HARMONIC MAPS AND REPRESENTATIONS OF NON-UNIFORM LATTICES OF $\mathrm{PU}(m, 1)$

VINCENT KOZIARZ AND JULIEN MAUBON

Abstract. We study representations of lattices of $\mathrm{PU}(m, 1)$ into $\mathrm{PU}(n, 1)$. We show that if a representation ρ is reductive and if $m \geq 2$, then there exists a finite energy harmonic ρ-equivariant map from $\mathbb{H}^m_{\mathbb{C}}$ to $\mathbb{H}^n_{\mathbb{C}}$. This allows us to give a differential geometric proof of rigidity results obtained by M. Burger and A. Iozzi. We also define a new invariant associated to representations into $\mathrm{PU}(n, 1)$ of non-uniform lattices of $\mathrm{PU}(1, 1)$, and more generally of fundamental groups of orientable surfaces of finite topological type and negative Euler characteristic. We prove that this invariant is bounded by a constant depending only on the Euler characteristic of the surface and we give a complete characterization of representations with maximal invariant, thus generalizing the results of D. Toledo for uniform lattices.

0. Introduction

Lattices in semi-simple Lie groups with no compact factor (say, defined over $\mathbb{R}$ and with trivial center) enjoy several rigidity properties. For example, with the exception of lattices in groups locally isomorphic to $\mathrm{PSL}(2, \mathbb{R})$, they all satisfy Mostow strong rigidity, which roughly means the following. Take two such Lie groups G and H, an irreducible lattice Γ in G, and a representation (that is, a homomorphism of groups) of Γ into H. Assume that the representation is faithful and that the image of Γ is also a lattice in H. Then the representation extends to a homomorphism from the ambient Lie group G to H (see [Mo73]). Another type of rigidity, known as Margulis superrigidity, provides the same kind of conclusion but with much weaker assumptions: the only hypothesis is that the image of Γ should be Zariski-dense in H. Superrigidity holds for lattices in Lie groups of rank at least 2 ([Ma91]) and for lattices of quaternionic or octonionic hyperbolic spaces (that is, lattices in the rank one Lie groups $\mathrm{Sp}(m, 1)$, $m \geq 2$, and $\mathrm{F}_4^{-20)}$) (see [Co92] and [GS92]). On the contrary, for lattices of real and complex hyperbolic spaces, namely, lattices in the other rank one Lie groups $\mathrm{PO}(m, 1)$ and $\mathrm{PU}(m, 1)$, superrigidity is known to fail in general.

In this paper, we will focus on lattices in $\mathrm{PU}(m, 1)$, the group of orientation-preserving isometries (or equivalently, of biholomorphisms) of complex hyperbolic m-space $\mathbb{H}^m_{\mathbb{C}} = \mathrm{PU}(m, 1)/\mathrm{U}(m)$. They are of particular interest because they lie somewhere in between the very flexible lattices of $\mathrm{PO}(m, 1)$ and those, superrigid, of the higher rank Lie groups.

In [GM87], W. M. Goldman and J. J. Millson studied representation spaces of *uniform* torsion-free lattices $\Gamma < \mathrm{SU}(m, 1)$ (which can be considered as lattices in $\mathrm{PU}(m, 1)$) into $\mathrm{PU}(n, 1)$, for $n > m \geq 2$. They proved that there are no non-trivial deformations of the standard representation of such a lattice. This means that all nearby representations are $\mathbb{C}$-*Fuchsian*, namely, they are discrete, faithful, and they stabilize a totally geodesic copy of $\mathbb{H}^m_{\mathbb{C}}$ in $\mathbb{H}^n_{\mathbb{C}}$. The case $m = 1$ was previously treated by Goldman in [Go85]. Note that the corresponding statement for lattices in $\mathrm{PO}(m, 1)$ is false (cf. for example [JM87]).

They also conjectured that a much stronger rigidity should hold. The *volume* of a representation ρ of a torsion-free uniform lattice $\Gamma < \mathrm{PU}(m, 1)$ into $\mathrm{PU}(n, 1)$ is defined by pulling-back

Date: March 2004.

the Kähler form of $\mathbb{H}_{\mathbb{C}}^n$ on $\mathbb{H}_{\mathbb{C}}^m$ via the representation, taking its m-th exterior power to obtain a de Rham cohomology class in $H_{DR}^{2m}(\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m)$ and evaluating it on the fundamental class of the compact quotient $\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m$. Observe that if $\Gamma < \mathrm{SU}(m,1)$ and if $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$, $n > m$, is the standard representation, then $\mathrm{vol}(\rho) = \mathrm{Vol}(\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m)$. Their conjecture then reads: any representation ρ such that $\mathrm{vol}(\rho) = \mathrm{Vol}(\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m)$ must be $\mathbb{C}$-Fuchsian. This was proved by K. Corlette in [Co88] for $m \geq 2$ and by D. Toledo in [To89] for $m = 1$. Remark that the volume assumption is needed precisely because lattices in $\mathrm{PU}(m,1)$ are not superrigid.

Recently, M. Burger and A. Iozzi proved in [BI01] (see also [Io02]) that the conjecture is also true for *non-uniform* lattices of $\mathrm{PU}(m,1)$, $m \geq 2$, if one suitably modifies the definition of the "volume" of the representation (indeed, with the former one, any representation of a non-uniform lattice has zero volume). We will explain precisely how this invariant is computed in section 3.1 but here we sketch its definition. Again, the Kähler form ω_n of $\mathbb{H}_{\mathbb{C}}^n$ is pulled-back to the quotient $\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m$ via the representation. It turns out that this gives a well-defined L^2-cohomology class in $H_{(2)}^2(\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m)$. Now, integrating a L^2-representative $\rho^\star\omega_n$ against the Kähler form ω_m of $\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m$, we get the Burger-Iozzi invariant (slightly modified from [BI01]):

$$\tau(\rho) := \frac{1}{2m}\int_{\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m} \langle \rho^\star\omega_n, \omega_m\rangle dV_m \ .$$

In complex dimension 1 and for uniform lattices, this invariant coincides with the invariant defined in [To89]. We can now state the main theorem of [BI01]:

Theorem A *Let Γ be a torsion-free lattice in $\mathrm{PU}(m,1)$, $m \geq 2$, and let $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$ be a representation. Then $|\tau(\rho)| \leq \mathrm{Vol}(\Gamma\backslash\mathbb{H}_{\mathbb{C}}^m)$ and equality holds if and only if there exists a totally geodesic isometric ρ-equivariant embedding of $\mathbb{H}_{\mathbb{C}}^m$ into $\mathbb{H}_{\mathbb{C}}^n$ (in particular, $\rho(\Gamma)$ seen as a subgroup of $\mathrm{PU}(m,1)$ is a lattice).*

Burger and Iozzi's proof heavily relies on the theory of bounded cohomology developed by Burger and N. Monod in [BM02]. As a corollary, they obtain the result of Goldman and Millson for a general lattice:

Corollary A' *Let Γ be a torsion-free lattice in $\mathrm{SU}(m,1)$, $m \geq 2$, and let $n > m$. Then there are no non-trivial deformations of the standard representation of Γ into $\mathrm{PU}(n,1)$.*

The aim of this paper is to use harmonic map techniques to give a new and more (differential) geometric proof of Theorem A and to extend this result to the case of complex dimension 1, that is, of non-uniform lattices of $\mathrm{PU}(1,1)$.

The over-all harmonic map strategy for proving rigidity results about representations of lattices in a Lie Group G to another Lie group H goes as follows. First, one has to know that there exists a harmonic map between the corresponding symmetric spaces, equivariant w.r.t. the representation. Then, one must prove, generally by using a Bochner-type formula, that there are additional constraints on the harmonic map, which force it to be pluriharmonic, holomorphic, totally geodesic, or isometric, depending on the situation.

For a uniform Γ and when the target symmetric space is non-positively curved (which will be assumed from now on), the existence results for harmonic maps go back to J. Eells and J. H. Sampson in [ES64] and have been extended by several authors, in particular by Corlette in [Co88]. The second step was pionneered by Y.-T. Siu in [Si80] where he proved a strenghtened version of Mostow strong rigidity theorem in the case of Hermitian locally symmetric spaces. This has later on been applied in different directions by many authors. We should mention the proof of the above conjecture of Goldman and Millson by Corlette in [Co88] and the geometric proof of Margulis superrigidity theorem in the Archimedean setting worked out by N. Mok, Y.-T. Siu and S.-K. Yeung in [MSY93].

When the lattice is not uniform, the only general existence theorem for harmonic maps is due to Corlette in [Co92], see Theorem 1.1 below. The main issue is that to apply this theorem, one needs to prove that there exists an equivariant map of *finite energy* (see section 1 for the definition). If this is the case, the harmonic map also has finite energy and the second step generally goes as if the lattice was uniform, but is technically more involved. The energy finiteness condition is very important, and in general difficult to prove. In some particular cases it is possible to obtain a harmonic map by other means (see for example [JZ97] and section 4 of this paper) but then its energy is infinite and the analysis that follows becomes much harder. These are the reasons why, for example, "geometric superrigidity" for non-uniform lattices is not yet proved.

Our paper is organized as follows. The first three sections are devoted to the proof of Theorem A. In section 1 we give the necessary definitions and we prove that Corlette's general theorem applies in our setting, so that we obtain our main existence theorem (cf. Theorem 1.2):

Theorem B *Let Γ be a torsion-free lattice in $\mathrm{PU}(m,1)$, $m \geq 2$, and $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$ be a representation such that $\rho(\Gamma)$ has no fixed point on the boundary at infinity of $\mathbb{H}^n_{\mathbb{C}}$. Then there exists a finite energy harmonic ρ-equivariant map from $\mathbb{H}^m_{\mathbb{C}}$ to $\mathbb{H}^n_{\mathbb{C}}$.*

In section 2 we prove that the harmonic map previously obtained is pluriharmonic and even holomorphic or antiholomorphic if its rank is high enough (at least 3 at some point). Section 3 is devoted to the precise definition of the Burger-Iozzi invariant and to the proof of Theorem A.

In section 4, we study the case of lattices of $\mathrm{PU}(1,1)$, that is, of fundamental groups of Riemann surfaces with a finite volume hyperbolic metric. The analogue of Theorem A for *uniform* lattices was proved by Toledo in [To89]. In [GP03] (see also [GP00]), N. Gusevskii and J. R. Parker claim that if one restricts to *type-preserving* representations, then the original definition of the Toledo invariant can be used to generalize Toledo's result to non-uniform lattices. However, it seems to us that this claim is not entirely exact (see for example the remark following Proposition 4.5).

There are mainly two reasons why the 1-dimensional case is different from the higher dimensional one. First of all, Toledo and/or Burger-Iozzi invariants are not defined for non-uniform lattices. Secondly, there are representations for which no equivariant map of *finite energy* exists. It should also be noted that Corollary A' fails in this case by a result of Gusevskii and Parker (cf. [GP00]).

As we shall see, it is in fact more natural to work in the general setting of fundamental groups of orientable surfaces of finite topological type, that is surfaces obtained by removing finitely many points from closed orientable surfaces. Using cohomology with compact support, we define at the beginning of section 4 a new invariant associated to representations of these fundamental groups into $\mathrm{PU}(n,1)$, which we again call τ. We obtain (see Theorem 4.3):

Theorem C *Let Γ be the fundamental group of a p-times punctured closed orientable surface M of negative Euler characterisic $\chi(M)$, and let $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$ be a representation. Then $|\tau(\rho)| \leq -2\pi\chi(M)$ and equality holds if and only if $\rho(\Gamma)$ stabilizes a complex geodesic L in $\mathbb{H}^n_{\mathbb{C}}$, ρ is faithful and discrete, and M is diffeomorphic to the quotient $\rho(\Gamma)\backslash L$.*

The proof relies on the fact that though there may be no equivariant map of finite energy, there exists an equivariant harmonic map whose energy density can be controlled. This control allows us to extend the proofs given in the finite energy case to this setting.

