sup\{|\lambda|;\ \lambda \in \sigma(A)\}$$

*is called the* **spectral radius** *(rayon spectral) of $A$.*

**Theorem 1.12.** *Let $A \in \mathcal{L}(\mathcal{H})$. We have:*

$$r(A) = \lim_{n \to +\infty} \|A^n\|^{\frac{1}{n}} \leq \|A\|.$$

*If $A$ is self-adjoint, $r(A) = \|A\|$.*

**Proposition 1.13.** *Let $A \in \mathcal{L}(\mathcal{H})$.*

(i) $\lambda \in \sigma(A) \Longleftrightarrow \overline{\lambda} \in \sigma(A^*)$
(ii) *For $\lambda \in \rho(A)$, we have $(R_A(\lambda))^* = R_{A^*}(\overline{\lambda})$*
(iii) $\lambda \in \sigma_r(A) \Longrightarrow \overline{\lambda} \in \sigma_p(A^*)$
(iv) $\lambda \in \sigma_p(A) \Longrightarrow \overline{\lambda} \in \sigma_r(A^*) \cup \sigma_p(A^*)$
(v) $\lambda \in \sigma_c(A) \Longleftrightarrow \overline{\lambda} \in \sigma_c(A^*)$

**Theorem 1.14.** *If $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint, then $\sigma(A) \subset \mathbb{R}$, $\sigma_r(A) = \emptyset$ and eigenvectors corresponding to distinct eigenvalues of $A$ (i.e. distintct elements of $\sigma_p(A)$) are orthogonal.*

## 2. SPECTRAL ANALYSIS OF COMPACT OPERATORS

### 2.1. Definition and properties of compact operators.

**Definition 2.1.** *An* **operator** $A \in \mathcal{L}(\mathcal{H})$ *is called* **compact** *if the set* $\{Ax;\ x \in \mathcal{H},\ \|x\| \leq 1\}$ *is relatively compact in* $\mathcal{H}$ *(i.e. whose closure is compact in* $\mathcal{H}$*).*
*Equivalently,* $A$ *is compact if and only if for every bounded sequence* $(x_n)_{n\in\mathbb{N}}$ *of* $\mathcal{H}$*,* $(Ax_n)_{n\in\mathbb{N}}$ *has a subsequence convergent in* $\mathcal{H}$*. We will denote by* $\mathcal{S}_\infty(\mathcal{H})$ **the set of compact operators** *on* $\mathcal{H}$*.*

**Example:** A finite rank bounded operator (i.e. whose range is finite dimensional) is compact:

$$A \in \mathcal{L}(\mathcal{H}),\ \dim(A(\mathcal{H})) < +\infty \quad \Longrightarrow \quad A \in \mathcal{S}_\infty(\mathcal{H}).$$

**Proposition 2.2.** *For* $A_1, A_2 \in \mathcal{S}_\infty(\mathcal{H})$*,* $B \in \mathcal{L}(\mathcal{H})$ *and* $\mu \in \mathbb{C}$*, we have:*

$$A_1 + \mu A_2 \in \mathcal{S}_\infty(\mathcal{H}); \quad A_1 B \in \mathcal{S}_\infty(\mathcal{H}); \quad BA_1 \in \mathcal{S}_\infty(\mathcal{H}).$$

**Proposition 2.3.** *The following properties are equivalents:*

(i) $A \in \mathcal{S}_\infty(\mathcal{H})$
(ii) *The operator* $A$ *maps weakly convergent sequences into norm convergent sequences*
(iii) *There exists a sequence of finite rank operators* $(A_n)_{n\in\mathbb{N}}$ *which converge to* $A$ *in* $\mathcal{L}(\mathcal{H})$
(iv) $A^* \in \mathcal{S}_\infty(\mathcal{H})$

**Corollary 2.4.** *Let* $K \in \mathcal{S}_\infty(\mathcal{H})$*.*

(i) *If the sequence of bounded operators* $(A_n)_n$ *converge weakly to* $A$*, then* $KA_n$ *converge strongly to* $KA$
(ii) *If the sequence of bounded operators* $(A_n)_n$ *converge strongly to* $A$*, then* $KA_n$ *converge in norm to* $KA$

### 2.2. Spectral properties of compact operators.

**Theorem 2.5** (Hilbert-Schmidt Theorem: Spectral Theorem for self-adjoint compact operators)**.** *Let* $K \in \mathcal{S}_\infty(\mathcal{H})$ *be a self-adjoint compact operator. Then, there is a complete orthonormal basis* $(\phi_n)_n$*, for* $\mathcal{H}$ *so that*

$$K\phi_n = \lambda_n \phi_n, \quad \text{and} \lim_{n\to+\infty} \lambda_n = 0.$$

