ON THE CARTWRIGHT-STEGER SURFACE

DONALD I. CARTWRIGHT, VINCENT KOZIARZ, SAI-KEE YEUNG

Abstract *In this article, we study various concrete algebraic and differential geometric properties of the Cartwright-Steger surface. In particular, we determine the genus of a generic fiber of the Albanese fibration, and deduce that the singular fibers are not totally geodesic, answering an open problem about fibrations of a complex ball quotient over a Riemann surface.*

0. Introduction

The Cartwright-Steger surface was found during work on the classification of fake projective planes completed in [PY] and [CS1]. A fake projective plane is a smooth surface with the same Betti numbers as the projective plane but not biholomorphic to it. It is known that a fake projective plane is a complex two ball quotient $\Pi\backslash B^2_{\mathbb{C}}$ with Euler number 3, where Π is an arithmetic lattice in $\mathrm{PU}(2,1)$, cf. [PY]. In the scheme of classification of fake projective planes carried out in [PY], it was conjectured but not proved in [PY] that the lattice Π associated to a fake projective plane cannot be defined over a pair of number fields $\mathcal{C}_{11} = (\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\zeta_{12}))$, where ζ_{12} is a 12-th root of unity. Such a Π would be of index 864 in a certain maximal arithmetic subgroup $\bar{\Gamma}$ of $\mathrm{PU}(2,1)$. As reported in [CS1], the authors showed using a lengthy computer search that there is no torsion free lattice Π of index 864 in this $\bar{\Gamma}$ with $b_1(\Pi) = 0$, but surprisingly there is one with $b_1(\Pi) = 2$. The surface $\Pi\backslash B^2_{\mathbb{C}}$ is the subject of study in this article.

The Cartwright-Steger surface is unique as a Riemannian manifold with the given Euler and first Betti numbers, but has two different biholomorphic structures given by complex conjugation. From an algebraic geometric point of view, the fake projective planes and the Cartwright-Steger surfaces are interesting since they have the smallest possible Euler number, namely 3, among smooth surfaces of general type, and constitute all such surfaces. From a differential geometric point of view, they are interesting since they constitute smooth complex hyperbolic space forms, or complex ball quotients, of smallest volume in complex dimension two. We refer the reader to [R], [Y1], and [Y2] for some general discussions related to the above facts. Unlike fake projective planes, whose lattices arise from division algebras of non-trivial degree as classified, the Cartwright-Steger surface is defined by Hermitian forms over the number fields mentioned above. It is realized among experts that such a surface is commensurable to a Deligne-Mostow surface, the type of surfaces which have been studied by Picard, Le Vavasseur, Mostow, Deligne-Mostow, Terada and many others, cf. [DM1].

Even though the lattice involved is described in [CS2], it is surprising that the algebraic geometric structures of the surface are far from being understood. A typical problem is to find out the genus of a generic fiber of the associated Albanese fibration. Conventional algebraic geometric techniques do not seem to be readily applicable to such a problem. The goal of this article is to develop tools and techniques which allow us to understand concrete surfaces such as the Cartwright-Steger surface. In particular, we recover algebraic geometric

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properties from a description of the fundamental group of the surface, using a combination of various algebraic geometric, differential geometric, group theoretical techniques and computer implementations.

Here are the results obtained in this paper.

Main Theorem *Let X be the Cartwright-Steger surface and $\alpha : X \to T$ the Albanese map.*

- (a) *The genus of a generic fiber of α is 19.*
- (b) *All fibers of α have multiplicity 1. The singular set of the fibration α consists of either three nodal singularities or one tacnode singularity.*
- (c) *The Albanese torus T is $\mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$, where ω is a cube root of unity.*
- (d) *The Picard number of X is 3, equal to $h^{1,1}(X)$, so that all the Hodge $(1,1)$ classes are algebraic. The Néron-Severi group is generated by three immersed totally geodesic curves we explicitly give.*
- (e) *The automorphism group Σ of X, isomorphic to $\mathbb{Z}_3$, has 9 fixed points, and induces a nontrivial action on T which has 3 fixed points. Three fixed points of Σ lie over each fixed point in T. Over one fixed point on T, the three fixed points of Σ are of type $\frac{1}{3}(1,1)$. The other 6 fixed points of Σ are of type $\frac{1}{3}(1,2)$.*

The Main Theorem follows from Theorem 3, Lemma 9, Corollary 1, Lemma 5 and Lemma 32.

As an immediate consequence, see Theorem 4, we have answered an open problem communicated to us by Ngaiming Mok on properties of fibrations on complex ball quotients.

Corollary *There exists a fibration of a smooth complex two ball quotient over a smooth Riemann surface with non-totally geodesic singular fibers.*

Apart from the results above, we have given a detailed analysis of the Albanese map in §5. Moreover, results on the surface parallel to an original construction of Livné [Li] on fibrations of a complex hyperbolic surface over a Riemann surface are explained in Section 6. As another application, we have used the surface to derive some interesting properties related to a question of Nori [N] on Lefschetz properties for singular ample curves on a projective algebraic surface in §7.

Here are a few words about the presentation of the article. To streamline our arguments and to make the results more understandable, we state and prove the geometric results of the article sequentially in the main parts of the article. Many of these results rely on computations in the groups Π and $\bar{\Gamma}$, often obtained with assistance of the algebra package Magma, and we present these in an appendix. More details can be found on the webpage of the first author at `http://www.maths.usyd.edu.au/u/donaldc/cs-surface/`.

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Since completing this paper, we were informed by Domingo Toledo that he, Fabrizio Catanese, JongHae Keum and Matthew Stover had independently proved some of our results in a paper they are preparing.

1. Basic facts

1.1. Let F be a Hermitian form on $\mathbb{C}^3$ with signature $(2,1)$. We denote by $\mathrm{U}(F) = \{g \in \mathrm{GL}(3,\mathbb{C}) \mid g^*Fg = F\}$ the subgroup of $\mathrm{GL}(3,\mathbb{C})$ preserving the form F, by $\mathrm{SU}(F)$ the subgroup of $\mathrm{U}(F)$ of elements with determinant 1, and by $\mathrm{PU}(F)$ their image in $\mathrm{PGL}(3,\mathbb{C})$. The group $\mathrm{PU}(F)$ is naturally identified with the group of biholomorphisms of the two-ball $B^2_{\mathbb{C}}(F) := \{[z] \in \mathbb{P}^2_{\mathbb{C}} = \mathbb{P}(\mathbb{C}^3) \mid F(z) < 0\}$.

Our aim is to study a special complex hyperbolic surface $X = \Pi\backslash B^2_{\mathbb{C}}(F)$ where Π is a cocompact torsion-free lattice in some $\mathrm{PU}(F)$. The group Π appears as a finite index subgroup of an arithmetic lattice $\bar{\Gamma}$ which can be easily described as follows.

Let $\zeta = \zeta_{12}$ be a primitive 12-th root of unity. Then $r = \zeta + \zeta^{-1}$ is a square root of 3. Let $\ell = \mathbb{Q}(\zeta)$ and $k = \mathbb{Q}(r) \subset \ell$. For real and complex calculations below, we take $\zeta = e^{\pi i/6}$, and then r is the positive square root of 3. We could define $\bar{\Gamma}$ to be the group of 3×3 matrices g' with entries in ℓ such that ${g'}^*F'g' = F'$, where

$$F' = \begin{pmatrix} r+1 & -1 & 0 \\ -1 & r-1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

such that g' has entries in $\mathbb{Z}[\zeta]$, modulo $Z = \{\zeta^j I : j = 0,\dots,11\}$.

However, it is convenient to work with a diagonal form instead of F'. Notice that $F' = (r-1)^{-1}\gamma_0^* F\gamma_0$ for

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-r \end{pmatrix}, \quad \text{and} \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1-r & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So we instead define $\bar{\Gamma}$ to be the group of matrices g, modulo Z, with entries in ℓ, which are unitary with respect to F for which $g' = \gamma_0^{-1}g\gamma_0$ has entries in $\mathbb{Z}[\zeta]$. Such g's have entries in $\frac{1}{r-1}\mathbb{Z}[\zeta] \subset \frac{1}{2}\mathbb{Z}[\zeta]$.

Since F is diagonal, it is easy to make the group $\mathrm{PU}(F)$ act on the standard unit two-ball, which we will just denote by $B^2_{\mathbb{C}}$: if $gZ \in \bar{\Gamma}$, the action of gZ on $B^2_{\mathbb{C}}$ is given by

$$(gZ).(z,w) = (z',w') \quad \text{if} \quad DgD^{-1}\begin{pmatrix} z \\ w \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} z' \\ w' \\ 1 \end{pmatrix},$$

for some $\lambda \in \mathbb{C}$, where D is the diagonal matrix with diagonal entries 1, 1 and $\sqrt{r-1}$.

We often ignore the distinction between matrices g and elements gZ of $\bar{\Gamma}$, though we sometimes need to carefully distinguish these two objects.

Now $\bar{\Gamma}$ contains a subgroup K of order 288 generated by the two matrices $u = \gamma_0 u'\gamma_0^{-1}$ and $v = \gamma_0 v'\gamma_0^{-1}$ where

$$u' = \begin{pmatrix} \zeta^3+\zeta^2-\zeta & 1-\zeta & 0 \\ \zeta^3+\zeta^2-1 & \zeta-\zeta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v' = \begin{pmatrix} \zeta^3 & 0 & 0 \\ \zeta^3+\zeta^2-\zeta-1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A presentation for K is given by the relations

$$u^3 = v^4 = 1, \text{ and } (uv)^2 = (vu)^2.$$

The elements of K are most neatly expressed if we use not only the generators u and v, but also $j = (uv)^2$, which is the diagonal matrix with diagonal entries ζ, ζ and 1, and which generates the center of K.

There is one further generator needed for $\bar{\Gamma}$, namely $b = \gamma_0 b'\gamma_0^{-1}$ for

$$b' = \begin{pmatrix} 1 & 0 & 0 \\ -2\zeta^3-\zeta^2+2\zeta+2 & \zeta^3+\zeta^2-\zeta-1 & -\zeta^3-\zeta^2 \\ \zeta^2+\zeta & -\zeta^3-1 & -\zeta^3+\zeta+1 \end{pmatrix}.$$

Theorem 1 ([CS2]). *A presentation of $\bar\Gamma$ is given by the generators u, v and b and the relations*

$$u^3 = v^4 = b^3 = 1,\ (uv)^2 = (vu)^2,\ vb = bv,\ (buv)^3 = (buvu)^2v = 1.$$

1.2. Let us record here the connection with a group which was first discovered by Mostow: the group $\bar\Gamma$ is isomorphic to a group generated by complex reflections, denoted by $\Gamma_{3,\frac{1}{3}}$ in the paper [Mo1] and by $\Gamma_{3,4}$ in [Pa], and whose presentation (see Parker [Pa]) is

$$\Gamma_{3,4} = \langle J, R_1, A_1\ :\ J^3 = R_1^3 = A_1^4 = 1,\ A_1 = (JR_1^{-1}J)^2,\ A_1R_1 = R_1A_1\rangle.$$

Defining $R_2 = JR_1J^{-1}$, it was shown in [Pa, Proposition 4.6] that the subgroup $\langle A_1, R_2\rangle$ of $\Gamma_{3,4}$ is finite, with order 288 (actually, it is isomorphic to K above). It has the simple presentation

$$\langle A_1, R_2\ :\ A_1^4 = R_2^3 = 1,\ A_1R_2A_1R_2 = R_2A_1R_2A_1\rangle.$$

The following result was communicated to us by John Parker.

Proposition 1. *There is an isomorphism $\psi : \bar\Gamma \to \Gamma_{3,4}$ such that*

$$\psi(u) = R_2,\ \psi(v) = A_1,\ \textit{and}\ \psi(b) = R_1.$$

It satisfies $\psi(K) = \langle A_1, R_2\rangle$, and its inverse satisfies

$$\psi^{-1}(R_1) = b,\ \psi^{-1}(A_1) = v,\ \psi^{-1}(J) = buv, \quad \textit{and}\ \ \psi^{-1}(R_2) = u.$$

1.3. It is also convenient to see $\bar\Gamma$ as a (Deligne-)Mostow group: it corresponds to item 8 in the paper of Mostow [Mo2, p. 102] whose associated weights $(2, 2, 2, 7, 11)/12$ satisfy the condition (ΣINT) in the notation of [Mo2]. We refer to [Mo2] and [DM2] for details on the description below.

The orbifold quotient $\bar\Gamma\backslash B^2_{\mathbb{C}}$ is a compactification of the moduli space of 5-tuples of distinct points $(x_0, x_1, x_2, x_3, x_4) \in (\mathbb{P}^1_{\mathbb{C}})^5$ modulo the diagonal action of $\mathrm{PGL}(2, \mathbb{C})$ and the action of the symmetric group on three letters Σ_3 on the three first points. The compactification can be described as follows. First, it can be easily seen that the moduli space Q of 5-tuples of distinct points $(x_0, x_1, x_2, x_3, x_4) \in (\mathbb{P}^1_{\mathbb{C}})^5$ modulo the diagonal action of $\mathrm{PGL}(2, \mathbb{C})$ can be realized as $\mathbb{P}^2_{\mathbb{C}}$ with a configuration of six lines removed. In homogeneous coordinates $[X_0 : X_1 : X_2]$ on $\mathbb{P}^2_{\mathbb{C}}$, these six lines correspond to the three lines of "type A" with equation $X_i = X_j$ ($1 \leqslant i < j \leqslant 2$) and the three lines of "type B" with equation $X_i = 0$ ($i = 0, 1, 2$). In fact, the compactification $\bar Q = \mathbb{P}^2_{\mathbb{C}}$ of Q is determined by the fact that we allow two or three of the points x_0, x_1 and x_2 to coincide ($x_0 = x_1$ corresponds to $X_0 = X_1$, $x_0 = x_2$ to $X_0 = X_2$ and $x_1 = x_2$ to $X_1 = X_2$) and we also allow one or two of the points x_0, x_1 and x_2 to coincide with x_3 ($x_0 = x_3$ corresponds to $X_0 = 0$, $x_1 = x_3$ to $X_1 = 0$ and $x_2 = x_3$ to $X_2 = 0$).

Then, as we mentioned above, the underlying topological space of $\bar\Gamma\backslash B^2_{\mathbb{C}}$ is a compactification R of Q/Σ_3 and actually is the weighted projective plane $\mathbb{P}(1, 2, 3) \cong \mathbb{P}^2_{\mathbb{C}}/\Sigma_3$ where the symmetric group on three letters Σ_3 acts by permutation of the homogeneous coordinates $[X_0 : X_1 : X_2]$ on $\mathbb{P}^2_{\mathbb{C}}$. There are two remarkable (irreducible) divisors on $\mathbb{P}(1, 2, 3)$: one is the image D_A of the divisors of type A, the other one is the image D_B of the divisors of type B. The divisor D_A has a cusp at the image P_1 of the point $[1 : 1 : 1]$ and the divisor D_B is smooth. These two divisors meet at two points: once at the image P_2 of the points $[1 : 0 : 0]$, $[0 : 1 : 0]$ or $[0 : 0 : 1]$ where they are tangent, once at the image P_3 of the points $[1 : 1 : 0]$, $[1 : 0 : 1]$ or $[0 : 1 : 1]$ where the intersection is transverse. There are also two singular points on $\mathbb{P}(1, 2, 3)$: one is a singularity of type A_1 and is the image $P_4 \in D_B$ of the points $[1 : -1 : 0]$, $[1 : 0 : -1]$ or $[0 : 1 : -1]$, the other one is a singularity of type A_2 and is the image P_5 of the points $[1 : \omega : \omega^2]$ or $[1 : \omega^2 : \omega]$ where ω is a primitive 3rd root of unity.

Remark 1. *In the book* [DM2, p. 111]*, the divisor D_A (resp. D_B) is denoted by D_{AA} (resp. D_{AB}) and the points $P_1, \dots, P_5$ simply by $1, \dots, 5$.*

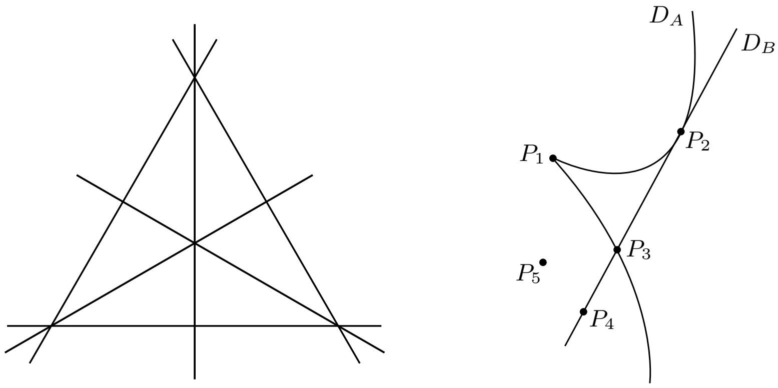


FIGURE 1. $\bar{Q} = \mathbb{P}^2_{\mathbb{C}}$ and $R = \mathbb{P}^2_{\mathbb{C}}/\Sigma_3$

There is a standard method to compute the weight of the orbifold divisors on $\bar{\Gamma}\backslash B^2_{\mathbb{C}}$ as well as the local groups at the orbifold points, according to the weights $(2, 2, 2, 7, 11)/12$. The weight of D_A is $3 = 2(1-(2+2)/12)^{-1}$ and the weight of D_B is $4 = (1-(2+7)/12)^{-1}$. This means that the preimage of D_A (resp. D_B) in $B^2_{\mathbb{C}}$ is a union of mirrors of complex reflections of order 3 (resp. 4). We will denote by $\mathcal{M}_A$ (resp. $\mathcal{M}_B$) the corresponding sets of mirrors. Said another way, the isotropy group at a generic point of some $M \in \mathcal{M}_A$ is isomorphic to $\mathbb{Z}_3$ and the isotropy group at a generic point of some $M \in \mathcal{M}_B$ is isomorphic to $\mathbb{Z}_4$, both generated by a complex reflection of the right order. This of course has to be compared with the description of $\bar{\Gamma}$ as $\Gamma_{3,4}$.

The isotropy group at a point above the transverse intersection P_3 of D_A and D_B is naturally isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_4$. As P_5 is a singularity of type A_2 but does not belong to any orbifold divisor, the local group at P_5 is isomorphic to $\mathbb{Z}_3$. But since $P_4 \in D_A$ is a singularity of type A_1, the local group at P_4 has order $8 = 2 \cdot 4$ and actually is isomorphic to $\mathbb{Z}_8$.

It is a little bit more difficult to determine the isotropy group above the points P_1 and P_2. It will also be useful to describe the stabilizer in $\bar{\Gamma}$ of a mirror. For this, one can use a method similar to the one in [Der1, Lemma 2.12] and obtain the following lemma which already appeared in an unpublished manuscript of Deraux and Yeung.

Lemma 1. *Let $\mathcal{M}_A$ (resp. $\mathcal{M}_B$) denote the set of mirrors of complex reflections of order 3 (resp. 4) in $\bar{\Gamma}$.*

Let $\mathcal{P} \subset B^2_{\mathbb{C}}$ denote the set of points above P_1 and $\mathcal{T} \subset B^2_{\mathbb{C}}$ denote the set of points above P_2. The following holds.

(a) *The group $\bar{\Gamma}$ acts transitively on $\mathcal{M}_A$, on $\mathcal{M}_B$, on $\mathcal{P}$ and on $\mathcal{T}$.*
(b) *For each point $x \in \mathcal{P}$, the stabilizer of x is the one labelled $\sharp 4$ in the Shephard-Todd list. It is a central extension of a $(2, 3, 3)$-triangle group, with center of order 2, and has order 24. There are precisely 4 mirrors in $\mathcal{M}_A$ through each such $x \in \mathcal{P}$.*
(c) *For each point $y \in \mathcal{T}$, the stabilizer of y is the one labelled $\sharp 10$ in the Shephard-Todd list. It is a central extension of a $(2, 3, 4)$-triangle group, with center of order 12, and has order 288. Through each such $y \in \mathcal{T}$, there are 8 elements of $\mathcal{M}_A$ and 6 elements of $\mathcal{M}_B$.*
(d) *The stabilizer of any element $M \in \mathcal{M}_A$ is a central extension of a $(2, 4, 12)$-triangle group, with center of order 3.*
(e) *The stabilizer of any element $M \in \mathcal{M}_B$ is a central extension of a $(2, 3, 12)$-triangle group, with center of order 4.*

Sketch of proof. (a) Follows from the above discussion.

(b) The point P_1 corresponds to $x_0 = x_1 = x_2$ so that the computation $3/2 = (1-(2+2)/12)^{-1}$ shows that the spherical triangle group associated to the projective action of the isotropy group at $x \in \mathcal{P}$ is $(2,3,3)$. Indeed, we have to consider the triangle with angles $(2\pi/3, 2\pi/3, 2\pi/3)$ and take the symmetry into account (i.e. dividing the triangle into six parts), so that we obtain a triangle with angles $(\pi/2, \pi/3, \pi/3)$. The center has order given by $2 = (1-(2+2+2)/12)^{-1}$. Comparing with [ST, Table 1], we see that the relevant group is the one labelled ♯4 in the Shephard-Todd list and the rest of the assertion follows.

(c) Similarly, the point P_2 corresponds for instance to $x_0 = x_1 = x_3$ and the additional computation $4 = (1-(2+7)/12)^{-1}$ shows that the spherical triangle group associated to the projective action of the isotropy group at $y \in \mathcal{T}$ is $(2,3,4)$. Indeed, we have to consider the triangle with angles $(\pi/4, \pi/4, 2\pi/3)$ and take the symmetry into account (i.e. dividing the triangle into two parts), so that we obtain a triangle with angles $(\pi/2, \pi/3, \pi/4)$. The center has order given by $12 = (1-(2+2+7)/12)^{-1}$. Comparing with [ST, Table 2], we see that the relevant group is the one labelled ♯10 in the Shephard-Todd list.

(d) Follows from the interpretation of the stabilizer of $M \in \mathcal{M}_A$ as a central extension with center of order 3 (corresponding to the order of the reflection with mirror M) of a Deligne-Mostow group with weights $(2,4,7,11)/12$ coming for instance from the collapsing of x_1 and x_2. The associated triangle group is $(2,4,12)$ since $2 = (1-(2+4)/12)^{-1}$, $4 = (1-(2+7)/12)^{-1}$ and $12 = (1-(4+7)/12)^{-1}$.

(e) Similarly, the stabilizer of $M \in \mathcal{M}_B$ is a central extension with center of order 4 (corresponding to the order of the reflection with mirror M) of a (Deligne-)Mostow group with weights $(2,2,9,11)/12$ coming for instance from the collapsing of x_2 and x_3. We have moreover to take care of the symmetry coming from the first two weights. The associated triangle group is $(2,3,12)$ since $3/2 = (1-(2+2)/12)^{-1}$ and $12 = (1-(2+9)/12)^{-1}$ so that we have to divide into two parts a triangle with angles $(2\pi/3, \pi/12, \pi/12)$. □

Remark 2. *The data concerning the isotropy groups can be recovered using calculations in* $\bar{\Gamma}$*, see Proposition A.8.*

1.4. Cartwright and Steger discovered a very interesting torsion-free subgroup Π of $\bar{\Gamma}$ with finite index. The surface $\Pi\backslash B^2_{\mathbb{C}}$ is called the Cartwright-Steger surface in this article.

Theorem 2 ([CS2])**.** *The elements*

$$a_1 = vuv^{-1}j^4buvj^2, \quad a_2 = v^2ubuv^{-1}uv^2j \quad \text{and} \quad a_3 = u^{-1}v^2uj^9bv^{-1}uv^{-1}j^8$$

of $\bar{\Gamma}$ *generate a torsion-free subgroup* Π *of index 864, with* $\Pi/[\Pi,\Pi] \cong \mathbb{Z}^2$.

Proof. Using the given presentation of $\bar{\Gamma}$, the Magma `Index` command shows that Π has index 864 in $\bar{\Gamma}$. We see that Π is torsion-free as follows. The 864 elements $b^\mu k$, for $\mu = 0, 1, -1$ and $k \in K$, form a set of representatives for the cosets Πg of Π in $\bar{\Gamma}$. One can verify this by a method we shall use repeatedly: for $g = b^\mu k$ and $g' = b^{\mu'}k'$, we check that $\Pi g \neq \Pi g'$ unless $\mu' = \mu$ and $k' = k$ by having Magma calculate the index in $\bar{\Gamma}$ of $\langle a_1, a_2, a_3, g'g^{-1}\rangle$.

If $1 \neq \pi \in \Pi$ has finite order, then $\pi = gtg^{-1}$ for one of the elements t given in the table of Proposition A.7, or the inverse of one of these. But then $(b^\mu k)t(b^\mu k)^{-1} \in \Pi$ for some $\mu \in \{0,1,-1\}$ and $k \in K$, and Magma's `Index` command shows that this is not the case.

The Magma `AbelianQuotientInvariants` command shows that $\Pi/[\Pi,\Pi] \cong \mathbb{Z}^2$. For any isomorphism $f : \Pi/[\Pi,\Pi] \to \mathbb{Z}^2$, the image under f of $a_1^3a_2^{-2}a_3^7$ is trivial. We can choose f so that it maps a_1, a_2 and a_3 to $(1,3)$, $(-2,1)$ and $(-1,-1)$, respectively. So $f(a_1a_2^{-1}a_3^2) = (1,0)$ and $f(a_1^{-1}a_2a_3^{-3}) = (0,1)$. □

Magma shows that the normalizer of Π in $\bar{\Gamma}$ contains Π as a subgroup of index 3, and is generated by Π and j^4. One may verify that

$$\begin{aligned} j^4a_1j^{-4} &= \zeta^3(a_3a_2^{-3}a_3^3a_1), \\ j^4a_2j^{-4} &= \zeta^{-1}a_3^{-1}, \quad \text{and} \\ j^4a_3j^{-4} &= \zeta^{-1}a_1^{-1}a_2^{-1}a_1a_2^2a_1^{-1}a_2^{-1}a_1a_3^{-1}a_1^{-1}a_2a_1. \end{aligned}$$

With the above isomorphism $f : \Pi/[\Pi,\Pi] \to \mathbb{Z}^2$,

$$f(\pi) = (m,n) \quad \implies f(j^4\pi j^{-4}) = (m,n)\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix} \quad \text{for all } \pi \in \Pi. \tag{1}$$

From now on, all the proofs involving computations with Magma will be given in the appendix (§A).

1.5. Cartwright and Steger noticed that the group Π can be exhibited as a congruence subgroup of $\bar{\Gamma}$: we have two reductions $r_2 : \mathbb{Z}[\zeta] \to \mathbb{F}_4 = \mathbb{F}_2[\omega]$ and $r_3 : \mathbb{Z}[\zeta] \to \mathbb{F}_9 = \mathbb{F}_3[i]$ defined by sending ζ to ω (resp. i) where $1+\omega+\omega^2 = 0$ (resp. $i^2 = -1$). They induce (surjective) group morphisms $\rho_2 : \bar{\Gamma} \to \mathrm{PU}(3,\mathbb{F}_4)$ and $\rho_3 : \bar{\Gamma} \to \mathrm{PU}(3,\mathbb{F}_9)$ (recall that $\mathrm{PU}(3,\mathbb{F}_4)$ and $\mathrm{PU}(3,\mathbb{F}_9)$ have respective cardinality 216 and 6048).

Note that for an element of $\mathrm{PU}(3,\mathbb{F}_4)$, the determinant is well defined since $\omega^3 = 1$. This enables us to define a (surjective) morphism $\det_2 = \det \circ \rho_2 : \bar{\Gamma} \to \mathbb{F}_4^*$. Let us denote the subgroup $\det_2^{-1}(1)$ of index 3 of $\bar{\Gamma}$ by Π_2.

Remark also that there exist subgroups of order 21 in $\mathrm{PU}(3,\mathbb{F}_9)$ (they are all conjugate) and let us denote one of them by G_{21}. Then, define $\Pi_3 := \rho_3^{-1}(G_{21})$: it is a subgroup of $\bar{\Gamma}$ of index $288 = 6048/21$.

Finally, one can check that $\Pi_2 \cap \Pi_3$ is a torsion-free subgroup of $\bar{\Gamma}$ of index $864 = 3 \cdot 288$ and that it is isomorphic to Π.

1.6.

Lemma 2. *The Cartwright-Steger surface $X = \Pi\backslash B^2_{\mathbb{C}}$ has the following numerical invariants:*

$$c_1^2 = 9, \quad c_2 = 3, \quad \chi(\mathcal{O}_X) = 1, \quad q := h^{1,0} = 1, \quad p_g := h^{2,0} = 1, \quad h^{1,1} = 3.$$

Proof. The orbifold $\bar{\Gamma}\backslash B^2_{\mathbb{C}}$ has orbifold Euler characteristic $1/288$ (see [PY] or [Sa] for instance) so that X has Euler characteristic $c_2(X) = 3 = 864/288$. Then, as it is a two-ball quotient, $c_1^2(X) = 9$ and thus its arithmetic genus is $\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2) = 1$. Since $\Pi/[\Pi,\Pi] \cong \mathbb{Z}^2$, we have $b_1 = 2q = 2$. So, from

$$\begin{aligned} 1 &= \chi(\mathcal{O}_X) = 1 - q + p_g, \\ 3 &= c_2(X) = 2b_0 - 2b_1 + b_2, \end{aligned}$$

we deduce that $p_g = 1$, $b_2 = 5$, and finally, $h^{1,1} = 3$. □

We will see later (Corollary 1) that the Picard number of X is actually 3. It is our purpose to understand the geometric properties of the surface X, especially using its Albanese map.

2. Summary of configurations of some totally geodesic divisors

Here we summarize results about configuration of totally geodesic divisors on the Cartwright-Steger surface $X = \Pi\backslash B^2_{\mathbb{C}}$. Let $\pi : X \to R = \bar{\Gamma}\backslash B^2_{\mathbb{C}}$ be the projection. We use the notation of §1.3. From the description of the local groups at P_1, P_2 and P_3, we know that $\pi^{-1}(P_2)$ consists of $3 = 864/288$ points $O_1 = \Pi(O), O_2 = \Pi(b \cdot O), O_3 = \Pi(b^{-1} \cdot O)$ on X, $\pi^{-1}(P_1)$ consists of $36 = 864/24$ points, and $\pi^{-1}(P_3)$ consists of $72 = 864/12$ points.

For the curves D_A and D_B, their preimages $\pi^{-1}(D_A)$ and $\pi^{-1}(D_B)$ consist of singular totally geodesic curves on X, denoted to be of types A and B respectively. The curves have simple crossings at $\pi^{-1}(P_i)$ for $i = 1,2,3$.

2.1. By Propositions A.9, A.10 and A.11, the (singular) totally geodesic curves on X of type B consist of three curves of geometric genus 4, denoted by E_1, E_2 and E_3 and associated with M_0, M_∞ and M_1 respectively (in the notation of Proposition A.10). These curves are specified by having multiplicities at O_1, O_2, O_3 given by $(3,1,2)$, $(2,1,3)$ and $(1,4,1)$ respectively and the points in $\pi^{-1}(P_2)$ are the only ones where they can intersect.