Remark. In an earlier version of this paper, Theorem C was proven in a weaker form, and only for what we call *tame* representations (see Definition 4.2). M. Burger and A. Iozzi then informed us that their methods should allow them to get rid of this tameness assumption. Later, they communicated us the text [BI03], where they define a "bounded Toledo number" and prove Theorem C.

Acknowledgments. We would like to thank J.-P. Otal who suggested that it would be interesting to have a more geometric proof of Burger and Iozzi's result. We also are grateful to F. Campana and J. Souto for helpful conversations. We finally thank M. Burger and A. Iozzi for their interest in our work and for having encouraged us to improve the first draft of Theorem C.

1. Existence of finite energy equivariant harmonic maps

In this section, we assume that $m \geq 2$.

Let Γ be a torsion-free lattice in $\mathrm{PU}(m,1)$, the group of biholomorphisms of complex hyperbolic m-space $\mathbb{H}^m_{\mathbb{C}}$ and let $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$ be a representation into the group of biholomorphisms of complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$.

We call M the quotient manifold $\Gamma\backslash\mathbb{H}^m_{\mathbb{C}}$. The representation ρ determines a flat bundle $M \times_\rho \mathbb{H}^n_{\mathbb{C}}$ over M with fibers isomorphic to $\mathbb{H}^n_{\mathbb{C}}$. Since $\mathbb{H}^n_{\mathbb{C}}$ is contractible, this bundle has global sections. This is equivalent to the existence of maps (belonging to the same homotopy class) from $\mathbb{H}^m_{\mathbb{C}}$ to $\mathbb{H}^n_{\mathbb{C}}$, equivariant w.r.t. the representation ρ. Let f be such a map (or section).

We can consider the differential $\mathrm{d}f$ of f as a $f^\star T\mathbb{H}^n_{\mathbb{C}}$-valued 1-form on $\mathbb{H}^m_{\mathbb{C}}$. There is a natural pointwise scalar product on such forms coming from the Riemannian metrics g_m and g_n (of constant holomorphic sectional curvature -1) on $\mathbb{H}^m_{\mathbb{C}}$ and $\mathbb{H}^m_{\mathbb{C}}$: if $(e_i)_{1\leq i\leq 2m}$ is a g_m-orthonormal basis of $T_x\mathbb{H}^m_{\mathbb{C}}$, then $\|\mathrm{d}f\|_x^2 := \sum_i g_n(\mathrm{d}f(e_i), \mathrm{d}f(e_i))$. Since f is ρ-equivariant and the action of Γ on $\mathbb{H}^n_{\mathbb{C}}$ via ρ is isometric, $\|\mathrm{d}f\|$ is a well-defined function on M. We say that f has *finite energy* if the energy density $e(f) := \frac{1}{2}\|\mathrm{d}f\|^2$ of f is integrable on M:

$$E(f) = \frac{1}{2}\int_M \|\mathrm{d}f\|^2 dV_m \,<\, +\infty \;,$$

where dV_m is the volume density of the metric g_m. When there is no risk of confusion, we will write e instead of $e(f)$ for the energy density of f.

There is also a natural connection ∇ on $f^\star T\mathbb{H}^n_{\mathbb{C}}$-valued 1-forms on $\mathbb{H}^m_{\mathbb{C}}$ coming from the Levi-Civita connections ∇^m and ∇^n of $\mathbb{H}^m_{\mathbb{C}}$ and $\mathbb{H}^n_{\mathbb{C}}$. If $\nabla^{f^\star T\mathbb{H}^n_{\mathbb{C}}}$ denotes the connection induced by ∇^n on the bundle $f^\star T\mathbb{H}^n_{\mathbb{C}} \longrightarrow \mathbb{H}^m_{\mathbb{C}}$, then $\nabla\mathrm{d}f(X,Y) = \nabla_X^{f^\star T\mathbb{H}^n_{\mathbb{C}}}\mathrm{d}f(Y) - \mathrm{d}f(\nabla^m_X Y)$. Since ∇^m and ∇^n are torsion-free, $\nabla\mathrm{d}f$ is a symmetric 2-tensor taking values in $f^\star T\mathbb{H}^n_{\mathbb{C}}$.

A map $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ is said to be *harmonic* if $\mathrm{tr}_{g_m}\nabla\mathrm{d}f = 0$.

The following theorem of Corlette ([Co92]) implies that if there exists a finite energy ρ-equivariant map from the universal cover $\mathbb{H}^m_{\mathbb{C}}$ of M to $\mathbb{H}^n_{\mathbb{C}}$, and under a very mild assumption on ρ, then there exists a harmonic ρ-equivariant map of finite energy from $\mathbb{H}^m_{\mathbb{C}}$ to $\mathbb{H}^n_{\mathbb{C}}$:

Theorem 1.1. *Let X be a complete Riemannian manifold and Y a complete simply-connected manifold with non-positive sectional curvature. Let $\rho : \pi_1(X) \longrightarrow \mathrm{Isom}(Y)$ be a representation such that the induced action of $\pi_1(X)$ on the sphere at infinity of Y has no fixed point (ρ is then called reductive). If there exists a ρ-equivariant map of finite energy from the universal* *cover $\widetilde{X}$ of X to Y, then there exists a harmonic ρ-equivariant map of finite energy from $\widetilde{X}$ to Y.*

Theorem B will therefore follow from the

Theorem 1.2. *Let Γ be a torsion-free lattice in* $\mathrm{PU}(m,1)$*, $m \geq 2$, and let ρ be a representation of Γ into* $\mathrm{PU}(n,1)$*. Then there exists a finite energy ρ-equivariant map $\mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$.*

Proof. Of course this is trivially true if the manifold is compact, that is, if Γ is a uniform lattice. To prove the theorem in the non-uniform case, we recall some known facts about the structure at infinity of the finite volume complex hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^m_{\mathbb{C}}$, cf. for example [Go99], or [Bi97] and [HS96].

We will work with the Siegel model of complex hyperbolic space:

$$\mathbb{H}^m_{\mathbb{C}} = \{(z,w) \in \mathbb{C}^{m-1} \times \mathbb{C} \ \mid \ 2\mathrm{Re}(w) > \langle\langle z,z\rangle\rangle\} \ ,$$

where $\langle\langle .,.\rangle\rangle$ is the standard Hermitian product on $\mathbb{C}^{m-1}$. We will call h the function given by $h(z,w) = 2\mathrm{Re}(w) - \langle\langle z,z\rangle\rangle$. The boundary at infinity of $\mathbb{H}^m_{\mathbb{C}}$ is the set $\{h=0\} \cup \{\infty\}$ and the horospheres in $\mathbb{H}^m_{\mathbb{C}}$ centered at ∞ are the level sets of h. The complex hyperbolic metric (of constant holomorphic sectional curvature -1) in the Siegel model of $\mathbb{H}^m_{\mathbb{C}}$ is given by

$$g_m = \frac{4}{h(z,w)^2}\Big[(dw - \langle\langle dz,z\rangle\rangle)(d\bar{w} - \langle\langle z,dz\rangle\rangle) + h(z,w)\langle\langle dz,dz\rangle\rangle\Big] \ .$$

The stabilizer P of ∞ in $\mathrm{PU}(m,1)$ is the semi-direct product $\mathcal{N}^{2m-1} \rtimes (\mathrm{U}(m-1) \times \{\phi_s\}_{s\in\mathbb{R}})$ where $\mathcal{N}^{2m-1}$ is the $(2m-1)$-dimensional Heisenberg group, $\mathrm{U}(m-1)$ is the unitary group and $\{\phi_s\}_{s\in\mathbb{R}}$ is the one-parameter group corresponding to the horocyclic flow associated to ∞. The group $\mathcal{N}^{2m-1}$ is a central extension of $\mathbb{C}^{m-1}$ and can be seen as $\mathbb{C}^{m-1}\times\mathbb{R}$ with product given by $(\xi_1,\nu_1)(\xi_2,\nu_2) = (\xi_1+\xi_2, \nu_1+\nu_2+2\mathrm{Im}\langle\langle\xi_1,\xi_2\rangle\rangle)$. This is a two-step nilpotent group which acts simply transitively and isometrically on horospheres. Its center $\mathcal{Z}$ is the group of "vertical translations": $\{(0,\nu),\ \nu\in\mathbb{R}\}$.

If we set $u+iv = 2\overline{w} - \langle\langle z,z\rangle\rangle$, we obtain the so-called *horospherical coordinates* $(z,v,u) \in \mathbb{C}^{m-1}\times\mathbb{R}\times\mathbb{R}^\star_+$, in which the action of P on $\mathbb{H}^m_{\mathbb{C}}$ is given by:

$$(\xi,\nu)A\phi_s.(z,v,u) = (Ae^{-s}z+\xi, e^{-2s}v+\nu+2\mathrm{Im}\langle\langle\xi, Ae^{-s}z\rangle\rangle, e^{-2s}u)$$

and the metric g_m takes the form

$$g_m = \frac{du^2}{u^2} + \frac{1}{u^2}\Big(-dv + 2\mathrm{Im}\langle\langle z,dz\rangle\rangle\Big)^2 + \frac{4}{u}\langle\langle dz,dz\rangle\rangle \ .$$

Replacing u by $t = \log u$, the metric tensor decomposes as:

$$g_m = dt^2 + e^{-2t}\Big(-dv + 2\mathrm{Im}\langle\langle z,dz\rangle\rangle\Big)^2 + 4e^{-t}\langle\langle dz,dz\rangle\rangle \ .$$

The coordinates $(z,v,t) \in \mathbb{C}^{m-1}\times\mathbb{R}\times\mathbb{R}$ will also be called horospherical coordinates.

A complex hyperbolic manifold M of finite volume is the union of a compact part and a finite number of disjoint cusps. Each cusp C of M is diffeomorphic to the product $N\times[0,+\infty)$, where N is a compact quotient of some horosphere HS in $\mathbb{H}^m_{\mathbb{C}}$. We can assume that HS is centered at ∞. The fundamental group Γ_C of C, hence of N, can be identified with the stabilizer in Γ of the horosphere HS: it is therefore equal to $\Gamma\cap(\mathcal{N}^{2m-1}\rtimes\mathrm{U}(m-1))$.

If we call β the 1-form $-dv+2\mathrm{Im}\langle\langle z,dz\rangle\rangle$ on $\mathbb{H}^m_{\mathbb{C}}$, it is easily checked that $\mathrm{d}^ct := J\mathrm{d}t = e^{-t}\beta$. Therefore, since $\mathcal{N}^{2m-1}\rtimes\mathrm{U}(m-1)$ preserves the horospheres, t, dt^2, and β are invariant by Γ_C. The decomposition of g_m hence goes down to the cusp C and we have:

$$g_m = dt^2 + e^{-2t}\beta^2 + e^{-t}g \ ,$$

where g is the image of $4\langle\langle dz, dz\rangle\rangle$.

Remark. The Kähler form ω_m, which we normalize so that $\omega_m(X, JX) \geq 0$, is of course exact on $\mathbb{H}^m_{\mathbb{C}}$. More precisely, $\omega_m = -\mathrm{dd}^c t = -\mathrm{d}(e^{-t}\beta)$. The invariance of t and β implies that this relation still holds in the cusps of M.