*In particular any nonzero* $\lambda \in \sigma(K)$ *is an eigenvalue of finite multiplicity (i.e. the corresponding eigenspace is finite dimensional) and for any* $\phi \in \mathcal{H}$*, we have:*

$$K\phi = \sum_{n=0}^{N} \lambda_n \langle \phi, \phi_n \rangle \, \phi_n,$$

*where* $N$ *is finite when* $K$ *is of finite rank.*

**Theorem 2.6** (Riesz-Schauder Theorem: Spectral Theorem for compact operators)**.** *Let* $K \in \mathcal{S}_\infty(\mathcal{H})$ *with dim*$\mathcal{H} = \infty$*. Then,* $\sigma(K) = \{0\} \cup \{z_n;\ n \in \mathbb{N}\}$*,* $\quad \lim_{n\to+\infty} z_n = 0$ *where* $z_n$ *are the nonzero eigenvalues of* $K$ *which have finite multiplicity (i.e. dim Ker*$(K - z_n I) < \infty$*).*
*Moreover there exist orthonormal sets* $(\phi_n)_{n\in\mathbb{N}}$ *and* $(\psi_n)_{n\in\mathbb{N}}$ *(in* $\mathcal{H}$*) and positive real numbers* $(\lambda_n)_n$*, called singular values of* $K$*, such that for any* $\phi \in \mathcal{H}$*, we have:*

$$K\phi = \sum_{n=0}^{\infty} \lambda_n \langle \phi, \phi_n \rangle \, \psi_n, \quad \lim_{n\to+\infty} \lambda_n = 0.$$

## 3. EXERCICES

**Exercice 1.** Soit $\varphi \in C^0([0,1],\mathbb{C})$ et $\mathcal{H} = L^2((0,1),\mathbb{C})$.

**1)** Montrer que $A$ défini par $A(u)(x) = \varphi(x)u(x)$ est un élément de $\mathcal{L}(\mathcal{H})$. Calculer sa norme.

**2)** Déterminer $\rho(A)$ l'ensemble résolvant de $A$.

**3)** Déterminer $A^*$ et donner une condition sur $\varphi$ pour que $A^* = A$.

**Exercice 2.** Soient $\mathcal{H}$ un espace de Hilbert, $F$ un sous espace vectoriel fermé non vide de $\mathcal{H}$ et $P$ la projection orthogonale sur $F$. Montrer que:

1) $P^2 = P$ 2) $\forall x, y \in H\ \ (Px/y) = (x/Py)$ 3) $\|P\| = 1$ 4) $H = KerP \oplus ImP$.

**Exercice 3.** Soit $A$ l'application linéaire définie par $(Au)(x) = \displaystyle\int_0^x u(t)dt$.

**1)** Montrer que $A \in \mathcal{L}(L^2(0,1))$. Déterminer $A^*$.

**2)** Montrer que $A$ est compact (on pourra utiliser le Théorème d'Ascoli).

**Exercice 4.** On considère l'espace de Hilbert $l^2(\mathbb{N}) := \{u = (u_n)_n \in \mathbb{R}^{\mathbb{N}};\ \sum |u_n|^2 < +\infty\}$. Soit $T$ l'application qui à $u = (u_n)_n \in l^2(\mathbb{N})$ associe $T(u) = (Tu)_n$ défini par $(Tu)_n = \frac{u_n}{n}$.

**1)** Montrer que $T \in \mathcal{L}(l^2(\mathbb{N}))$ et calculer sa norme.

**2)** Montrer que $T$ est un opérateur auto-adjoint compact.

**3)** Déterminer le spectre de $T$.

**Exercice 5.** Montrer la Proposition 1.5.

**Exercice 6.** Soit $\mathcal{H}$ un espace de Hilbert et $A \in \mathcal{L}(\mathcal{H})$ tel que $\forall f \in \mathcal{H},\quad \langle Af, f\rangle \geq 0$.

**1)** Montrer que $\sigma_p(A) \subset [0,+\infty)$.

**2)** Montrer que $A$ est auto-adjoint.

**3)** Montrer que $\sigma(A) \subset [0,+\infty)$ (on pourra utiliser le Théorème de Lax-Milgram).

**Exercice 7.** Soit $U \in \mathcal{L}(\mathcal{H})$ un opérateur unitaire, c'est-à-dire tel que $U^*U = UU^* = I$

**1)** Montrer que $\sigma_p(U) \subset S^1 = \{z \in \mathbb{C};\ |z| = 1\}$ et que $\sigma_p(U^*) \subset S^1$.

**2)** Montrer que $\sigma_r(U) \subset S^1$ et que $\sigma(U) \subset S^1$.

**Exercice 8.** Soit $\mathcal{H}$ un espace de Hilbert et $A \in \mathcal{L}(\mathcal{H})$. On note $N(A) := \{\langle Af, f\rangle; \|f\| = 1\}$. Montrer que $\sigma(A) \subset \overline{N(A)}$.

**Exercice 9.** Les opérateurs suivants sont-ils prolongeables en des opérateurs bornés sur $L^2(\mathbb{R})$?

**1)** $\mathcal{F} : f \mapsto \hat{f}$ défini pour $f \in L^1(\mathbb{R})$ par $\hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\mathbb{R}} e^{-ix\xi} f(x)dx$.

**2)** Pour $V \in L^\infty(\mathbb{R})$, $M$ l'opérateur de multiplication par $V$, $M : u \mapsto V\, u$.

**3)** $T : u \mapsto -u''$ défini pour $u \in H^2(\mathbb{R})$.

**Exercice 10.** Soit $(u_n)_n$ une suite d'éléments de $H^2(\mathbb{R}^d)$ telle que dans $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$

$$\lim_{n\to+\infty} (u_n, \Delta u_n) = (u, v).$$

Montrer que nécessairement $\Delta u = v$ et $u \in H^2(\mathbb{R}^d)$.