2.2. By Propositions A.13, A.14, A.15 and A.16, the (singular) totally geodesic curves on X of type A consist of four curves denoted by C_1, C_2, C_3 and C_4, associated with $b(M_c)$, $b^{-1}(M_c)$, M_c and M_{-c} respectively (in the notation of Proposition A.14), and where the geometric genera are given by 4, 4, 10 and 10 respectively. The curves of type A may intersect at points in $\pi^{-1}(P_1)$ and $\pi^{-1}(P_2)$, and nowhere else. In the following discussions, we shall mainly focus on C_1 and C_2, both of which cross O_1, O_2, O_3 with multiplicities $(0, 1, 2)$ respectively. The corresponding multiplicities for C_3 and C_4 are $(4, 3, 2)$. The curve C_1 passes precisely once through exactly 18 of the 36 points in $\pi^{-1}(P_1)$, as does C_2. The curves C_1 and C_2 intersect once at 12 of those 36 points.

2.3. A curve of type A and one of type B may intersect at points of $\pi^{-1}(P_3)$, apart from the intersections at O_j, $j = 1, 2, 3$, mentioned by the data above. From Proposition A.18, we get the following data. The curve E_1 intersects each of $C_i, i = 1, 2, 3, 4$ once in normal crossing in 6 of the 72 points in $\pi^{-1}(P_3)$. The curve E_2 has no intersection with C_1 and C_2, but intersects once with each of C_3, C_4 at 12 of the points of $\pi^{-1}(P_3)$. The curve E_3 intersects each of C_1 and C_2 once at three of the points of $\pi^{-1}(P_3)$, and intersects each of the curves C_3 and C_4 once at 9 of the points of $\pi^{-1}(P_3)$.

Remark 3. *It can be checked with Magma that the normalizations of the three curves E_i are orbifold coverings of degree 72 of the orbifold $\mathbb{P}^1_{\mathbb{C}}$ endowed with three orbifold points of respective multiplicities $(2, 3, 12)$ hence by the Riemann-Hurwitz formula, the genus of E_i is indeed*

$$g(E_i) = \frac{72}{2}\left(-2 + \frac{2-1}{2} + \frac{3-1}{3} + \frac{12-1}{12}\right) + 1 = 4.$$

Note that $864 = 4 \cdot 3 \cdot 72$ where 4 is the order of the reflections of type B and 3 the number of curves of type B.

In the same way, the normalizations of C_1 and C_2 (resp. C_3 and C_4) are orbifold coverings of degree 36 (resp. 108) of the orbifold $\mathbb{P}^1_{\mathbb{C}}$ endowed with three orbifold points of respective multiplicities $(2, 3, 12)$ so that $g(C_1) = g(C_2) = 4$ and $g(C_3) = g(C_4) = 10$. Here again, $864 = 3(2 \cdot 36 + 2 \cdot 108)$ where 3 is the order of the reflections of type A.

All these computations are consistent with Lemmas 1(d) and (e).

2.4. We have seen that $H_1(X, \mathbb{Z}) = \mathbb{Z}e_1 + \mathbb{Z}e_2 \cong \mathbb{Z}^2$ in terms of a basis e_1 and e_2. Let D be a smooth curve of genus 4. A presentation of $\pi_1(D)$ can be given as

$$\langle u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4 \mid \prod_{i=1}^{4}[u_i, v_i] = 1\rangle.$$

For each of the curves D of genus 4, $E_i, i = 1, 2, 3$ and $C_j, j = 1, 2$, abusing notation we denote by $f : H_1(\widehat{D}, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \cong \mathbb{Z}^2$ the homomorphism induced by the normalization of the immersed image of D in X. Using Magma, we have found explicitly a basis of such elements $u_i, v_i, i = 1, \ldots, 4$ in $\pi_1(\widehat{D})$, and computed their images $f(u_i), f(u_j)$ in $H_1(X, \mathbb{Z})$ in terms of e_1, e_2 (see Propositions A.12 and A.17). This is summarized as follows for E_1, E_2 and C_1, which is all we need for later computations.

D	$f(u_1)$	$f(v_1)$	$f(u_2)$	$f(v_2)$	$f(u_3)$	$f(v_3)$	$f(u_4)$	$f(v_4)$
E_1	$(-5,-2)$	$(-2,7)$	$(-2,1)$	$(0,0)$	$(1,4)$	$(3,-6)$	$(2,5)$	$(-1,-4)$
E_2	$(-1,2)$	$(2,-1)$	$(-2,1)$	$(0,0)$	$(-3,0)$	$(-1,2)$	$(-2,1)$	$(3,0)$
C_1	$(0,-2)$	$(-2,0)$	$(-4,0)$	$(0,2)$	$(-4,2)$	$(4,0)$	$(2,0)$	$(0,-2)$

3. Picard number

3.1.

Lemma 3. *Suppose D is a reduced (not necessarily irreducible) totally geodesic curve on a smooth complex two-ball quotient X self-intersecting only at $P_1, \dots, P_k$ with simple multiplicities given by $(b_1, \cdots, b_k)$ and let us denote by D_i $(i = 1, \dots, n)$ its irreducible components, $\widehat{D}_i$ their normalization. Let $\nu : \widehat{D} = \cup_i \widehat{D}_i \to D$ be the normalization of D. Then*

$$K_X \cdot D = 3\sum_{i=1}^{n}(g(\widehat{D}_i) - 1) \quad \textit{and} \quad D \cdot D = \frac{1}{2}e(\widehat{D}) + \widetilde{\delta}, \quad \textit{where} \quad \widetilde{\delta} = \sum_{i=1}^{k} b_i(b_i - 1)$$

and $e(\widehat{D})$ is the Euler characteristic of $\widehat{D}$.

Proof. Note that we are in the case of a (non necessarily connected) immersed smooth curve in a surface, with singularities given by intersections of transversal local branches. Moreover, it is well known that for a totally geodesic curve D in a two-ball quotient, $c_1(K_{\widehat{D}}) = \frac{2}{3}\nu^* c_1(K_X)$ (this is a simple computation involving the curvature form on $B^2_{\mathbb{C}}$). As a consequence, by the adjunction formula,

$$K_X \cdot D = \int_D c_1(K_X) = \frac{3}{2}\sum_i \int_{\widehat{D}_i} c_1(K_{\widehat{D}_i}) = 3\sum_{i=1}^{n}(g(\widehat{D}_i) - 1).$$

Recall moreover from [BHPV, §II.11] that

$$g(D) = g(\widehat{D}) + \delta^{\mathrm{an}}(D), \quad \text{where } g(\widehat{D}) = 1 + \sum_i (g(\widehat{D}_i) - 1) \text{ and } \delta^{\mathrm{an}}(D) = \sum_{x \in D} \dim_{\mathbb{C}}(\nu_* \mathcal{O}_{\widehat{D}} / \mathcal{O}_D)$$

(here, the genus of a singular curve is its arithmetic genus). From the adjunction formula for embedded curves, $2(g(D) - 1) = K_X \cdot D + D \cdot D$ and therefore,

$$D \cdot D = 2(g(D) - 1) - K_X \cdot D = 2(g(\widehat{D}) + \delta^{\mathrm{an}}(D) - 1) - 3(g(\widehat{D}) - 1) = (1 - g(\widehat{D})) + 2\delta^{\mathrm{an}}(D).$$

Finally, observe that in the case at hand, $\delta^{\mathrm{an}}(D) = \frac{1}{2}\sum_{i=1}^{k} b_i(b_i - 1) = \frac{1}{2}\widetilde{\delta}$. □

3.2.

Lemma 4. *We have the following intersection numbers.*

(a) *For $i = 1, 2, 3$, we have $K_X \cdot E_i = 9$. Moreover, for $i = 1, 2$, $E_i \cdot E_i = 5$, $E_i \cdot E_3 = 9$ and $E_1 \cdot E_2 = 13$. We also have $E_3 \cdot E_3 = 9$.*
(b) *Denote by C either C_1 or C_2. Then $K_X \cdot C = 9$, $C \cdot C = -1$, $E_1 \cdot C = 11$, $E_2 \cdot C = 7$ and $E_3 \cdot C = 9$.*

Proof. The results follow immediately from Lemma 3 (here, all the involved curves are irreducible) and the summary in §2.

First, note that since the normalizations of the curves in (a) and (b) all have genus 4, their intersection with K_X is always 9 by Lemma 3. We leave the other computations to the reader and just observe that:
– a curve E_i can only intersect a curve E_j at $\pi^{-1}(P_2)$,
– two local branches of a curve C can only intersect at $\pi^{-1}(P_2)$,
– a curve C can only intersect a curve E_i at $\pi^{-1}(P_2)$ and $\pi^{-1}(P_3)$. □

3.3. From now on, for any two divisors D and D' on X, $D \equiv D'$ will mean that D and D' are numerically equivalent.

Lemma 5. *E_1, E_2 and C represent numerically linearly independent elements in the Néron-Severi group, where $C = C_1$ or C_2.*

Proof. Assume that E_1, E_2 and C satisfy numerically an identity

$$aE_1 + bE_2 + cC \equiv 0.$$

By considering the intersection of the above identity with E_1, E_2 and C respectively, we conclude that

$$\begin{aligned} 0 &= 5a + 13b + 11c \\ 0 &= 13a + 5b + 7c \\ 0 &= 11a + 7b - c \end{aligned}$$

The determinant of the above linear system is $1296 \neq 0$. Hence $a = b = c = 0$. □

3.4.

Corollary 1. *The Picard number of X is* 3.

Proof. It follows from the previous lemma that the Picard number is at least 3, given by the classes of E_1, E_2 and C. On the other hand, $h^{1,1}(X) = 3$ by Lemma 2. Since the Picard number is bounded from above by $h^{1,1}$, we conclude that the Picard number is 3. □

3.5. The following fact is a corollary of the earlier discussions.

Proposition 2. *The canonical line bundle K_X and E_3 give rise to the same class in the Néron-Severi group. Moreover, $K_X \equiv E_3 \equiv \frac{1}{2}E_1 + \frac{1}{2}E_2$.*

Proof. From the discussions in the previous section, we know that E_1, E_2 and $C = C_1$ form a basis of the Néron-Severi group (which is torsion free since $H_1(X, \mathbb{Z}) = \mathbb{Z}^2$ is torsion free).

Hence we may write

$$K_X \equiv aE_1 + bE_2 + cC$$

for some rational numbers a, b and c. By pairing with E_1, E_2 and C respectively, we arrive at

$$\begin{aligned} 9 &= 5a + 13b + 11c \\ 9 &= 13a + 5b + 7c \\ 9 &= 11a + 7b - c \end{aligned}$$

Solving the above system of equations, we obtain

$$K_X \equiv \frac{1}{2}E_1 + \frac{1}{2}E_2.$$

The same computation leads to $E_3 \equiv \frac{1}{2}(E_1 + E_2)$ since $E_3 \cdot E_i = K_X \cdot E_i$ for $i = 1, 2$ and $E_3 \cdot C = K_X \cdot C$. □

Remark 4. *By the previous proposition, we also have $K_X \equiv \frac{2}{3}(\frac{1}{2}E_1 + \frac{1}{2}E_2) + \frac{1}{3}E_3 = \frac{1}{3}(E_1 + E_2 + E_3)$. This fact can be recovered directly from the description of X as an orbifold covering of $R = \bar{\Gamma}\backslash B^2_{\mathbb{C}}$ as in §2.*

We use the notation of §1.3. Let $q : \bar{Q} = \mathbb{P}^2_{\mathbb{C}} \to R = \mathbb{P}^2_{\mathbb{C}}/\Sigma_3$ be the projection. First, we compute the canonical divisor K_R of R. We have $K_R = aD_A = 2aD_B$ for some $a \in \mathbb{Q}$ (see [DM2, §11.4 and Proposition 11.5] *for a description of* Pic(R)*). If we denote by $L = \mathcal{O}(1)$ the positive generator of* Pic$(\mathbb{P}^2_{\mathbb{C}})$*, we have $-3L = K_{\mathbb{P}^2_{\mathbb{C}}} = q^*K_R + 3L = 6aL + 3L$ as q branches at order 2 along D_A, and D_A has three lines as a preimage in $\mathbb{P}^2_{\mathbb{C}}$. Hence $K_R = -D_A = -2D_B$.*

Now, the orbifold canonical divisor of $\bar{\Gamma}\backslash B^2_{\mathbb{C}}$ is $K_R + \frac{3-1}{3}D_A + \frac{4-1}{4}D_B = (-1 + \frac{2}{3} + \frac{3}{8})D_A = \frac{1}{24}D_A = \frac{1}{12}D_B$. In particular, as $\pi^*D_B = 4(E_1 + E_2 + E_3)$, we get the result.

4. Geometry of a generic fiber of the Albanese map

4.1. Let $\alpha : X \to T$ be the Albanese map of X. From $\Pi/[\Pi,\Pi] \cong \mathbb{Z}^2$, we know that T is an elliptic curve, and in particular, α is onto. Moreover, note that since the image of α is a curve, the fibers of α are connected (see [U, Proposition 9.19]). Let D be a curve on X. The mapping α induces a mapping $\alpha|_D : D \to T$. Suppose F is the generic fiber of α. Then the degree of $\alpha|_D$ is given by $D \cdot F$.

Lemma 6. *Let m, n, p be the degrees of E_1, E_2, and $C = C_1$, respectively, over the Albanese torus T of X. The generic fiber F of the Albanese fibration of X satisfies*

$$F \equiv \frac{1}{72}\big((-3m+5n+2p)E_1 + (5m-7n+6p)E_2 + 2(m+3n-4p)C\big).$$

Proof. From Lemma 5, we may write numerically

$$F \equiv aE_1 + bE_2 + cC$$

for some rational numbers a, b, c. By pairing with E_1, E_2 and C respectively, we arrive at

$$\begin{aligned} m &= 5a + 13b + 11c \\ n &= 13a + 5b + 7c \\ p &= 11a + 7b - c \end{aligned}$$

The lemma follows from solving the above system of equations. □

4.2.

Lemma 7. *The degrees of $E_1, E_2, C = C_1$ over the Albanese torus T of X are given by*

$$m = 60,\ n = 12,\ p = 24.$$

Proof. Let D represent one of the curves E_1, E_2, C, $\nu : \hat{D} \to D$ the normalization of D and $\hat{\alpha} = \alpha \circ \nu$. Let ω be a positive $(1,1)$ form on T. Then the degree of D over T is given by $\deg(D) = \frac{\int_D \alpha^*\omega}{\int_T \omega}$. The key is to find the degree from the information of the explicit curves that we have. For this purpose, we use an analogue of the Riemann bilinear relations. Let η be a holomorphic 1-form on the smooth Riemann surface $\hat{D}$. Let $\{u_i, v_i\}$ be a basis of $\pi_1(\hat{D})$ as studied in §2.4. Then the Riemann bilinear relation (cf. [GH, p. 231]) states that

$$\int_{\hat{D}} \sqrt{-1}\eta \wedge \overline{\eta} = \sqrt{-1}\sum_{i=1}^{4}\left[\int_{u_i}\eta\overline{\int_{v_i}\eta} - \int_{v_i}\eta\overline{\int_{u_i}\eta}\right]$$

where we use the same notation for an element of $\pi_1(\hat{D})$ and its image in $H_1(\hat{D}, \mathbb{Z})$. Let us write $T = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ where $\operatorname{Im}\tau > 0$. Let $\omega_T = \sqrt{-1}dz \wedge \overline{dz}$ be the standard $(1,1)$ form on $\mathbb{C}$ and hence T. The above formula gives

$$\int_T \omega_T = \sqrt{-1}(\overline{\tau} - \tau). \tag{2}$$

Pulling back to D, the above formula gives

$$\begin{aligned} \int_D \alpha^*\omega_T &= \int_{\hat{D}} \hat{\alpha}^*\omega_T \\ &= \int_{\hat{D}} \sqrt{-1}\hat{\alpha}^*dz \wedge \hat{\alpha}^*\overline{dz} \\ &= \sqrt{-1}\sum_{i=1}^{4}\left[\int_{u_i}\hat{\alpha}^*dz\overline{\int_{v_i}\hat{\alpha}^*dz} - \int_{v_i}\hat{\alpha}^*dz\overline{\int_{u_i}\hat{\alpha}^*dz}\right] \\ &= \sqrt{-1}\sum_{i=1}^{4}\left[\int_{\hat{\alpha}_*(u_i)}dz\overline{\int_{\hat{\alpha}_*(v_i)}dz} - \int_{\hat{\alpha}_*v_i}dz\overline{\int_{\hat{\alpha}_*u_i}dz}\right]. \end{aligned} \tag{3}$$

In the above, $\hat{\alpha}_* : H_1(\hat{D},\mathbb{Z}) \to H_1(T,\mathbb{Z}) \cong H_1(X,\mathbb{Z}) \cong \mathbb{Z}^2$ refers to the map on 1-cycles induced by $\hat{\alpha}$. Hence the right-hand side of the above expression in terms of the notation in §2.4 is (up to sign)

$$\sqrt{-1}\sum_{i=1}^{4}\left[\int_{f(u_i)} dz \overline{\int_{f(v_i)} dz} - \int_{f(v_i)} dz \overline{\int_{f(u_i)} dz}\right] = \left[\sum_{i=1}^{4}\det(f(u_i),f(v_i))\right]\sqrt{-1}(\overline{\tau}-\tau), \quad (4)$$

where $\det(f(u_i),f(v_i))$ stands for the determinant of the two by two matrix formed by the two vectors $f(u_i)$ and $f(v_i)$ from the table in §2.4. Notice that the resulting number will be positive if and only if the orientation on $\hat{D}$ coming from the choice of $(u_1,v_1,\dots,u_4,v_4)$ as a symplectic basis of $H_1(\hat{D},\mathbb{Z})$, and the orientation on T induced by the choice of the basis (e_1,e_2) of $H_1(T,\mathbb{Z})$ are compatible (i.e. both are the same, or the opposite, as the one induced by the respective complex structures).

Substituting into (3) and (4) the values of $f(u_i)$ and $f(v_i)$ from the table in §2.4, we conclude the values of $-60,-12,-24$ for the values of $\sum_{i=1}^{4}\det(f(u_i),f(v_i))$ in the case of E_1,E_2 and C respectively. We conclude from (2), (3) and (4) that the degrees m,n,p are given by $60, 12$ and 24 respectively, and that the orientation on $\hat{D}$ and T are not compatible (we will say more on this below, see §5.5). □

4.3.

Theorem 3. *A fiber of the Albanese map $\alpha : X \to T$ represents the same numerical class as $-E_1+5E_2$, and the genus of a generic fiber F is* 19.

Proof. Substituting the values of m,n,p from the previous lemma into Lemma 6, we conclude that F represents the same class as $-E_1+5E_2$ in the Néron-Severi group. Hence

$$F\cdot K_X = -E_1\cdot K_X + 5E_2\cdot K_X = 36.$$

On the other hand, from the adjunction formula,

$$2(g-1) = (K_X+F)\cdot F = K_X\cdot F.$$

Hence $g=19$. □

5. Geometry of the Albanese fibration

5.1. Consider the Albanese fibration $\alpha : X \to T$. First, recall that the fibers of α are connected (see §4.1). Let X_s be the fiber of α at $s\in T$. It is connected (see §4.1). Now $g(X_s)\geqslant 2$, because X has negative holomorphic sectional curvature. Although we will not need this in the sequel, we observe that the fibration cannot be locally holomorphically trivial. Otherwise there is a smooth non-trivial family of holomorphic mappings from X_s (where $s\in T$ is generic) to X. However, a holomorphic map is harmonic with respect to any Kähler metric on X_s and the Poincaré metric on X. As the Poincaré metric on X is strictly negative, it follows from uniqueness of harmonic maps to a negatively curved Kähler manifold in its homotopy class that the family is actually a singleton, a contradiction.

5.2. The result below is just a rewriting of Proposition X.10 in [Be]. As usual, if D is a (not necessarily reduced) curve, we denote by $g(D)$ its arithmetic genus (see [BHPV, §II.11]).

Proposition 3. *Let X (resp. C) be a smooth complex surface (resp. curve) and $\pi : X \to C$ a surjective morphism with connected fibers. Let $D=\sum_{i=1}^{k} m_iD_i$, $(m_i\geqslant 1)$ be a singular fiber of π and let $D^{\mathrm{red}}=\sum_{i=1}^{k} D_i$ be the reduced divisor associated to D. Let $\nu : \widehat{D^{\mathrm{red}}} \to D^{\mathrm{red}}$ be the normalization. For any x in the support of D^{red}, we define $\delta_x^{\mathrm{top}} := \dim_{\mathbb{C}}(\nu_*\mathbb{C}_{\widehat{D^{\mathrm{red}}}}/\mathbb{C}_{D^{\mathrm{red}}}) = \sharp\nu^{-1}(x)-1$ the number of (local) irreducible components of*

D^{red} *at* x *minus* 1 *and* $\delta_x^{\mathrm{an}} := \dim_{\mathbb{C}}(\nu_* \mathcal{O}_{\widehat{D^{\mathrm{red}}}}/\mathcal{O}_{D^{\mathrm{red}}})$ *so that* $\mu_x := 2\delta_x^{\mathrm{an}} - \delta_x^{\mathrm{top}}$ *is the Milnor number of* D^{red} *at* x. *We also set* $\mu = \sum_{x \in D^{\mathrm{red}}} \mu_x$. *Then, we have*

$$e(D^{\mathrm{red}}) - e(X_s) = \mu + \Big(\sum_{i=1}^{k} (m_i - 1)\big(2(g(D_i) - 1) - D_i^2\big)\Big) - \big(D^{\mathrm{red}}\big)^2. \tag{5}$$

Proof. From Lemma VI.5 and the proof of Proposition X.10 in [Be], we immediately get

$$e(D^{\mathrm{red}}) = \mu + 2\chi(\mathcal{O}_{D^{\mathrm{red}}}) = \mu + e(X_s) + 2(\chi(\mathcal{O}_{D^{\mathrm{red}}}) - \chi(\mathcal{O}_D)),$$

where we used the fact that the arithmetic genus of the fibers of a morphism from a surface onto a curve is constant. Now, since $D^2 = 0$,

$$\begin{aligned} 2(\chi(\mathcal{O}_{D^{\mathrm{red}}}) - \chi(\mathcal{O}_D)) &= (K_X + D) \cdot D - (K_X + D^{\mathrm{red}}) \cdot D^{\mathrm{red}} \\ &= K_X \cdot (D - D^{\mathrm{red}}) - \big(D^{\mathrm{red}}\big)^2 \\ &= \sum_{i=1}^{k} (m_i - 1)(K_X + D_i) \cdot D_i - \sum_{i=1}^{k} (m_i - 1) D_i^2 - \big(D^{\mathrm{red}}\big)^2. \end{aligned}$$

That $2\delta_x^{\mathrm{an}} - \delta_x^{\mathrm{top}}$ is the Milnor number of D^{red} at x is proved in [BG, Proposition 1.2.1]. □

Remark 5. *In the notation of Proposition 3,* $\mu_x = 0$ *if and only if* D^{red} *is smooth at* x *and if* $\mu_x = 1$ *it is easily seen that the singularity of* D^{red} *at* x *is nodal (see Lemmas 1.2.1 and 1.2.4 in* [BG] *for instance).*

Corollary 2. *Let* $I \subset T$ *be the set of singular values of the Albanese fibration* α. *Then*

(a) $\sum_{s_o \in I} \big(e(X_{s_o}) - e(X_s)\big) = 3$ *where* X_s *is a generic fiber,*
(b) *the cardinality of* I *is at most* 3,
(c) α *has no multiple fiber, and therefore* $(X_{s_0})^{\mathrm{red}}$ *is singular for at least one* $s_0 \in I$,
(d) *the total number of singular points in the fibers is at most* 3 *and if equality holds, the three singularities are nodal and the fibration is stable. More precisely,*

$$\sum_{s_o \in I} \Big(\sum_{x \in X_{s_0}} \mu_x \Big) = 3. \tag{6}$$

Proof. Note first that there are no rational or elliptic curves in X since the holomorphic sectional curvature of a ball quotient is negative.

(a) From the standard formula for the Euler number of a holomorphic fibration (see [Be, Lemma VI.4] or [BHPV, Proposition III.11.4]), we have

$$3 = e(X) = e(T) \cdot e(X_s) + \sum_{s_o \in I} n_{s_o} = \sum_{s_o \in I} n_{s_o},$$

where $n_{s_o} = e(X_{s_o}) - e(X_s)$ for $s \in T_o := T - I$. Here we used the fact that the Euler characteric of T vanishes.

(b) It is well known (see [BHPV, Remark III.11.5]), and it can be easily recovered from Proposition 3, that $n_{s_o} \geqslant 0$ with equality if and only if X_{s_o} is a multiple fiber with $(X_{s_o})^{\mathrm{red}}$ smooth elliptic. But as we noticed above, this is impossible in our case thus $n_{s_o} > 0$ for any $s_o \in I$. Since $\sum_{s_o \in I} n_{s_o} = 3$, we conclude in particular that $|I| \leqslant 3$ (and each $n_{s_o} \leqslant 3$).

(c) Assume first that a fiber D might be written $D = mD^{\mathrm{red}}$ with $m \geqslant 2$. Then, by (a) and formula (5), $3 \geqslant e(D^{\mathrm{red}}) - e(X_s) \geqslant (m-1)\sum_{i=1}^{k}(g(D_i) - 1)$ and the only possibility is that $k = 1$, $m = 2$ and $g(D_1) = 2$. However, by Theorem 3, $18 = g(D) - 1 = m(g(D_1) - 1) = 2$, a contradiction.

Now, assume that $D = \sum_{i=1}^{k} m_i D_i$ with $k \geqslant 2$, $m_i \geqslant 1$ and $m_1 \geqslant 2$. Recall that by Zariski's lemma (see [BHPV, Lemma III.8.2]) the self intersection of any effective cycle supported on D^{red} must be nonpositive, and it is equal to zero if and only if it is proportional to D (in particular $D_1^2 < 0$). Therefore by formula (5), $3 \geqslant e(D^{\mathrm{red}}) - e(X_s) \geqslant 3 - \big(D^{\mathrm{red}}\big)^2$. But $\big(D^{\mathrm{red}}\big)^2 = 0$ if and only if $D = mD^{\mathrm{red}}$, a case which has already been ruled out.

(d) is a consequence of the previous points, equation (5) and Remark 5. □

5.3.

Lemma 8. $\deg(\alpha_*\omega_{X|T}) = 1$.

Proof. Note that we do not know a priori that the fibration α is stable. The lemma is a direct consequence of [X, Chapter 1], where it is shown that $\alpha_*\omega_{X|T}$, the direct image of the relative dualizing sheaf $\omega_{X|T}$, is locally free of rank $g = g(X_s)$, where $s \in T$ is a generic point (as in the classical case of a stable fibration). As a consequence, this is also the case of $R^1\alpha_*\mathcal{O}_X$ which is the dual sheaf of $\alpha_*\omega_{X|T}$.

Then, using the Leray spectral sequence and the Riemann-Roch formula, we get

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_T) - \chi(R^1\alpha_*\mathcal{O}_X) = -\deg(R^1\alpha_*\mathcal{O}_X) + (g-1)(g(T)-1) = \deg \alpha_*\omega_{X|T}$$

since $\deg \alpha_*\omega_{X|T} = -\deg(R^1\alpha_*\mathcal{O}_X)$ and $g(T) = 1$. As $\chi(\mathcal{O}_X) = 1$, the result follows. □

5.4. Recall from §1.4 (see also §A.5) that the normalizer N of Π in $\bar{\Gamma}$ is generated by the element j^4 of order 3 and Π, and the automorphism group Σ of X is given by the group N/Π, which has order 3. Denote by σ the automorphisms of $B^2_{\mathbb{C}}$ and of X induced by j^4. If $\xi = (z_1, z_2) \in B^2_{\mathbb{C}}$, then $\sigma(\xi) = (\omega z_1, \omega z_2)$ where $\omega = \zeta^4$ is a non trivial cube root of unity.

The Albanese map $\alpha : X \to T = \mathbb{C}/\Lambda$ can be lifted to a holomorphic map $\alpha_0 : B^2_{\mathbb{C}} \to \mathbb{C}$ so that $\alpha_0(O) = 0$ (choosing $\Pi O \in X$ as base point when defining α):

$$\begin{array}{ccccc} & & & & \mathbb{C} \\ & & \overset{\alpha_0}{\dashrightarrow} & & \downarrow \\ B^2_{\mathbb{C}} & \longrightarrow & X & \xrightarrow{\alpha} & T \end{array}$$

If $\pi \in \Pi$, then $\alpha_0(\pi\xi) - \alpha_0(\xi) \in \Lambda$ is independent of $\xi \in B^2_{\mathbb{C}}$, and so there is a map $\theta_0 : \Pi \to \Lambda$ such that $\alpha_0(\pi\xi) = \alpha_0(\xi) + \theta_0(\pi)$ for all $\xi \in B^2_{\mathbb{C}}$ and $\pi \in \Pi$. Since θ_0 is a homomorphism, it factors through our abelianization map $f : \Pi \to \mathbb{Z}^2$, see §1.4. So there is a homomorphism $\theta : \mathbb{Z}^2 \to \Lambda$ such that

$$\alpha_0(\pi\xi) = \alpha_0(\xi) + \theta(f(\pi)) \quad \text{for all } \xi \in B^2_{\mathbb{C}} \text{ and } \pi \in \Pi. \tag{7}$$

By the universal property of the Albanese map, there is an automorphism $\sigma_T : T \to T$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & T \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_T} \\ X & \xrightarrow{\alpha} & T \end{array} \tag{8}$$

If the automorphism is trivial, then $\alpha_0(\sigma(\xi)) - \alpha_0(\xi) \in \Lambda$ for all $\xi \in B^2_{\mathbb{C}}$, and so is constant. Since $\sigma(O) = O$, $\alpha_0(j^4\xi) = \alpha_0(\xi)$ for all ξ, and this implies that $\theta(f(j^4\pi j^{-4})) = \theta(f(\pi))$ for all $\pi \in \Pi$. But then (1) implies that $\theta = 0$, because $I - \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ is non-singular hence $\Pi\xi \mapsto \alpha_0(\xi)$ is a holomorphic function $X \to \mathbb{C}$, and so is constant because X is compact, contradicting surjectivity of α.

As a consequence, Σ acts non trivially on T and since $\sigma(O) = O$, the action of Σ fixes the point $\alpha(\Pi O) = 0 + \Lambda$. From this and $|\Sigma| = 3$, it follows immediately that the elliptic curve has to be $T = \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$, and the vertical map σ_T on the right in (8) is $z + \Lambda \mapsto \omega^i z + \Lambda$ with $i = 1$ or 2. Indeed, the automorphism σ_T which fixes $0 + \Lambda$ is induced by a nontrivial $\mathbb{C}$-linear automorphism of $\mathbb{C}$ preserving Λ (see [Be, Proposition V.12] for instance). Since it has order 3, it must be multiplication by ω^i, where $i = 1$ or 2. Hence Λ contains 1 and ω (after renormalization of the lattice).