For lattices in $\mathrm{Sp}(m, 1)$, $m \geq 2$, or in F_4^{-20}, Corlette proves in [Co92] a simple lemma that allows him to deduce the existence of finite energy equivariant maps. Here, the same idea will only provide the result for $m \geq 3$:

Lemma 1.3. *Assume $m \geq 3$. Then there exists a finite energy retraction of $M = \Gamma\backslash\mathbb{H}^m_{\mathbb{C}}$ onto a compact subset of M.*

Proof. It is enough to construct the retraction on a cusp $C = N \times [0, +\infty)$ of M: we define $r : N \times [0, +\infty) \longrightarrow N \times \{0\}$ obviously by $r(x, t) = (x, 0)$.

If $(\frac{\partial}{\partial t}, \frac{\partial}{\partial v}, e_3, \ldots, e_{2m})$ is an orthonormal basis of $T_{(x,0)}C$ compatible with the splitting of g_m, then $(\frac{\partial}{\partial t}, e^t\frac{\partial}{\partial v}, e^{t/2}e_3, \ldots, e^{t/2}e_{2m})$ is such a basis of $T_{(x,t)}C$. Hence, $\|\mathrm{d}r\|^2_{(x,t)} = e^{2t}+(2m-2)e^t$. If we call dV_N the volume element of $N \times \{0\}$, the volume element of $N \times \{t\}$ is given by $e^{\frac{1}{2}(-2t-(2m-2)t)}dV_N = e^{-mt}dV_N$. Hence, the energy of r on C is

$$\frac{1}{2}\int_C \|\mathrm{d}r\|^2\, dV_m = \frac{1}{2}\int_0^{+\infty}\int_N (2(m-1)e^t + e^{2t})e^{-mt}\, dV_N\, dt\ .$$

This is clearly finite if $m \geq 3$. □

This retraction lifts to a map $\tilde{r} : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^m_{\mathbb{C}}$, invariant by Γ. Therefore, if $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ is any ρ-equivariant map, so is $f \circ \tilde{r}$, and its energy is finite. The theorem is therefore proved if $m \geq 3$.

In the case $m = 2$, the energy density of the retraction r grows like e^{2t} when t goes to infinity whereas the volume element grows like e^{-2t}: the energy of r is infinite and we need a deeper analysis of the situation in the cusps.

We fix a cusp $C = N \times [0, +\infty)$ of M and we look for a finite energy map from the universal cover $HS \times [0, +\infty)$ of C to $\mathbb{H}^n_{\mathbb{C}}$, equivariant w.r.t. the fundamental group Γ_C of C (equivalently, a section of the restriction of the flat bundle $M \times_\rho \mathbb{H}^n_{\mathbb{C}}$ to $C \subset M$).

As we said, Γ_C can be seen as a subgroup of $\mathcal{N} \rtimes \mathrm{U}(1)$, where now $\mathcal{N} := \mathcal{N}^3$ is just $\mathbb{C} \times \mathbb{R}$. It follows from L. Auslander's generalization of Bieberbach's theorem (cf. [Au60]) that $\Gamma_{\mathcal{N}} := \Gamma_C \cap \mathcal{N}$ is a discrete uniform subgroup of $\mathcal{N}$, of finite index in Γ_C. Therefore ([Au60], Lemma 1.3.), $\Gamma_{\mathcal{N}}$ cannot be contained in any proper analytic subgoup of $\mathcal{N}$. From this, it is easy to deduce that there exists $\varepsilon > 0$ such that, for all $\gamma = (\xi_\gamma, \nu_\gamma) \in \Gamma_{\mathcal{N}}$, $|\xi_\gamma| > \varepsilon$ as soon as $\xi_\gamma \neq 0$. In other words, the image of the homomorphism of groups $T : \Gamma_{\mathcal{N}} \longrightarrow \mathbb{C}$, $\gamma \longmapsto \xi_\gamma$, is a lattice in $\mathbb{C}$. Let $\gamma_1 = (\xi_1, \nu_1)$ and $\gamma_2 = (\xi_2, \nu_2)$ be two elements of $\Gamma_{\mathcal{N}}$ such that ξ_1 and ξ_2 generate the lattice $T(\Gamma_{\mathcal{N}})$. A straightforward computation yields $[\gamma_1, \gamma_2] := \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} = (0, -2\mathrm{Im}(\xi_1\overline{\xi}_2))$. Since ξ_1 and ξ_2 are linearly independent (over $\mathbb{R}$), $\mathrm{Im}(\xi_1\overline{\xi}_2) \neq 0$ and hence the subgroup $\Gamma_{\mathcal{Z}} := \Gamma_C \cap \mathcal{Z}$ of $\Gamma_{\mathcal{N}}$ is non trivial. It is therefore isomorphic to $\mathbb{Z}$ and we call γ_0 its generator. Note that γ_0, γ_1 and γ_2 generate $\Gamma_{\mathcal{N}}$.

The construction of the equivariant map will depend on the type of $\rho(\gamma_0)$. Recall that an isometry of $\mathbb{H}^n_{\mathbb{C}}$ can be of one of the following (exclusive) 3 types:

- *elliptic* if it has a fixed point in $\mathbb{H}^n_{\mathbb{C}}$;
- *parabolic* if it has exactly one fixed point on the sphere at infinity of $\mathbb{H}^n_{\mathbb{C}}$ and no fixed points in $\mathbb{H}^n_{\mathbb{C}}$;

- *hyperbolic* if it has exactly two fixed points on the sphere at infinity of $\mathbb{H}^n_{\mathbb{C}}$ and no fixed points in $\mathbb{H}^n_{\mathbb{C}}$. In this case, the isometry acts by translation on the geodesic joining its fixed points at infinity.

Claim 1.4. *$\rho(\gamma_0)$ can not be a hyperbolic isometry of $\mathbb{H}^n_{\mathbb{C}}$.*

Proof. Assume that $\rho(\gamma_0)$ is a hyperbolic isometry of $\mathbb{H}^n_{\mathbb{C}}$ and call A_0 its axis (the geodesic joining its fixed points). Then, since γ_1 and γ_2 commute with γ_0, their images by ρ commute with $\rho(\gamma_0)$, hence they must fix A_0 and act on it by translations: there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that $\rho(\gamma_1)A_0(t) = A_0(t+\tau_1)$ and $\rho(\gamma_2)A_0(t) = A_0(t+\tau_2)$. This implies that $\rho([\gamma_1,\gamma_2])$ acts trivialy on A_0. But $[\gamma_1,\gamma_2] = {\gamma_0}^p$ for some $p \in \mathbb{Z}^\star$ and $\rho(\gamma_0)$ does not act trivially on A_0. This is a contradiction. $\square$

Hence $\rho(\gamma_0)$ is either elliptic or parabolic. In both cases we will start by constructing an equivariant map from the universal cover $HS \simeq \mathcal{N}$ of N and then we shall extend it to the universal cover of the whole cusp.

Case 1: $\rho(\gamma_0)$ is parabolic. The idea is to find an equivariant map from HS to a horosphere in $\mathbb{H}^n_{\mathbb{C}}$ centered at the fixed point of $\rho(\gamma_0)$ on the sphere at infinity $\partial\mathbb{H}^n_{\mathbb{C}}$ of $\mathbb{H}^n_{\mathbb{C}}$ and then to extend it to $HS \times [0,+\infty)$ using the horocyclic flow defined by the fixed point. Roughly speaking, when t goes to infinity, the image of $HS \times \{t\}$ must go to infinity in $\mathbb{H}^n_{\mathbb{C}}$ fast enough so that the decay of the metric in $\mathbb{H}^n_{\mathbb{C}}$ prevents the energy density of the map from growing too quickly.

Using again the Siegel model for $\mathbb{H}^n_{\mathbb{C}}$, we may assume that the fixed point of $\rho(\gamma_0)$ is ∞. Since γ_0 is in the center of Γ_C, the whole group $\rho(\Gamma_C)$ must fix ∞, and therefore must be contained in its stabilizer in PU$(n,1)$. Moreover, $\rho(\Gamma_C)$ must stabilize each horosphere centered at ∞. For, if this was not the case, there would be an element $\gamma \in \Gamma_C$ such that $\rho(\gamma)$ is hyperbolic. But then, since γ_0 commutes with γ, $\rho(\gamma_0)$ would fix the axis of $\rho(\gamma)$. This is impossible since we assumed that $\rho(\gamma_0)$ is parabolic.

We see $\mathbb{H}^n_{\mathbb{C}}$ as $\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{R}$ with horospherical coordinates $(z',v',t'=\log u')$. The metric g_n at a point (z',v',t') is given by $g_n = dt'^2 + e^{-2t'}(-dv' + 2\mathrm{Im}\langle\langle z', dz'\rangle\rangle)^2 + 4e^{-t'}\langle\langle dz', dz'\rangle\rangle$.

Let $HS' \subset \mathbb{H}^n_{\mathbb{C}}$ be the horosphere $\mathbb{C}^{n-1}\times\mathbb{R}\times\{0\}$. The representation ρ can be seen as a homomorphism from the fundamental group of N to the isometry group of HS'. Since HS' is contractible, there exists a ρ-equivariant map φ from the universal cover $HS \subset \mathbb{H}^m_{\mathbb{C}}$ of N to the horosphere $HS' \subset \mathbb{H}^n_{\mathbb{C}}$. Now, define a ρ-equivariant map f from the universal cover $HS \times [0,+\infty)$ of the cusp $C = N \times [0,+\infty)$ to $\mathbb{H}^n_{\mathbb{C}}$ by:

$$\begin{array}{rcl} f : HS \times [0,+\infty) & \longrightarrow & HS' \times [0,+\infty) \ \subset \mathbb{H}^n_{\mathbb{C}} \\ (x,t) & \longmapsto & (\varphi(x), 2t) \end{array}$$

Using the same notation as in Lemma 1.3, the energy density of f can be estimated as follows:

$$\begin{array}{rcl} \|\mathrm{d}f\|^2_{(x,t)} & = & |\mathrm{d}f(\frac{\partial}{\partial t})|^2_{(\varphi(x),t)} + e^{2t}|\mathrm{d}f(\frac{\partial}{\partial v})|^2_{(\varphi(x),t)} + e^t\sum_{k=3}^4 |\mathrm{d}f(e_k)|^2_{(\varphi(x),t)} \\ & \leq & 4 + e^{-2t}\Big(e^{2t}|\mathrm{d}\varphi(\frac{\partial}{\partial v})|^2_{(\varphi(x),0)} + e^t\sum_{k=3}^4 |\mathrm{d}\varphi(e_k)|^2_{(\varphi(x),0)}\Big) \\ & \leq & 4 + \|\mathrm{d}\varphi\|^2_x \end{array}$$

where $\|\mathrm{d}\varphi\|$ denotes the norm of the differential of $\varphi : HS \longrightarrow HS'$ computed with the metrics of HS and HS' induced from g_m and g_n.

The energy of f in the cusp C is therefore finite since:

$$E_C(f) = \frac{1}{2}\int_C \|\mathrm{d}f\|^2 dV_m \leq \frac{1}{2}\int_0^{+\infty}\int_N \Big(4 + \|\mathrm{d}\varphi\|^2\Big)e^{-2t}dV_N dt < +\infty\ .$$

Case 2: $\rho(\gamma_0)$ is elliptic. In this case, there is no canonical "direction" in which to send the slices $HS \times \{t\}$ to infinity in $\mathbb{H}^n_{\mathbb{C}}$. Once the equivariant map f is constructed on $HS \times \{0\}$, the most natural way to define it on $HS \times \{t\}$ is to set $f_{|HS\times\{t\}} = f_{|HS\times\{0\}}$. Therefore, the growth of the energy density in the cusp cannot be controled by some decay of the metric in $\mathbb{H}^n_{\mathbb{C}}$, and we must control it at the start. We shall achieve this by demanding the equivariant map $HS \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ to be constant in the "vertical direction" $\mathbb{R}$ of $HS = \mathbb{C}\times\mathbb{R}$.

As mentionned before, $\Gamma_{\mathcal{N}}$ is a finite index subgroup of Γ_C and we have the tower of coverings:

$$HS = \mathbb{C}\times\mathbb{R} \xrightarrow{\Gamma_{\mathcal{N}}} \widehat{N} \xrightarrow{\Gamma_C/\Gamma_{\mathcal{N}}} N \ ,$$

where $\widehat{N} = (\mathbb{C}\times\mathbb{R})/\Gamma_{\mathcal{N}}$ is a circle bundle over the 2-torus $\mathbb{T} = \mathbb{C}/T(\Gamma_{\mathcal{N}})$, and $\Gamma_C/\Gamma_{\mathcal{N}}$ can be seen as a finite subgroup of $\mathrm{U}(1)$, acting freely on this bundle.

The group $\Gamma_C/\Gamma_{\mathcal{N}}$ is generated by a primitive p-th root of unity a and its action on $\mathbb{C}$ preserves the lattice $T(\Gamma_{\mathcal{N}}) \subset \mathbb{C}$. This implies that a is a root of a degree 2 polynomial with integer coefficients and hence the possible values of a are 1, -1, $e^{i\frac{2\pi}{3}}$, i, or $e^{i\frac{\pi}{3}}$. On the other hand, the number of possible lattices is also restricted:

- if $a = 1$ or $a = -1$, $T(\Gamma_{\mathcal{N}})$ can be any lattice of $\mathbb{C}$;
- if $a = i$, $T(\Gamma_{\mathcal{N}})$ must be a square lattice, meaning that we can choose the first two generators $\gamma_1 = (\xi_1, \nu_1)$ and $\gamma_2 = (\xi_2, \nu_2)$ of $\Gamma_{\mathcal{N}}$ so that $\xi_2 = i\xi_1$;
- if $a = e^{i\frac{2\pi}{3}}$ or $a = e^{i\frac{\pi}{3}}$, $T(\Gamma_{\mathcal{N}})$ must be an equilateral triangle lattice, meaning that we can choose γ_1 and γ_2 so that $\xi_2 = e^{i\frac{\pi}{3}}\xi_1$.

We start with the case $a = 1$, namely $\Gamma_{\mathcal{N}} = \Gamma_C$. We want to define a map φ from $\mathbb{C}$ to $\mathbb{H}^n_{\mathbb{C}}$ and then to extend φ to $\mathbb{C}\times\mathbb{R}$ by $\varphi(z,v) = \varphi(z)$, so that this extended map is equivariant w.r.t. the action of Γ_C. An obvious necessary condition is that $\varphi : \mathbb{C} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ must be equivariant w.r.t. the action of $T(\Gamma_C)$ on $\mathbb{C}$. Another necessary condition is that φ should send $\mathbb{C}$ to the fixed points set Fix_0 of $\rho(\gamma_0)$ in $\mathbb{H}^n_{\mathbb{C}}$. Indeed, for any $z\in\mathbb{C}$, $\gamma_0(z,0)$ belongs to $\{z\}\times\mathbb{R}$ and φ maps $\{z\}\times\mathbb{R}$ to the point $\varphi(z)$. These two conditions are also clearly sufficient.

So let $x_0 \in \mathbb{H}^n_{\mathbb{C}}$ be a fixed point of $\rho(\gamma_0)$ and set $\varphi(0) = x_0$. Since $\gamma_i = (\xi_i, \nu_i)$, $i = 1$ or 2, commutes with γ_0, the point $x_i = \rho(\gamma_i)x_0$ must also be fixed by $\rho(\gamma_0)$. Let σ_{0i} be the geodesic arc in $\mathbb{H}^n_{\mathbb{C}}$ joining x_0 to x_i. Note that Fix_0 is a convex subset of $\mathbb{H}^n_{\mathbb{C}}$ and hence σ_{0i} is included in Fix_0. Let φ map the segment $[0,\xi_i]$ onto σ_{0i}. We then map the segment $[\xi_1, \xi_1+\xi_2]$ to $\rho(\gamma_1)\sigma_{02}$ and the segment $[\xi_2, \xi_1+\xi_2]$ to $\rho(\gamma_2)\sigma_{01}$. This is well defined since $\rho(\gamma_1)(x_2) = \rho(\gamma_1\gamma_2)(x_0) = \rho(\gamma_2\gamma_1)\rho(\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2)x_0 = \rho(\gamma_2\gamma_1)\rho(\gamma_0^k)x_0 = \rho(\gamma_2\gamma_1)x_0 = \rho(\gamma_2)(x_1)$. Moreover, because of the commutation of γ_1 and γ_2 with γ_0, $\rho(\gamma_1)\sigma_{02}$ and $\rho(\gamma_2)\sigma_{01}$ are included in Fix_0.

Hence we get an equivariant map φ from the boundary of a fundamental domain of $T(\Gamma_C)$ in $\mathbb{C}$ to Fix_0 (φ can be made smooth, for example by taking it constant near 0, ξ_1 and ξ_2). We can therefore extend φ to a $T(\Gamma_C)$-equivariant map from $\mathbb{C}$ to Fix_0.

Define now $f : HS\times[0,+\infty) = \mathbb{C}\times\mathbb{R}\times[0,+\infty) \longrightarrow \mathrm{Fix}_0 \subset \mathbb{H}^n_{\mathbb{C}}$ by $f(z,v,t) = \varphi(z)$. The map f is ρ-equivariant and its energy density is:

$$\|\mathrm{d}f\|^2_{(x,t)} = |\mathrm{d}f(\frac{\partial}{\partial t})|^2_{\varphi(x)} + e^{2t}|\mathrm{d}f(\frac{\partial}{\partial v})|^2_{\varphi(x)} + e^t\sum_{k=3}^{4}|\mathrm{d}f(e_k)|^2_{\varphi(x)} = 0 + 0 + e^t\|\mathrm{d}\varphi\|^2_x$$

where $\|\mathrm{d}\varphi\|$ denotes the norm of the differential of $\varphi : \mathbb{C} \longrightarrow \mathrm{Fix}_0$ computed with the metrics of $\mathbb{C}\times\{0\} \subset HS \subset \mathbb{H}^m_{\mathbb{C}}$ and $\mathrm{Fix}_0 \subset \mathbb{H}^n_{\mathbb{C}}$ induced from g_m and g_n.

Therefore, $E_C(f) = \frac{1}{2}\int_0^{+\infty}\int_N \|\mathrm{d}\varphi\|^2 e^{-t}dV_N dt < +\infty$.

Now, consider the cases where $a \neq 1$. We want to proceed as we just did, namely, we want to first construct a map φ from $\mathbb{C}$ to $\mathbb{H}^n_{\mathbb{C}}$ and then extend it to HS by requiring that

$\varphi(z,v) = \varphi(z)$. The two conditions we mentioned are of course still necessary but we need to be more careful because of the action of $\Gamma_C/\Gamma_{\mathcal{N}}$.

Let γ_3 be an element of Γ_C such that $\gamma_3\Gamma_{\mathcal{N}} = a$. Then $\gamma_3 = (\xi_3, \nu_3, a)$ for some $\xi_3 \in \mathbb{C}$ and $\nu_3 \in \mathbb{R}$. It is easy to check that γ_0, γ_1, γ_2 and γ_3 generate Γ_C.

The first thing to notice is that the point $\zeta = \frac{\xi_3}{1-a}$ is fixed by the action of γ_3 on the $\mathbb{C}$-factor. Since we want φ to be constant on $\{\zeta\} \times \mathbb{R}$, φ must send ζ to a fixed point of $\rho(\gamma_3)$. This can be done because of the:

Claim 1.5. *Let $\gamma = (\xi, \nu, b) \in \Gamma_C$ be such that $b \neq 1$. Then $\rho(\gamma)$ and $\rho(\gamma_0)$ have a common fixed point in $\mathbb{H}^n_{\mathbb{C}}$.*

Proof. Since γ and γ_0 commute, $\rho(\gamma)$ stabilizes the totally geodesic submanifold Fix_0 of $\mathbb{H}^n_{\mathbb{C}}$. Let q be such that $b^q = 1$. Computing, we get $\gamma^q = (\xi, \nu, b)^q = ((\sum_{k=0}^{q-1} b^k)\xi, v, b^q) = (0, v, 1)$ for some $v \in \mathbb{R}$. Hence γ^q belongs to $\Gamma_{\mathcal{Z}}$: γ^q is a power of γ_0. The orbit under the group generated by $\rho(\gamma)$ of any point in Fix_0 must therefore be finite and this implies that the action of $\rho(\gamma)$ on Fix_0 has a fixed point. $\square$

With this in mind, it is now possible to complete the proof by constructing φ on the boundary ∂F of a fundamental domain F of the action of Γ_C on the $\mathbb{C}$-factor. Since γ_3 acts on $\mathbb{C}$ by rotation around its fixed point ζ, we can choose a fundamental domain G of the action of $T(\Gamma_{\mathcal{N}})$ on $\mathbb{C}$, centered at ζ and invariant by γ_3. For F we then take a fundamental domain for the action of γ_3 on G.

We do it in the case $a = e^{i\frac{\pi}{3}}$, the other cases are handled similarly.

The lattice $T(\Gamma_{\mathcal{N}})$ is generated by ξ_1 and $\xi_2 = e^{i\frac{\pi}{3}}\xi_1$. Let G be the regular hexagon centered at ζ with one vertex at the point $\zeta + \frac{1}{3}(\xi_1 + \xi_2)$. G is a fundamental domain for the action of $T(\Gamma_{\mathcal{N}})$ and it is invariant by γ_3. Let then F be the quadrilateral whose vertices are ζ, $\zeta + \frac{1}{2}\xi_1$, $\zeta + \frac{1}{3}(\xi_1 + \xi_2)$ and $\zeta + \frac{1}{2}\xi_2$. F is clearly a fundamental domain for the action of Γ_C on $\mathbb{C}$. See Fig. 1 for a picture.

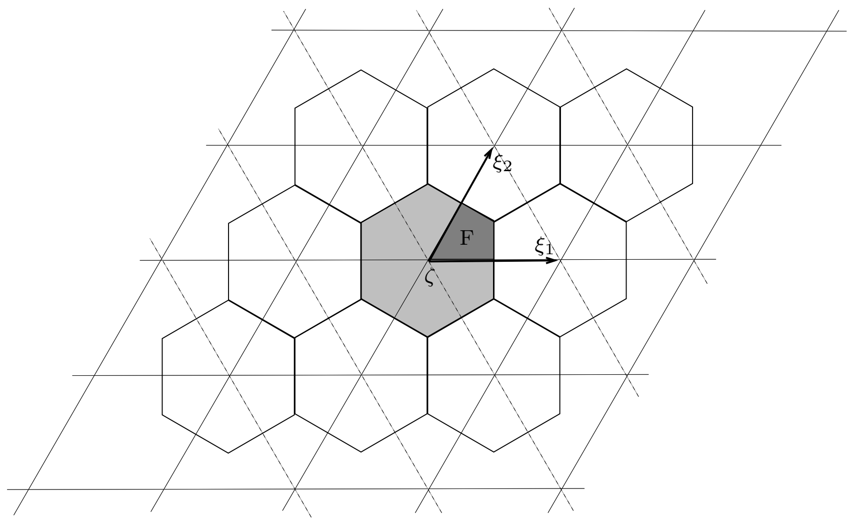

Fig. 1

Let now $x_0 \in \mathbb{H}^n_{\mathbb{C}}$ be a fixed point of both γ_0 and γ_3 (such a point exists by Claim 1.5). Set $\varphi(\zeta) = x_0$.