It follows that there are precisely 3 fixed points of Σ on T: a fundamental domain of T consists of two equilateral triangles and the fixed points are given by a vertex and the centroid of each of the two triangles i.e. are the points $p_\nu = \nu(2+\omega)/3 + \Lambda$, $\nu = 0, 1, -1$ (notice that $(1-\omega)^{-1} = (2+\omega)/3$). In particular, we have proved

Lemma 9. *The action of Σ descends to a non-trivial action of T. There are three fixed points in the action of Σ on T. The elliptic curve T is isomorphic to $\mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z})$.*

5.5. We still use the notation from §5.4. Note that by definition of the Albanese map, $\theta \circ f : \Pi \to \Lambda$ is onto. In other words, $a + b\omega := \theta(1,0)$ and $c + d\omega := \theta(0,1)$ (where $a, b, c, d \in \mathbb{Z}$) generate Λ over $\mathbb{Z}$ i.e. $ad - bc = \pm 1$. We wish to determine whether σ_T acts on T by ω or ω^2.

The automorphism σ is induced by the action of j^4 on $B^2_{\mathbb{C}}$, and recall that j^4 normalizes Π. For any $\pi \in \Pi$, we have

$$\alpha_0(j^4\pi O) = \alpha_0(j^4\pi j^{-4} j^4 O) = \alpha_0(j^4 O) + \theta(f(j^4\pi j^{-4})) = \alpha_0(O) + \theta(f(j^4\pi j^{-4}))$$

since O is fixed by j^4, and

$$\alpha_0(\pi O) = \alpha_0(O) + \theta(f(\pi)).$$

Clearly, σ_T acts on T by ω^i if and only if $\alpha_0(j^4\xi) = \omega^i\alpha_0(\xi)$ for all $\xi \in B^2_{\mathbb{C}}$. In particular, for all $\pi \in \Pi$, $\alpha_0(j^4\pi O) = \omega^i\alpha_0(\pi O)$. It follows from the above relations that for all $\pi \in \Pi$,

$$\theta(f(j^4\pi j^{-4})) = \omega^i\theta(f(\pi)).$$

Since f is surjective, (1) shows that $\theta(n, -m-n) = \omega^i\theta(m,n)$ for all $m, n \in \mathbb{Z}$, and taking $(m,n) = (1,0)$ we get $-c - d\omega = \omega^i(a + b\omega)$. When $i = 1$, this implies that $c = b$ and $d = b - a$, so that $ad - bc = -(a^2 - ab + b^2)$, and therefore $ad - bc = -1$ and $a + b\omega$ is a power $(-\omega)^\nu$ of $-\omega$, and $c + d\omega = -\omega(a + b\omega)$. When $i = 2$, it implies that $c = a - b$ and $d = a$, so that $ad - bc = a^2 - ab + b^2$, and therefore $ad - bc = +1$ and $a + b\omega$ is again a power $(-\omega)^\nu$ of $-\omega$, and this time $c + d\omega = -\omega^2(a + b\omega)$.

Finally, notice that we can multiply α_0 by $(-\omega)^{-\nu}$, and the new θ we get satisfies $\theta(1,0) = 1$ in both cases, but the new $\theta(0,1)$ is $-\omega$ when $i = 1$, and $-\omega^2$ when $i = 2$. To sum up, we have the following

Lemma 10. *The action of σ_T on T is by ω (resp. ω^2) if and only if $(\theta(1,0), \theta(0,1))$ is equal (up to a rotation) to $(1, -\omega)$ (resp. $(1, -\omega^2)$).*

In order to decide between the two possibilities for the action of σ_T on T, we will use the restriction of the Albanese map α to the curve E_1 (we could have chosen any of the other totally geodesic curves in X). Recall that in the course of the proof of Lemma 7, we noticed that the orientation on $\hat{E}_1$ and the one on T induced by $(\theta(1,0), \theta(0,1))$ were not compatible. First, we will determine the orientation on $\hat{E}_1$ induced by the complex structure on X. For this purpose, we compute the intersection form on $H_1(\hat{E}_1, \mathbb{Z})$ (where E_1 is the curve associated with the mirror M_0 such that $\pi_1(\hat{E}_1) \cong \Pi_0$) in the basis $(\delta_i)_{1 \leqslant i \leqslant 8}$ induced by the generators g_i which satisfy the relation

$$g_1g_2g_3g_4g_5g_6g_7g_8g_1^{-1}g_3^{-1}g_5^{-1}g_7^{-1}g_2^{-1}g_4^{-1}g_6^{-1}g_8^{-1} = 1$$

(see the proof of Proposition A.9).

The loops δ_i are the images in $\hat{E}_1$ of the axes of the generators g_i, seen as hyperbolic elements in SU_0 (see Lemmas A.20 and A.21) which are depicted in figure 2, where the point labelled i represents the attractive point at infinity of the axis of g_i. The dashed geodesics are the axes of the elements $g_9 = (g_1g_2)^{-1}$, $g_{10} = (g_3g_4)^{-1}$, $g_{11} = (g_5g_6)^{-1}$ and $g_{12} = (g_7g_8)^{-1}$ that will also be needed.

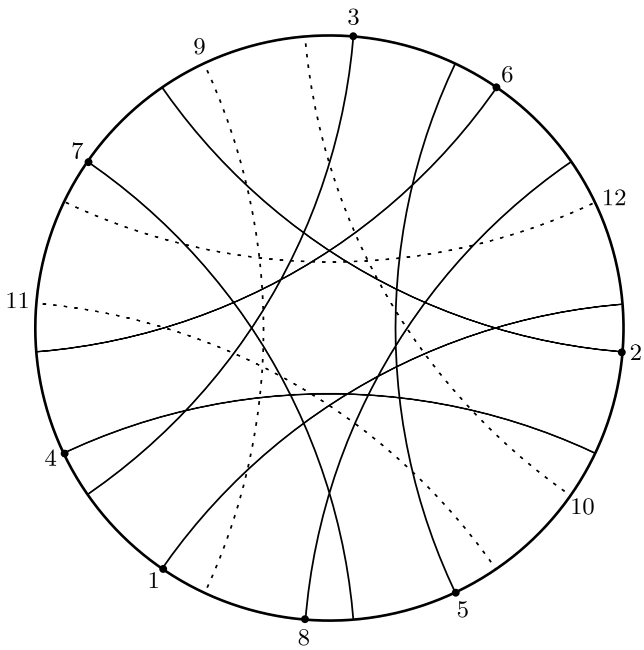


FIGURE 2. Axes of the generators of Π_0

Figure 2 was drawn with Maple, using the expression of $\psi_0(g_i) \in \mathrm{SU}_0$ in the proof of Proposition A.9. The matrices in SU_0 are unitary with respect to the diagonal form with diagonal entries 1 and $1-r$ (see Lemma A.21), and have the form

$$h = \begin{pmatrix} a & (r-1)b \\ \bar{b} & \bar{a} \end{pmatrix}, \tag{9}$$

where $a, b \in \mathbb{Z}[\zeta]$ and $|a|^2 - (r-1)|b|^2 = 1$. Conjugating them by the diagonal matrix with diagonal entries 1 and $\sqrt{r-1}$, we get elements of $\mathrm{SU}(1,1)$. So SU_0 acts on the unit disc $B(\mathbb{C})$ in $\mathbb{C}$ and its closure $\bar{B}(\mathbb{C})$. Assuming $b \neq 0$, the fixed points in $\bar{B}(\mathbb{C})$ of the h in (9) are

$$w = \frac{a - \bar{a} \pm \sqrt{(a+\bar{a})^2 - 4}}{2\bar{b}\sqrt{r-1}}. \tag{10}$$

These fixed points w satisfy $|w| = 1$ when $|a + \bar{a}| > 2$.

We find that the $\psi_0(g_i)$, $i = 1, \ldots, 12$, are the conjugates by the powers of $z = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$ of a single matrix (9), where $a = \zeta^2 + 3\zeta + 2$ and $b = 3 + 2r$. In fact, $\psi_0(g_i) = z^{-n_i} h z^{n_i}$ for $(n_1, \ldots, n_{12}) = (7, 11, 2, 6, 9, 1, 4, 8, 3, 10, 5, 0)$. So the fixed points of $\psi_0(g_i)$ are $e^{i\theta}\zeta^{n_i}$ and $e^{i(\pi-\theta)}\zeta^{n_i}$, where $\theta = \tan^{-1}\big(\sqrt{(2r-1)/11}\big)$, $e^{i\theta}$ being the fixed point (10) for this a and b, with the plus sign. An easy calculation shows that $e^{i\theta}\zeta^{n_i}$ and $e^{i(\pi-\theta)}\zeta^{n_i}$ are the attracting and repulsing fixed points, respectively, of $\psi_0(g_i)$.

In figure 3, we drew a fundamental domain for the action of Π_0, whose boundary is a 24-gon. The sides of this 24-gon are pairwise identified by the elements g_i ($i = 1, \ldots, 12$). We preferred to use the generators g_i's of Π_0 instead of the u_i's and the $v_i's$ because their axes pass closer to the origin of the disc and hence the picture is much clearer.

The δ_i's are all oriented in the same way, e.g. from the repulsive point to the attractive point. As all the geodesics in figure 2 actually meet inside the fundamental domain of figure 3, we deduce from the picture that the matrix of the intersection form $\langle\, , \,\rangle$ on $H_1(\hat{E}_1, \mathbb{Z})$ in the

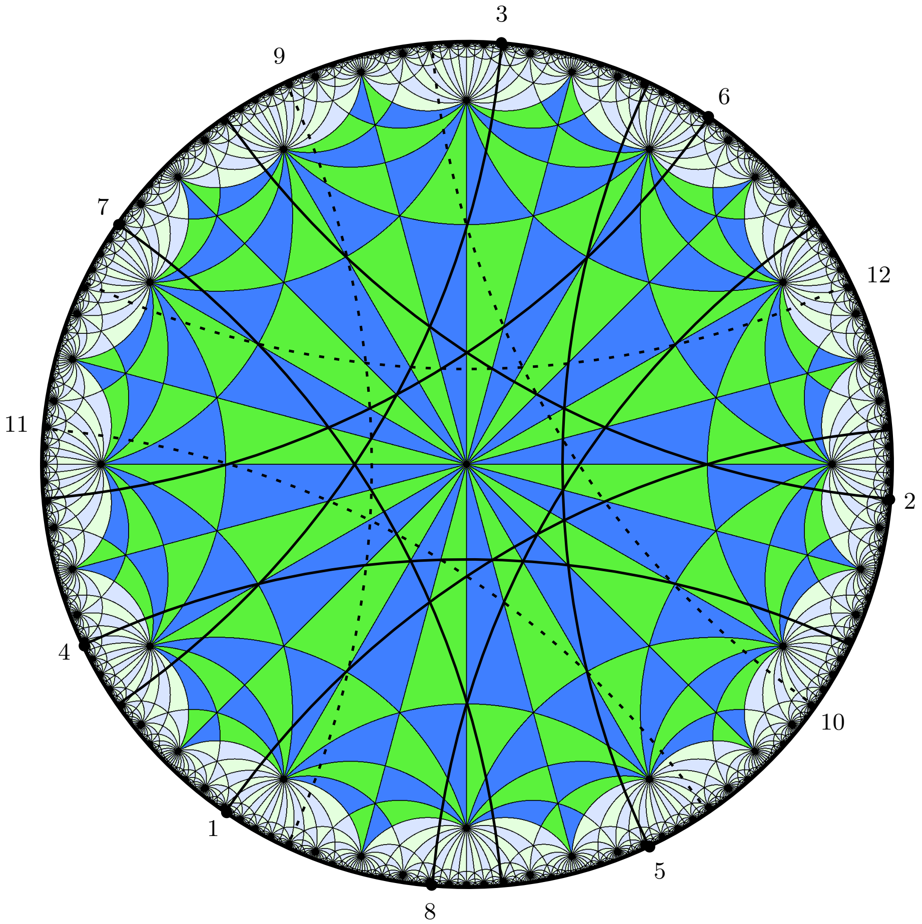


FIGURE 3. Fundamental domain for the action of Π_0

basis (δ_i) and with respect to the usual orientation of the disc is

$$I_\delta = \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \end{pmatrix}$$

where the entry in row i and column j is $\langle \delta_i, \delta_j \rangle$. Now, we cut the curve $\hat{E}_1$ along the loops δ_i in order to obtain a 16-gon Δ and consider the dual basis (δ_i^*) of (δ_i), i.e. $\langle \delta_i, \delta_j^* \rangle = \delta_{ij}$, as depicted in figure 4. Actually, there are a priori two choices for the orientation of the boundary of Δ and the one pictured in figure 4 is the good one since the matrix of the

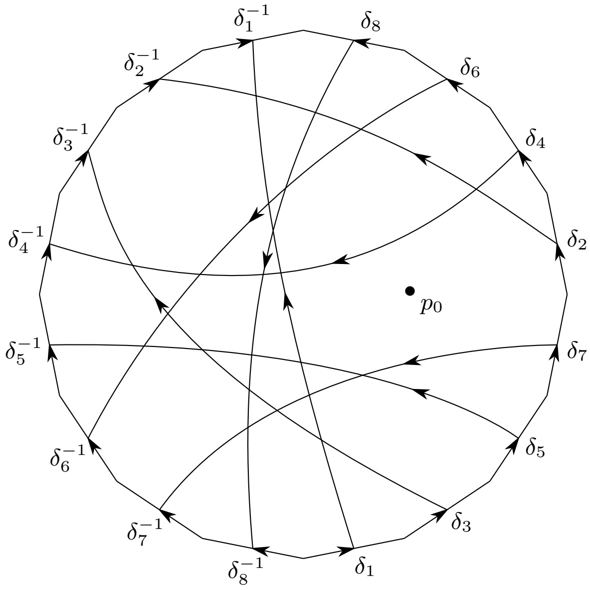


FIGURE 4. The basis (δ_i^*) in the 16-gon Δ

intersection form in the basis (δ_i^*) has to be the transpose of the inverse

$$I_{\delta^*} = {}^tI_\delta^{-1} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

of the one in the basis (δ_i), which is indeed the case as can be checked on the figure.

Since we do not use a standard presentation of the group Π_0, we need a generalized Riemann bilinear relation, a quick proof of which we now give, following [GH, pp. 229–231].

Let p_0 be a base point in the interior of Δ and η a holomorphic 1-form on $\hat{E}_1$. We define the holomorphic function $h(p) = \int_{p_0}^p \eta$ on the closure of Δ (which is simply connected), so that $dh = \eta$. Let p be a point on δ_i and p' the corresponding point of δ_i^{-1}. Then $\int_p^{p'} \eta = h(p') - h(p) = \sum_j \langle \delta_i^*, \delta_j^* \rangle \int_{\delta_j} \eta$ which is independent of p (and p'). Therefore,

$$\int_{\delta_i+\delta_i^{-1}} h\,\bar\eta = \int_{\delta_i} (h(p)-h(p'))\bar\eta = -\Big[\sum_j \langle \delta_i^*, \delta_j^* \rangle \int_{\delta_j} \eta\Big] \int_{\delta_i} \bar\eta = \Big[\sum_j \langle \delta_j^*, \delta_i^* \rangle \int_{\delta_j} \eta\Big] \int_{\delta_i} \bar\eta.$$

Now,

$$\sqrt{-1}\int_{\hat{E}_1} \eta\wedge\bar\eta = \sqrt{-1}\int_\Delta dh\wedge\bar\eta = \sqrt{-1}\int_{\partial\Delta} h\,\bar\eta = \sqrt{-1}\sum_i\sum_j \langle \delta_j^*, \delta_i^* \rangle \int_{\delta_j}\eta\int_{\delta_i}\bar\eta.$$

We shall apply this formula to $\eta = \hat\alpha^* dz$, as in the proof of Lemma 7, using the expression of the g_i's in terms of the generators a_i of Π (see the proof of Proposition A.9). We find using Lemma 10 that

$$V := \Big(\int_{\delta_i} \hat\alpha^* dz\Big)_{i=1,\dots,8} = (1+4\kappa, 4-5\kappa, -2-5\kappa, -5+7\kappa, 5-\kappa, -1-4\kappa, -7+2\kappa, 2+5\kappa)$$

where $\kappa = -\omega$ (resp. $-\omega^2$) if the action of σ_T is by ω (resp. ω^2), since $\int_{\delta_i} \hat\alpha^* dz = \theta(f(g_i))$. The coordinates of V are easily computed using the expression of each g_i in terms of a_1,

a_2 and a_3 as given in the first lines of the proof of Proposition A.9, and the computations in §1.4. Let us just give one example. For instance, $g_5 = \zeta^3 j^4 a_2 a_1 j^8 a_2^{-1} a_3^3 a_1^2$, hence

$$\begin{aligned} f(g_5) &= f(j^4 a_2 j^{-4}) + f(j^4 a_1 j^{-4}) - f(a_2) + 3f(a_3) + 2f(a_1) \\ &= (1,1) + (3,-4) - (-2,1) + 3(-1,-1) + 2(1,3) \\ &= (5,-1), \end{aligned}$$

hence the 5-th component of V is $5-\kappa$.

As $\hat{\alpha}$ is holomorphic, $\sqrt{-1}\int_{\hat{E}_1} \hat{\alpha}^* dz \wedge \hat{\alpha}^* d\bar{z} = \sqrt{-1}{}^t V I_{\delta^*} \bar{V}$ must be positive and we find that it is equal to $60r$ (resp. $-60r$) if $\kappa = -\omega$ (resp. $\kappa = -\omega^2$). Therefore, we conclude

Proposition 4. *The action of σ_T on T is by ω.*

5.6. Let $p_\nu = \nu(2+\omega)/3 + \Lambda$, $\nu = 0, 1, -1$ be the fixed points of Σ on T, as given by Lemma 9.

Lemma 11. (a) *There are altogether nine fixed points of* $\mathrm{Aut}(X)$ *on X.*
- (b) *The points O_1, O_2 and O_3 mentioned in §2 are fixed points of Σ, all lie in the same fiber $\alpha^{-1}(p_0)$.*
- (c) *The other fixed points are 6 of the 288 points lying in $\pi^{-1}(P_5)$ (see §2).*
- (d) *Each of the fibers $\alpha^{-1}(p_j)$ for $j = 1, 0, -1$ contains exactly three of the nine fixed points of* $\mathrm{Aut}(X)$.
- (e) *The fixed points O_i, $i = 1, 2, 3$ are of type $\frac{1}{3}(1,1)$, and the other six fixed points are of type $\frac{1}{3}(1,2)$.*

Proof. (a) follows from Lemma A.32. This corresponds to the case of Proposition 1.2 (2)(b) in Keum [K], the latter follows from Lefschetz fixed point formula and holomorphic Lefschetz fixed point formula. (b), (c) and (d) follow from Proposition A.19. The type of singularities follows from Lemma A.33, which is also stated as one of the cases in [K, Proposition 1.2], and was observed by Igor Dolgachev as well. □

5.7.

Lemma 12. *Let $O = O_i$ for $i = 1, 2, 3$. Then α is smooth at O.*

Proof. By Lemma 11 and Proposition 4, there exist coordinates (x, y) centered at $O \in X$ and a coordinate z centered at $\alpha(O) \in T$ such that $\alpha \circ \sigma(x, y) = \alpha(\omega x, \omega y) = \omega\alpha(x, y) = \sigma_T \circ \alpha(x, y)$. In terms of our local coordinates, we write

$$\alpha(x, y) = \sum_{i,j \geqslant 0} a_{ij} x^i y^j$$

and we have

$$\sum_{i,j \geqslant 0} a_{ij} \omega^{i+j} x^i y^j = \sum_{i,j \geqslant 0} a_{ij} \omega x^i y^j.$$

Since the above is true for all x, y, we conclude that for those i, j with $a_{ij} \neq 0$, we actually have $i + j \equiv 1 \pmod 3$, hence we may write

$$\alpha(x, y) = (a_{10}x + a_{01}y) + \Big(\sum_{i+j=4} a_{ij} x^i y^j\Big) + \sum_{i+j=3n+1, n \geqslant 2} a_{ij} x^i y^j.$$

The fiber through O is smooth at O if the first expression is non-zero. If $a_{10} = a_{01} = 0$ then α vanishes at order at least 4, hence α_x and α_y vanish at order at least 3, so that $\mathcal{O}(x, y)/\langle \alpha_x, \alpha_y \rangle$ has length at least 6 (i.e. the Milnor number of the fiber through O is at least 6). This violates formula (6) in Corollary 2 that the sum of the Milnor numbers at the singularities is at most 3. Hence α is smooth at O. □

5.8. Let $Q = Q_i, i = 1, \ldots, 6$ be one of the fixed points of σ other than $O_j, j = 1, 2, 3$ so that the local action of σ is of type $\frac{1}{3}(1, 2)$ at Q_i.

Lemma 13. *One of the following happens*

(i) *α is smooth at Q,*
(ii) *Q is a point of Milnor number 3 i.e. a tacnode.*

Proof. By Lemma 11 and Proposition 4, there exist coordinates (x, y) centered at $Q \in X$ and a coordinate z centered at $\alpha(Q) \in T$ such that $\alpha \circ \sigma(x, y) = \alpha(\omega x, \omega^2 y) = \omega\alpha(x, y) = \sigma_T \circ \alpha(x, y)$. As above, we write in terms of our local coordinates

$$\alpha(x, y) = \sum_{i,j \geqslant 0} a_{ij} x^i y^j$$

and we have

$$\sum_{i,j \geqslant 0} a_{ij} \omega^{i+2j} x^i y^j = \sum_{i,j \geqslant 0} a_{ij} \omega x^i y^j.$$

We conclude that for those i, j with $a_{ij} \neq 0$, we actually have $i + 2j \equiv 1 \pmod 3$, hence we may write

$$\alpha(x, y) = (a_{10}x) + (a_{02}y^2 + a_{21}x^2y + a_{13}xy^3 + a_{40}x^4) + \text{terms of order at least } 5.$$

It is smooth at $(0, 0)$ if $a_{10} \neq 0$. This is case (i).

Assume now that $a_{10} = 0$. First remark that $\mu_Q \geq 2$. Indeed,

$$\begin{aligned} \alpha_x &= 2a_{21}xy + \text{terms of order at least } 3 \\ \alpha_y &= 2a_{02}y + a_{21}x^2 + \text{terms of order at least } 3 \end{aligned}$$

hence 1 and x are linearly independent in $\mathcal{O}(x, y)/\langle \alpha_x, \alpha_y \rangle$. However, the case $\mu_Q = 2$ cannot occur since in this situation there would be exactly one more singular point P on X with Milnor number 1 by formula (6) in Corollary 2. But we saw in Lemma 12 that none of the O_i's is singular and we have just seen that none of the Q_i's can be a singularity with Milnor number 1. Therefore, P would not be a fixed point of σ and then $\mu_P = \mu_{\sigma(P)}$ which is a contradiction.

Finally, we recall (see [AGV, p. 183]) that a singularity with Milnor number 3 is holomorphically equivalent to a tacnode whose equation is $(x^2 - y)(x^2 + y) = 0$ (or $y(y - x^2) = 0$) and we note for instance that both these expressions are a priori admissible in our situation. □

5.9. From the previous two lemmas we deduce the following result about the singularities of the Albanese map α.

Proposition 5. *There are three mutually exclusive possibilities for the singularities of the Albanese map:*

(i) *α has exactly one singularity which is a tacnode at some Q_i ($i = 1, \ldots, 6$). The unique singular fiber is then irreducible and has geometric genus 17.*
(ii) *α has exactly one singular fiber which is one of the three (globally) fixed fibers by σ, with exactly three nodal singularities, and none of them is a fixed point of σ. The unique singular fiber might be reducible and its normalization has genus 16.*
(iii) *α has exactly three singular fibers with exactly one nodal singularity on each of them and the singular points are the elements of a σ-orbit. In this case, each singular fiber is irreducible and has geometric genus 18.*

Proof. The fact that only one of these three possibilities can occur is a straightforward consequence of Lemmas 12 and 13 together with formula (6). The genera are easily computed using formula (5). □

5.10. Ngaiming Mok has kindly drawn to our attention the following problem which was open and is interesting to geometric study of complex ball quotient.

Question 1. *Does there exists a homomorphism $f : X \to R$ from a smooth complex ball quotient X to a Riemann surface R with a non-totally geodesic singular fiber?*

There are very few explicit examples of mappings from a complex ball quotient to a Riemann surface. The known ones described by Deligne-Mostow, Mostow, Livné, Toledo and Deraux all have totally geodesic singular fibers, cf. [DM2], [T] or [Der2] and the references therein.

In the following we show that the surface studied in this note provides such an example.

Theorem 4. *None of the singular fibers of the Albanese fibration $\alpha : X \to T$ is totally geodesic.*

Proof. Let E be a singular fiber of α and let $\widehat{E}$ be the normalization of E. Assume for the sake of proof by contradiction that E is totally geodesic. According to Lemma 3,

$$E \cdot E = \frac{1}{2}e(\widehat{E}) + 2\delta^{\mathrm{an}}(E)$$

and moreover, $g = g(\widehat{E}) + \delta^{\mathrm{an}}(E)$ and $E \cdot E = 0$ since E is a fiber of the fibration, hence $1 - g + 3\delta^{\mathrm{an}}(E) = 0$. Since we have shown that $g = 19$ in Theorem 3, this leads to $\delta^{\mathrm{an}}(E) = 6$. However, for a node $\delta^{\mathrm{an}} = 1$ and for a tacnode $\delta^{\mathrm{an}} = 2$. Hence the result follows from Proposition 5. Note that we could have a priori ruled out the case of a tacnode since totally geodesic curves have simple crossings. □

6. A Livné-like rational fibration

In his PhD thesis [Li], R. Livné constructed two-ball quotients by taking branched coverings of some generalized universal elliptic curves with level structure and by construction, these surfaces admit a fibration onto a curve. In the case of the Cartwright-Steger surface, the Albanese fibration does not appear in the same fashion but one can exhibit another (rational) fibration appearing in a quite similar way to Livné's. Our starting point is the description by Deligne and Mostow of Livné's fibrations in [DM2, Chapter 16] from which one can deduce the following (which is only implicit in the book).

6.1. Let $\hat{R}$ be the surface obtained by blowing up the point $P_1 \in R \cong \mathbb{P}(1, 2, 3)$, see §1.3. Let $N \geq 3$ be an integer. We endow $\hat{R}$ with an orbifold structure: the ramification divisors are the strict transforms of D_A, D_B that we still denote in the same way and the exceptional curve that we denote by E with respective weights $(N, d, 2)$, and we denote this orbifold by $\hat{R}_{N,d,2}$. We also endow $\mathbb{P}^1_{\mathbb{C}}$ with an orbifold structure: there are 3 orbifold points, say p_1, p_2, p_3, with respective weights $(2, 3, N)$ and we denote this orbifold by $\mathbb{P}^1_{2,3,N}$.

Then there exists an orbifold morphism $\Phi : \hat{R}_{N,d,2} \to \mathbb{P}^1_{2,3,N}$ such that D_A is sent onto p_3, $\Phi(D_B) = \Phi(E) = \mathbb{P}^1_{\mathbb{C}}$, and the fibers of Φ above p_1 and p_2 have multiplicity 2 and 3 respectively. The generic fiber of Φ meets E once and D_B three times.

6.2. When $d = 2$, this fibration can be seen as the orbifold quotient of the universal generalized elliptic curve with structure of level N by the group $\mathrm{SL}(2, \mathbb{Z}_N) \rtimes (\mathbb{Z}_N)^2$. In this setting, it is natural to take $p_1 = 1728$, $p_2 = 0$ and $p_3 = \infty$, Φ can then be seen as the j-invariant (the fibers of Φ are rational curves which are the quotient of the corresponding elliptic curves by ± 1, the image of 0 is on E, the image of 2-torsion points on D_B), and $\mathbb{C} = \mathbb{P}^1_{\mathbb{C}} \backslash \{\infty\}$ is the set of values of the j-invariant. Above the point at infinity we have a "special curve" D_A, and the ramifications 2 and 3 at 1728 and 0 respectively are due to the fact that the corresponding elliptic curves have additional automorphisms.

The case we are interested in is $\hat{R}_{4,3,2} = \widehat{\overline{\Gamma} \backslash B^2_{\mathbb{C}}}$, i.e. we have $N = 3$ and $d = 4$ (it can be checked that indeed, E has weight -2, the minus sign meaning that it can be contracted, see [DM2, §17.9]) and $\mathbb{P}^1_{2,3,N} = \mathbb{P}^1_{2,3,3}$ is the orbifold attached to the tetrahedron group.

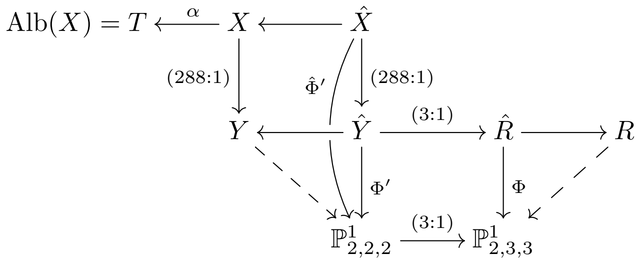
6.3. Let us consider Π_2 (see §1.5): we saw that it is a subgroup of index 3 of $\bar{\Gamma}$. We define $Y := \Pi_2 \backslash B^2_{\mathbb{C}}$ and $\hat{Y}$ its blow up at the preimage of P_1 by the natural morphism $Y \to R$ so that we have a ramified covering $\hat{Y} \to \hat{R}$ whose branch locus is D_A (of order 3). Then $\hat{Y}$ has a natural orbifold structure. The ramification divisors are the preimage D'_B of D_B with weight $d = 4$ and the preimage E' of E with weight -2. Both D'_B and E' are irreducible.

The orbifold $\mathbb{P}^1_{2,3,3}$ also admits an orbifold covering $\mathbb{P}^1_{2,2,2} \to \mathbb{P}^1_{2,3,3}$ of order 3 whose branch locus consists of the two points of weight 3 in $\mathbb{P}^1_{2,3,3}$ and where $\mathbb{P}^1_{2,2,2}$ is the quotient of $\mathbb{P}^1_{\mathbb{C}}$ by the subgroup of the tetrahedron group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The 3 orbifold points in $\mathbb{P}^1_{2,2,2}$ have weight 2 and are the points above the orbifold point of weight 2 in $\mathbb{P}^1_{2,3,3}$.