The point $\zeta + \frac{1}{2}\xi_1$ is fixed by $\gamma_1\gamma_3^3$, hence it must be sent by φ to a fixed point of $\rho(\gamma_1\gamma_3^3)$. It follows from Claim 1.5 that $\rho(\gamma_1\gamma_3^3)$ and $\rho(\gamma_0)$ have a common fixed point, say x_1. Let φ send the vertex $\zeta + \frac{1}{2}\xi_1$ to x_1 and the edge $[\zeta, \zeta + \frac{1}{2}\xi_1]$ of F to the geodesic arc σ_{01} joining x_0 to x_1 in Fix_0. Similarly, the vertex $\zeta + \frac{1}{3}(\xi_1 + \xi_2)$ is a fixed point of $\gamma_2\gamma_3^4$ and we let φ map it to a fixed point x_2 of $\rho(\gamma_2\gamma_3^4)$ in Fix_0. We map the edge $[\zeta + \frac{1}{2}\xi_1, \zeta + \frac{1}{3}(\xi_1 + \xi_2)]$ to the geodesic arc σ_{12} joining x_1 and x_2 in Fix_0.

Now the edge $[\zeta, \zeta+\frac{1}{2}\xi_2]$ is the image of $[\zeta, \zeta+\frac{1}{2}\xi_1]$ under γ_3 so we must map it to $\rho(\gamma_3)(\sigma_{01})$. In the same way, $[\zeta + \frac{1}{2}\xi_2, \zeta + \frac{1}{3}(\xi_1 + \xi_2)]$ is the image of $[\zeta + \frac{1}{2}\xi_1, \zeta + \frac{1}{3}(\xi_1 + \xi_2)]$ by $\gamma_2\gamma_3^4$ and we must therefore map it to $\rho(\gamma_2\gamma_3^4)(\sigma_{12})$. These definitions of φ agree at the point $\frac{1}{2}\xi_2$. Indeed, a simple computation shows that there exists q such that $\gamma_2\gamma_3 = \gamma_0^q\gamma_3\gamma_1$ and therefore, $\rho(\gamma_2\gamma_3^4)x_1 = \rho(\gamma_2\gamma_3\gamma_3^3)x_1 = \rho(\gamma_0^q\gamma_3\gamma_1\gamma_3^3)x_1 = \rho(\gamma_0^q\gamma_3)\rho(\gamma_1\gamma_3^3)x_1 = \rho(\gamma_3)\rho(\gamma_0^q)x_1 = \rho(\gamma_3)x_1$. Hence φ is well defined on ∂F. By construction, φ is equivariant w.r.t. γ_0 and the face-pairings γ_3 and $\gamma_2\gamma_3^4$ which generate the whole group Γ_C.

The construction of φ and f then goes on as in the case $a = 1$.

In this way we obtain a section f_i of the bundle $M \times_\rho \mathbb{H}^n_{\mathbb{C}}$ on each cusp C_i of M. This section can be extended to a section f defined on the whole manifold M and since the energy of $f_i : C_i \longrightarrow M \times_\rho \mathbb{H}^n_{\mathbb{C}}$ is finite for each i, the energy of $f : M \longrightarrow M \times_\rho \mathbb{H}^n_{\mathbb{C}}$ is finite and we are done. □

2. Pluriharmonicity and consequences

In this section, we study the properties of finite energy harmonic maps $\mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ which are equivariant w.r.t. a representation ρ of a torsion-free lattice $\Gamma < \mathrm{PU}(m, 1)$ into $\mathrm{PU}(n, 1)$.

2.1. **Pluriharmonicity.**

Theorem 2.1. *Let $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ be a ρ-equivariant harmonic map of finite energy. Then f is pluriharmonic, namely, the J-invariant part $(\nabla \mathrm{d}f)^{1,1}$ of $\nabla \mathrm{d}f$ vanishes identically. Moreover the complexified sectional curvature of $\mathbb{H}^n_{\mathbb{C}}$ is zero on $\mathrm{d}f(T^{1,0}\mathbb{H}^m_{\mathbb{C}})$.*

We first prove a general Bochner-type formula due to Mok, Siu and Yeung (cf. [MSY93]) in case Γ is a uniform lattice. We state it in the case of maps $\mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ but it is valid in the more general setting of equivariant maps from an irreducible (rank 1) symmetric space of non-compact type to a negatively curved manifold, as can be seen from the proof. Our exposition follows [Pa95].

Let R^m and R^n be the curvature tensors of g_m and g_n, and Q be any parallel tensor of curvature type on $\mathbb{H}^m_{\mathbb{C}}$. For h a symmetric 2-tensor with values in a vector bundle over $\mathbb{H}^m_{\mathbb{C}}$, define $(\overset{\circ}{Q}\, h)(X, Y) = \mathrm{tr}(W \longmapsto h(Q(W, X)Y, W))$.

Remark that if f is a ρ-equivariant map $\mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$, then, since Q is parallel and $\rho(\Gamma)$ acts by isometries on $\mathbb{H}^n_{\mathbb{C}}$, the $\mathbb{R}$-valued functions $\langle Q, f^\star R^n\rangle$ and $\langle \overset{\circ}{Q}\, \nabla \mathrm{d}f, \nabla \mathrm{d}f\rangle$ on $\mathbb{H}^m_{\mathbb{C}}$ are in fact ρ-invariant and hence can be considered as functions on $M = \Gamma\backslash\mathbb{H}^m_{\mathbb{C}}$:

Proposition 2.2. *Let f be a ρ-equivariant harmonic map of finite energy from $\mathbb{H}^m_{\mathbb{C}}$ to $\mathbb{H}^n_{\mathbb{C}}$ and Q a parallel tensor of curvature type on $\mathbb{H}^m_{\mathbb{C}}$. Then,*

$$(\diamondsuit) \qquad \int_M \left[\langle \overset{\circ}{Q}\, \nabla \mathrm{d}f, \nabla \mathrm{d}f\rangle - \frac{1}{2}\langle Q, f^\star R^n\rangle\right] dV_m = -\frac{1}{4m}\int_M \langle Q, R^m\rangle \|\mathrm{d}f\|^2\, dV_m\ ,$$

where if Γ is non-uniform, that is if M is non-compact, the left-hand side should read $\lim_{R\to\infty}\int_M \eta_R[\langle \overset{\circ}{Q}\ \nabla \mathrm{d}f, \nabla\mathrm{d}f\rangle - \frac{1}{2}\langle Q, f^\star R^n\rangle]dV_m$*, for* $\{\eta_R\}$ *a well-chosen family of cut-off functions on* M*.*

Proof. Let us first assume that M is compact. All the computations will be made in a normal coordinates system.

By definition, $(\overset{\circ}{Q}\ \nabla\mathrm{d}f)(X,Y) = \sum_k (\nabla\mathrm{d}f)(Q(e_k,X)Y,e_k)$. Since $\mathrm{d}f$ is closed, i.e. $\nabla\mathrm{d}f$ is symmetric, and Q is parallel, we have in fact

$$\overset{\circ}{Q}\ \nabla\mathrm{d}f(X,Y) = \sum_k (\nabla_{e_k}\mathrm{d}f)(Q(e_k,X)Y) = \sum_k (\nabla_{e_k}\mathrm{d}f\circ Q)(e_k,X)Y = -\nabla^\star(\mathrm{d}f\circ Q)(X,Y)\ ,$$

where $\nabla^\star$ is the formal adjoint of ∇: if T is a $(p+1)$-tensor, $(\nabla^\star T)(X_1,\dots,X_p) := -\mathrm{tr}(W\longmapsto (\nabla_W T)(W,X_1,\dots,X_p))$.

Integrating this relation over M (we assumed M compact), we get

$$\int_M \langle \overset{\circ}{Q}\ \nabla\mathrm{d}f,\nabla\mathrm{d}f\rangle dV_m = -\int_M \langle \nabla^\star(\mathrm{d}f\circ Q),\nabla\mathrm{d}f\rangle dV_m = -\int_M \langle \mathrm{d}f\circ Q,\nabla^2\mathrm{d}f\rangle dV_m$$

where $\nabla^2\mathrm{d}f$ is the 3-tensor $\nabla(\nabla\mathrm{d}f)$.

Using that Q, and hence $\mathrm{d}f\circ Q$, is skew-symmetric in its first two variables, one checks that

$$\langle \mathrm{d}f\circ Q,\nabla^2\mathrm{d}f\rangle = -\frac{1}{2}[\langle \mathrm{d}f\circ Q, f^\star R^n\rangle - \langle \mathrm{d}f\circ Q,\mathrm{d}f\circ R^m\rangle]\ .$$

We have $\langle \mathrm{d}f\circ Q, f^\star R^n\rangle = \langle Q, f^\star R^n\rangle$, where in the r.h.s. $f^\star R^n$ and Q are considered as (4,0)-tensors. Moreover, computations show that

$$\langle \mathrm{d}f\circ Q,\mathrm{d}f\circ R^m\rangle = \sum_{a,b}\frac{1}{2}\Big(\langle \iota_{e_a}Q,\iota_{e_b}R^m\rangle + \langle \iota_{e_b}Q,\iota_{e_a}R^m\rangle\Big) f^\star g_n(e_a,e_b)\ ,$$

where ι denotes interior product. Now, since M is locally symmetric, the symmetric 2-tensor θ given by

$$\theta(X,Y) = \frac{1}{2}(\langle \iota_X Q,\iota_Y R^m\rangle + \langle \iota_Y Q,\iota_X R^m\rangle)$$

is parallel. Thus it must be proportionnal to g_m (M is locally irreducible): $\theta = \frac{1}{2m}(\mathrm{tr}_{g_m}\theta)\, g_m$. Now, $\mathrm{tr}_{g_m}\theta = \langle Q,R^m\rangle$ and $\langle g_m, f^\star g_n\rangle = \|\mathrm{d}f\|^2$, so that $\langle \mathrm{d}f\circ Q,\mathrm{d}f\circ R^m\rangle = \frac{1}{2m}\langle Q,R^m\rangle\|\mathrm{d}f\|^2$ and hence

$$\int_M \langle \overset{\circ}{Q}\ \nabla\mathrm{d}f,\nabla\mathrm{d}f\rangle dV_m = \frac{1}{2}\int_M \left[\langle Q, f^\star R^n\rangle - \frac{1}{2m}\langle Q,R^m\rangle\|\mathrm{d}f\|^2\right] dV_m\ .$$

This ends the proof in the compact case.

Now assume M is non-compact of finite volume. The only global step in the preceding proof is the initial integration by parts. Thus we only have to show that this can be done in the finite volume case. We mimic the argument given by Corlette in [Co92].

As mentionned earlier, M is the union of a compact manifold with boundary M_0 and of a finite number of pairwise disjoint cusps C_i, each diffeomorphic to a compact $(2m-1)$-manifold N_i times $[0,+\infty)$. For each i, let t_i be the parameter in the $[0,+\infty)$ factor.

For $R>1$, we define a cut-off function η_R on M in the following manner. Take a smooth function η on $[0,+\infty)$ identically equal to 1 on $[0,1]$ and to 0 on $[2,+\infty)$. Set

$$\eta_R(x) = \begin{cases} 1 & \text{if } x\in M_0,\\ \eta\Big(\dfrac{t_i}{R}\Big) & \text{if } x\in C_i.\end{cases}$$

Since η_R is a horofunction along each cusp (cf. [Hel84], II.3.8), the absolute value $|\Delta\eta_R|$ of its Laplacian is bounded independently of R. Moreover, the norm $\|\mathrm{d}\eta_R\|$ of its differential is bounded by a constant times $\frac{1}{R}$.

Introducing η_R in the integration by parts, we obtain

$$\begin{aligned}
\int_M \eta_R\,\langle \overset{\circ}{Q}\,\nabla \mathrm{d}f, \nabla \mathrm{d}f\rangle dV_m &= -\int_M \langle \nabla^\star(\mathrm{d}f\circ Q), \eta_R\,\nabla\mathrm{d}f\rangle dV_m \\
&= -\int_M \langle \mathrm{d}f\circ Q, \eta_R\,\nabla^2\mathrm{d}f + \mathrm{d}\eta_R\otimes\nabla\mathrm{d}f\rangle dV_m \\
&= -\int_M \eta_R\,\langle \mathrm{d}f\circ Q, \nabla^2\mathrm{d}f\rangle dV_m - \int_M \langle \mathrm{d}f\circ Q, \mathrm{d}\eta_R\otimes\nabla\mathrm{d}f\rangle dV_m \\
&= \frac{1}{2}\int_M \eta_R\,\left[\langle Q, f^\star R^n\rangle - \frac{1}{2m}\,\langle Q, R^m\rangle\|\mathrm{d}f\|^2\right] dV_m \\
&\qquad - \int_M \langle \mathrm{d}f\circ Q, \mathrm{d}\eta_R\otimes\nabla\mathrm{d}f\rangle dV_m\ .
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_M \eta_R\left[\langle \overset{\circ}{Q}\,\nabla\mathrm{d}f, \nabla\mathrm{d}f\rangle - \frac{1}{2}\langle Q, f^\star R^n\rangle\right] dV_m &= -\frac{1}{4m}\int_M \eta_R\langle Q, R^m\rangle\|\mathrm{d}f\|^2 dV_m \\
&\qquad - \int_M \langle \mathrm{d}f\circ Q, \mathrm{d}\eta_R\otimes\nabla\mathrm{d}f\rangle dV_m\ .
\end{aligned}$$

The tensors Q and R^m are parallel and hence $\langle Q, R^m\rangle$ is constant on M. Therefore, $\langle Q, R^m\rangle\|\mathrm{d}f\|^2$ is integrable and the first term in the r.h.s. goes to $-\frac{1}{4m}\int_M\langle Q, R^m\rangle\|\mathrm{d}f\|^2 dV_m$ as R goes to infinity. On the other hand we have $\|\mathrm{d}\eta_R\| \leq \frac{C}{R}$ for some constant C independent of R and hence

$$\begin{aligned}
\left(\int_M \langle \mathrm{d}f\circ Q, \mathrm{d}\eta_R\otimes\nabla\mathrm{d}f\rangle dV_m\right)^2 &\leq \left(\int_M \|\mathrm{d}f\circ Q\|^2 dV_m\right)\left(\int_M \|\mathrm{d}\eta_R\otimes\nabla\mathrm{d}f\|^2\,dV_m\right) \\
&\leq \left(\int_M \frac{1}{2m}\,\|Q\|^2\|\mathrm{d}f\|^2 dV_m\right)\left(\int_M 2m\,\|\mathrm{d}\eta_R\|^2\,\|\nabla\mathrm{d}f\|^2\,dV_m\right) \\
&\leq \frac{C^2}{R^2}\left(\int_M \|Q\|^2\|\mathrm{d}f\|^2 dV_m\right)\left(\int_M \|\nabla\mathrm{d}f\|^2\,dV_m\right)\ .
\end{aligned}$$

Since Q is parallel, $\|Q\|$ is constant and $\int_M \|Q\|^2\|\mathrm{d}f\|^2 dV_m$ is finite.

The next lemma implies that $\lim_{R\to\infty}\int_M\langle \mathrm{d}f\circ Q, \mathrm{d}\eta_R\otimes\nabla\mathrm{d}f\rangle dV_m = 0$ and therefore ends the proof of Proposition 2.2. □

Lemma 2.3. $\|\nabla\mathrm{d}f\|$ *belongs to* $L^2(M)$*:* $\int_M \|\nabla\mathrm{d}f\|^2\,dV_m < +\infty$.

Proof. Because the energy density e of f is integrable on M, and using Green's formula, we see that

$$\int_M (\Delta e)\,\eta_R\,dV_m = \int_M e\,(\Delta\eta_R)\,dV_m$$

is bounded independently of R. Now, since we assumed that f is harmonic, the Bochner-type formula of Eells-Sampson ([ES64]) reads:

$$\Delta e = -2\|\nabla\mathrm{d}f\|^2 + \mathrm{Scal}(f^\star R^n) - \langle \mathrm{d}f\circ\mathrm{Ric}^m, \mathrm{d}f\rangle\ ,$$

where $\mathrm{Scal}(f^\star R^n)$ denotes the scalar contraction of the curvature tensor $f^\star R^n$ and Ric^m is the Ricci tensor of g_m seen as an endomorphism of $T\mathbb{H}^m_{\mathbb{C}}$. Since $\mathbb{H}^n_{\mathbb{C}}$ is negatively curved and $\mathrm{Ric}^m = -\frac{1}{2}(m+1)\mathrm{Id}$, we get $2\|\nabla\mathrm{d}f\|^2 \leq -\Delta e + (m+1)\,e$ and thus $\|\nabla\mathrm{d}f\|^2$ is integrable. □

Let us call I, resp. $I_{\mathbb{C}}$, the (3,1)-tensor of curvature type on $M = \Gamma\backslash\mathbb{H}^m_{\mathbb{C}}$ (or on $\mathbb{H}^m_{\mathbb{C}}$) given by $I(X,Y)Z = g_m(X,Z)Y - g_m(Y,Z)X$, resp. $I_{\mathbb{C}}(X,Y)Z = \frac{1}{4}(I(X,Y)Z + I(JX,JY)Z +$

$2g_m(JX,Y)JZ)$, for all $X,Y,Z \in T\mathbb{H}^m_{\mathbb{C}}$. The curvature tensor R^m of M (or of $\mathbb{H}^m_{\mathbb{C}}$) is just $-I_{\mathbb{C}}$. Both I and $I_{\mathbb{C}}$ are parallel tensors, and in fact they form a basis of the space of parallel tensors of curvature type on M. I and $I_{\mathbb{C}}$ will also denote the corresponding (4,0)-tensors.

We will apply the Bochner-type formula ($\diamondsuit$) to the parallel tensor of curvature type $Q = I_{\mathbb{C}} - I$.

Lemma 2.4. *Let $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ be a harmonic map and let $Q = I_{\mathbb{C}} - I$. Then $\langle \overset{\circ}{Q}\, \nabla \mathrm{d}f, \nabla \mathrm{d}f\rangle = -\frac{3}{2}\|(\nabla \mathrm{d}f)^{(1,1)}\|^2$, where $(\nabla \mathrm{d}f)^{(1,1)}$ is the J-invariant part of $\nabla \mathrm{d}f$: for all $X,Y \in T\mathbb{H}^m_{\mathbb{C}}$, $(\nabla \mathrm{d}f)^{(1,1)}(X,Y) := \frac{1}{2}[\nabla \mathrm{d}f(X,Y) + \nabla \mathrm{d}f(JX,JY)]$.*

Proof. A straightforward computation shows that for h a symmetric 2-tensor taking values in $f^\star T\mathbb{H}^n_{\mathbb{C}}$, $\overset{\circ}{I}\, h = h - g_m \operatorname{tr}_{g_m} h$ and $\overset{\circ}{I_{\mathbb{C}}}\, h(X,Y) = \frac{1}{4}[h(X,Y) - 3h(JX,JY) - g_m(X,Y)\operatorname{tr}_{g_m} h]$. Therefore, since $\operatorname{tr}_{g_m}\nabla \mathrm{d}f = 0$, $\overset{\circ}{Q}\, \nabla \mathrm{d}f = -\frac{3}{2}(\nabla \mathrm{d}f)^{(1,1)}$. The decomposition of a 2-tensor in J-invariant and J-skew-invariant parts is orthogonal, hence the result. □

Lemma 2.5. *$I_{\mathbb{C}}$ is the orthogonal projection of I onto the space of Kähler curvature type tensors, namely, the space of tensors of curvature type T such that $T(X,Y)JZ = J(T(X,Y)Z)$, for all $X,Y,Z \in T\mathbb{H}^m_{\mathbb{C}}$.*

Proof. Since $I_{\mathbb{C}}$ is clearly of Kähler curvature type, it remains to show that $I_{\mathbb{C}} - I$ is orthogonal to all tensors of Kähler curvature type. Simple computations show that if T is any tensor of curvature type, $\langle I,T\rangle = 2\,\mathrm{Scal}(T)$, whereas

$$\langle I_{\mathbb{C}},T\rangle = \frac{1}{2}\mathrm{Scal}(T) - \frac{1}{2}\sum_{k,l=1}^{2m}\Big(T(e_k,Je_l,Je_k,e_l) + T(e_k,Je_k,Je_l,e_l)\Big),$$

for $\{e_k\}$ an orthonormal basis of TM. It is then easy to check that if T is moreover of Kähler type, this last formula reduces to $\langle I_{\mathbb{C}},T\rangle = 2\,\mathrm{Scal}(T)$, hence the result. □

Let us recall what the *complexified sectional curvature* of a Hermitian manifold (N,g,J) is: if E and F are two vectors of the complexified tangent space $T^{\mathbb{C}}N = TN \otimes_{\mathbb{R}} \mathbb{C}$ of N then the complexified sectional curvature of the 2-plane they span is defined to be $R^N(E,F,\overline{E},\overline{F})$ where R^N is the curvature tensor of g extended by $\mathbb{C}$-linearity to $T^{\mathbb{C}}N$. Despite its name, the complexified sectional curvature takes real values.

If T is a tensor of curvature type, we define its *complexified scalar curvature* $\mathrm{Scal}_{\mathbb{C}}(T)$ as follows: $\mathrm{Scal}_{\mathbb{C}}(T) := \sum_{k,l=1}^m T(\zeta_k,\zeta_l,\overline{\zeta}_k,\overline{\zeta}_l)$, for $\{\zeta_k\}$ an orthonormal basis of the $(1,0)$-part of $T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}}$.