The fibration Φ then lifts to an orbifold fibration $\Phi' : \hat{Y} \to \mathbb{P}^1_{2,2,2}$ such that the divisor E' is a section of Φ' and D'_B has order 3 over the base. In other words, the generic fiber of Φ' is an orbifold $\mathbb{P}^1_{\mathbb{C}}$ with 4 orbifold points of weights $(2, 4, 4, 4)$. There is one special fiber which is the preimage D'_A of D_A and which is an orbifold of type $(2, 4, 12)$ and there are also 3 multiple fibers of order 2 (above the 3 orbifold points of $\mathbb{P}^1_{2,2,2}$).

6.4. Finally, there exists an orbifold cover $\hat{X}$ of $\hat{Y}$ of order 288 with $36 = 864/24$ exceptional curves (where 24 is the order of the isotropy group in $\bar{\Gamma}$ of a point $x \in \mathcal{P}$) and once these curves are contracted, we obtain the surface $X = \Pi \backslash B^2_{\mathbb{C}} = (\Pi_2 \cap \Pi_3) \backslash B^2_{\mathbb{C}}$.

We thus have the following diagram



We see in particular that the elliptic curve $\mathrm{Alb}(X)$ should be the rigid part of the Jacobian of the curves of the fibration $\hat{\Phi}' : \hat{X} \to \mathbb{P}^1_{2,2,2}$ which are ramified coverings of $\mathbb{P}^1_{\mathbb{C}}$ of type $(2, 4, 4, 4)$.

This point of view is also confirmed by the computation of the genus of the curves of type A. Indeed, the general fiber of $\hat{\Phi}'$ is a ramified covering of $\mathbb{P}^1_{\mathbb{C}}$ of order 288 and type $(2, 4, 4, 4)$ so that by the Riemann-Hurwitz formula its genus is

$$\frac{288}{2}\left(-2 + \frac{2-1}{2} + 3 \cdot \frac{4-1}{4}\right) + 1 = 109.$$

On the other hand, the arithmetic genus is constant on the fibration of a smooth surface onto a curve. Let us compute the arithmetic genus of the fiber of type A using the same method as in the proof of Lemma 3. Recall from §2.2 that the fiber has four irreducible components, two have genus 4 and two have genus 10, and moreover there are exactly three singular points, each of the same type, namely eight local branches crossing transversally. Then this curve has arithmetic genus

$$1 + 2(4-1) + 2(10-1) + 3\left(\frac{8 \cdot 7}{2}\right) = 109$$

which is the expected number.

7. Lefschetz type question

7.1. The goal of this section is to show that the Cartwright-Steger surface X provides examples for some natural questions related to Lefschetz properties of ample hypersurfaces in projective algebraic manifolds. In studying Lefschetz properties, Nori posted in [N] the following problem.

Question 2. *Let D be an effective divisor on a surface X with $D \cdot D > 0$. Let N be the normal subgroup of $\pi_1(X)$ generated by the images of the fundamental group of the non singular models of all the curves in D. Is $[\pi_1(X) : N]$ finite?*

The N defined above is the normal closure of the images of the fundamental group of the non singular models of all the curves in D.

Nori in [N] answered the above question affirmatively in the special case that D has only nodal singularities and satisfies the assumption that $D \cdot D > 2r(D)$, where $r(D)$ is the number of nodes. Some special cases of hyperellitpic fibrations have also been confirmed by Gurjar-Paul-Purnaprajna [GPP]. The question has attracted a lot of attention from studies of properties of fundamental groups of algebraic surfaces and function properties of their universal coverings, such as Lefschetz type properties or holomorphic convexity of the universal coverings.

7.2. We show that the Cartwright-Steger surface provides an interesting example to illustrate the problem.

Proposition 6. *Let X be the Cartwright-Steger surface. Let $D = E_1$ be the genus 4 curve of type B having multiplicities $(3, 1, 2)$ at the points O_i defined in §2.1. Let $i : D \to X$ be the inclusion map, $\rho : \widehat{D} \to D$ the normalization of D, and N the normal closure of $(i \circ \rho)_* \pi_1(\widehat{D})$ in $\pi_1(X)$. Then*

(a) $D \cdot D > 0$,
(b) $[\pi_1(X) : (i \circ \rho)_* \pi_1(\widehat{D})] = \infty$,
(c) $[\pi_1(X) : N] = 21$,
(d) $\pi_1(X) = i_* \pi_1(D)$.

Proof. (a) follows from Lemma 4 where we computed $D \cdot D = 5 > 0$.

(b). We recall results and use notation of §A.2. The curve D is irreducible by construction and the universal covering of D is a totally geodesic curve M_0 on the universal covering $B^2_{\mathbb{C}}$ of the Cartwright-Steger surface X. The stabilizer $\Pi_0 < \Pi$ of M_0 as a set in $B^2_{\mathbb{C}}$ is then a Fuchsian group of $M_0 \cong \Delta$, the unit disk. Since Π is torsion-free, so is the action of Π_0 on M_0. However, the image of $\Pi_0 \backslash M_0$ in X has self-intersection singularities on X since there are elements $g \in \Pi - \Pi_0$ such that $g \cdot M_0 \cap M_0 \neq \emptyset$.

In our situation, a smooth model of D is a normalization $\widehat{D}$ of D and is simply given by $\Pi_0 \backslash M_0$. Hence from construction, the fundamental group of a smooth model of D is Π_0. In fact, it suffices for us to know that the fundamental group is commensurable to Π_0. Clearly, the fundamental group $\pi_1(\widehat{D}) = \Pi_0$ has infinite index in Π, since the cohomology dimension of Π_0 is 2 and the corresponding one for Π is 4.

Part (c) follows from Lemma A.28.

Part (d) follows from Corollary A.3. □

Appendix A. Calculations in the group $\bar{\Gamma}$

A.1. The action of $\bar{\Gamma}$ on $B^2_{\mathbb{C}}$. The elements u and v of $\bar{\Gamma}$ are complex reflections of order 3 and 4, respectively. For $\alpha \in \mathbb{C}$, define

$$M_\alpha = \{(z, w) \in B^2_{\mathbb{C}} : z = \alpha w\}.$$

We also let $M_\infty = \{(z, w) \in B^2_{\mathbb{C}} : w = 0\}$. Setting $c = (r-1)(\zeta^3 - 1)/2 = \zeta^2 - \zeta$, one can check that u fixes each point of M_c, and v fixes each point of M_0. Let $\mathcal{M}_A = \{g(M_c) : g \in \bar{\Gamma}\}$ and $\mathcal{M}_B = \{g(M_0) : g \in \bar{\Gamma}\}$. We refer to these sets as *mirrors* of types A and B, respectively, since $g(M_c)$ and $g(M_0)$ are the sets of points of $B^2_{\mathbb{C}}$ fixed by the complex reflection gug^{-1}, and gvg^{-1}, respectively.

Note that the powers of $(\zeta^{-1} j)^\nu$, $\nu = 1, \dots, 11$, are complex reflections, but in their action on $B^2_{\mathbb{C}}$ they fix only the origin $O = (0, 0)$.

Proposition 7. *The non-trivial elements of finite order in $\bar\Gamma$ are all conjugate to one of the elements in the following table, or the inverse of one of these.*

d	*Representatives of elements of order d*
2	$v^2,\ j^6,\ (bu^{-1})^2$
3	$u,\ j^4,\ uj^4,\ buv$
4	$v,\ j^3,\ vj^3,\ v^2j^3,\ bu^{-1}$
6	$j^2,\ v^2j^2,\ v^2uj,\ v^2uj^5,\ bv^2u^{-1}j,\ bv^2$
8	$uvj,\ \zeta^{-1}bj,\ (\zeta^{-1}bj)^3$
12	$j,\ j^5,\ uv^{-1}j^2,\ uv^{-1}j^3,\ uv^{-1}j^6,\ uv^{-1}j^{-1},\ v^2j,\ uv^2,\ uj,\ uj^3,\ bv,\ (bv)^{-5}$
24	$uv,\ vuj^2$

The elements v and v^2 fix each point of M_0, while u fixes each point of M_c. The remaining elements in the table each fix just one point of $B^2_{\mathbb{C}}$.

Proof. Elements $gZ \in \bar\Gamma$ which fix points of $B^2_{\mathbb{C}}$ must have finite order, because $\bar\Gamma$ acts discontinuously on $B^2_{\mathbb{C}}$. Conversely (see [CS2, Lemma 3.3]) any element of finite order in $\bar\Gamma$ fixes at least one point of $B^2_{\mathbb{C}}$, and is conjugate to an element of $K \cup bK \cup bu^{-1}bK$. One can easily list the nontrivial elements of finite order in this last set (there are 408 of them, 76 in bK and 45 in $bu^{-1}bK$), all having order dividing 24. Routine calculations show that any such element (and hence each nontrivial element of finite order in $\bar\Gamma$) has a matrix representative g conjugate to one of the elements in the above table, or its inverse. One may verify that, with the exception of the elements conjugate to buv or its inverse, each element gZ of order d in $\bar\Gamma$ has a matrix representative g such that $g^d = I$. Note that $(buv)^3 = \zeta^{-1}I$.

To check that the elements in the table other than u, v and v^2 fix only one point of $B^2_{\mathbb{C}}$, note that $gZ \in \bar\Gamma$ fixes $(z, w) \in B^2_{\mathbb{C}}$ if and only if $(z, w, 1/\sqrt{r-1})^T$ is an eigenvector of g. In each case, we find that there is only one eigenvalue λ of g having an eigenvector $(v_1, v_2, v_3)^T$ satisfying $|v_1|^2 + |v_2|^2 < (r-1)|v_3|^2$, corresponding to a fixed point (z, w) with $z = v_1/(v_3\sqrt{r-1})$ and $w = v_2/(v_3\sqrt{r-1})$. See also the proof of Proposition 8 below. ☐

For $\alpha \in \mathbb{C} \cup \{\infty\}$ and for $\xi \in B^2_{\mathbb{C}}$, let

$$\bar\Gamma_\alpha = \{g \in \bar\Gamma : g(M_\alpha) = M_\alpha\} \quad \text{and} \quad \bar\Gamma_\xi = \{g \in \bar\Gamma : g.\xi = \xi\}$$

denote the stabilizer of M_α and ξ, respectively. We next describe the ξ for which $\bar\Gamma_\xi \neq \{1\}$. Two points are particularly important: the origin O, and

$$P = \Big(\frac{c(\zeta-1)}{\sqrt{r-1}}, \frac{\zeta-1}{\sqrt{r-1}}\Big). \tag{11}$$

As observed in [CS2, Lemma 3.1], $\bar\Gamma_O = K$. For P we have the following:

Lemma 14. *The subgroup $\bar\Gamma_P$ of $\bar\Gamma$ has order 24, and centre of order 2. It is generated by elements $f_z = (bu^{-1})^2$, $f_2 = bu^{-1}$, $f_3 = jbv^{-1}j$ and $f_3' = u$, and has a presentation*

$$f_2^2 = f_z,\ f_3^3 = 1,\ {f_3'}^3 = 1,\ f_z^2 = 1,\ f_3'f_3f_2 = 1,\ [f_2, f_z] = [f_3, f_z] = [f_3', f_z] = 1.$$

The subgroup $\bar\Gamma_P \cap \bar\Gamma_c$ equals $\langle f_z, f_3'\rangle$, and has order 6. Let $r_1 = 1$, $r_2 = f_2$, $r_3 = f_3$ and $r_4 = f_3'f_2$. Then $P \in r_\nu(M_c)$ for $\nu = 1, 2, 3, 4$, and the $r_\nu(M_c)$ are distinct.

We can find 36 elements $k_1, \dots, k_{36}$ of K such that

$$\bar\Gamma = \bigcup_{i=1}^{36} \Pi k_i \bar\Gamma_P, \quad \textit{a disjoint union.} \tag{12}$$

Proof. Suppose that $g \in \bar\Gamma$ and $g.P = P$. Then

$$d(g.O, O) \le d(g.O, g.P) + d(g.P, P) + d(P, O) = 2d(P, O) = \log\Big(\frac{1 + \|P\|}{1 - \|P\|}\Big).$$

Here $\|P\|^2 = |c|^2(2-r)|/(r-1) + (2-r)/(r-1) = 2r-3$. Since the squared Hilbert-Schmidt norm $\|g\|_{HS}^2$ of any $g \in U(2,1)$ is $3 + 4\sinh^2(d(g.O, O))$ ([CS2, Lemma 3.2]), this gives $\|g\|_{HS}^2 \leq 8r + 15$. Now the $g \in \bar\Gamma$ such that $\|g\|_{HS}^2 \leq 8r+15$ consist of the three double cosets K, KbK and $Kbu^{-1}bK$ ([CS2, §3]). So it is enough to run through the elements g of these double cosets (48672 in all) checking the condition $g.P = P$. This search found 24 elements with this property. In particular it found the elements f_z, f_2, f_3 and f_3' above. One may check that they satisfy the given relations. The abstract group generated by elements f_z, f_2, f_3 and f_3' satisfying these relations has order 24. So the stabilizer of P in $\bar\Gamma$ has order 24, and has the presentation given above.

The statements about $\bar\Gamma_P \cap \bar\Gamma_c$ and $r_1, \dots, r_4$ are easily verified.

Magma verifies that the 36 elements $k'_\nu, k'_\nu j^4, k'_\nu j^8$, for the following 12 elements k'_ν of K, is a set of representatives for the distinct double cosets $\Pi g \bar\Gamma_P$ in $\bar\Gamma$:

$$v,\ v^2,\ vuv^{-1},\ vu^{-1}v^2u,\ v^{-1},\ uv^2,\ j,\ j^2,\ 1,\ j^3,\ uv,\ u^{-1}v^{-1}. \tag{13}$$

The order of the k'_ν has be chosen to make the tables in the proof of Proposition 16 tidier. □

Routine calculations show that the fixed points of $\gamma_3 = buv$, $\gamma_8 = \zeta^{-1}bj$ and $\gamma_{12} = bv$ are

$$\xi_3 = \Big(\frac{c_1}{\sqrt{r-1}}, \frac{c_2}{\sqrt{r-1}}\Big), \quad \xi_8 = \Big(0, \frac{(1-2\sin(\pi/12))\zeta^3}{\sqrt{r-1}}\Big) \quad \text{and} \quad \xi_{12} = \Big(0, \frac{\zeta - 1}{\sqrt{r-1}}\Big), \tag{14}$$

respectively, where for $\lambda = e^{-\pi i/18}$,

$$c_1 = \zeta^3 - \zeta^2 - \zeta + 1 + (\zeta^2 - \zeta + 1)\lambda + (-\zeta^3 + \zeta^2 - 1)\lambda^2, \quad \text{and} \quad c_2 = \zeta^3 - (\zeta - 1)\lambda^2.$$

Lemma 15. *For $d = 3$, 8 and 12, the group $\bar\Gamma_{\xi_d}$ is cyclic of order d, generated by γ_d.*

Proof. By the method used in the proof of Lemma 14, we see that in each of these three cases, $\bar\Gamma_\xi \subset K \cup KbK$, and then search this set for the elements fixing ξ. □

For $\xi \in B_{\mathbb{C}}^2$, let $\mathcal{M}_A(\xi)$, respectively $\mathcal{M}_B(\xi)$ denote the set of distinct mirrors M, of type A and B, respectively, containing ξ.

Lemma 16. *The groups $\bar\Gamma_c$ and $\bar\Gamma_0$ are the commutators in $\bar\Gamma$ of u and v, respectively. For each $\xi \in B_{\mathbb{C}}^2$, $|\mathcal{M}_A(\xi)|$, respectively $|\mathcal{M}_B(\xi)|$, is equal to the number of elements of $\bar\Gamma_\xi$ conjugate to u, respectively v.*

Proof. Suppose that $g \in \bar\Gamma$ commutes with u. If $\xi \in M_c$, then $u.(g.\xi) = g.(u.\xi) = g.\xi$, so that $g.\xi$ is one of the points of $B_{\mathbb{C}}^2$ fixed by u, and so is in M_c. Thus $g \in \bar\Gamma_c$. Conversely, if $g \in \bar\Gamma_c$, then gug^{-1} fixes each point of M_c. A simple calculation shows that the $h \in \bar\Gamma$ fixing each point of M_c are just the powers of u. Considering traces and determinants, we find that u is not conjugate to its inverse. Hence $gug^{-1} = u$. The proof for v is similar.

If $\xi \in g(M_c)$, then $g^{-1}.\xi \in M_c$, and so $u.g^{-1}.\xi = g^{-1}.\xi$. Hence $gug^{-1} \in \bar\Gamma_\xi$. If $g, g' \in \bar\Gamma$ and $gug^{-1} = g'ug'^{-1} \in \bar\Gamma_\xi$, then $g^{-1}g'$ commutes with u, and so is in $\bar\Gamma_c$, so that $g(M_c) = g'(M_c)$. So $|\mathcal{M}_A(\xi)|$ is the number of distinct conjugates of u belonging to $\bar\Gamma_\xi$. The calculation of $|\mathcal{M}_B(\xi)|$ is the same. □

Lemma 17. *The orbit under the finite group K of M_c consists of the eight mirrors M_α for $\alpha = c_{\pm\pm\pm} = \pm(r \pm 1)(i \pm 1)/2$ (so that for example $c = c_{+--}$), and $\mathcal{M}_A(O)$ is the set of these M_α's. The 8 elements $k_\alpha \in K$ in the proof below form a set of representatives of the cosets gK_c in K, for $K_c = K \cap \bar\Gamma_c = \langle u, j\rangle$.*

Proof. We know that u fixes each point of M_c. Also, $j.(\alpha w, w) = (\alpha\zeta w, \zeta w)$ for any w and α, and so $j(M_\alpha) = M_\alpha$ for any α. So the 36 elements of the subgroup $K_c = \langle u, j\rangle$ of K fix the set M_c. For $\alpha = c_{+--}$, c_{--+}, c_{---} and c_{+-+}, let $k_\alpha = 1, v, v^2$ and v^3, respectively, and then $k_\alpha.(cw, w) = (\alpha w', w') \in M_\alpha$ for $w' = w$. For $\alpha = c_{-++}$, c_{-+-}, c_{+++} and c_{++-}, let $k_\alpha = u^{-1}v^2u$, $vu^{-1}v^2u$, $v^2u^{-1}v^2u$ and $v^3u^{-1}v^2u$, respectively, and then $k_\alpha.(cw, w) = (\alpha w', w') \in M_\alpha$ for $w' = -cw$. So the eight elements k_α lie in distinct cosets gK_c. To see that $\mathcal{M}_A(O)$ consists just of these M_α's, we apply Lemma 16. Let $k \in K$ be a conjugate

gug^{-1} of u for some $g \in \bar{\Gamma}$. Then k fixes each point of $g(M_c)$. But the 8 elements $k_\alpha u k_\alpha^{-1}$ are the only elements of order 3 in K fixing more than one point of $B^2_{\mathbb{C}}$. □

Lemma 18. *The orbit under the finite group K of M_0 consists of the six mirrors M_α for $\alpha \in \{0, 1, -1, i, -i, \infty\}$, and $\mathcal{M}_B(O)$ is the set of these M_α's. The 6 elements $k_\alpha \in K$ in the proof below form a set of representatives of the cosets gK_0 in K, for $K_0 = K \cap \bar{\Gamma}_0 = \langle v, j\rangle$.*

Proof. We know that v fixes each point of M_0 and that $j(M_0) = M_0$. So the 48 elements of the subgroup $K_0 = \langle v, j\rangle$ of K fix the set M_0. For $k_0 = 1$ and $k_\infty = u^{-1}v^2uj^6$, we have $k_0.(0, w) = (0, w) \in M_0$ and $k_\infty.(0, w) = (w, 0) \in M_\infty$. For $\alpha = i, -1, -i, 1$, let $k_\alpha = uj$, vuj, v^2uj and v^3uj, respectively, and then $k_\alpha.(0, w) = (\alpha w', w') \in M_\alpha$ for $w' = (i + 1)w/2$. The last statement is now clear. The proof that $\mathcal{M}_B(O)$ consists of these M_α's is similar to that of the corresponding statement in Lemma 17. □

By a *generic* element of M_c, respectively, M_0, we mean a point $\xi \in M_c$, respectively M_0, which is not in the $\bar{\Gamma}$-orbit of O, P, ξ_8 or ξ_{12}. We shall see that no point in the $\bar{\Gamma}$-orbit of ξ_3 belongs to M_c or M_0.

Proposition 8. *The $\xi \in B^2_{\mathbb{C}}$ for which $\bar{\Gamma}_\xi \neq \{1\}$ are either in the $\bar{\Gamma}$-orbit of a generic point of M_c or M_0, or in the $\bar{\Gamma}$-orbit of one of O, P, ξ_3, ξ_8 or ξ_{12}. With notation as above, we record the following data for these points:*

ξ	$\bar{\Gamma}_\xi$	$\lvert\bar{\Gamma}_\xi\rvert$	$\lvert\mathcal{M}_A(\xi)\rvert$	$\lvert\mathcal{M}_B(\xi)\rvert$
O	K	288	8	6
P	$\langle f_z, f_3, f_3'\rangle$	24	4	0
ξ_3	$\langle\gamma_3\rangle$	3	0	0
ξ_8	$\langle\gamma_8\rangle$	8	0	1
ξ_{12}	$\langle\gamma_{12}\rangle$	12	1	1
generic M_c	$\langle u\rangle$	3	1	0
generic M_0	$\langle v\rangle$	4	0	1

Proof. By assumption, there is a non-trivial element of $\bar{\Gamma}$ fixing ξ, and this element must be of finite order, and so is conjugate to one of the elements in the table of Proposition 7. So we may assume that ξ is fixed by one of the elements in that table. If ξ is fixed by one of the elements in the table belonging to K, other than u, v and v^2, then $\xi = O$. There are 9 elements in the table which do not belong to K. By Lemma 15, if ξ is fixed by buv, then $\xi = \xi_3$. If ξ is fixed by $\zeta^{-1}bj$ or $(\zeta^{-1}bj)^3$, then $\xi = \xi_8$. If ξ is fixed by bu^{-1} or $(bu^{-1})^2$, then $\xi = P$, by Lemma 14. If ξ is fixed by $bv^2u^{-1}j$, then it is fixed by $(bv^2u^{-1}j)^3 = v^{-1}f_zv$, where f_z is as in Lemma 14, and so ξ is in the K-orbit of P. Since b and v commute, $bv^2 = (bv)^{-2}$, and so the points fixed by $(bv)^{-5}$, bv and bv^2 are all the same, and equal to ξ_{12}. If ξ is fixed by one of the elements u, v and v^2, but is not fixed by any other element in the table, then ξ is a generic point of either M_c or M_0.

We have already seen in Lemmas 17 and 18 that $|\mathcal{M}_A(O)| = 8$ and $|\mathcal{M}_B(O)| = 6$.

We calculated $\bar{\Gamma}_P$ in Lemma 14. It is easy to verify that it contains eight elements of order 3, namely $r_\nu u^{\pm 1} r_\nu^{-1}$, $\nu = 1, \ldots, 4$, for r_ν as in Lemma 14. So $|\mathcal{M}_A(P)| = 4$. Also, $\bar{\Gamma}_P$ contains six elements of order 4, but all are conjugate to bu^{-1} or its inverse. So $\bar{\Gamma}_B$ contains no conjugates of v, so that $|\mathcal{M}_B(P)| = 0$.

Since $\gamma_3 = buv$ is not conjugate to $u^{\pm 1}$, $\bar{\Gamma}_{\xi_3} = \langle\gamma_3\rangle$ contains no conjugates of u, and clearly none of v, and so $|\mathcal{M}_A(\xi_3)| = |\mathcal{M}_B(\xi_3)| = 0$.

Now $\langle\zeta^{-1}bj\rangle$ contains two elements of order 4, namely $(\zeta^{-1}bj)^2 = \zeta^{-3}v$ and its inverse. As v is not conjugate to v^{-1}, we see that $\langle\zeta^{-1}bj\rangle$ contains just one conjugate of v, so that $|\mathcal{M}_B(\xi_8)| = 1$. Since $\bar{\Gamma}_{\xi_8}$ contains no elements of order 3, we have $|\mathcal{M}_A(\xi_8)| = 0$.

The elements of order 3 and 4 in $\langle bv\rangle$ are $(bv)^{\pm 4} = b^{\pm 1}$ and $(bv)^{\pm 3} = v^{\mp 1}$, respectively. Using $b = (ub)u(ub)^{-1}$, we see that $\langle bv\rangle$ contains just one conjugate of each of u and v. So $|\mathcal{M}_A(\xi_{12})| = |\mathcal{M}_B(\xi_{12})| = 1$. □

We next want to describe the groups $\bar{\Gamma}_c$ and $\bar{\Gamma}_0$ of elements fixing M_c and M_0, respectively.

Lemma 19. *For any $\alpha \in \mathbb{C}$, a 3×3 matrix $g = (g_{ij})$ with complex entries which is unitary with respect to F satisfies $g(M_\alpha) = M_\alpha$ if and only if*

(a) $g_{13} = \alpha g_{23}$, *and*
(b) $g_{12} = \alpha(\alpha g_{21} - g_{11} + g_{22})$.

Proof. This is straightforward. □

Lemma 20. *If $gZ \in \bar{\Gamma}$, then $gZ \in \bar{\Gamma}_0$ if and only if we can write*

$$g = \theta' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & (r-1)b \\ 0 & \bar{b} & \bar{a} \end{pmatrix} \tag{15}$$

where $a, b \in \mathbb{Z}[\zeta]$, $\theta, \theta' \in \{\zeta^k : k = 0, \ldots, 11\}$, $|a|^2 - (r-1)|b|^2 = 1$, and $a - 1 \in (r-1)\mathbb{Z}[\zeta]$. This expression for g is unique, with $\theta' = g_{11}$ and ${\theta'}^3\theta = \det(g)$.

Proof. Suppose that $gZ \in \bar{\Gamma}_0$. Applying Lemma 19 for $\alpha = 0$ to g and to

$$g^{-1} = F^{-1}g^*F = \begin{pmatrix} \bar{g}_{11} & \bar{g}_{21} & -(r-1)\bar{g}_{31} \\ \bar{g}_{12} & \bar{g}_{22} & -(r-1)\bar{g}_{32} \\ -\bar{g}_{13}/(r-1) & -\bar{g}_{23}/(r-1) & \bar{g}_{33} \end{pmatrix}, \tag{16}$$

we see that $g_{12} = g_{13} = g_{21} = g_{31} = 0$. The condition that $\gamma_0^{-1}g\gamma_0$ has entries in $\mathbb{Z}[\zeta]$ tells us that g_{11}, g_{22}, g_{33}, g_{32}, $(g_{11} - g_{22})/(r-1)$ and $g_{23}/(r-1)$ are in $\mathbb{Z}[\zeta]$.

Now $g^*Fg = F$ implies that $|g_{11}|^2 = 1$. This and $g_{11} \in \mathbb{Z}[\zeta]$ implies that g_{11} is a power of ζ, and so replacing g by $g_{11}^{-1}g$, we may suppose that $g_{11} = 1$. Also, $a = g_{22}$ and $b = g_{23}/(r-1)$ are in $\mathbb{Z}[\zeta]$. Now $\det(g) = \det(\gamma_0^{-1}g\gamma_0) \in \mathbb{Z}[\zeta]$, and $g^*Fg = F$ implies that $|\det(g)| = 1$. So $\theta = \det(g)$ is also a power of ζ. Using the fact that $F^{-1}g^*F$ equals $\theta^{-1}\mathrm{Adj}(g)$, we see that $g_{33} = \bar{a}\det(g)$ and $g_{32} = \bar{b}\det(g)$, and then that $|a|^2 - (r-1)|b|^2 = 1$. Finally, it is easy to check that $\gamma_0^{-1}g\gamma_0$ has entries in $\mathbb{Z}[\zeta]$ if and only if $a - 1 \in (r-1)\mathbb{Z}[\zeta]$. □

Let U_0 denote the group of matrices with entries in $\mathbb{Z}[\zeta]$ which are unitary with respect to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1-r \end{pmatrix}.$$

If SU_0 is the subgroup of U_0 consisting of its elements of determinant 1, then U_0 is the semidirect product of SU_0 and the group of order 12 generated by the matrix $z = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$. We define an embedding of $\bar{\Gamma}_0$ into U_0 as follows. If $gZ \in \bar{\Gamma}_0$, write g as in (15), and set

$$\psi_0(gZ) = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} a & (r-1)b \\ \bar{b} & \bar{a} \end{pmatrix}. \tag{17}$$

Lemma 21. *The group SU_0 is generated by*

$$d = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} \zeta^3 + \zeta^2 - 1 & -\zeta^2 - \zeta + 1 \\ \zeta^3 + \zeta^2 - 1 & -\zeta^3 - \zeta^2 \end{pmatrix},$$

and has the following presentation with respect to these generators:

$$\langle d, x \mid d^{12} = x^3 = 1, (dx^2)^3 = d^6, d^6x = xd^6\rangle.$$

We get a presentation of U_0 by adding the generator z and the relations $z^{12} = 1$, $zdz^{-1} = d$ and $zxz^{-1} = d^6x^{-1}d$. The subgroup $H_0 = \langle xd, dx, d^3\rangle$ of SU_0 has index 3.

Proof. Given a field $\mathbb{F}$ not of characteristic 2, and $\alpha,\beta\in\mathbb{F}^\times$, the *quaternion algebra* $(\alpha,\beta)_{\mathbb{F}}$ consists of elements $\xi = x_0+x_1i+x_2j+x_3k$, where $x_0,\dots,x_3\in\mathbb{F}$, with an associative multiplication satisfying $ij=k=-ji$ and $i^2=\alpha$, $j^2=\beta$. The *reduced norm* $N(\xi)=N_A(\xi)$ of ξ is $x_0^2-\alpha x_1^2-\beta x_2^2+\alpha\beta x_3^2$. If $\xi,\xi'\in A$, then $N(\xi\xi')=N(\xi)N(\xi')$. Writing $a=x_0+x_1i$ and $b=x_2+x_3i$, we can think of $(\alpha,\beta)_{\mathbb{F}}$ as consisting of elements $a+bj$, where $a,b\in\mathbb{F}(\sqrt{\alpha})$, $j^2=\beta$, and $ja=\bar{a}j$ for the automorphism $^- : x_0+x_1i\mapsto x_0-x_1i$ of $\mathbb{F}(\sqrt{\alpha})$. The classical Hamiltonian quaternion algebra is $\mathbb{H}=(-1,-1)_{\mathbb{R}}$. We have $N_{\mathbb{H}}(a+bj)=|a|^2+|b|^2$ for $a,b\in\mathbb{R}(\sqrt{-1}\,)=\mathbb{C}$.