Using the formulae given in the proof of the previous lemma, one gets

Lemma 2.6. *$\langle I_{\mathbb{C}} - I, T\rangle = -6\,\mathrm{Scal}_{\mathbb{C}}(T)$.*

We are now ready to prove Theorem 2.1. Recall that $Q = I_{\mathbb{C}} - I$. First, Lemma 2.5 implies that the right-hand side in the Bochner-type formula ($\diamondsuit$) vanishes. Next, it follows from Lemma 2.4 and Lemma 2.6 that

$$\int_M \eta_R\left[\langle \overset{\circ}{Q}\, \nabla \mathrm{d}f, \nabla \mathrm{d}f\rangle - \frac{1}{2}\langle Q, f^\star R^n\rangle\right] dV_m = -\frac{3}{2}\int_M \eta_R\left[\|(\nabla \mathrm{d}f)^{(1,1)}\|^2 - 2\,\mathrm{Scal}_{\mathbb{C}}(f^\star R^n)\right] dV_m$$

for any $R > 1$. Thus, formula ($\diamondsuit$) reads:

$$\lim_{R\to\infty}\int_M \eta_R\left[\|(\nabla \mathrm{d}f)^{(1,1)}\|^2 - 2\,\mathrm{Scal}_{\mathbb{C}}(f^\star R^n)\right] dV_m \;=\; 0\;.$$

It is known that, since the sectional curvature of $(\mathbb{H}^n_{\mathbb{C}}, g_n)$ is pinched between -1 and $-\frac{1}{4}$, its complexified sectional curvature is non-positive (see for example [Her91]). Therefore, $\mathrm{Scal}_{\mathbb{C}}(f^\star R^n)$ being a mean of complexified sectional curvatures of $\mathbb{H}^n_{\mathbb{C}}$, it is non-positive. Thus $R \longmapsto \int_M \eta_R \left[\|(\nabla \mathrm{d}f)^{(1,1)}\|^2 - 2\,\mathrm{Scal}_{\mathbb{C}}(f^\star R^n)\right] dV_m$ is a non-negative non-decreasing function whose limit as R goes to infinity is zero. It follows at once that $(\nabla \mathrm{d}f)^{(1,1)}$ vanishes identically, that is, f is pluriharmonic. Finally, we also have $\mathrm{Scal}_{\mathbb{C}}(f^\star R^n) = 0$ everywhere and this implies that $R^n(f_\star\zeta_k, f_\star\zeta_l, \overline{f_\star\zeta_k}, \overline{f_\star\zeta_l}) = 0$ for all $k, l = 1, \ldots, n$. Theorem 2.1 is proved.

2.2. **Holomorphicity of "high" rank harmonic maps.**

Let $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ be a finite energy harmonic map, equivariant w.r.t a representation of a torsion-free lattice $\Gamma < \mathrm{PU}(m,1)$ into $\mathrm{PU}(n,1)$. In this section, we exploit the full strength of Theorem 2.1 to prove a result that was first obtained by J. A. Carlson and D. Toledo in [CT89] in the case Γ is cocompact and $\rho(\Gamma)$ is discrete in $\mathrm{PU}(n,1)$ or in a more general target Lie group G. Their proof relies on a careful study of maximal abelian subalgebras of the complexification of the Lie algebra of G. In our setting, the simple form of the curvature tensor of $(\mathbb{H}^n_{\mathbb{C}}, g_n)$ allows a more elementary proof that we give for completeness.

Proposition 2.7. *Let $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ be a finite energy harmonic map equivariant w.r.t. a representation ρ of the torsion-free lattice $\Gamma < \mathrm{PU}(m,1)$ in $\mathrm{PU}(n,1)$. If the real rank of f is at least 3 at some point, then f is holomorphic or anti-holomorphic.*

Before proving this proposition, we introduce some notations that will be needed in the proof and later on in the paper.

For $l = m, n$, let $T^{\mathbb{C}}\mathbb{H}^l_{\mathbb{C}} = T\mathbb{H}^l_{\mathbb{C}} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $T\mathbb{H}^l_{\mathbb{C}}$ and $T^{\mathbb{C}}\mathbb{H}^l_{\mathbb{C}} = T^{1,0}\mathbb{H}^l_{\mathbb{C}} \oplus T^{0,1}\mathbb{H}^l_{\mathbb{C}}$ be its decomposition in (1,0) and (0,1) part. We extend the differential of f by $\mathbb{C}$-linearity and still write $\mathrm{d}f : T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}} \longrightarrow T^{\mathbb{C}}\mathbb{H}^n_{\mathbb{C}}$ (if the distinction is necessary we will use $\mathrm{d}^{\mathbb{C}}f$). Its components are

$$\partial^{1,0} f : T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}} \longrightarrow T^{1,0}\mathbb{H}^n_{\mathbb{C}} \ ,$$
$$\partial^{0,1} f : T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}} \longrightarrow T^{0,1}\mathbb{H}^n_{\mathbb{C}} \ .$$

We extend g_n by $\mathbb{C}$-linearity to $T^{\mathbb{C}}\mathbb{H}^n_{\mathbb{C}}$. We will sometimes write $(X,Y) = g_n(X,Y)$ and $|X|^2 = (X, \overline{X})$ for $X, Y \in T^{\mathbb{C}}\mathbb{H}^n_{\mathbb{C}}$.

If $(e_i)_{1\leq i\leq 2m} = (e_\alpha, Je_\alpha)_{1\leq\alpha\leq m}$ is an orthonormal $\mathbb{R}$-basis of $T\mathbb{H}^m_{\mathbb{C}}$, we set $z_\alpha = \frac{1}{2}(e_\alpha - iJe_\alpha)$. $(z_\alpha)_{1\leq\alpha\leq m}$ is an orthogonal $\mathbb{C}$-basis of $T^{1,0}\mathbb{H}^m_{\mathbb{C}}$.

To lighten the notations, we will sometimes use f_α instead of $\partial^{1,0}f(z_\alpha)$ and $f_{\overline{\beta}}$ instead of $\partial^{1,0}f(\bar{z}_\beta)$, so that $\partial^{0,1}f(\bar{z}_\alpha) = \bar{f}_\alpha$ and $\partial^{0,1}f(z_\beta) = \bar{f}_{\overline{\beta}}$.

In the sequel, we will often restrict $\partial^{1,0}f$, resp. $\partial^{0,1}f$, to $T^{1,0}\mathbb{H}^m_{\mathbb{C}}$ and consider them as sections of $\mathrm{Hom}(T^{1,0}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{1,0}\mathbb{H}^n_{\mathbb{C}})$, resp. $\mathrm{Hom}(T^{1,0}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{0,1}\mathbb{H}^n_{\mathbb{C}})$. We will call $e'(f)$, resp. $e''(f)$, the square of the norm of $\partial^{1,0}f$, resp. $\partial^{0,1}f$, namely:

$$\begin{aligned} e'(f) &:= \|\partial^{1,0}f\|^2 = 2\sum_{\alpha=1}^m g_n(\partial^{1,0}f(z_\alpha), \overline{\partial^{1,0}f(z_\alpha)}) = 2\sum_{\alpha=1}^m |f_\alpha|^2 \ , \\ e''(f) &:= \|\partial^{0,1}f\|^2 = 2\sum_{\alpha=1}^m g_n(\partial^{0,1}f(z_\alpha), \overline{\partial^{0,1}f(z_\alpha)}) = 2\sum_{\alpha=1}^m |f_{\overline{\alpha}}|^2 \ . \end{aligned}$$

Note that with these definitions, the energy density of f is given by $e(f) := \frac{1}{2}\|\mathrm{d}f\|^2 = e'(f) + e''(f)$. Again, we will often abreviate $e'(f)$ and $e''(f)$ to e' and e'' when no confusion is possible.

Proof of Proposition 2.7. Theorem 2.1 shows that $f^\star R^n(z_\alpha, z_\beta, \bar z_\alpha, \bar z_\beta) = 0$ for all $\alpha, \beta \in \{1, \ldots, m\}$. Let us see what this implies in more details.

$$\begin{aligned}
f^\star R^n(z_\alpha, z_\beta, \bar z_\alpha, \bar z_\beta) &= R^n(f_\alpha + \bar f_{\overline{\alpha}}, f_\beta + \bar f_{\overline{\beta}}, \bar f_\alpha + f_{\overline{\alpha}}, \bar f_\beta + f_{\overline{\beta}}) \\
&= -\tfrac{1}{2}\Big[|f_\alpha|^2|f_{\overline{\beta}}|^2 + |f_{\overline{\alpha}}|^2|f_\beta|^2 - (f_{\overline{\alpha}}, \bar f_{\overline{\beta}})(f_\alpha, \bar f_\beta) \\
&\quad -(\bar f_\beta, \bar f_\alpha)(f_{\overline{\beta}}, \bar f_{\overline{\alpha}}) - (f_\alpha, \bar f_{\overline{\beta}})(f_{\overline{\alpha}}, \bar f_\beta) - (f_\beta, \bar f_{\overline{\alpha}})(f_{\overline{\beta}}, \bar f_\alpha) \\
&\qquad +(f_\beta, \bar f_{\overline{\alpha}})(f_{\overline{\alpha}}, \bar f_\beta) + (f_\alpha, \bar f_{\overline{\beta}})(f_{\overline{\beta}}, \bar f_\alpha)\Big] \\
&= -\tfrac{1}{2}\big(\|f_\alpha \wedge \bar f_{\overline{\beta}} - f_\beta \wedge \bar f_{\overline{\alpha}}\|^2 + |(f_\alpha, \bar f_{\overline{\beta}}) - (f_\beta, \bar f_{\overline{\alpha}})|^2\big)
\end{aligned}$$

Therefore, for all α and β, we have $f_\alpha \wedge \bar f_{\overline{\beta}} = f_\beta \wedge \bar f_{\overline{\alpha}}$.

Suppose that the families $(f_\alpha)_{1\le\alpha\le m}$ and $(f_{\overline{\beta}})_{1\le\beta\le m}$ are both of rank less than or equal to 1. We may assume that for all α there exits λ_α such that $f_\alpha = \lambda_\alpha f_1$ and that for some k and for all β there exists μ_β such that $f_{\overline{\beta}} = \mu_\beta f_{\overline{k}}$.

If $f_1 = 0$, then for all α, $\mathrm{d}^{\mathbb{C}} f(z_\alpha) = \bar f_{\overline{\alpha}}$ and $\mathrm{d}^{\mathbb{C}} f(\bar z_\alpha) = f_{\overline{\alpha}}$. Therefore the complex rank of $\mathrm{d}^{\mathbb{C}} f$ is at most 2, namely the real rank of $\mathrm{d}f$ is at most 2. The same is true if $f_{\overline{k}} = 0$.

If both f_1 and $f_{\overline{k}}$ are non zero, then from the fact that $f_1 \wedge \bar f_{\overline{\alpha}} = f_\alpha \wedge \bar f_{\overline{1}}$ we deduce that $\overline{\mu_\alpha} f_1 \wedge \bar f_{\overline{k}} = \lambda_\alpha \overline{\mu_1} f_1 \wedge \bar f_{\overline{k}}$, that is, $\overline{\mu_\alpha} = \lambda_\alpha \overline{\mu_1}$. Then,

$$\mathrm{d}^{\mathbb{C}} f(z_\alpha) = f_\alpha + \bar f_{\overline{\alpha}} = \lambda_\alpha f_1 + \overline{\mu_\alpha} \bar f_{\overline{k}} = \lambda_\alpha f_1 + \lambda_\alpha \overline{\mu_1} \bar f_{\overline{k}} = \lambda_\alpha (f_1 + \bar f_{\overline{1}}) = \lambda_\alpha\, \mathrm{d}^{\mathbb{C}} f(z_1)\ .$$

Hence the family $(\mathrm{d}^{\mathbb{C}} f(z_\alpha))_{1\le\alpha\le m}$ has rank ≤ 1. This also holds for the family $(\mathrm{d}^{\mathbb{C}} f(\bar z_\beta))_{1\le\beta\le m}$ and we conclude that the real rank of f is again less than or equal to 2.

In any case, we see that if the real rank of f is at least 3 at some point, then the rank of one of the families $(f_\alpha)_{1\le\alpha\le m}$ and $(f_{\overline{\beta}})_{1\le\beta\le m}$ is at least 2 at this point.

Suppose now that at some point of $\mathbb{H}^m_{\mathbb{C}}$, the rank of the family $(f_\alpha)_{1\le\alpha\le m}$ is at least 2, for example $f_1 \wedge f_2 \neq 0$. Then $f_1 \wedge \bar f_{\overline{2}} = f_2 \wedge \bar f_{\overline{1}}$ implies $f_{\overline{1}} = f_{\overline{2}} = 0$. From $f_1 \wedge \bar f_{\overline{\gamma}} = f_\gamma \wedge \bar f_{\overline{1}}$, we conclude that $f_{\overline{\gamma}} = 0$ for all $1 \le \gamma \le m$, i.e. $e'' = 0$ at this point.