Let $A=(-1,r-1)_{\mathbb{Q}(r)}$. Identifying $i\in A$ with $\zeta^3\in\ell=\mathbb{Q}(\zeta)$, we see that $A=\{a+bj : a,b\in\ell\}$, and that $N(a+bj)=|a|^2-(r-1)|b|^2$. Let $\mathcal{O}=\{a+bj\in A : a,b\in\mathbb{Z}[\zeta]\}$. Then $\mathcal{O}$ is a subring of A, closed under (left) multiplication by $\mathbb{Z}[r]$, and so is an *order* in A. In fact, it is a *maximal* order. Clearly SU_0 is isomorphic to the group $\mathcal{O}^1$ of elements of $\mathcal{O}$ having reduced norm 1. The group $\mathbb{Z}[r]^\times$ of units in $\mathbb{Z}[r]$ consists of the elements $m+nr$, where $m,n\in\mathbb{Z}$ and $m^2-3n^2=\pm1$. Now $m^2-3n^2=-1$ never holds, and $m^2-3n^2=1$ if and only if $m+nr=(2+r)^k$ for some $k\in\mathbb{Z}$ (see [NZM, §7.8], for example). So $\mathbb{Z}[r]^\times$ is generated by -1 and $2+r$. If $\xi\in\mathcal{O}^\times$, then $N(\xi)\in\mathbb{Z}[r]^\times$. In fact, $N(\xi)$ is never equal to -1, for if $\epsilon:\mathbb{Q}(r)\to\mathbb{R}$ is the field embedding mapping r to $-\sqrt{3}$, then

$$f : x_0+x_1i+x_2j+x_3k\mapsto \epsilon(x_0)+\epsilon(x_1)\,i+\epsilon(x_2)\sqrt{\sqrt{3}+1}\;j+\epsilon(x_3)\sqrt{\sqrt{3}+1}\;k$$

is an embedding of A into $\mathbb{H}$ satisfying $\epsilon(N_A(\xi))=N_{\mathbb{H}}(f(\xi))$. Now $\mathcal{O}^1\subset\mathcal{O}^\times$. Since $2+r=N(\zeta+1)$, if $\xi\in\mathcal{O}^\times$ and $N(\xi)=(2+r)^k$, then $\xi/(\zeta+1)^k\in\mathcal{O}^1$. Since $(\zeta+1)^2=\zeta(2+r)$, we see that $\mathcal{O}^1/\{1,-1\}$ embeds as an index 2 subgroup of $\mathcal{O}^\times/\mathbb{Z}[r]^\times$. Magma has routines for finding a presentation of $\mathcal{O}^\times/\mathbb{Z}[r]^\times$. As these may be less familiar to the reader, we give some details. We set up $\mathbb{Q}(\zeta)$, $\mathbb{Q}(r)$ and A with the commands

```
L<z> := CyclotomicField(12);
K<r> := sub< L | z + 1/z>;
A<i,j,k> := QuaternionAlgebra< K | -1, r - 1>;
```

As $\zeta=(r+\zeta^3)/2$, we set `zz := (r + i)/2;` and `O := Order([1,zz,j,zz*j]);`. Now the commands `G := FuchsianGroup(O);` and `u,m := Group(G);` and `u;` give a presentation for $\mathcal{O}^\times/\mathbb{Z}[r]^\times$. The command `[A!Quaternion(m(U.i)) : i in [1..2]];` makes the generators u_1, u_2 explicit. We find that $u_1=(2+r-i)/2$ and $u_2=-(r+2)(i+k)$. These satisfy $u_1^{12}=u_2^2=(u_1u_2)^3=1 \pmod{\mathbb{Z}[r]^\times}$. Note that $N(u_1)=N(u_2)=2+r$. Magma verifies that the subgroup of the abstract group $\langle u_1,u_2\mid u_1^{12}=u_2^2=(u_1u_2)^3=1\rangle$ has a single index 2 subgroup, and it is generated by $g_1=u_2u_1^{-1}$ and $g_2=u_1^2$, and the relations $g_1^3=(g_1g_2)^3=g_2^6=1$ give a presentation. For the given concrete $u_1,u_2\in A$, we set $g_1=u_2u_1^{-1}$ and $g_2=(2-r)u_1^2$. Then $g_1,g_2\in\mathcal{O}^1$ generate $\mathcal{O}^1/\{-1,1\}$ and satisfy $g_1^3=1$, $(g_1g_2)^3=-1=g_2^6$. The given elements d and x are just g_2^{-1} and $g_2^{-3}g_1g_2^{-3}$. So they and the given relations form a presentation of SU_0.

The remaining assertions are routine to verify. $\square$

Lemma 22. *The image under ψ_0 of $\bar\Gamma_0$ is $\langle z\rangle H_0$.*

Proof. The elements xd, dx, d^3 and z are all in $\psi_0(\bar\Gamma_0)$, being respectively the images of the elements gZ of $\bar\Gamma_0$ for the following g's:

$$\zeta^{-4}j^{-3}bj^7,\quad \zeta^{-4}j^{-1}bj^5,\quad \zeta^{-3}v^{-1}j^6,\quad\text{and } \zeta j^{-1}.$$

Now $d\notin\psi_0(\bar\Gamma_0)$ since $\zeta-1\notin(r-1)\mathbb{Z}[\zeta]$. So we have $\langle z\rangle H_0\subset\psi_0(\bar\Gamma_0)\subsetneqq \mathrm{U}_0$. Since H_0 has index 3 in SU_0 we have $\langle z\rangle H_0=\psi_0(\bar\Gamma_0)$. $\square$

We now describe $\bar\Gamma_c$. Recall that $c=(r-1)(\zeta^3-1)/2=\zeta^2-\zeta$.

Lemma 23. *If $gZ\in\bar\Gamma$, then $gZ\in\bar\Gamma_c$ if and only if we can write*

$$g=\theta'\begin{pmatrix}1&0&0\\0&1&0\\0&0&\theta\end{pmatrix}\begin{pmatrix}(a(2-r)+1)/(3-r) & c(a-1)/(3-r) & bc\\ (a-1)\bar{c}/(3-r) & (a+2-r)/(3-r) & b\\ \bar{b}\bar{c}/(r-1) & \bar{b}/(r-1) & \bar{a}\end{pmatrix}\tag{18}$$

where $a, b \in \mathbb{Z}[\zeta]$, $\theta, \theta' \in \{\zeta^k : k = 0, \dots, 11\}$, $|a|^2 - r|b|^2 = 1$, and $a - 1 \in (\zeta^4 - 1)\mathbb{Z}[\zeta]$. This expression for g is unique, with $\theta' = g_{11} - cg_{21}$ and ${\theta'}^3\theta = \det(g)$.

Proof. Suppose that $gZ \in \bar{\Gamma}_0$. Applying Lemma 19 for $\alpha = c$ to g and to g^{-1}, we have $g_{13} = cg_{23}$, $g_{12} = c(cg_{21} - g_{11} + g_{22})$, $\overline{g_{31}} = c\overline{g_{32}}$, and $\overline{g_{21}} = c(c\overline{g_{12}} - \overline{g_{11}} + \overline{g_{22}})$. From the second and fourth of these equations, we find that $\bar{c}g_{12} = cg_{21}$.

Using Lemma 19 again, we see that the map $g \mapsto g_{11} - cg_{21}$ is multiplicative on the group of matrices satisfying $g(M_c) = M_c$. So we get $1 = \big(g_{11} - cg_{21}\big)\big(\overline{g_{11}} - c\overline{g_{12}}\big) = |g_{11} - cg_{21}|^2$ by applying this to g and g^{-1}, and so $\theta' = g_{11} - cg_{21}$ has modulus 1. The condition that $\gamma_0^{-1} g\gamma_0$ has entries in $\mathbb{Z}[\zeta]$ implies in particular that $g_{11}, g_{21} \in \frac{1}{r-1}\mathbb{Z}[\zeta]$, so that $\theta' \in \frac{1}{r-1}\mathbb{Z}[\zeta]$. This and $|\theta'| = 1$ imply that $\theta' \in \{\zeta^k : k = 0, \dots, 11\}$. So replacing g by ${\theta'}^{-1}g$, if necessary, we may suppose that $g_{11} - cg_{21} = 1$. We can now express g_{11}, g_{12} and g_{21} in terms of g_{22}. Now let $N = F^{-1}g^*F - \mathrm{Adj}(g)/\theta = (n_{ij})$, where $\theta = \det(g)$. By (16), this is zero. We solve $cn_{11} + n_{12} = 0$ for g_{22}, obtaining $g_{22} = (|c|^2 + \theta\overline{g_{33}})/(|c|^2 + 1)$. Now solving $n_{31} = 0$, we get $g_{32} = \overline{g_{23}}/((r-1)\bar{\theta}) = \theta\overline{g_{23}}/(r-1)$, using $|\theta| = 1$. Write $a = \theta\overline{g_{33}}$ and $b = g_{23}$. Then (18) holds. There is just one remaining condition on a and b to ensure that $N = 0$, namely $|a|^2 - r|b|^2 = 1$. This equation is also the condition that the determinant of the last matrix on the right in (18) is 1. So taking determinants, we see that $\det(g) = {\theta'}^3\theta$. As in Lemma 20, $\det(g) \in \mathbb{Z}[\zeta]$, and so $\theta \in \{\zeta^k : k = 0, \dots, 11\}$ too. Finally, by considering $g - I$, it is routine to check that $\gamma_0^{-1} g\gamma_0$ has entries in $\mathbb{Z}[\zeta]$ if and only if $a - 1 \in (\zeta^4 - 1)\mathbb{Z}[\zeta]$. □

Let U_c be the group of matrices with entries in $\mathbb{Z}[\zeta]$ which are unitary with respect to

$$\begin{pmatrix} 1 & 0 \\ 0 & -r \end{pmatrix}.$$

If SU_c is the subgroup of U_c consisting of its elements of determinant 1, then U_c is the semidirect product of SU_c and the group of order 12 generated by the above matrix z. We define an embedding of $\bar{\Gamma}_c$ into U_c as follows. If $gZ \in \bar{\Gamma}_c$, write g as in (18), and set

$$\psi_c(gZ) = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} a & rb \\ \bar{b} & \bar{a} \end{pmatrix}. \tag{19}$$

Then ψ_c is an injective homomorphism $\bar{\Gamma}_c \to \mathrm{U}_c$.

Lemma 24. *The group SU_c has generators*

$$d = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},\ q = \begin{pmatrix} r+1 & r/c \\ 1/\bar{c} & r+1 \end{pmatrix},\ \textit{and } s = \begin{pmatrix} \zeta^3(r+1) & r/c \\ 1/\bar{c} & \zeta^{-3}(r+1) \end{pmatrix},$$

and has the following presentation with respect to these generators:

$$\mathrm{SU}_c = \langle d, q, s \mid d^{12} = 1, s^2 = (qd^3)^2 = (qd^2sd^2)^2 = d^6 \rangle.$$

A presentation for U_c is obtained to adding to the above presentation of SU_c the generator z and the relations $z^{12} = 1$, $zd = dz$, $zsz^{-1} = dqd^2$ and $zqz^{-1} = d^{-2}sd^{-1}$. The subgroup $H_c = \langle sd, ds, q \rangle$ has index 4 in SU_c.

Proof. The proof is similar to that of Lemma 21. We use the quaternion algebra $A = (-1, r)_{\mathbb{Q}(r)}$ and the maximal order $\mathcal{O} = \{a + bj : a, b \in \mathbb{Z}[\zeta]\}$. Since $N(a + bj) = |a|^2 - r|b|^2$, we have $\mathrm{SU}_c \cong \mathcal{O}^1$. Again $\mathcal{O}^1/\{-1, 1\}$ embeds as a subgroup of index 2 in $\mathcal{O}^\times/\mathbb{Z}[r]^\times$ (we exclude $N(\xi) = -1$ in the same way, with $3^{1/4}$ in place of $\sqrt{\sqrt{3} + 1}$ in the definition of the embedding $A \to \mathbb{H}$). This time we get a presentation for $\mathcal{O}^\times/\mathbb{Z}[r]^\times$ with generators $u_1 = (r+2-i)/2$ and $u_2 = (r+1-(3r+5)i-2(r+2)k)/2$ satisfying $u_1^{12} = u_2^4 = (u_1u_2)^2 = 1$ (mod $\mathbb{Z}[r]^\times$). The elements $g_1 = u_2u_1^{-1}$, $g_2 = u_1^2$ and $g_3 = u_1u_2$ generate one of the three index 2 subgroups of the abstract group $\langle u_1, u_2 \mid u_1^{12} = u_2^4 = (u_1u_2)^2 = 1 \rangle$, and this subgroup has presentation $(g_1g_2)^2 = (g_1g_3)^2 = g_3^2 = g_2^6 = 1$. For the given concrete $u_1, u_2 \in A$, we set $g_1 = u_2u_1^{-1}$ and $g_2 = (2-r)u_1^2$ and $g_3 = (2-r)u_1u_2$. Then $g_1, g_2, g_3 \in \mathcal{O}^1$ generate

$\mathcal{O}^1/\{-1,1\}$ and satisfy $(g_1g_2)^2 = (g_1g_3)^2 = g_3^2 = g_2^6 = -1$. We have $g_1 = -ds$, $g_2 = d^{-1}$ and $g_3 = -d^2qd$. The result follows. □

Lemma 25. *The image of* $\bar\Gamma_c$ *in* U_c *is* $\langle z\rangle H_c$.

Proof. Now $H_c \subset \psi_c(\bar\Gamma_c)$, since for the following elements g of $\bar\Gamma$:

$$\zeta^{-4}j^7bu^{-1}buj^7, \quad \zeta^{-4}j^{-3}bu^{-1}buj^5, \quad j^4bu^{-1}bu^{-1}j^2,$$

we have $\det(g) = 1$, $g_{11} - cg_{21} = 1$, $g_{13} = cg_{23}$ and $g_{12} = c(cg_{21} - g_{11} + g_{22})$, while $\psi_c(gZ)$ equals sd, ds and q, respectively. Also, $z = \psi_c(gZ)$ for $g = \zeta j^{-1}$. Hence $\langle z\rangle H_c \subset \psi_c(\bar\Gamma_c)$. Now $d, d^2, d^3 \notin \psi_c(\bar\Gamma_c)$, since $\zeta^i - 1 \notin (\zeta^4 - 1)\mathbb{Z}[\zeta]$ for $i = 1, 2, 3$, and so the index of $\psi_c(\bar\Gamma_c)$ in U_c is at least 4. Since $[\mathrm{SU}_c : H_c] = 4$, we must have $\psi_c(\bar\Gamma_c) = \langle z\rangle H_c$. □

The subgroup Π of $\bar\Gamma$ is torsion-free, and so the set $X = \Pi\backslash B^2_{\mathbb{C}}$ is a smooth compact complex surface. Let $\varphi : B^2_{\mathbb{C}} \to X$ be the natural map. If M is a mirror of type A or B, let $\bar\Gamma_M$ denote the stabilizer of M (so $\bar\Gamma_\alpha = \bar\Gamma_{M_\alpha}$) The group $\Pi_M = \{\pi \in \Pi : \pi(M) = M\} = \Pi \cap \bar\Gamma_M$ acts on M, and is the fundamental group of $\mathcal{C}^1_M := \Pi_M\backslash M$. We denote by φ_M the map $\Pi_M\xi \mapsto \Pi\xi$ from $\mathcal{C}^1_M$ to X, and write Π_α and φ_α instead of Π_{M_α} and φ_{M_α}, respectively.

A.2. The groups Π_M when M is a mirror of type B. As at the end of the last section, $\Pi_0 = \Pi_{M_0} = \{\pi \in \Pi : \pi(M_0) = M_0\} = \Pi \cap \bar\Gamma_0$.

Proposition 9. *The group* Π_0 *has a presentation*

$$\langle u_1, \dots, u_4, v_1, \dots, v_4 \ : \ [u_1, v_1][u_2, v_2][u_3, v_3][u_4, v_4] = 1\rangle, \tag{20}$$

with explicit generators u_i, v_i, *given below, and so* $\Pi_0\backslash M_0$ *is a curve of genus 4. The image under* ψ_0 *of* Π_0 *is a normal subgroup of* SU_0 *which is an index 24 subgroup of* $H_0 = \langle xd, dx, d^3\rangle$.

Proof. Using the fact that j^4 normalizes Π, we can define $g_1, \dots, g_8 \in \Pi$ by setting $g_1 = \zeta^5a_3^{-3}a_1^{-1}a_2a_1$, $g_3 = \zeta^{-4}a_2a_1^{-2}a_3^{-3}a_1^{-1}$, $g_5 = \zeta^3j^4a_2a_1j^8a_2^{-1}a_3^3a_1^2$, and $g_7 = j^4a_1^{-1}a_2^{-1}j^4a_2a_1j^4$, and then $g_{2\nu} = j^4g_{2\nu-1}j^{-4}$ for $\nu = 1, 2, 3, 4$. With the given scalar factors, each g_j has determinant 1 and $(1,1)$-entry 1. They satisfy the relation:

$$g_1g_2g_3g_4g_5g_6g_7g_8g_1^{-1}g_3^{-1}g_5^{-1}g_7^{-1}g_2^{-1}g_4^{-1}g_6^{-1}g_8^{-1} = 1. \tag{21}$$

The g_j were found by a search amongst the short words in the generators of Π. We show that $g_1, \dots, g_8$ generate Π_0. Each g_j has determinant 1, and has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}[\zeta]$. Hence $G = \langle g_1, \dots, g_8\rangle$ is contained in Π_0, and ψ_0 embeds G in SU_0. With $h_1 = xd$, $h_2 = dx$ and $h_3 = d^3$ the generators of H_0, we find that

$$\begin{aligned}
&\psi_0(g_1) = h_3h_2h_3^{-1}h_1^{-1}, && \psi_0(g_4) = h_3^{-1}h_2h_1h_2, && \psi_0(g_7) = h_3h_2^{-1}h_3^{-1}h_1,\\
&\psi_0(g_2) = h_1h_2h_1h_3^{-1}, && \psi_0(g_5) = h_1^{-1}h_2^{-1}h_3h_2^{-1}, && \psi_0(g_8) = h_1^{-1}h_3h_1^{-1}h_2^{-1}.\\
&\psi_0(g_3) = h_2h_1^{-2}h_2^{-2}h_3^{-1}, && \psi_0(g_6) = h_2h_3h_1^{-1}h_3^{-1},
\end{aligned}$$

Magma tells us that $\psi_0(G)$ is normal in SU_0 and has index 24 in H_0, which has index 12 in $\langle z\rangle H_0 = \psi_0(\bar\Gamma_0)$. So G has index 288 in $\bar\Gamma_0$. The group $K_0 = K \cap \bar\Gamma_0 = \langle u, j\rangle$ has order 48, and acts freely on any transversal of G in $\bar\Gamma_0$, since G is torsion-free. So we can find $6 = 288/48$ elements $t_1, \dots, t_6$ in $\bar\Gamma_0$ so that

$$\bar\Gamma_0 = \bigcup_{i=1}^{6} Gt_iK_0. \tag{22}$$

For example, if τ_1, τ_2 and τ_3 are the elements of $\bar\Gamma_0$ given in the proof of Lemma 22 satisfying $\psi_0(\tau_i) = h_i$ for $i = 1, 2, 3$, then we can take $t_1, \ldots, t_6$ to be

$$1,\ \tau_1,\ j\tau_1,\ j^2\tau_1,\ j^3\tau_1,\ \text{and}\ \tau_1\tau_2^{-1}.$$

If G were strictly contained in Π_0, then there would be a transversal element $t_ik \neq 1$, where $i \in \{1, \ldots, 6\}$ and $k \in K_0$, such that $t_ik \in \Pi_0$. But Magma verifies that if $t_ik \neq 1$, then $\langle a_1, a_2, a_3, t_ik\rangle$ has index less than 864 in $\bar\Gamma$, so that $t_ik \notin \Pi$. So Π_0 is generated by $g_1, \ldots, g_8$, and Magma's `Rewrite` command shows that these generators and the single relation (21) form a presentation of Π_0.

We now replace $g_1, \ldots, g_8$ by generators $u_1, \ldots, v_4$ satisfying (20). Our thanks go to Jonathan Hillman for showing us this method. The word W on the left in (21) is a product of 16 letters $g_i^{\pm1}$, with exactly one of each letter. Moreover, W has the form $AdeBd^{-1}Ce^{-1}$ where d and e are letters, and A, B and C are words not involving the letters d, d^{-1}, e, e^{-1}. Notice that

$$AdeBd^{-1}Ce^{-1} = [D_1, E_1] \cdot E_1ACBE_1^{-1} \quad \text{for } D_1 = Ad \text{ and } E_1 = eB. \tag{23}$$

The word $W' = ACB$, which is a product of 12 letters $g_i^{\pm1}$, $i = 1, \ldots, 6$, with exactly one of each letter, again has the form $A'd'e'B'd'^{-1}C'e'^{-1}$, and so we can repeat this manoevre, obtaining

$$A'd'e'B'd'^{-1}C'e'^{-1} = [D_2, E_2] \cdot E_2A'C'B'E_2^{-1} \quad \text{for } D_2 = A'd' \text{ and } E_2 = e'B'.$$

Once again, $W'' = A'C'B'$ has the form of the word on the left in (23), and we can repeat the manoevre, and then once more. In this way we obtain words $D_1, \ldots, D_4$ and $E_1, \ldots, E_4$ so that $W = [u_1, v_1][u_2, v_2][u_3, v_3][u_4, v_4]$ for

$$\begin{aligned} u_1 &= D_1, & v_1 &= E_1,\\ u_2 &= E_1D_2E_1^{-1}, & v_2 &= E_1E_2E_1^{-1},\\ u_3 &= E_1E_2D_3E_2^{-1}E_1^{-1}, \quad \text{and} & v_3 &= E_1E_2E_3E_2^{-1}E_1^{-1},\\ u_4 &= E_1E_2E_3D_4E_3^{-1}E_2^{-1}E_1^{-1}, & v_4 &= E_1E_2E_3E_4E_3^{-1}E_2^{-1}E_1^{-1}. \end{aligned} \tag{24}$$

The words D_i and E_i are easily read off from the original word W. Explicitly:

$$\begin{aligned} D_1 &= g_1g_2g_3g_4g_5g_6g_7, & E_1 &= g_8g_1^{-1}g_3^{-1}g_5^{-1},\\ D_2 &= g_1g_2g_3g_4, & E_2 &= g_5g_6g_2^{-1},\\ D_3 &= g_1, \quad \text{and} & E_3 &= g_2g_3g_6^{-1},\\ D_4 &= g_3^{-1}, & E_4 &= g_6. \end{aligned} \tag{25}$$

This procedure can easily be reversed, by first expressing the g_i's in terms of the D_j's and E_j's, and then these in terms of $u_1, v_1, \ldots, u_4, v_4$. We give the results of these calculations explicitly:

$$\begin{aligned} g_1 &= v_1^{-1}v_2^{-1}u_3v_2v_1, & g_5 &= v_1^{-1}v_4u_4v_4^{-1}v_3v_2v_1,\\ g_2 &= v_1^{-1}v_2^{-1}v_4u_4v_3v_2v_1, & g_6 &= v_1^{-1}v_2^{-1}v_3^{-1}v_4v_3v_2v_1,\\ g_3 &= v_1^{-1}v_2^{-1}v_3^{-1}u_4^{-1}v_3v_2v_1, & g_7 &= v_1^{-1}v_2^{-1}v_3^{-1}u_4^{-1}v_4^{-1}u_2^{-1}v_1u_1,\\ g_4 &= v_1^{-1}v_2^{-1}v_3^{-1}v_4^{-1}u_3^{-1}v_2u_2v_1, & g_8 &= v_4u_4v_4^{-1}u_4^{-1}v_3u_3v_2v_1. \end{aligned}$$

Hence Π_0 has the presentation (20) for the u_i's and v_i's given in (24). □

We now consider Π_M for the other mirrors M of type B.

Proposition 10. *If $g \in \bar\Gamma$ and $M = g(M_0)$ is a mirror of type B, then*

(a) *There is a $\pi \in \Pi$ such that $\pi(M) = M_0$, M_1 or M_∞.*
(b) *Correspondingly, Π_M is conjugate in Π to either Π_0, Π_1 or Π_∞.*
(c) $\Pi_M = g\Pi_0g^{-1}$.

(d) $h(\Pi_M)h^{-1} = \Pi_{h(M)}$ *for any* $h \in \bar\Gamma$.

In particular, it follows from (c) that for any mirror M *of type* B, $\Pi_M\backslash M \cong \Pi_0\backslash M_0$

Proof. (a) Since the elements $b^\mu k$, $\mu = 0, 1, -1$ and $k \in K$, form a set of coset representatives of Π in $\bar\Gamma$, and since, by Lemma 18, the k_α, $\alpha \in \{0, 1, -1, i, -i, \infty\}$, form a set of coset representatives of $K_0 = K \cap \bar\Gamma_0$ in K, we may assume that $M = b^\mu(M_\alpha)$ for some $\mu \in \{0, 1, -1\}$ and $\alpha \in \{0, 1, -1, i, -i, \infty\}$. For the cases with $\mu = 0$, we have

$$a_1^{-1}a_2^{-1}(M_{-1}) = a_1^{-1}a_2^{-1}a_1(M_i) = M_0, \quad \text{and} \quad a_2^{-1}(M_{-i}) = M_\infty.$$

Here is a table of elements $\pi \in \Pi$ such that $\pi(bM_\alpha) = M_\beta \in \{M_0, M_1, M_\infty\}$:

α	0	1	-1	i	$-i$	∞
π	1	a_2^{-2}	$a_3^3a_1^2a_2^{-1}$	$a_1^{-1}a_3^{-3}$	a_2^{-1}	$a_2^{-1}a_1a_2^{-1}$
β	0	1	1	1	1	∞

Here is a table of elements $\pi \in \Pi$ such that $\pi(b^{-1}M_\alpha) = M_\beta \in \{M_0, M_1, M_\infty\}$:

α	0	1	-1	i	$-i$	∞
π	1	$a_3^{-1}a_1^{-1}a_2^{-1}$	$a_3^{-1}a_1^{-1}a_2^{-2}$	a_3^{-2}	a_2^{-2}	$a_1^{-2}a_3^{-3}$
β	0	∞	∞	1	∞	0

This proves (a), and (b) follows immediately, since if $M' = \pi(M)$ with $\pi \in \Pi$, we have $\Pi_{M'} = \pi\Pi_M\pi^{-1}$.

(c) We first show that $h\Pi_0h^{-1} \subset \Pi$ for each $h \in \bar\Gamma$. We may assume that $h = b^\mu k$ for some $\mu \in \{0, 1, -1\}$ and some $k \in K$, and for such h, we must check that $hg_jh^{-1} \in \Pi$ for each of the 8 generators g_j of Π_0 given in the proof of Proposition 9. We do this as usual by having Magma check that $\langle a_1, a_2, a_3, hg_jh^{-1}\rangle$ has index 864 in $\bar\Gamma$. It follows, in particular, that $h\Pi_0h^{-1} = \Pi_0$ for each $h \in \bar\Gamma_0$.

We next prove (c) in the cases $g = k_\beta$, $\beta = 1, \infty$. Now $g\Pi_0g^{-1} \subset \Pi$ and so $k_\beta\Pi_0k_\beta^{-1} \subset \Pi_\beta$ for both $\beta = 1, \infty$. To see that $k_\beta\Pi_0k_\beta^{-1} = \Pi_\beta$, note that by Proposition 9, $\Pi_0 \subset k_\beta^{-1}\Pi_\beta k_\beta \subset \bar\Gamma_0$. We saw in the proof of Proposition 9 that the elements t_ik, $i = 1, \dots, 6$, $k \in K_0$, form a transversal of Π_0 in $\bar\Gamma_0$. We show that $\Pi_0 = k_\beta^{-1}\Pi_\beta k_\beta$ by checking that $t_ik \notin k_\beta^{-1}\Pi_\beta k_\beta$ unless $t_ik = 1$, and this is done by Magma checking that the index in $\bar\Gamma$ of $\langle a_1, a_2, a_3, k_\beta(t_ik)k_\beta^{-1}\rangle$ is less than 864.

Now we know that $k_\beta\Pi_0k_\beta^{-1} = \Pi_\beta$ for $\beta = 0, 1, \infty$, we use (a) to see that for our given g, there is a $\pi \in \Pi$ so that $g(M_0) = \pi(M_\beta)$ for one of these β's. Then $h = k_\beta^{-1}\pi^{-1}g$ is in $\bar\Gamma_0$, so that $h\Pi_0h^{-1} = \Pi_0$. Then $(\pi^{-1}g)\Pi_0(\pi^{-1}g)^{-1} = \Pi_\beta$ by the case $g = k_\beta$ of (c) we have already proved. Finally $g\Pi_0g^{-1} = \pi(\Pi_{M_\beta})\pi^{-1} = \Pi_{\pi(M_\beta)} = \Pi_M$.

Part (d) follows immediately from (c). □

It is a consequence of Proposition 11 below that the three possibilities in (a) are mutually exclusive.

For any mirror M, the embedding $M \hookrightarrow B^2_{\mathbb{C}}$ induces an immersion $\varphi_M : \Pi_M\backslash M \to X$. Whenever M is of type B, it follows from Proposition 10(c) and (a) that $\Pi_M\backslash M \cong \Pi_0\backslash M_0$, and that the image of φ_M is equal to the image of either φ_{M_0}, φ_{M_1} or φ_{M_∞}.

We want to find out how the curves $\varphi_M(\Pi_M\backslash M) = \varphi(M)$ self-intersect.

Lemma 26. *Suppose that* $x \in X$ *is the image under* φ_M *of two or more distinct elements of* $\Pi_M\backslash M$. *If* M *is of type* B, *then* x *must be one of the three points* $\Pi(O)$, $\Pi(b.O)$ *and* $\Pi(b^{-1}.O)$. *If* M *is of type* A, *then* x *is either one of these three points or one of the 36 points* $\Pi(k_i.P)$, *where the* k_i *are as in* (12). *If* $\xi \in M$, *then* $\varphi_M(\Pi_M\xi)$ *is one of the three* *points* $\Pi(b^\mu.O)$, $\mu = 0,1,-1$, *if and only if* ξ *is in the* $\bar\Gamma$*-orbit of* O*, and it is one of the 36 points* $\Pi(k_i.P)$ *if and only if* ξ *is in the* $\bar\Gamma$*-orbit of* P.

Proof. Suppose that $\xi,\xi' \in M$ and that $\varphi_M(\Pi_M\xi) = \varphi_M(\Pi_M\xi') = x$, with $\Pi_M\xi \neq \Pi_M\xi'$. Then $\xi' = \pi\xi$ for some $\pi \in \Pi$. If $M = g(M_0)$ is of type B, then both $\xi, \pi\xi \in M$ are fixed by gvg^{-1}, so that both gvg^{-1} and $\pi^{-1}gvg^{-1}\pi$ are in $\bar\Gamma_\xi$, and so either $\pi^{-1}gvg^{-1}\pi = gvg^{-1}$ or $|\mathcal{M}_B(\xi)| \geq 2$, by Lemma 16. Now $\pi^{-1}gvg^{-1}\pi = gvg^{-1}$ means that $g^{-1}\pi g$ commutes with v, and so is in $\bar\Gamma_0$ by the same lemma, and so $\pi \in g\bar\Gamma_0 g^{-1} = \bar\Gamma_M$. But then $\pi \in \Pi \cap \bar\Gamma_M$, so that $\Pi_M\xi' = \Pi_M\xi$, contrary to hypothesis. Hence $|\mathcal{M}_B(\xi)| \geq 2$, and so ξ is in the $\bar\Gamma$-orbit of O, by Proposition 8. Since the elements $b^\mu k$, $\mu = 0,1,-1$ and $k \in K$, form a set of coset representatives of Π in $\bar\Gamma$, we can write $\xi = \pi b^\mu.O$ for some $\pi \in \Pi$ and $\mu \in \{0,1,-1\}$. So $x = \varphi_M(\Pi_M\xi) = \Pi\xi = \Pi(b^\mu.O)$.

If $M = g(M_c)$ is of type A, and $\xi,\xi' \in M$ satisfy $\varphi_M(\Pi_M\xi) = \varphi_M(\Pi_M\xi') = x$, with $\Pi_M\xi \neq \Pi_M\xi'$, we similarly show that $|\mathcal{M}_A(\xi)| \geq 2$, but now Proposition 8 shows that ξ is in the $\bar\Gamma$-orbit of either O or P. The last statement, in the case when ξ is in the $\bar\Gamma$-orbit of P, follows from (12). □

It is a consequence of Proposition 16 below that when M is of type A, the 12 points $\Pi(k_i.P)$ for $i = 25,\dots,36$, are each the image under φ_M of just one element of $\Pi_M\backslash M$.

Lemma 27. *For each mirror* M *of type* B*, there are exactly six distinct* $\Pi_M\xi \in \Pi_M\backslash M$ *such that* $\xi \in M$ *is in the* $\bar\Gamma$*-orbit of* O.

Proof. Write $M = g(M_0)$. If $\xi \in M_0$ is in the $\bar\Gamma$-orbit of O, then $g.\xi \in M$ is in the $\bar\Gamma$-orbit of O, and conversely. Also, if $\xi,\xi' \in M_0$, then $\Pi_0\xi = \Pi_0\xi'$ if and only if $\Pi_M(g.\xi) = \Pi_M(g.\xi')$, by Proposition 10(c). So we may assume that $M = M_0$. So suppose that $\xi \in M_0$ is in the $\bar\Gamma$-orbit of O. Writing $\xi = g.O$, we have $O \in g^{-1}(M_0)$. By Lemma 18, the distinct mirrors of type B containing O are the $k_\beta(M_0)$, $\beta \in \{0,1,-1,i,-i,\infty\}$. So $g^{-1}(M_0) = k(M_0)$, for some $k \in K$. Hence $\xi = g.O = (gk).O = h.O$ for some $h \in \bar\Gamma_0$. Since G in (22) equals Π_0, (22) implies that $\xi = \pi_0 t_i.O$ for some $\pi_0 \in \Pi_0$ and some $i \in \{1,\dots,6\}$. So $\Pi_0\xi$ is one of the six elements $\Pi_0(t_i.O)$, $i = 1,\dots,6$, and these are evidently distinct. So there are exactly 6 distinct $\Pi_0\xi$'s in $\Pi_0\backslash M_0$ with $\xi \in M_0$ in the $\bar\Gamma$-orbit of O. □

For any mirror M, and any $\mu \in \{0,1,-1\}$, let

$$n_\mu(M) = \sharp\{\Pi_M\xi \in \Pi_M\backslash M : \varphi_M(\Pi_M\xi) = \Pi(b^\mu.O)\}.$$

By the last lemma, $n_0(M) + n_1(M) + n_{-1}(M) = 6$ if M is of type B.

Proposition 11. *If* M *is a mirror of type* B*, then according to the three possibilities in Proposition 10(a),* $(n_0(M), n_1(M), n_{-1}(M))$ *is either* $(3,1,2)$*,* $(1,4,1)$ *or* $(2,1,3)$*, respectively.*

Proof. For any mirror M, $n_\mu(M)$ equals

$$\sharp\{\Pi_M\pi \in \Pi_M\backslash\Pi : \pi(b^\mu.O) \in M\}, \tag{26}$$

for if $\xi \in M$, $\varphi_M(\Pi_M\xi) = \Pi(b^\mu.O)$ if and only if there is a $\pi \in \Pi$ such that $\pi b^\mu.O = \xi$. If $\pi b^\mu.O = \xi$ and $\pi' b^\mu.O = \xi'$, with $\pi,\pi' \in \Pi$ and $\xi,\xi' \in M$, then $\Pi_M\xi = \Pi_M\xi'$ if and only if $\pi' b^\mu.O = \pi_M\pi b^\mu.O$ for some $\pi_M \in \Pi_M$, or equivalently, $(\pi_M\pi b^\mu)^{-1}(\pi' b^\mu) \in K$. Since Π is torsion-free, this holds if and only if $\pi' = \pi_M\pi$. So $\Pi_M\xi = \Pi_M\xi'$ if and only if $\Pi_M\pi = \Pi_M\pi'$.

If $M' = \pi(M)$ for some $\pi \in \Pi$, then clearly $n_\mu(M') = n_\mu(M)$ for $\mu = 0,1,-1$, so we need only calculate $n_\mu(M_\alpha)$ for $\alpha = 0,1,\infty$. A search amongst the short words in the generators a_1, a_2 and a_3 of Π, looking for $\pi \in \Pi$ such that $\pi.(b^\mu.O) \in M_\alpha$, found the elements in the following table:

Π_α coset representatives of $\pi \in \Pi$ such that $\pi.(b^\mu.O) \in M_\alpha$			
α	$\mu = 0$	$\mu = 1$	$\mu = -1$
0	$1,\ a_1^{-1}a_2^{-1},\ a_1^{-1}a_2^{-1}a_1$	1	$1,\ a_1^{-2}a_3^{-3}$
1	1	$a_2^{-1},\ a_2^{-2},\ a_1^{-1}a_3^{-3},\ a_3^3a_1^2a_2^{-1}$	a_3^{-2}
∞	$1,\ a_2^{-1}$	$a_2^{-1}a_1a_2^{-1}$	$a_2^{-2},\ a_3^{-1}a_1^{-1}a_2^{-1},\ a_3^{-1}a_1^{-1}a_2^{-2}$

It is easy to check that distinct elements π_1, π_2 in the same cell of this table satisfy $\pi_2\pi_1^{-1} \notin \bar\Gamma_\alpha$, and so belong to different Π_α-cosets. Since there are six elements given in each row of the table, it follows from Lemma 26 that the table gives a complete list of coset representatives. $\square$

Corollary 3. *The subgroup of* Π *generated by* $\{\pi \in \Pi : \pi.O \in M_0\}$ *equals* Π.

Proof. Denote the subgroup by S. From the $\alpha = 0$, $\mu = 0$ cell in the table in the proof of Proposition 11, we see that $a_1, a_2 \in S$. As $j^4a_2j^{-4} = \zeta^{-1}a_3^{-1}$, and S is closed under conjugation by j^4, we have $a_3 \in S$ too. So $S = \Pi$. $\square$

The fact that $n_0(M_0)$, $n_0(M_1)$ and $n_0(M_\infty)$ are different shows that if $\alpha, \beta \in \{0, 1, \infty\}$ are distinct, then there is no $\pi \in \Pi$ such that $\pi(M_\alpha) = M_\beta$. Equivalently, it shows that the images of φ_{M_0}, φ_{M_1} and φ_{M_∞} are distinct. So the cases in Proposition 10(a) are mutually exclusive.

Lemma 28. *The normal closure* N_0 *of* Π_0 *in* Π *has index 21 in* Π *and is normal in* $\bar\Gamma$. *For any mirror* M *of type B, the normal closure* N_M *of* Π_M *in* Π *is equal to* N_0.

Proof. Let $g_1, \ldots, g_8$ be the eight generators of Π_0 used in the proof of Proposition 9. Then N_0 contains, as well as the g_ν's, all conjugates of the g_ν's by elements of Π. Magma verifies that the g_ν's and their conjugates $a_ig_\nu a_i^{-1}$ and $a_i^{-1}g_\nu a_i$, $i = 1, 2, 3$, generate a normal subgroup N of $\bar\Gamma$ of index 21×864, and so $N_0 = N$.

To prove the second statement, by Proposition 10(a) and (b), it is enough to check this when $M = M_\alpha$, $\alpha = 1, \infty$. Now $\Pi_\alpha = k_\alpha\Pi_0k_\alpha^{-1} \subset k_\alpha N_0k_\alpha^{-1} = N_0$ because N_0 is normal in $\bar\Gamma$. Because $N_0 \subset \Pi$, we have $N_\alpha \subset N_0$. Magma verifies that the 24 elements $k_\alpha g_\nu k_\alpha^{-1}$ and $a_2^{\pm1}k_\alpha g_\nu k_\alpha^{-1}a_2^{\mp1}$, $\nu = 1, \ldots, 8$, generate a subgroup in $\bar\Gamma$ of index 21×864. This subgroup is contained in N_α, and so $N_\alpha = N_0$. $\square$

We conclude this section with some calculations involving the abelianization map $f : \Pi \to \Pi/[\Pi, \Pi] \cong \mathbb{Z}^2$ (see just after Theorem 1), which are needed in Section 2.4.

Proposition 12. *The images under the abelianization map* f *of the generators* u_i *and* v_i *of* Π_0 *are as follows:*

$$f(u_1) = (-5, -2),\quad f(u_2) = (-2, 1),\quad f(u_3) = (1, 4),\quad f(u_4) = (2, 5),$$
$$f(v_1) = (-2, 7),\quad f(v_2) = (0, 0),\quad f(v_3) = (3, -6),\quad f(v_4) = (-1, -4).$$

Presentations (20) *of the groups* Π_1 *and* Π_∞, *and calculations of the corresponding* $f(u_i)$ *and* $f(v_i)$ *are given below. The image under* f *of* Π_M *for any mirror of type B is equal to* $\{(m, n) \in \mathbb{Z}^2\ :\ m - n$ *is divisible by* $3\}$.

Proof. In the notation of the proof of Proposition 9, $f(u_i) = f(D_i)$ and $f(v_i) = f(E_i)$ for $i = 1, \ldots, 4$, and so it is routine to calculate these from the given expressions (25) for D_i and E_i, and from this we read off $f(\Pi_0)$.

Next we consider Π_1 and Π_∞. If $g_1, \ldots, g_8$ are the generators of Π_0 given in the proof of Proposition 9, then Π_1 and Π_∞ have generators $g_i' = k_1g_ik_1^{-1}$ and $g_i'' = k_\infty g_ik_\infty^{-1}$, respectively, which satisfy the same relation (21) as do the g_i's. So we get generators u_i and v_i for these groups by using (24), and by using (25) with the g_i's replaced by g_i''s and g_i'''s, respectively. To calculate the $f(u_i)$ and $f(v_i)$'s, we need to express the g_i''s and g_i'''s in terms of the generators of Π. One may verify that:

$$g_1' = \zeta^2 a_2^{-3} a_3^3 a_1 a_2 a_1 a_3 a_1 a_2, \qquad g_5' = \zeta^5 j^8 (a_1^{-1} a_3^{-1} a_1 a_2^2 a_1^{-1} a_2^{-1} a_1 a_3 a_3) j^4,$$
$$g_3' = \zeta^3 j^4 (a_2^{-2} a_1^{-1} a_3^{-1} a_1 a_2^2 a_1^{-1} a_2^{-1} a_1) j^8, \qquad g_7' = \zeta^{-5} j^4 (a_1^{-2} a_3^{-3} a_1^{-1} a_3^{-1}) j^8,$$

and

$$g_1'' = \zeta^{-4} j^4 (a_1^{-1} a_3^{-2} a_1^{-1}) j^8 a_1^{-1} a_2^{-1}, \quad g_5'' = \zeta^{-1} j^8 (a_2^{-1} a_3^{-1}) j^4,$$
$$g_3'' = \zeta^{-2} j^8 (a_3 a_1 a_2 a_1^{-1} a_2^{-1}) j^4, \qquad g_7'' = \zeta^{-2} j^4 (a_1 a_3 a_1^{-1} a_3^{-2}) j^8,$$

and $g_{2\nu}' = j^4 g_{2\nu-1}' j^{-4}$ and $g_{2\nu}'' = j^4 g_{2\nu-1}'' j^{-4}$ for $\nu = 1,2,3,4$. So in the case Π_1 we get

$$f(u_1) = (-3,0), \quad f(u_2) = (2,-1), \quad f(u_3) = (1,4), \quad f(u_4) = (0,3),$$
$$f(v_1) = (0,3), \quad f(v_2) = (-4,2), \quad f(v_3) = (7,-8), \quad f(v_4) = (-3,0),$$

while in the case Π_∞, we get

$$f(u_1) = (-1,2), \quad f(u_2) = (-2,1), \quad f(u_3) = (-3,0), \quad f(u_4) = (-2,1),$$
$$f(v_1) = (2,-1), \quad f(v_2) = (0,0), \quad f(v_3) = (-1,2), \quad f(v_4) = (3,0).$$

For any mirror M of type B, the image under f of Π_M is the image of the normal closure of Π_M, and so is the same as that of Π_0. □

A.3. The groups Π_M when M is a mirror of type A. Recall that $c = (r-1)(\zeta^3-1)/2$, and that by Lemma 25 we have an injective homomorphism $\psi_c : \bar{\Gamma}_c \to \mathrm{U}_c$ with image $\langle z \rangle H_c$.

Proposition 13. *The group Π_c has a presentation*

$$\langle u_1, \dots, u_{10}, v_1, \dots, v_{10} \ : \ [u_1,v_1][u_2,v_2]\cdots[u_9,v_9][u_{10},v_{10}] = 1 \rangle, \tag{27}$$

with explicit generators u_i, v_i, given below, and so $\Pi_c \backslash M_c$ is a curve of genus 10. The image under ψ_c of Π_c is a normal subgroup of SU_c which is an index 27 subgroup of $H_c = \langle sd, ds, q \rangle$.

Proof. The proof is very similar to that of Proposition 9. We recall that j^4 normalizes Π, and define 20 elements $g_1, \dots, g_{20}$ of Π by setting

$$g_1 = j^8 a_1^{-1} a_2 a_1 a_3 a_1^{-1} j^4 a_2 a_1, \qquad g_{12} = (-1) a_2^{-1} a_1 a_3 a_1^{-1} a_3^{-1} j^4 a_3 a_1 a_2^2 a_1^{-1} a_2^{-1} j^8,$$
$$g_3 = \zeta j^4 a_2 a_1 a_2^{-2} a_1^{-1} a_3 j^4 a_3^3 j^4, \qquad g_{15} = \zeta j^4 a_1 j^4 a_2 a_3 a_1^{-1} j^4,$$
$$g_5 = \zeta^4 j^8 a_1^{-1} j^4 a_2 a_1 j^4 a_3 a_2^{-1} a_1 a_3 a_1^{-1} j^8, \quad g_{17} = \zeta^{-2} j^8 a_1^{-2} a_2^{-1} j^4 a_3 a_1 a_2 a_1,$$
$$g_7 = \zeta^{-5} j^8 a_2 a_1 j^4 a_3^{-1} j^4 a_2 a_1^{-1} a_2^{-1} a_3^{-3} j^8, \quad g_{19} = \zeta^{-1} a_2^{-1} a_1 a_3 a_1^{-1} a_3^{-2} j^4 a_1 a_2 j^4 a_1^{-1} a_2^{-1} j^4,$$
$$g_9 = \zeta^4 j^8 a_1^{-1} a_2^{-2} a_1^{-1} a_3^{-1} j^8 a_1^{-1} a_2^{-1} j^8,$$

and also $g_{\nu+1} = j^4 g_\nu j^{-4}$ for $\nu \in \{1,3,5,7,9,10,12,13,15,17,19\}$. Each $h = g_j$ satisfies $h_{13} = ch_{23}$ and $h_{12} = c(ch_{21} - h_{11} + h_{22})$, and so is in Π_c, by Lemma 19. With the given scalar factors, each has determinant 1 and satisfies $h_{11} - ch_{21} = 1$ (cf. Lemma 23). The g_j's satisfy

$$\begin{aligned} &g_4 g_{14}^{-1} g_2^{-1} g_{17}^{-1} g_9 g_{19} g_{20} g_{14} g_7^{-1} g_{10}^{-1} g_5^{-1} g_{16}^{-1} g_3^{-1} g_{12}^{-1} g_1 g_2 g_{18}^{-1} g_{10} g_{19}^{-1} g_{12} \\ &\times g_8^{-1} g_{11}^{-1} g_6^{-1} g_{15} g_{16} g_4^{-1} g_{13}^{-1} g_1^{-1} g_{17} g_{18} g_{11} g_{20}^{-1} g_{13} g_7 g_8 g_9^{-1} g_5 g_6 g_{15}^{-1} g_3 = 1. \end{aligned} \tag{28}$$

The elements g_j were found by a search for elements of Π_c amongst the short words in the generators of Π. The conjugates by j^4 and by j^8 of the elements found were added to the output, and then products of pairs of all these elements were formed, retaining those of small Hilbert-Schmidt norm (cf. [CS2, Lemma 3.2]).

Hence $G = \langle g_1, \dots, g_{20} \rangle$ is contained in Π_c, and ψ_c embeds G in SU_c. With $h_1 = sd$, $h_2 = ds$ and $h_3 = q$ the generators of H_c, we find that

$$
\begin{aligned}
\psi_c(g_1) &= h_1^{-1}h_2^{-1}h_3^{-1}h_2h_1h_3, & \psi_c(g_{11}) &= h_1^{-1}h_2^{-1}h_1^{-3}h_2h_1,\\
\psi_c(g_2) &= h_3^{-1}h_1h_3h_2^2h_1, & \psi_c(g_{12}) &= h_2^{-1}h_3^{-1}h_2^{-1}h_3^{-1}h_1^{-1}h_3^{-1}h_1^{-1},\\
\psi_c(g_3) &= h_1h_3h_2h_3^{-1}h_2h_1, & \psi_c(g_{13}) &= h_3^{-1}h_2h_1h_3^{-1}h_2h_1h_3^{-1}h_2h_1,\\
\psi_c(g_4) &= h_2h_1^2h_3h_2h_3^{-1}, & \psi_c(g_{14}) &= h_2h_1h_3^{-1}h_2h_1h_3^{-1}h_2h_1h_3^{-1},\\
\psi_c(g_5) &= h_1^{-1}h_2h_1h_2^{-1}, & \psi_c(g_{15}) &= h_1h_3h_2h_1^{-1}h_2h_1h_2^{-1}h_3^{-1}h_2h_1,\\
\psi_c(g_6) &= h_2^2h_1^2h_2h_1, & \psi_c(g_{16}) &= h_1^{-1}h_2^{-1}h_3h_2^2h_1h_2^{-1}h_3^{-1},\\
\psi_c(g_7) &= h_2h_1^2h_2^2h_1, & \psi_c(g_{17}) &= h_1^{-2}h_2^{-1}h_3^{-1}h_1h_3,\\
\psi_c(g_8) &= h_1^{-1}h_2^{-1}h_1h_2, & \psi_c(g_{18}) &= h_2h_1^{-1}h_2^{-1}h_3^{-1}h_1h_3h_1^{-1}h_2^{-1},\\
\psi_c(g_9) &= h_1^{-3}, & \psi_c(g_{19}) &= h_2^{-1}h_3^{-1}h_2^2h_1h_3,\\
\psi_c(g_{10}) &= h_2h_1^{-3}h_2^{-1}, & \psi_c(g_{20}) &= h_2h_1h_2^{-1}h_3^{-1}h_1h_3h_2h_1^{-1}h_2h_1.
\end{aligned}
$$

Magma tells us that $\psi_c(G)$ is normal in SU_c and has index 27 in H_c, which has index 12 in $\langle z \rangle H_c = \psi_c(\bar{\Gamma}_c)$. So G has index 324 in $\bar{\Gamma}_c$. The group $K_c = K \cap \bar{\Gamma}_c = \langle v, j \rangle$ has order 36, and acts freely on any transversal of G in $\bar{\Gamma}_c$, since G is torsion-free. So we can find $9 = 324/36$ elements $t_1, \dots, t_9$ in $\bar{\Gamma}_c$ so that

$$\bar{\Gamma}_c = \bigcup_{i=1}^{9} G t_i K_c. \tag{29}$$

For example, if τ_1, τ_2 and τ_3 are the elements of $\bar{\Gamma}_c$ given in the proof of Lemma 25 satisfying $\psi_c(\tau_i) = h_i$ for $i = 1, 2, 3$, then we can take $t_1, \dots, t_9$ to be

$$1,\ \tau_1,\ j\tau_1,\ j^2\tau_1,\ j^3\tau_1,\ \tau_1\tau_3,\ j\tau_1\tau_3,\ j^2\tau_1\tau_3,\ \text{and } j^3\tau_1\tau_3.$$

If G were strictly contained in Π_c, then there would be a transversal element $t_ik \neq 1$, where $i \in \{1, \dots, 9\}$ and $k \in K_c$, such that $t_ik \in \Pi_c$. But Magma verifies that if $t_ik \neq 1$, then $\langle a_1, a_2, a_3, t_ik \rangle$ has index less than 864 in $\bar{\Gamma}$, so that $t_ik \notin \Pi$. So Π_c is generated by $g_1, \dots, g_{20}$, and Magma's `Rewrite` command shows that these generators and the single relation (28) form a presentation of Π_c.

We now replace this presentation by a presentation (27). The method used in the proof of Proposition 9 extends to this case, and we can write the word on the left in (28) as a product $[u_1, v_1][u_2, v_2] \cdots [u_9, v_9][u_{10}, v_{10}]$, where for each i, we have $u_i = E_1 \cdots E_{i-1} D_i E_{i-1}^{-1} \cdots E_1^{-1}$ and $v_i = E_1 \cdots E_{i-1} E_i E_{i-1}^{-1} \cdots E_1^{-1}$, where

$$
\begin{aligned}
D_1 &= g_4g_{14}^{-1}g_2^{-1}g_{17}^{-1}g_9g_{19}g_{20}g_{14}g_7^{-1}g_{10}^{-1}g_5^{-1}g_{16}^{-1},\\
D_2 &= g_4g_{14}^{-1}g_2^{-1}g_{17}^{-1}g_9g_{19}g_{20}g_{14}g_7^{-1}g_{10}^{-1}g_5^{-1}g_4^{-1}g_{13}^{-1}g_1^{-1}g_{17}g_{18}g_{11}g_{20}^{-1}g_{13}g_7g_8g_9^{-1}g_5g_6,\\
D_3 &= g_4g_{14}^{-1}g_2^{-1}g_{17}^{-1}g_9g_{19}g_{20}g_{14}g_7^{-1}g_{10}^{-1}g_5^{-1}g_4^{-1}g_{13}^{-1}g_1^{-1}g_{17}g_{18},\\
D_5 &= g_4g_7^{-1}g_{10}^{-1}g_5^{-1}g_4^{-1}g_{13}^{-1}g_1^{-1}g_{17}g_{10}g_{19}^{-1}g_{12}g_8^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
D_4 &= g_4g_{14}^{-1}, & D_7 &= g_{13}^{-1}g_1^{-1}g_{17}g_{10}g_{19}^{-1}g_{12}g_9^{-1}, & D_9 &= g_{13}^{-1}g_{10}g_{19}^{-1},\\
D_6 &= g_4, & D_8 &= g_{13}^{-1}g_1^{-1}, & D_{10} &= g_{13}^{-1},
\end{aligned}
$$

and also

$$\begin{aligned}
E_1 &= g_3^{-1}g_{12}^{-1}g_1g_2g_{18}^{-1}g_{10}g_{19}^{-1}g_{12}g_8^{-1}g_{11}^{-1}g_6^{-1}g_{15}, & E_6 &= g_7^{-1}g_{10}^{-1}g_5^{-1},\\
E_2 &= g_{15}^{-1}g_{12}^{-1}g_1g_2g_{18}^{-1}g_{10}g_{19}^{-1}g_{12}g_8^{-1}g_{11}^{-1}, & E_7 &= g_5g_{12}^{-1}g_1g_{17}^{-1},\\
E_3 &= g_{11}g_{20}^{-1}g_{13}g_7g_8g_9^{-1}g_5g_{12}^{-1}g_1g_2, & E_8 &= g_{17}g_{10}g_{19}^{-1}g_{12}g_{19}g_{13}g_{10}^{-1}g_{12}^{-1},\\
E_4 &= g_2^{-1}g_{17}^{-1}g_9g_{19}g_{20}, & E_9 &= g_{12},\\
E_5 &= g_{20}^{-1}g_{13}g_7, & E_{10} &= g_{10}.
\end{aligned}$$

The generators $g_1, \dots, g_{20}$ can be expressed in terms of $u_1, v_1, \dots, u_{10}, v_{10}$ by first expressing them in terms of the D_i's and E_i's, as in the proof of Proposition 9. Hence Π_c has the presentation (27) for the given u_i's and v_i's. □

We now consider Π_M for the other mirrors M of type A. As well as $c = (r-1)(z^3-1)/2 = c_{+--}$, the parameter $-c = c_{---}$ is important in the next result.

Proposition 14. *If $g \in \bar{\Gamma}$ and $M = g(M_c)$ is a mirror of type A, then*

(a) *There is a $\pi \in \Pi$ such that $\pi(M) = M'$, where $M' \in \{M_c, M_{-c}, b(M_c), b^{-1}(M_c)\}$.*
(b) *If M' is as in (a), then Π_M is conjugate in Π to $\Pi_{M'}$.*
(c) *$\Pi_M = g\Pi_c g^{-1}$ in the first two cases of (a), and in particular if $g = k_\alpha$ for any $\alpha \in \{c_{+++}, \dots, c_{---}\}$, so that $\Pi_\alpha = k_\alpha \Pi_c k_\alpha^{-1}$ for all these α's.*
(d) *In the other two cases of (a), $g\Pi_c g^{-1}$ has index 3 in Π_M.*

Proof. (a) Since the elements $b^\mu k$, $\mu = 0, 1, -1$ and $k \in K$, form a set of coset representatives of Π in $\bar{\Gamma}$, and since the k_α, $\alpha \in \{c_{+++}, \dots, c_{---}\}$, form a set of coset representatives of $K_c = K \cap \bar{\Gamma}_c$ in K, by Lemma 17, we may assume that $M = b^\mu(M_\alpha)$ for some $\mu \in \{0, 1, -1\}$ and $\alpha \in \{c_{+++}, \dots, c_{---}\}$.

The next three tables list elements $\pi \in \Pi$ and $M' \in \{M_c, M_{-c}, b(M_c), b^{-1}(M_c)\}$ such that $\pi(b^\mu(M_\alpha)) = M'$ for each of these α's, and for $\mu = 0$, 1 and -1, respectively.

α	c_{+++}	c_{++-}	c_{+-+}	c_{+--}	c_{-++}	c_{-+-}	c_{--+}	c_{---}
π	a_2^2	$a_1a_3^{-1}$	$a_3^{-1}a_1a_2^2$	1	$a_1a_3^{-1}a_1a_3$	$a_2^2a_1^{-1}a_3^{-1}$	$a_1^{-1}a_2^{-2}a_1^{-1}$	1
M'	M_{-c}	M_c	M_{-c}	M_c	M_c	M_{-c}	M_c	M_{-c}

α	c_{+++}	c_{++-}	c_{+-+}	c_{+--}	c_{-++}	c_{-+-}	c_{--+}	c_{---}
π	a_3^{-3}	π^*	$a_2a_1^{-1}a_2^{-3}$	1	$a_1^{-1}a_2^{-1}a_1a_3a_2^{-1}$	$a_3a_1a_2^{-1}$	$a_2a_1^{-1}a_2^{-1}$	$a_1^{-1}a_2^{-2}$
M'	M_c	M_{-c}	M_{-c}	$b(M_c)$	M_c	$b^{-1}(M_c)$	M_{-c}	M_c

α	c_{+++}	c_{++-}	c_{+-+}	c_{+--}	c_{-++}	c_{-+-}	c_{--+}	c_{---}
π	$a_2a_1^{-2}a_3^{-3}$	$a_1^{-2}a_3^{-3}a_2^{-1}$	$a_2^2a_1a_3$	1	$a_3^3a_1a_2^{-1}$	$\pi^\dagger$	$a_1^{-1}a_2^{-3}$	$a_1^{-1}a_2^{-1}$
M'	M_{-c}	M_{-c}	$b^{-1}(M_c)$	$b^{-1}(M_c)$	$b(M_c)$	$b(M_c)$	M_c	M_c

where $\pi^* = a_1^{-1}a_2^{-1}a_3^{-2}a_1^2a_2^{-1}$ and $\pi^\dagger = a_2a_1^{-2}a_3^{-1}a_1a_3^{-1}a_1^{-1}a_2^{-2}$. This proves (a), and (b) follows immediately, since $\Pi_{\pi(M)} = \pi\Pi_M\pi^{-1}$ for any $\pi \in \Pi$.

(c) We first show that $h\Pi_c h^{-1} \subset \Pi$ for each $h \in \bar{\Gamma}$. We may assume that $h = b^\mu k$ for some $\mu \in \{0, 1, -1\}$ and some $k \in K$, and for such h, we must check that $hg_jh^{-1} \in \Pi$ for each of the 20 generators g_j of Π_c given in Proposition 13. We do this as usual by having Magma check that $\langle a_1, a_2, a_3, hg_jh^{-1}\rangle$ has index 864 in $\bar{\Gamma}$. It follows, in particular, that $h\Pi_c h^{-1} = \Pi_c$ for each $h \in \bar{\Gamma}_c$.

We next prove (c) in the case $g = k_{-c}$, and (d) in the cases $g = b$ and $g = b^{-1}$. Now $g\Pi_c g^{-1} \subset \Pi$ and so $g\Pi_c g^{-1} \subset \Pi_M$ for $M = g(M_c)$. So

$$\Pi_c \subset g^{-1}\Pi_M g \subset \bar{\Gamma}_c.$$

We saw in the proof of Proposition 13 that the elements t_ik, $i = 1,\dots,9$, $k \in K_c$, form a transversal of Π_c in $\bar\Gamma_c$. Now $t_ik \in g^{-1}\Pi_M g$ if and only if $gt_ikg^{-1} \in \Pi$, and so if and only if the index in $\bar\Gamma$ of $\langle a_1, a_2, a_3, gt_ikg^{-1}\rangle$ equals 864. We find that if $g = k_{-c}$, then $t_ik \in g^{-1}\Pi_M g$ only if $t_ik = 1$. It follows that $\Pi_c = g^{-1}\Pi_M g$ if $g = k_{-c}$, proving (c) in that case. However when $g = b$, we find that, as well as $t_ik = 1$, also $t_6u^2j^8$ and $t_8u^2j^6$ are in $g^{-1}\Pi_M g$, and that when $g = b^{-1}$, as well as $t_ik = 1$, also $t_3u^2j^7$ and t_5j^9 are in $g^{-1}\Pi_M g$. Explicitly,

$$\begin{aligned} b\tau_1\tau_3u^2j^8b^{-1} &= a_3^3a_1a_3^2j^8a_2^{-2}a_3^3a_1j^4, & b^{-1}j\tau_1u^2j^7b &= -j^8a_1a_3^{-1}a_1^{-1}a_2^{-2}j^8a_2a_1j^8,\\ bj^2\tau_1\tau_3u^2j^6b^{-1} &= \zeta^3a_2^3a_1^{-1}a_3^{-1}j^8a_2^{-2}a_1^{-1}j^4, & b^{-1}j^3\tau_1j^9b &= \zeta^3j^4a_1^{-1}a_2^{-1}a_3^{-3}a_1^{-1}a_2a_1j^8, \end{aligned} \tag{30}$$

are in Π. Magma checks that no other $gt_ikg^{-1} \neq 1$ are in Π. So for both $g = b$ and $g = b^{-1}$, $g\Pi_cg^{-1}$ has index 3 in Π_M, proving (d) in these cases.

Now we know that $k_\beta\Pi_ck_\beta^{-1} = \Pi_\beta$ for $\beta = c, -c$, suppose that $g \in \bar\Gamma$, and write $M = g(M_c)$. Suppose there is a $\pi \in \Pi$ so that $\pi(M) = M_\beta$ for one of these β's. Then $h = k_\beta^{-1}\pi g$ is in $\bar\Gamma_c$, so that $h\Pi_ch^{-1} = \Pi_c$. Then $(\pi g)\Pi_c(\pi g)^{-1} = \Pi_\beta$ by the case $g = k_\beta$ of (c) we have already proved. Finally $g\Pi_cg^{-1} = \pi^{-1}(\Pi_{M_\beta})\pi = \Pi_{\pi^{-1}(M_\beta)} = \Pi_M$. This completes the proof of (c).

To prove (d), suppose that $M = g(M_c)$ and that there is a $\pi \in \Pi$ such that $\pi(M) = b^\mu(M_c)$, for $\mu = 1$ or -1. Then $h = b^{-\mu}\pi g \in \bar\Gamma_c$ and so $h\Pi_ch^{-1} = \Pi_c$, and therefore $(\pi g)\Pi_c(\pi g)^{-1} = b^\mu\Pi_cb^{-\mu}$, which has index 3 in $\Pi_{b^\mu(M_c)}$, by the cases $g = b$ and b^{-1} of (d). So $g\Pi_cg^{-1}$ has index 3 in $\pi^{-1}(\Pi_{b^\mu(M_c)})\pi = \Pi_{\pi^{-1}(b^\mu(M_c))} = \Pi_M$. □

It is a consequence of Proposition 16 below that the four possibilities in Proposition 14(a) are mutually exclusive. If M is a mirror of type A, then by Proposition 14(a), the image of the immersion $\varphi_M : \Pi_M\backslash M \to X$ is equal to the image of $\varphi_{M'}$ for $M' = M_c$, M_{-c}, $b(M_c)$ or $b^{-1}(M_c)$. By Proposition 16 again, these images are distinct.

If there is a $\pi \in \Pi$ such that $\pi(M) = M_c$ or M_{-c} (in particular if $M = M_\alpha$ for some $\alpha \in \{c_{+++},\dots,c_{---}\}$), Proposition 14(c) shows that $\Pi_M\backslash M \cong \Pi_c\backslash M_c$, so that $\Pi_M\backslash M$ is a surface of genus 10. For the other two possibilities in Proposition 14(a), things are very different, as we now see.

Proposition 15. *If M is a mirror of type A, and if there is a $\pi \in \Pi$ such that $\pi(M) = b(M_c)$ or $b^{-1}(M_c)$, then $\Pi_M\backslash M$ is a surface of genus 4.*

Proof. We may assume that $M = b^\mu(M_c)$ for $\mu = 1$ or -1. As we saw in the proof of Proposition 14, $b^{-\mu}\Pi_Mb^\mu$ is the union of three cosets Π_ct_ik of Π_c in $\bar\Gamma_c$. Recall that we have an injective homomorphism $\psi_c : \bar\Gamma_c \to \mathrm{U}_c$, and $\psi_c(\Pi_c)$ has index 27 in $H_c = \langle h_1, h_2, h_3\rangle \subset \mathrm{U}_c$. We find that

$$\begin{aligned} \psi_c(\tau_1\tau_3u^2j^8) &= h_1h_3h_1^{-1}h_2^{-1}, & & \psi_c(j\tau_1u^2j^7) = h_2h_1h_3,\\ \psi_c(j^2\tau_1\tau_3u^2j^6) &= h_2h_1^{-1}h_2^{-1}h_3^{-1}, &\text{and}\quad & \psi_c(j^3\tau_1j^9) = h_3^{-1}h_1^{-1}h_2^{-1}. \end{aligned} \tag{31}$$

So $\psi_c(\Pi_c) \subset \psi_c(b^{-\mu}\Pi_Mb^\mu) \subset H_c$, and $\psi_c(b^{-\mu}\Pi_Mb^\mu)$ has index 9 in H_c, and is generated by $\psi(\Pi_c)$ and two more elements, which are given in (31) (cf. (30)). We find that in both cases, $\psi_c(b^{-\mu}\Pi_Mb^\mu)$ is generated by eight elements satisfying a single relation, and have abelianization $\mathbb{Z}^8$. So the same is true of Π_M.

Let us record here generators of $\Pi \cap b\bar\Gamma_cb^{-1}$ ($= \Pi_M$ for $M = b(M_c)$):

$$\begin{aligned} p_1 &= \zeta^{-1}a_2^3a_1^{-1}a_3^{-1}j^8a_2^{-2}a_1^{-1}j^4, & p_5 &= \zeta^{-4}a_3^3a_1a_3^2j^4a_1^{-1}j^8a_3^2a_1a_2^{-3},\\ p_2 &= a_3^3a_1a_3^2a_2a_1j^4a_3^{-1}j^8a_3^{-2}a_1^{-1}a_3^{-3}, & p_6 &= \zeta^4a_3^3a_1a_2a_1a_3a_2^{-3},\\ p_3 &= \zeta^4j^8a_1^{-1}a_3^{-3}a_2^2j^4a_3^{-2}a_1^{-1}a_3^{-3}, & p_7 &= \zeta{-1}a_3^3a_1j^8a_1a_2^{-2}a_1^{-1}a_3^2j^4,\\ p_4 &= \zeta^{-2}j^8a_2a_1a_2^{-2}a_1^{-1}j^4a_3^3a_1^2a_2^{-1}, & p_8 &= \zeta^{-2}j^4a_3^{-2}j^8a_2a_1a_2a_1a_2^{-2}, \end{aligned}$$

the scalar factors arranged so that each $h = b^{-1}p_\nu b$ satisfies $h_{11} - ch_{21} = 1 = \det(h)$. These satisfy the relation

$$p_5^{-1}p_2^{-1}p_5p_1p_3p_8^{-1}p_4p_1^{-1}p_7^{-1}p_6^{-1}p_7p_2p_3^{-1}p_8p_4^{-1}p_6 = 1.$$

Here are the images under ψ_c of the $b^{-1}p_\nu b$'s in H_c:

$$\begin{aligned}
\psi_c(b^{-1}p_1b) &= h_2h_1^{-1}h_2^{-1}h_3^{-1}, & \psi_c(b^{-1}p_5b) &= h_1h_3h_2h_1^{-1}h_3,\\
\psi_c(b^{-1}p_2b) &= h_2h_3^{-1}h_1^{-1}h_2^{-1}, & \psi_c(b^{-1}p_6b) &= h_2h_1^2h_2h_1h_3,\\
\psi_c(b^{-1}p_3b) &= h_2h_1h_3^{-1}h_1^{-1}, & \psi_c(b^{-1}p_7b) &= h_2h_1h_2^{-1}h_3^{-1}h_2^{-1},\\
\psi_c(b^{-1}p_4b) &= h_1^{-1}h_2^{-1}h_3h_2h_1^2, & \psi_c(b^{-1}p_8b) &= h_1^{-1}h_2^{-1}h_3h_2h_1^{-1}.
\end{aligned}$$

Following the same procedure as in the proof of Proposition 9, we obtain a presentation (20) for $\Pi \cap b\bar{\Gamma}_c b^{-1}$, with $u_i = E_1 \cdots E_{i-1}D_iE_{i-1}^{-1} \cdots E_1^{-1}$ and $v_i = E_1 \cdots E_{i-1}E_iE_{i-1}^{-1} \cdots E_1^{-1}$ for

$$\begin{aligned}
D_1 &= p_5^{-1}p_2^{-1}p_5p_1p_3p_8^{-1}p_4p_1^{-1}p_7^{-1}, & & E_1 = p_6^{-1},\\
D_2 &= p_5^{-1}p_2^{-1}p_5p_1p_3p_8^{-1}, & \text{and} \quad & E_2 = p_4p_1^{-1}p_2p_3^{-1},\\
D_3 &= p_5^{-1}p_2^{-1}p_5p_1, & & E_3 = p_3,\\
D_4 &= p_5^{-1}, & & E_4 = p_2^{-1}.
\end{aligned}$$

Let us also record here generators of $\Pi \cap b^{-1}\bar{\Gamma}_c b$ ($= \Pi_M$ for $M = b^{-1}(M_c)$):

$$\begin{aligned}
m_1 &= \zeta^2 j^8 a_1a_3^{-1}a_1^{-1}a_2^{-2}j^8a_2a_1j^8, & m_5 &= \zeta^4 j^4a_1^{-2}a_3^{-3}j^4a_1^{-1}a_2^{-1}j^4a_2^{-1}a_3^3a_1,\\
m_2 &= \zeta^{-3}j^8a_1a_3^3j^4, & m_6 &= \zeta^{-4}j^4a_1^{-1}a_2^{-1}a_1a_3^3a_1a_2^{-1}j^4a_1^{-1}a_2^{-2}a_3^3j^4,\\
m_3 &= \zeta^{-5}j^4a_2^{-2}a_1^{-1}a_3^{-1}j^8a_1a_3^{-2}a_1^{-1}a_2^{-2}, & m_7 &= \zeta^2a_3a_1a_2j^4a_2a_1j^8,\\
m_4 &= -j^8a_1a_3^{-1}a_1^{-1}a_2a_1^{-1}a_2^{-1}j^4a_2^{-2}a_3^3a_1, & m_8 &= -j^4a_1^{-1}a_2^{-1}a_1a_2^2a_1j^8a_2a_3^{-1}a_1^{-1}a_2^{-2},
\end{aligned}$$

the scalar factors arranged so that each $h = bm_\nu b^{-1}$ satisfies $h_{11} - ch_{21} = 1 = \det(h)$. These satisfy the relation

$$m_3m_8^{-1}m_4m_5m_7^{-1}m_2m_3^{-1}m_1m_5^{-1}m_7m_4^{-1}m_1^{-1}m_6m_2^{-1}m_6^{-1}m_8 = 1.$$

Here are the images under ψ_c of the $bm_\nu b^{-1}$'s in H_c:

$$\begin{aligned}
\psi_c(bm_1b^{-1}) &= h_2h_1h_3, & \psi_c(bm_5b^{-1}) &= h_2h_1h_2^{-1}h_1^{-1}h_2h_1h_3,\\
\psi_c(bm_2b^{-1}) &= h_1^{-1}h_2^{-1}h_3h_1^{-1}h_2^{-1}, & \psi_c(bm_6b^{-1}) &= h_2^2h_1^2h_2h_1,\\
\psi_c(bm_3b^{-1}) &= h_1^{-1}h_2^{-1}h_3^{-1}, & \psi_c(bm_7b^{-1}) &= h_1^{-1}h_2^{-1}h_3h_2h_1^2h_2^2h_1h_3,\\
\psi_c(bm_4b^{-1}) &= h_2h_1^{-3}h_2^{-1}, & \psi_c(bm_8b^{-1}) &= h_2h_1^{-1}h_2^{-1}h_1.
\end{aligned}$$

In the same way, we obtain a presentation (20) for $\Pi \cap b^{-1}\bar{\Gamma}_c b$, with generators $u_i = E_1 \cdots E_{i-1}D_iE_{i-1}^{-1} \cdots E_1^{-1}$ and $v_i = E_1 \cdots E_{i-1}E_iE_{i-1}^{-1} \cdots E_1^{-1}$ for

$$\begin{aligned}
D_1 &= m_3, & E_1 &= m_8^{-1}m_4m_5m_7^{-1}m_2,\\
D_2 &= m_1m_5^{-1}m_7m_4^{-1}m_1^{-1}m_6, & E_2 &= m_2^{-1},\\
D_3 &= m_1m_5^{-1}, & E_3 &= m_7m_4^{-1}m_1^{-1}m_4,\\
D_4 &= m_1, & E_4 &= m_4^{-1}.
\end{aligned}$$

□

We want to find out how these curves $\varphi_M(\Pi_M\backslash M) = \varphi(M)$ self-intersect. See Lemma 26.

Lemma 29. *Suppose that M is a mirror of type A, and that there is a $\pi \in \Pi$ such that $\pi(M) = M_c$ or M_{-c}, respectively such that $\pi(M) = b(M_c)$ or $b^{-1}(M_c)$. There are exactly 9 (respectively 3) distinct $\Pi_M\xi \in \Pi_M\backslash M$ such that $\xi \in M$ is in the $\bar{\Gamma}$-orbit of O. There are exactly 54 (respectively 18) distinct $\Pi_M\xi \in \Pi_M\backslash M$ such that $\xi \in M$ is in the $\bar{\Gamma}$-orbit of P.*

Proof. Write $M = g(M_c)$. Suppose first that there is a $\pi \in \Pi$ such that $\pi(M) = M_\beta$ for $\beta = c$ or $-c$. Then by Proposition 14, we have $\Pi_M = g\Pi_c g^{-1}$ and so the number of distinct $\Pi_M\xi$, with $\xi \in M$ in the $\bar\Gamma$-orbit of O (respectively of P) is the same as the number of distinct $\Pi_c\xi$, with $\xi \in M_c$ in the $\bar\Gamma$-orbit of O (respectively of P). So we may suppose that $M = M_c$. If $\xi \in M_c$ is in the $\bar\Gamma$-orbit of O then writing $\xi = g.O$, we have $O \in g^{-1}(M_c)$. By Lemma 17, the distinct mirrors of type A containing O are the $k_\alpha(M_c)$, $\alpha \in \{c_{+++}, \dots, c_{---}\}$. So $g^{-1}(M_c) = k(M_c)$ for some $k \in K$. Hence $\xi = g.O = (gk).O = h.O$ for some $h \in \bar\Gamma_c$. By (29) (where now $G = \Pi_c$) $\Pi_c\xi$ is one of the 9 elements $\Pi_c(t_i.O)$, $i = 1, \dots, 9$, and these are evidently distinct.

If $\xi \in M_c$ is in the $\bar\Gamma$-orbit of P, then using the fact that the four distinct mirrors of type A containing P are the $r_\nu(M_c)$, $\nu = 1, 2, 3, 4$, where $r_\nu \in \bar\Gamma_P$ are given in Lemma 14, we similarly find that $\xi = h(P)$ for some $h \in \bar\Gamma_c$. This time the group $\bar\Gamma_P \cap \bar\Gamma_c$ has order 6, and acts freely on any transversal of Π_c in $\bar\Gamma_c$. So there are $54 = 324/6$ elements $s_1, \dots, s_{54}$ of $\bar\Gamma_c$ such that $\bar\Gamma_c$ is the disjoint union of the double cosets $\Pi_c s_i(\bar\Gamma_P \cap \bar\Gamma_c)$. So $\xi = h.P$, with $h \in \bar\Gamma_c$, implies that $\Pi_c\xi$ is one of the 54 elements $\Pi_c(s_i.P)$ of $\Pi_c\backslash M_c$, and these are evidently distinct.

If instead there is a $\pi \in \Pi$ such that $\pi(M) = b^\mu(M_c)$ for $\mu = 1$ or -1, we may suppose that $M = b^\mu(M_c)$. Then $\Pi_c \subset b^{-\mu}\Pi_M b^\mu \subset \bar\Gamma_c$, and Π_c is of index 3 in $\tilde\Pi_c = b^{-\mu}\Pi_M b^\mu$. So $\tilde\Pi_c$ has index 108 in $\bar\Gamma_c$. Since $\tilde\Pi_c$ is torsion-free, the group $K_c = K \cap \bar\Gamma_c$ acts freely on any transversal of $\tilde\Pi_c$ in $\bar\Gamma_c$, and so we can find $3 = 108/36$ elements $u_1, u_2, u_3 \in \bar\Gamma_c$ such that

$$\bar\Gamma_c = \bigcup_{i=1}^{3} \tilde\Pi_c u_i K_c, \quad \text{a disjoint union.}$$

So if $\xi \in M$ is in the $\bar\Gamma$-orbit of O, we find that $\xi = b^\mu h.O$ for some $h \in \bar\Gamma_c$ and then that $\Pi_M\xi$ is one of the three elements $\Pi_M(b^\mu u_i.O)$. Similarly, we can write Π_c as the union of $18 = 108/6$ double cosets $\tilde\Pi_c v_i(\bar\Gamma_P \cap K_c)$, and if $\xi \in M$ is in the $\bar\Gamma$-orbit of P, then we can write $\xi = (b^\mu h).P$ for some $h \in \bar\Gamma_c$, and $\Pi_M\xi$ is one of the points $\Pi_M(b^\mu v_i.P)$. □

We now calculate for mirrors M of type A, the numbers $n_\nu(M)$, $\nu = 0, 1, -1$, as well as the numbers

$$m_i(M) = \sharp\{\Pi_M\xi \in \Pi_M\backslash M : \varphi_M(\Pi_M\xi) = \Pi(k_i.P)\}$$

for $i = 1, \dots, 36$. Here the $k_i \in K$ are as in (12) and (13). If M and M' are two such mirrors, and if $M' = \pi(M)$ for some $\pi \in \Pi$, then $n_\nu(M') = n_\nu(M)$ and $m_i(M') = m_i(M)$ for each ν and i, and so by Proposition 14(a), we need only do the calculation for the four mirrors M_c, M_{-c}, $b(M_c)$ and $b^{-1}(M_c)$.

Proposition 16. *For mirrors M of type A, the numbers $n_\nu(M)$ are as follows:*

M	$n_0(M)$	$n_1(M)$	$n_{-1}(M)$
M_c	*4*	*3*	*2*
M_{-c}	*4*	*3*	*2*
$b(M_c)$	0	1	2
$b^{-1}(M_c)$	0	1	2

The numbers $m_i = m_i(M)$ are as follows:

M	$m_1 \dots, m_{12}$	$m_{13}, \dots, m_{18}$	$m_{19}, \dots, m_{24}$	$m_{25}, \dots, m_{36}$
M_c	*2*	*0*	*3*	*1*
M_{-c}	*2*	*3*	*0*	*1*
$b(M_c)$	*0*	*0*	*1*	*1*
$b^{-1}(M_c)$	*0*	*1*	*0*	*1*

Proof. Using (26), we read off the numbers $n_\nu(M)$ by counting the elements in cells of the next three tables: It is easy to check that distinct elements π_1, π_2 in the same cell of this table satisfy $\pi_2\pi_1^{-1} \notin \bar{\Gamma}_M$, and so belong to different Π_M-cosets.

The next three tables list, for $\mu = 0$, 1 and -1, respectively, Π_M-coset representatives $\pi \in \Pi$ such that $\pi(b^\mu.O) \in M'$ for each $M' \in \{M_c, M_{-c}, b(M_c), b^{-1}(M_c)\}$.

M_c	$1,\quad a_1a_3^{-1},\quad a_1a_3^{-1}a_1a_3,\quad a_1^{-1}a_2^{-2}a_1^{-1}$
M_{-c}	$1,\quad a_2^2,\quad a_2^2a_1^{-1}a_3^{-1},\quad a_3^{-1}a_1a_2^2$
$b(M_c)$	—
$b^{-1}(M_c)$	—

M_c	$a_1^{-1}a_2^{-2},\quad a_3^{-3},\quad a_1^{-1}a_2^{-1}a_1a_3a_2^{-1}$
M_{-c}	$a_2a_1^{-1}a_2^{-1},\quad a_2a_1^{-1}a_2^{-3},\quad a_1^{-1}a_2^{-1}a_3^{-2}a_1^2a_2^{-1}$
$b(M_c)$	1
$b^{-1}(M_c)$	$a_3a_1a_2^{-1}$

M_c	$a_1^{-1}a_2^{-1},\quad a_1^{-1}a_2^{-3}$
M_{-c}	$a_1^{-2}a_3^{-3}a_2^{-1},\quad a_2a_1^{-2}a_3^{-3}$
$b(M_c)$	$a_3^3a_1a_2^{-1}$, $a_2a_1^{-2}a_3^{-1}a_1a_3^{-1}a_1^{-1}a_2^{-2}$
$b^{-1}(M_c)$	1, $a_2^2a_1a_3$

For these three tables, there are in total 9 elements given in the first row, 9 in the second row, 3 in the third row and 3 in the fourth row. So it follows from Lemma 29 that the tables give complete lists of coset representatives.

For $i = 1, \ldots, 36$, $m_i(M) = \sharp\{\Pi_M\pi \in \Pi_M\backslash\Pi : (\pi k_i).P \in M\}$, which is proved as was (26). If $k_j = k_ij^4$, then $m_j(M) = m_i(M)$ for each $M \in \{M_c, M_{-c}, b(M_c), b^{-1}(M_c)\}$. For if $M = M_c$ or M_{-c} and $(\pi k_i).P \in M$, then $((j^4\pi j^{-4})(k_ij^4)).P = (j^4\pi k_i).P \in j^4(M) = M$. If also $(\pi' k_i).P \in M$, then $\pi'\pi^{-1} \in \Pi_M$ if and only if $(j^4\pi' j^{-4})(j^4\pi j^{-4})^{-1} \in \Pi_M$ because j^4 normalizes Π_M in these cases. To see that $m_j(M) = m_i(M)$ when $k_j = k_ij^4$ and $M = b^\mu(M_c)$ for $\mu = 1, -1$, notice first that for $\mu = 0, 1, -1$,

$$b^\mu j^4 b^{-\mu} j^{-4} = \pi_\mu \in \Pi, \text{ for } \pi_0 = 1,\ \pi_1 = \zeta^{-4}a_2a_1^{-2}a_3^{-3}a_1^{-1} \text{ and } \pi_{-1} = a_2^2a_1a_3a_1^{-1}. \quad (32)$$

If $(\pi k_i).P \in b^\mu(M_c)$, then

$$((\pi_\mu j^4\pi j^{-4})(k_ij^4)).P = \pi_\mu j^4((\pi k_i).P)) \in \pi_\mu j^4(b^\mu(M_c)) = b^\mu j^4(M_c) = b^\mu(M_c).$$

If also $(\pi' k_i).P \in M = b^\mu(M_c)$, then $\pi'\pi^{-1} \in \Pi_M$ if and only if $(\pi_\mu j^4\pi' j^{-4})(\pi_\mu j^4\pi j^{-4})^{-1}$ is in Π_M, because we see from $b^\mu j^4 = \pi_\mu j^4 b^\mu$ that $\pi_\mu j^4$ normalizes Π_M. So $\pi \mapsto \pi_\mu j^4\pi j^{-4}$ induces a bijection between the two sets we are counting.

So writing $k_i = k'_\nu j^{4\alpha}$, with the k'_ν as in (13), the numbers $m_i(M)$ depend only on ν, and can be read off by counting the elements π in the cells of the following tables:

k'_ν	M_c	M_{-c}
v	$a_1^{-1}a_2^{-2}a_1^{-1},\quad a_1^{-1}a_2^{-3}$	$a_2a_1^{-1}a_2^{-1},\quad a_1^{-1}a_3^{-3}a_2$
v^2	$a_1^{-1}a_2^{-1},\quad a_1^{-1}a_2^{-2}$	$1,\quad a_1^{-1}a_3^{-3}a_2$
vuv^{-1}	$a_1a_3^{-1}a_1a_3,\quad a_1^{-1}a_2^{-3}$	$a_1^{-1}a_2^{-1}a_1a_2^2,\quad a_1^{-1}a_2^{-1}a_1a_2^2a_1^{-1}a_2^{-1}$
$vu^{-1}v^2u$	$a_1^{-1}a_2,\quad a_1a_3^{-1}a_2^{-1}$	$a_2a_1^{-1}a_2^{-1},\quad a_2^2a_1^{-1}a_3^{-1}$
v^{-1}	—	$a_3^{-1}a_1a_2^2,\quad a_1^{-1}a_3^{-3}a_2,\quad a_2a_1^{-1}a_2^{-3}$
uv^2	—	$a_3^{-1},\quad a_2^2,\quad a_1^{-1}a_2^{-1}a_3^{-3}$
j	$1,\quad a_1^{-1}a_2^{-3},\quad a_2^{-3}a_3^2$	—
j^2	$1,\quad a_1a_3^{-1}a_1a_3a_2^{-1},\quad a_1^{-1}a_2^{-2}a_1^{-2}a_2^{-1}$	—
1	1	$a_1^{-1}a_3^{-3}a_2$
j^3	1	$a_1^{-1}a_2^{-1}$
uv	$a_1a_3^{-1}a_2^{-1}a_1a_3^{-1}$	$a_3^{-1}a_1a_2^2$
$u^{-1}v^{-1}$	$a_1^{-1}a_2^{-2}a_1^{-1}$	$a_1^{-1}a_2^{-1}a_1a_2^2$

k'_ν	$b(M_c)$	$b^{-1}(M_c)$
v	—	—
v^2	—	—
vuv^{-1}	—	—
$vu^{-1}v^2u$	—	—
v^{-1}	—	$a_2^2a_1a_3$
uv^2	—	1

k'_ν	$b(M_c)$	$b^{-1}(M_c)$
j	$a_3^3a_1$	—
j^2	$a_3^3a_1a_3^2$	—
1	1	1
j^3	$a_2a_1^{-2}a_3^{-1}a_1a_3^{-1}a_1^{-1}a_2^{-1}$	a_3
uv	$a_3^3a_1$	1
$u^{-1}v^{-1}$	1	a_3a_1

Notice that writing $k_i = k'_\nu j^{4\alpha}$, the numbers of coset representatives given are 2 (for $\nu = 1,\dots,4$), 0 (for $\nu = 5,6$), 3 (for $\nu = 7,8$), and 1 for $\nu = 9,\dots,12$, adding up to 18 for each given $\alpha \in \{0,1,2\}$, and thus adding up to 54 in total. So by Lemma 29, the table is complete. Similarly for $M = M_{-c}$, the numbers given add up to 54. On the other hand, for $M = b(M_c)$, the numbers of coset representatives given are 0 (for $\nu = 1,\dots,6$) and 1 (for $\nu = 7,\dots,12$), adding up to 6 for each given $\alpha \in \{0,1,2\}$, and thus adding up to 18 in total. Again by Lemma 29, the table is complete. Similarly for $M = b^{-1}(M_c)$. □

Let us make a few remarks about the above numbers $n_\nu(M)$ and $m_i(M)$:

(a) From $n_0(M) = 0$ for $M = b(M_c)$ and $b^{-1}(M_c)$, we see that the mirrors M of type A for which $\Pi_M\backslash M$ is a surface of genus 10 are just the M for which the point ΠO is in the image of φ_M.

(b) The numbers $n_\nu(M)$ alone are not sufficient to distinguish the cases for which there is a $\pi \in \Pi$ such that $\pi(M) = M_c$ and $\pi(M) = M_{-c}$, nor between the cases $\pi(M) = b(M_c)$ and $\pi(M) = b^{-1}(M_c)$. The numbers $m_i(M)$ do make these distinctions.

(c) We can refine Lemma 26 as follows: For $i = 25,\dots,36$, we have $m_i(M) = 1$ for each mirror of type A and so these points $x = \Pi(k_i.P)$ of X are all in the image of φ_M, but there is no self-intersecting of the curves there. For the M for which there is a $\pi(M) = M_c$ or $\pi(M) = M_{-c}$, only 30 of the points $x = \Pi(k_i.P)$ are in the image of φ_M, and self-intersecting happens at only 18 of them. For the M for which there is a $\pi(M) = b(M_c)$ or $\pi(M) = b^{-1}(M_c)$, only 18 of the points $x = \Pi(k_i.P)$ are in the image of φ_M, and self-intersecting happens at none of them. In fact, for these M, self-intersections happen only at $x = \Pi(b^{-1}.O)$.

Lemma 30. *The normal closure N_c of Π_c in Π has index 84 in Π, and is normal in $\bar\Gamma$. For any mirror M such that there is a $\pi \in \Pi$ so that $\pi(M) = M_c$ or M_{-c}, the normal closure N_M of Π_M in Π is equal to N_c.*

Proof. Consider the generators $g_1, \dots, g_{20}$ of Π_c given in Proposition 13. The 140 elements g_j, $a_i g_j a_i^{-1}$, and $a_i^{-1} g_j a_i$, for $j = 1, \dots, 20$ and $i = 1, 2, 3$, must lie in any normal subgroup of Π containing Π_c. If L is the subgroup that they generate, then the Magma `Index` command shows that L has index $72576 = 84 \times 864$ in $\bar{\Gamma}$, and the `IsNormal` command shows that L is normal in $\bar{\Gamma}$. It is then clear that this L must equal N_c.

By Proposition 14, in proving the second statement, we may assume that $M = M_{-c}$. Magma verifies that $k_{-c} g_j k_{-c}^{-1}$, $a_i k_{-c} g_j k_{-c}^{-1} a_i^{-1}$ and $a_i^{-1} k_{-c} g_j k_{-c}^{-1} a_i$, for $j = 1, \dots, 20$ and $i = 1, 2, 3$, generate a normal subgroup of $\bar{\Gamma}$ of index 84×864. The result follows. ☐

Lemma 31. *If M is a mirror and if there is a $\pi \in \Pi$ such that $\pi(M) = b(M_c)$ or $b^{-1}(M_c)$, then the normal closure in Π of Π_M is of index 4 in Π, and is independent of M. It is not normal in $\bar{\Gamma}$.*

Proof. We need only consider the cases $M = b(M_c)$ and $M = b^{-1}(M_c)$.

(a) For $M = b(M_c)$, consider the following 8 elements x_i of Π. Magma verifies that $\langle x_1, \dots, x_8 \rangle$ is a normal subgroup of Π of index 4.

$$x_1 = a_3^3 a_1^2 a_2^{-1} a_3 a_1 a_2^2 a_1^{-1} a_3^{-1} j^8 a_2 a_1^{-1} a_3^{-1} j^4, \quad x_5 = a_3 a_1 a_2^{-2} a_3^2,$$
$$x_2 = a_3 a_1^{-1} a_3^{-3} a_2^2 a_1 a_2^{-2} a_1^{-1} a_3 a_1 a_3^{-1}, \quad x_6 = a_1^{-1} a_2^{-2} a_3^3 a_1^2,$$
$$x_3 = a_1 a_2^{-2} a_3^3, \quad x_7 = j^8 a_1^{-1} a_2^{-1} j^4 a_1^{-1},$$
$$x_4 = a_2^{-2} a_3^3 a_1, \quad x_8 = a_3 a_2^{-2} a_3^3 a_1 a_3^{-1}.$$

For the following 8 elements y_i of Π, one may verify using Lemma 19 that each $b^{-1} y_i^{-1} x_i y_i b$ is in $\bar{\Gamma}_c$.

$$y_1 = a_1 a_2^{-1} a_3 a_2^{-3}, \quad y_3 = a_1^2 a_3^2 a_1 a_3^{-1} a_1^{-1} a_2^{-2}, \quad y_5 = a_3 a_1^2 a_3^2 a_1 a_3^{-1} a_1^{-1} a_2^{-2}, \quad y_7 = a_1 a_2^{-2} a_3 a_2^{-3},$$
$$y_2 = a_3 a_1 a_2^{-1} a_3 a_2^{-3}, \quad y_4 = a_1 a_3^2 a_1 a_3^{-1} a_1^{-1} a_2^{-2}, \quad y_6 = a_3^2 a_1 a_3^{-1} a_1^{-1} a_2^{-2}, \quad y_8 = a_3^{-1} a_1^{-1} a_3^{-3} a_2^2.$$

So each $y_i^{-1} x_i y_i$ is in $\Pi_M = \Pi \cap b\bar{\Gamma}_c b^{-1}$, so that each x_i is in the normal closure of Π_M. This proves the result for $M = b(M_c)$.

(b) For $M = b^{-1}(M_c)$, consider the following 7 elements x_i of Π. Magma verifies that $\langle x_1, \dots, x_7 \rangle$ is a normal subgroup of Π of index 4, and equals the normal closure calculated in (a):

$$x_1 = a_3^{-1} a_1^{-1} a_2 a_3^{-1} a_1^{-1} a_2 a_1^{-1} a_3^{-3}, \quad x_5 = j^8 a_2 a_1 j^4 a_3^{-3},$$
$$x_2 = a_3^{-1} a_1^{-1} a_2 a_1^{-1} a_3^{-4} a_1^{-1} a_2, \quad x_6 = j^8 a_3 a_1 j^4 a_1 a_3^{-2} a_1^{-1},$$
$$x_3 = a_1^{-1} j^4 a_2^3 a_3^{-1} a_1^{-1} j^8, \quad x_7 = j^8 a_1 a_3 a_1^{-1} a_3 a_2^{-1} j^8 a_1^{-1} j^4 a_3^3 a_1 a_2^2 j^4.$$
$$x_4 = j^4 a_2^3 a_3^{-1} a_1^{-1} j^8 a_1^{-1},$$

For the following 7 elements y_i of Π, one may verify using Lemma 19 that each $b y_i^{-1} x_i y_i b^{-1}$ is in $\bar{\Gamma}_c$.

$$y_1 = a_1^{-1} a_3^{-3} a_1^{-1} a_3^{-1}, \quad y_3 = a_1^{-1} a_3^{-3} a_1^{-1} a_3^{-1}, \quad y_5 = a_1^{-1} a_3^{-1}, \quad y_7 = a_1 a_3^{-1} a_1^{-1} a_2^{-1} a_3^3 a_1^2.$$
$$y_2 = a_1 a_3 a_2^{-3} a_3^2, \quad y_4 = a_3^{-3} a_1^{-1} a_3^{-1}, \quad y_6 = a_3^{-1},$$

So each $y_i^{-1} x_i y_i$ is in $\Pi_M = \Pi \cap b^{-1}\bar{\Gamma}_c b$, so that each x_i is in the normal closure of Π_M. This proves the result for $M = b^{-1}(M_c)$. ☐

We conclude this section with some calculations involving the abelianization map which will are needed in Section 2.4.

Proposition 17. *If M is a mirror of type A, and $\Pi_M \backslash M$ has genus 10, then the image under f of Π_M is $\{(m, n) \in \mathbb{Z}^2 : m - n \text{ is divisible by } 6 \text{ and } n \text{ is divisible by } 2\}$. If $\Pi_M \backslash M$ has genus 4, the image of Π_M is $\{(m, n) \in \mathbb{Z}^2 : m, n \text{ are divisible by } 2\}$. The images under f of the generators u_i and v_i for $M = M_c$, M_{-c}, $b(M_c)$ and $b^{-1}(M_c)$ are given below.*

Proof. For $M = M_c$, in the notation of the proof of Proposition 13, $f(u_i) = f(D_i)$ and $f(v_i) = f(E_i)$ for $i = 1,\dots,10$, and so it is routine to calculate these from the given expressions for D_i and E_i, and from this we read off $f(\Pi_c)$. We find that

$$f(u_1) = (4,-2),\ f(u_2) = (2,-4),\ f(u_3) = (2,-4),\quad f(u_4) = (2,-4),\ f(u_5) = (-6,6),$$
$$f(v_1) = (4,-2),\ f(v_2) = (-6,6),\ f(v_3) = (-2,-2),\ f(v_4) = (2,-4),\ f(v_5) = (2,2),$$

and that

$$f(u_6) = (-2,-2),\ f(u_7) = (-4,2),\quad f(u_8) = (-8,4),\ f(u_9) = (-6,0),\ f(u_{10}) = (-2,4),$$
$$f(v_6) = (-4,2),\quad f(v_7) = (-2,-2),\ f(v_8) = (8,2),\quad f(v_9) = (2,2),\quad f(v_{10}) = (6,-6).$$

For $M = M_{-c}$, Π_M has generators $g_i' = k_{-c}g_ik_{-c}^{-1}$, which satisfy the same relation (28) as do the g_i's. So we get generators u_i and v_i for these groups by defining elements D_i and E_i as in the proof of Proposition 13, with the g_i's there replaced by g_i''s, then defining $u_i = E_1\cdots E_{i-1}D_iE_{i-1}^{-1}\cdots E_1^{-1}$ and $v_i = E_1\cdots E_{i-1}E_iE_{i-1}^{-1}\cdots E_1^{-1}$ for $i = 1,\dots,10$. To calculate these $f(u_i)$ and $f(v_i)$'s, we need to express the g_i''s in terms of the generators of Π. We find that

$$g_1' = j^4a_1^{-1}a_3^{-3}a_2^2j^4a_2^{-2}a_3^3a_1j^4,\qquad g_9' = \zeta^{-2}j^8a_1^{-1}a_2^{-1}a_3^{-2}a_1^{-1}j^8a_2^{-2}a_1^{-1}j^8,$$
$$g_3' = \zeta^2j^8a_2^2a_3^2j^4,\qquad g_{12}' = j^4a_1^{-1}a_3^{-3}a_2^2j^8a_1^{-1}j^4a_2a_1^{-1}a_3j^8,$$
$$g_5' = \zeta^{-4}j^8a_1^{-1}a_2^{-1}a_1a_2^2j^8a_1^{-1}a_3^{-3}j^8a_2a_1,\qquad g_{15}' = \zeta^4j^8a_2a_1a_3a_1^{-1}j^4a_1^{-1}a_3^{-3}a_1^{-2}a_3^{-3},$$
$$g_7' = \zeta^{-3}j^8a_1a_2^2a_1^{-1}a_2^{-1}j^8a_3^{-1}a_1^{-1}a_2a_1a_2^{-1}j^8,\quad g_{17}' = \zeta^{-2}j^8a_1^{-1}a_2^{-1}j^4a_1^{-1}a_3^{-3}j^8a_1a_2^{-1}a_3^3a_1j^4,$$

and

$$g_{19}' = \zeta^{-2}j^4a_1a_2a_1^{-2}a_3^{-2}j^4a_2a_1^{-1}a_2^{-1}j^8a_2^2a_1^{-1}a_2^{-1}j^8a_2^{-1},$$

and $g_{\nu+1}' = j^4g_\nu'j^{-4}$ for $\nu \in \{1,3,5,7,9,10,12,13,15,17,19\}$.

It is then routine to calculate the $f(u_i)$ and $f(v_i)$, and we obtain

$$f(u_1) = (-4,8),\ f(u_2) = (-8,4),\ f(u_3) = (-6,6),\ f(u_4) = (-4,2),\ f(u_5) = (-8,4),$$
$$f(v_1) = (8,-4),\ f(v_2) = (2,2),\quad f(v_3) = (4,-8),\ f(v_4) = (2,2),\quad f(v_5) = (2,2),$$

and

$$f(u_6) = (-6,0),\ f(u_7) = (-4,-4),\ f(u_8) = (-6,0),\ f(u_9) = (-4,-4),\ f(u_{10}) = (-4,2),$$
$$f(v_6) = (-6,6),\ f(v_7) = (2,-4),\quad f(v_8) = (6,0),\quad f(v_9) = (-2,4),\quad f(v_{10}) = (4,-2).$$

For $M = b(M_c)$ and $M = b^{-1}(M_c)$, generators u_i and v_i were given in the proof of Proposition 15. For $M = b(M_c)$ we read off

$$f(u_1) = (0,-2),\quad f(u_2) = (-4,0),\quad f(u_3) = (-4,2),\quad f(u_4) = (2,0),$$
$$f(v_1) = (-2,0),\quad f(v_2) = (0,2),\quad f(v_3) = (4,0),\quad f(v_4) = (0,-2),$$

and for $M = b^{-1}(M_c)$, we read off

$$f(u_1) = (2,0),\quad f(u_2) = (4,0),\quad f(u_3) = (0,0),\quad f(u_4) = (0,2),$$
$$f(v_1) = (-2,4),\quad f(v_2) = (-2,2),\quad f(v_3) = (2,-2),\quad f(v_4) = (-4,2).$$

□

A.4. The points of X coming from the $\bar\Gamma$-orbit of ξ_{12}. Recall that the point $\xi_{12} \in X$ was defined in (14). It is the point of $B_{\mathbb{C}}^2$ fixed by $\gamma_{12} = bv$. By Proposition 8, it belongs to exactly one mirror of type A (namely $g(M_c)$ for $g = bu$, since $(ub)u(ub)^{-1} = b = (bv)^4 = \gamma_{12}^4$ fixes ξ_{12}), and exactly one mirror of type B (namely M_0).

Proposition 18. *There are exactly 72 distinct points $\Pi\xi$ in X such that ξ is in the $\bar\Gamma$-orbit of ξ_{12}. The set of these points may be partitioned into three subsets of size 24, consisting of the points in the images of M_0, M_1 and M_∞, respectively. For $\alpha = 0,1,\infty$, the set of 24 points belonging to the image of M_α is partitioned into sets of n_1, n_2, n_3 and n_4 points in the images of M_c, M_{-c}, $b(M_c)$ and $b^{-1}(M_c)$, respectively, where*

(a) *for* $\alpha = 0$, $(n_1, n_2, n_3, n_4) = (6, 6, 6, 6)$,
(b) *for* $\alpha = 1$, $(n_1, n_2, n_3, n_4) = (9, 9, 3, 3)$,
(c) *for* $\alpha = \infty$, $(n_1, n_2, n_3, n_4) = (12, 12, 0, 0)$.

Proof. Recall that $T = \{b^\mu k : \mu \in \{0, 1, -1\}$ and $k \in K\}$ is a set of representatives for the set of 864 distinct cosets Πg, $g \in \bar{\Gamma}$. Since Π is torsion-free, the group $\langle \gamma_{12} \rangle$ acts freely on T, and so we can find $72 = 864/12$ elements $s_1, \ldots, s_{72}$ of T such that $\bar{\Gamma} = \bigcup_{i=1}^{72} \Pi s_i \langle \gamma_{12} \rangle$, a disjoint union. Because $\bar{\Gamma}_{\xi_{12}} = \langle \gamma_{12} \rangle$, as we saw in Lemma 15, the points $\Pi(s_i.\xi_{12}) \in X$ are distinct, and consist of the $\Pi\xi$ in X such that ξ is in the $\bar{\Gamma}$-orbit of ξ_{12}. Magma verifies that we can take $s_1, \ldots, s_{72}$ to be the elements s'_ν, $s'_\nu j^4$ and $s'_\nu j^8$, where $s'_1, \ldots, s'_{24}$ are the elements in the first column of the table below. Since $|\mathcal{M}_A(\xi_{12})| = 1 = |\mathcal{M}_B(\xi_{12})|$ by Proposition 8, each $\Pi(s_i.\xi_{12})$ belongs to the image of exactly one of M_0, M_1 and M_∞, and to the image of exactly one of M_c, M_{-c}, $b(M_c)$ and $b^{-1}(M_c)$. For each i, we can find $\pi, \pi' \in \Pi$ so that $\pi s_i \xi_{12} \in M$ and $\pi' s_i \xi_{12} \in M'$, where $M \in \{M_0, M_1, M_\infty\}$ and $M' \in \{M_c, M_{-c}, b(M_c), b^{-1}(M_c)\}$.

If $\pi \in \Pi$, $s_i = b^\mu k$, and $\pi s_i \xi_{12} \in M_\alpha$, where $\mu \in \{0, 1, -1\}$, $k \in K$, and $\alpha \in \{0, 1, \infty, c, -c\}$, then with $\pi_\mu \in \Pi$ as in (32),

$$s_i j^4 \xi_{12} = b^\mu j^4 k \xi_{12} = \pi_\mu j^4 b^\mu k \xi_{12} = \pi_\mu j^4 \pi^{-1}(\pi b^\mu k \xi_{12}) \in \pi_\mu j^4 \pi^{-1}(M_\alpha) = \pi_\mu j^4 \pi^{-1} j^{-4}(M_\alpha),$$

so that $\tilde{\pi} s_i j^4 \xi_{12} \in M_\alpha$ for $\tilde{\pi} = j^4 \pi j^{-4} \pi_\mu^{-1}$. Similarly, if $\pi s_i \xi_{12} \in b^\nu(M_c)$, then

$$\pi_\mu j^4 \pi^{-1}(\pi b^\mu k \xi_{12}) \in \pi_\mu j^4 \pi^{-1}(b^\nu(M_c)) = \pi_\mu j^4 \pi^{-1} b^\nu j^{-4}(M_c) = \pi_\mu j^4 \pi^{-1} \pi_{-\nu} j^{-4} b^\nu(M_c)$$

shows that $\tilde{\pi} s_i j^4 \xi_{12} \in b^\nu(M_c)$ for $\tilde{\pi} = j^4 \pi_{-\nu}^{-1} \pi j^{-4} \pi_\mu^{-1}$.

So it is enough to find, for $\nu = 1, \ldots, 24$, $\pi, \pi' \in \Pi$ so that $\pi s'_\nu \xi_{12} \in M$ and $\pi' s'_\nu \xi_{12} \in M'$, where $M \in \{M_0, M_1, M_\infty\}$ and $M' \in \{M_c, M_{-c}, b(M_c), b^{-1}(M_c)\}$. Suitable π, π' are listed in the next table.

s'_ν	π	M	π'	M'
j^2	1	M_0	$a_1^{-1}a_2^{-2}a_1^{-2}a_2^{-1}$	M_c
uj	$a_1^{-1}a_2^{-1}a_1$	M_0	$a_2^{-3}a_3^2$	M_c
1	1	M_0	$a_1^{-1}a_3^{-3}a_2$	M_{-c}
uj^3	$a_1^{-1}a_2^{-1}a_1$	M_0	$a_1^{-1}a_2^{-1}$	M_{-c}
j	1	M_0	$a_3^3a_1$	$b(M_c)$
uj^2	$a_1^{-1}a_2^{-1}a_1$	M_0	$a_3^3a_1a_3^2$	$b(M_c)$
j^3	1	M_0	a_3	$b^{-1}(M_c)$
u	$a_1^{-1}a_2^{-1}a_1$	M_0	1	$b^{-1}(M_c)$
$v^{-1}uj$	1	M_1	$a_1a_3^{-1}a_2^{-1}a_1a_3^{-1}$	M_c
$v^{-1}uj^3$	1	M_1	$a_1a_3^{-1}a_1a_3a_2a_3^{-1}$	M_c
bu	$a_1^{-1}a_3^{-3}$	M_1	1	M_c
$v^{-1}uj^2$	1	M_1	a_3^{-1}	M_{-c}
buj	$a_1^{-1}a_3^{-3}$	M_1	$a_1^{-1}a_2^{-1}a_1a_2^2$	M_{-c}
$bvuj$	$a_3^3a_1^2a_2^{-1}$	M_1	a_2	M_{-c}
$bv^{-1}uj$	a_2^{-2}	M_1	$a_2^3a_3^{-1}a_1^{-1}a_3^{-3}$	$b(M_c)$
$v^{-1}u$	1	M_1	$a_2^2a_1a_3$	$b^{-1}(M_c)$
$vu^{-1}j$	a_2^{-1}	M_∞	$a_1^{-1}a_2^{-1}$	M_c
$vu^{-1}j^2$	a_2^{-1}	M_∞	$a_1^{-1}a_2^{-2}a_1^{-1}a_2$	M_c
$uv^{-1}uj$	1	M_∞	$a_3a_2^{-3}a_3^3a_1$	M_c
$b^{-1}vuj^2$	$a_3^{-1}a_1^{-1}a_2^{-2}$	M_∞	$a_1a_3^{-1}a_1a_3a_2^{-2}$	M_c
vu^{-1}	a_2^{-1}	M_∞	$a_2a_1^{-1}a_2^{-1}$	M_{-c}
$vu^{-1}j^3$	a_2^{-1}	M_∞	$a_2^2a_1^{-1}a_3a_1$	M_{-c}
$uv^{-1}u$	1	M_∞	$a_1a_2a_1^{-2}a_3^{-3}a_2a_1$	M_{-c}
$bvu^{-1}v^2uj$	$a_2^{-1}a_1a_2^{-1}$	M_∞	$a_2a_1^{-1}a_2^{-2}$	M_{-c}

For each pair (M, M') we can read off from this table the i such that $\pi s_i\xi_{12} \in M$ and $\pi' s_i\xi_{12} \in M'$ for some $\pi, \pi' \in \Pi$. □

A.5. The fixed points of the automorphisms of X. As we saw in §1, the normalizer of Π in $\bar\Gamma$ contains Π as a subgroup of index 3, and is generated by Π and j^4. Denote by σ the automorphisms of $B_{\mathbb{C}}^2$ and of X induced by j^4. If $\xi = (z_1, z_2) \in B_{\mathbb{C}}^2$, then $\sigma(\xi) = (\omega z_1, \omega z_2)$.

Lemma 32. *The automorphism σ of X has exactly 9 fixed points. These are the three points $\Pi b^\mu O$, $\mu = 0, 1, -1$, and six points $\Pi h_i\xi_3$, $i = 1, \ldots, 6$, where ξ_3, as in* (14)*, is the fixed point of $\gamma_3 = buv$.*

Proof. If $\Pi\xi$ is fixed by σ, then $\Pi j^4\xi = \Pi\xi$, and so $\pi j^4\xi = \xi$ for some $\pi \in \Pi$. This implies that πj^4 has finite order. It cannot be trivial, since Π is torsion free. So there is an element t, in the list of representative nontrivial elements of finite order in $\bar\Gamma$ given in Proposition 7, or the inverse of one of these, such that $\pi j^4 = gtg^{-1}$ for some $g \in \bar\Gamma$. Thus $gtg^{-1}j^{-4} \in \Pi$. Since the elements $b^\mu k$, $\mu = 0, 1, -1$ and $k \in K$, form a set of coset representatives for Π in $\bar\Gamma$, and since j^4 normalizes Π, we can assume that $g = b^\mu k$ for some μ and k.

So we search through the finite set of elements $b^\mu ktk^{-1}b^{-\mu}j^{-4}$, checking which are in Π (by the remark below, we need only consider the cases $t = j^4$, $t = buv$ and $t = (buv)^{-1}$). We find that $b^\mu ktk^{-1}b^{-\mu}j^{-4} \in \Pi$ only happens for $t = j^4$ and $t = buv$. When $t = j^4$, we have $b^\mu ktk^{-1}b^{-\mu}j^{-4} = b^\mu j^4b^{-\mu}j^{-4}$, independent of k. We find that these three elements are in Π. Explicitly, $b^\mu j^4b^{-\mu}j^{-4} = \pi_\mu$ for π_μ given in (32). and these equations mean that the three points $\Pi(b^\mu.O)$ are fixed by σ.

For $t = buv$, we find that $b^\mu ktk^{-1}b^{-\mu}j^{-4} \in \Pi$ for only 18 pairs (μ, k). This means that σ fixes $\Pi(b^\mu k.\xi_3)$ for these 18 (μ, k)'s. If (μ, k) satisfies $b^\mu ktk^{-1}b^{-\mu}j^{-4} \in \Pi$, then so does

(μ, kj^4), since we can write $b^\mu j^4 = \pi_\mu j^4 b^\mu$ for some $\pi_\mu \in \Pi$, as we have just seen. Moreover, $\Pi(b^\mu k j^4.\xi_3) = \Pi(b^\mu k.\xi_3)$, since $kj^4 = j^4 k$ and so

$$\Pi(b^\mu k j^4.\xi_3) = \Pi(\pi_\mu j^4 b^\mu k.\xi_3) = \Pi(j^4 b^\mu k.\xi_3) = \sigma\big(\Pi(b^\mu k.\xi_3)\big) = \Pi(b^\mu k.\xi_3).$$

So we need only consider six of the (μ, k)'s, and correspondingly setting

$$h_1 = b^{-1}vuj^3, \quad h_2 = u^{-1}vj, \quad h_3 = buv^2j^2,$$
$$h_4 = b^{-1}v^2uj^3, \quad h_5 = vj^2, \quad h_6 = bvu^{-1}v,$$

we have $h_i(buv)h_i^{-1}j^{-4} = \pi_i' \in \Pi$ for $i = 1, \dots, 6$; explicitly,

$$\pi_1' = \zeta^4 a_2^2 a_1 a_3^3, \quad \pi_2' = j^8 a_1 j^4, \quad \pi_3' = \zeta^2 j^8 a_1 a_2^3 j^4 a_2 a_1 a_2^{-2} a_1^{-1}.$$
$$\pi_4' = \zeta^{-5} a_3^3 a_1^2 a_3^3, \quad \pi_5' = \zeta^{-1} j^4 a_1^{-1} a_2^{-1} j^8, \quad \pi_6' = \zeta a_2 a_1^{-1}.$$

These six points $\Pi h_i \xi_3$ are distinct, as we see by checking that $(b^{\mu'}k')(buv)^\epsilon(b^\mu k)^{-1}$ is not in Π for $\epsilon = 0, 1, 2$ when (μ', k') and (μ, k) are distinct pairs in the above list. □

Remark 6. *If $\pi \in \Pi$, then $\pi' = (\pi j^4)^3 = (\pi)(j^4\pi j^8)(j^8\pi j^4)$ is also in Π. Since the possible orders of the elements of $\bar\Gamma$ are the divisors of 24, if πj^4 has finite order, then $1 = (\pi j^4)^{24} = (\pi')^8$, so π' must be 1, so that $(\pi j^4)^3$ must be 1. Obviously, $\pi j^4 \neq 1$, and so πj^4 must have order 3. Write $\pi j^4 = gtg^{-1}$ for some $g \in \bar\Gamma$, where $t^3 = 1$ and t or t^{-1} is in the table in Proposition 7. We know from* (32) *that for each $\mu \in \{0, 1, -1\}$, there is a $\pi_\mu \in \Pi$ such that $b^\mu j^4 b^{-\mu} j^{-4} = \pi_\mu$. Using this and writing $g = \pi' b^\mu k$, where $\pi' \in \Pi$, $\mu \in \{0, 1, -1\}$, and $k \in K$, we get*

$$\begin{aligned}\pi j^4 = \pi' b^\mu k t k^{-1} b^{-\mu} \pi'^{-1} &= \pi' b^\mu k t k^{-1}(j^{-4} b^{-\mu} \pi_\mu j^4)\pi'^{-1} \\ &= \pi'(b^\mu k)(tj^{-4})(b^\mu k)^{-1}(\pi_\mu j^4 \pi'^{-1} j^{-4}) j^4.\end{aligned}$$

So $(b^\mu k)(tj^{-4})(b^\mu k)^{-1}$ is in Π, and therefore either $t = j^4$ or tj^{-4} has infinite order. In particular, apart from $t = j^4$, our t cannot be in K, and so must be buv or $(buv)^{-1}$.

Lemma 33. *In the notation of Lemma 32, the six points $\Pi h_i \xi_3$ are of type $\frac{1}{3}(1,2)$, while the three points $\Pi b^\mu O$ are of type $\frac{1}{3}(1,1)$.*

Proof. If $\gamma \in \bar\Gamma$, then writing $\gamma.(z_1, z_2) = (w_1, w_2)$, a routine calculation shows that

$$\begin{pmatrix} \frac{\partial w_1}{\partial z_1} & \frac{\partial w_2}{\partial z_1} \\ \frac{\partial w_1}{\partial z_2} & \frac{\partial w_2}{\partial z_2} \end{pmatrix},$$

evaluated at $\xi = (z_1, z_2)$, equals

$$\frac{\zeta^2/(r-1)}{(\gamma_{31}\kappa z_1 + \gamma_{32}\kappa z_2 + \gamma_{33})^2}\begin{pmatrix} \kappa z_2 \overline{\gamma}_{23} + (r-1)\,\overline{\gamma}_{22} & -(\kappa z_2 \overline{\gamma}_{13} + (r-1)\,\overline{\gamma}_{12}) \\ -(\kappa z_1 \overline{\gamma}_{23} + (r-1)\,\overline{\gamma}_{21}) & \kappa z_1 \overline{\gamma}_{13} + (r-1)\,\overline{\gamma}_{11} \end{pmatrix},$$

where $\kappa = \sqrt{r-1}$. Taking $\gamma = h_i \gamma_3 h_i^{-1}$ and $\xi = \xi_3 = (c_1/\kappa, c_2/\kappa)$ as given in (14), we find that this matrix has eigenvalues $e^{\pm 2\pi i/3}$. If instead we take $\gamma = b^\mu j^4 b^{-\mu}$, and $\xi = b^\mu O$, for $\mu = 0, 1, -1$, we find that the matrix is $e^{2\pi i/3} I$. □

Proposition 19. *With the notation of §5.4, three of the nine fixed points of σ are mapped by α to each of p_0, p_1 and p_{-1}. Moreover, $\alpha(\Pi bO) = \alpha(\Pi b^{-1}O) = \alpha(\Pi O)$, $\alpha(\Pi h_1 \xi_3) = \alpha(\Pi h_2 \xi_3) = \alpha(\Pi h_3 \xi_3)$ and $\alpha(\Pi h_4 \xi_3) = \alpha(\Pi h_5 \xi_3) = \alpha(\Pi h_6 \xi_3)$.*

Proof. Writing $\alpha(\Pi\xi) = \alpha_0(\xi) + \Lambda$, as before, where $\alpha_0(O) = 0$, we proved in Lemma 10 that

$$\alpha_0(j^4\xi) = \omega\alpha_0(\xi) \quad \text{for all } \xi \in B^2_{\mathbb{C}}.$$

Now $bj^4b^{-1} = \pi_1 j^4$ for π_1 as in (32), and $f(\pi_1) = (-2, -5)$, so by Lemma 10 and Proposition 4,

$$\alpha_0(bj^4b^{-1}\xi) = \alpha_0(\pi_1 j^4 \xi) = \alpha_0(j^4\xi) - 2 + 5\omega = \omega\alpha_0(\xi) - 2 + 5\omega.$$

In particular, taking $\xi = bO$, we have $\alpha_0(bO) = \omega\alpha_0(bO) - 2 + 5\omega$, so that

$$\alpha_0(bO) = \frac{2+\omega}{3}(-2+5\omega) = \omega - 3 \in \Lambda.$$

Hence $\alpha(\Pi b.O) = \alpha(\Pi O) = p_0$.

Similarly, $b^{-1}j^4 b = \pi_{-1}j^4$ for π_{-1} as in (32), and $f(\pi_{-1}) = (-5, 1)$, so that

$$\alpha_0(b^{-1}j^4 b\xi) = \alpha_0(\pi_{-1}j^4\xi) = \alpha_0(j^4\xi) + \theta(f(\pi_{-1})) = \omega\alpha_0(\xi) - 5 - \omega.$$

So taking $\xi = b^{-1}O$, we have $\alpha_0(b^{-1}O) = \omega\alpha_0(b^{-1}O) - 5 - \omega$, so that

$$\alpha_0(b^{-1}O) = \frac{2+\omega}{3}(-5-\omega) = -3 - 2\omega \in \Lambda.$$

Hence $\alpha(\Pi b^{-1}.O) = \alpha(\Pi O) = p_0$ too.

Recall now that $h_i(buv)h_i^{-1}j^{-4} = \pi'_i \in \Pi$ for $i = 1, \ldots, 6$, and so

$$\alpha_0(h_i(buv)h_i^{-1}\xi) = \alpha_0(\pi'_i j^4\xi) = \alpha_0(j^4\xi) + \theta(f(\pi'_i)) = \omega\alpha_0(\xi) + \theta(f(\pi'_i)).$$

In particular, taking $\xi = h_i\xi_3$, we get $\alpha_0(h_i\xi_3) = \omega\alpha_0(h_i\xi_3) + \theta(f(\pi'_i))$, so that

$$\alpha_0(h_i\xi_3) = \frac{2+\omega}{3}\theta(f(\pi'_i)).$$

Calculating

$$\begin{aligned} &f(\pi'_1) = (-6, 2), \quad f(\pi'_2) = (-4, 1), \quad f(\pi'_3) = (1, -6),\\ &f(\pi'_4) = (-4, 0), \quad f(\pi'_5) = (-4, 3), \quad f(\pi'_6) = (-3, -2), \end{aligned}$$

we have

$$\begin{aligned} \theta(f(\pi'_1)) &= -6 - 2\omega \equiv 1 - (1-\omega) & & \theta(f(\pi'_4)) = -4 \equiv -1\\ \theta(f(\pi'_2)) &= -4 - \omega \equiv 1 + (1-\omega) & \text{and} \quad & \theta(f(\pi'_5)) = -4 - 3\omega \equiv -1\\ \theta(f(\pi'_3)) &= 1 + 6\omega \equiv 1 & & \theta(f(\pi'_6)) = -3 + 2\omega \equiv -1 + (1-\omega), \end{aligned}$$

where the congruences are modulo 3. Hence $\alpha(\Pi h_i\xi_3) = \frac{2+\omega}{3} + \Lambda$ for $i = 1, 2, 3$ and $\alpha(\Pi h_i\xi_3) = -\frac{2+\omega}{3} + \Lambda$ for $i = 4, 5, 6$. □

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(Donald I. Cartwright) University of Sydney, Sydney NSW 2006 Australia
E-mail address: `donald.cartwright@sydney.edu.au`

(Vincent Koziarz) Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France
E-mail address: `vkoziarz@math.u-bordeaux1.fr`

(Sai-Kee Yeung) Purdue University, West Lafayette, IN 47907, USA
E-mail address: `yeung@math.purdue.edu`