In the same way, if the rank of the family $(f_{\overline{\beta}})_{1\le\beta\le m}$ is at least 2, then $e' = 0$.

Finally, since f is pluriharmonic and the complexified sectional curvature of $\mathbb{H}^n_{\mathbb{C}}$ is zero on $\mathrm{d}f(T^{1,0}\mathbb{H}^m_{\mathbb{C}})$ (see Theorem 2.1 above), and because $\mathbb{H}^n_{\mathbb{C}}$ is a Kähler symmetric space, it is known that $\partial^{1,0} f$, resp. $\partial^{0,1} f$, are holomorphic sections of the holomorphic bundles $\mathrm{Hom}(T^{1,0}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{1,0}\mathbb{H}^n_{\mathbb{C}})$, resp. $\mathrm{Hom}(T^{1,0}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{0,1}\mathbb{H}^n_{\mathbb{C}})$ ([CT89], Theorem 2.3). So they have a generic rank on $\mathbb{H}^m_{\mathbb{C}}$. Therefore, if for example the family $(f_\alpha)_{1\le\alpha\le m}$ has rank at least 2 at some point, it has rank at least 2 on a dense open subset of $\mathbb{H}^m_{\mathbb{C}}$ and so $e'' = 0$ on a dense open subset of $\mathbb{H}^m_{\mathbb{C}}$, hence everywhere, and f is holomorphic. Similarly, if $\mathrm{rk}(f_{\overline{\beta}},\ 1 \le \beta \le m) \ge 2$ at some point, f is antiholomorphic. $\square$

2.3. **Some technical lemmas.**

If $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ is a ρ-equivariant pluriharmonic map whose rank is at most 2 everywhere, f needs not be holomorphic nor antiholomorphic. Nevertheless, pluriharmonicity has other consequences that will be useful later. Namely, if $e'(f)$ and $e''(f)$ are the previously defined squared norms of $\partial^{1,0} f$ and $\partial^{0,1} f$, we have

$$\langle f^\star \omega_n, \omega_m \rangle := \sum_{i,j=1}^{2m} f^\star \omega_n(e_i, e_j)\omega_m(e_i, e_j) = 2(e'(f) - e''(f))\ ,$$

as it is easy to check. Because of this, some results on the energies $e'(f)$ and $e''(f)$ will be needed in the proof of Theorem 3.1 and we shall prove them in this section.

The results stated here were obtained in complex dimension 1 in [To79] and [Wo79]. The proofs in the general case (see also [Lic70]) go almost exactly as in the case $m = 1$ and they are given only for completeness and to fix the notations.

We will work on the complexifications of the tangent spaces of $\mathbb{H}^m_{\mathbb{C}}$ and $\mathbb{H}^n_{\mathbb{C}}$ and therefore we extend all needed sections, tensors and operators defined on the real tangent spaces by $\mathbb{C}$-linearity to these complexifications.

Since $\partial^{1,0}f$ can be considered as a section of $\mathrm{Hom}(T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{\mathbb{C}}\mathbb{H}^n_{\mathbb{C}})$, we can define its covariant derivative $\nabla\partial^{1,0}f \in \mathrm{Hom}(T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}} \otimes T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{\mathbb{C}}\mathbb{H}^n_{\mathbb{C}})$. It follows easily from the fact that $(\mathbb{H}^n_{\mathbb{C}}, g_n)$ is Kähler that $\nabla\partial^{1,0}f$ belongs in fact to $\mathrm{Hom}(T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}} \otimes T^{\mathbb{C}}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{1,0}\mathbb{H}^n_{\mathbb{C}})$. We will call $\nabla'\partial^{1,0}f \in \mathrm{Hom}(T^{1,0}\mathbb{H}^m_{\mathbb{C}} \otimes T^{1,0}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{1,0}\mathbb{H}^n_{\mathbb{C}})$ its restriction to $T^{1,0}\mathbb{H}^m_{\mathbb{C}} \otimes T^{1,0}\mathbb{H}^m_{\mathbb{C}}$. We define $\nabla'\partial^{0,1}f \in \mathrm{Hom}(T^{1,0}\mathbb{H}^m_{\mathbb{C}} \otimes T^{1,0}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{0,1}\mathbb{H}^n_{\mathbb{C}})$ and $\nabla'\mathrm{d}f = \nabla'\partial^{1,0}f + \nabla'\partial^{0,1}f \in \mathrm{Hom}(T^{1,0}\mathbb{H}^m_{\mathbb{C}} \otimes T^{1,0}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{\mathbb{C}}\mathbb{H}^n_{\mathbb{C}})$ similarly. Note that $\|\nabla'\mathrm{d}f\|$, $\|\nabla'\partial^{1,0}f\|$ and $\|\nabla'\partial^{0,1}f\|$ belong to $L^2(M)$ because $\|\nabla\mathrm{d}f\|$ does (Lemma 2.3).

In the entire section, f denotes a ρ-equivariant *pluriharmonic* map $\mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$.

Lemma 2.8. *We have*

$$\frac{1}{4}\Delta e' = -\frac{1}{2}\|\nabla'\partial^{1,0}f\|^2 - 2R' + \frac{m+1}{4}\,e' \quad \text{and} \quad \frac{1}{4}\Delta e'' = -\frac{1}{2}\|\nabla'\partial^{0,1}f\|^2 - 2R'' + \frac{m+1}{4}\,e''$$

where

$$\begin{aligned} R' &= \textstyle\sum_{\alpha,\beta} R^n(\mathrm{d}f(z_\alpha), \mathrm{d}f(\bar z_\alpha), \partial^{1,0}f(z_\beta), \overline{\partial^{1,0}f(z_\beta)}) \ , \\ R'' &= \textstyle\sum_{\alpha,\beta} R^n(\mathrm{d}f(z_\alpha), \mathrm{d}f(\bar z_\alpha), \partial^{0,1}f(z_\beta), \overline{\partial^{0,1}f(z_\beta)}) \ . \end{aligned}$$

Proof. We make the computation for $\Delta e'$, and we use normal coordinates:

$$\frac{1}{4}\Delta e' = -\sum_\alpha \nabla\mathrm{d}e'(\bar z_\alpha, z_\alpha) = -\sum_\alpha \bar z_\alpha . z_\alpha . e' \ .$$

Now, $z_\alpha . \langle \partial^{1,0}f, \overline{\partial^{1,0}f}\rangle = \langle \nabla_{z_\alpha}\partial^{1,0}f, \overline{\partial^{1,0}f}\rangle + \langle \partial^{1,0}f, \nabla_{z_\alpha}\overline{\partial^{1,0}f}\rangle$. The map f is pluriharmonic and therefore $\nabla\mathrm{d}f(Z,\bar W) = \nabla\mathrm{d}f(\bar Z, W) = 0$ for all Z, W in $T^{1,0}\mathbb{H}^m_{\mathbb{C}}$. Since $\mathbb{H}^m_{\mathbb{C}}$ is Kähler, $\nabla_{\bar z_\alpha}\partial^{1,0}f = \left(\nabla_{\bar z_\alpha}\mathrm{d}f\right)^{1,0}$ and hence vanishes identically on $T^{1,0}\mathbb{H}^m_{\mathbb{C}}$. It follows that $\nabla_{z_\alpha}\overline{\partial^{1,0}f} = \overline{\nabla_{\bar z_\alpha}\partial^{1,0}f} = 0$. Thus,

$$\begin{aligned} \frac{1}{4}\Delta e' &= -\sum_\alpha \bar z_\alpha . \langle \nabla_{z_\alpha}\partial^{1,0}f, \overline{\partial^{1,0}f}\rangle \\ &= -\sum_\alpha \left[\langle \nabla_{\bar z_\alpha}\nabla_{z_\alpha}\partial^{1,0}f, \overline{\partial^{1,0}f}\rangle + \langle \nabla_{z_\alpha}\partial^{1,0}f, \overline{\nabla_{z_\alpha}\partial^{1,0}f}\rangle \right] \\ &= -\frac{1}{2}\|\nabla'\partial^{1,0}f\|^2 - \sum_\alpha \langle \nabla_{\bar z_\alpha}\nabla_{z_\alpha}\partial^{1,0}f, \overline{\partial^{1,0}f}\rangle \ . \end{aligned}$$

Therefore, since $(\nabla_{\bar z_\alpha}\nabla_{z_\alpha}\partial^{1,0}f)(z_\beta) = (\nabla_{\bar z_\alpha}\nabla_{z_\alpha}\partial^{1,0}f)(z_\beta) - (\nabla_{z_\alpha}\nabla_{\bar z_\alpha}\partial^{1,0}f)(z_\beta)$, we have

$$\begin{aligned} \sum_\alpha (\nabla_{\bar z_\alpha}\nabla_{z_\alpha}\partial^{1,0}f)(z_\beta) &= \sum_\alpha R^n(\mathrm{d}f(z_\alpha), \mathrm{d}f(\bar z_\alpha))\partial^{1,0}f(z_\beta) - \sum_\alpha \partial^{1,0}f(R^m(z_\alpha, \bar z_\alpha)z_\beta) \\ &= \sum_\alpha R^n(\mathrm{d}f(z_\alpha), \mathrm{d}f(\bar z_\alpha))\partial^{1,0}f(z_\beta) + \frac{1}{2}\,\partial^{1,0}f(\mathrm{Ric}^m(z_\beta)) \ . \end{aligned}$$

The result follows since the Ricci curvature tensor of g_m is $-\frac{m+1}{2}g_m$. □

We also have

Lemma 2.9. *At each point of $\mathbb{H}^m_{\mathbb{C}}$ where $e' \neq 0$, resp. $e'' \neq 0$,*

$$\frac{1}{4}\Delta \log e' = -\alpha' - 2\,\frac{R'}{e'} + \frac{m+1}{4} \ , \ \text{resp.} \ \ \frac{1}{4}\Delta \log e'' = -\alpha'' - 2\,\frac{R''}{e''} + \frac{m+1}{4} \ ,$$

where

$$\alpha' = \frac{1}{2e'^2}\Big(\|\nabla'\partial^{1,0}f\|^2 e' - \|\langle\nabla'\partial^{1,0}f,\overline{\partial^{1,0}f}\rangle\|^2\Big)$$

and

$$\alpha'' = \frac{1}{2e''^2}\Big(\|\nabla'\partial^{0,1}f\|^2 e'' - \|\langle\nabla'\partial^{0,1}f,\overline{\partial^{0,1}f}\rangle\|^2\Big)$$

are both nonnegative by Cauchy-Schwarz's inequality.

Proof. Again, we make this (easy) computation only for $\frac{1}{4}\Delta\log e' = \frac{1}{4e'}\Delta e' + \frac{1}{4e'^2}\|\mathrm{d}e'\|^2$. Now,

$$\frac{1}{4}\|\mathrm{d}e'\|^2 = \sum_\alpha \mathrm{d}e'(z_\alpha)\mathrm{d}e'(\bar z_\alpha) = \sum_\alpha \langle\nabla_{z_\alpha}\partial^{1,0}f,\overline{\partial^{1,0}f}\rangle\langle\partial^{1,0}f,\nabla_{\bar z_\alpha}\overline{\partial^{1,0}f}\rangle = \frac{1}{2}\|\langle\nabla'\partial^{1,0}f,\overline{\partial^{1,0}f}\rangle\|^2$$

where $\langle\nabla'\partial^{1,0}f,\overline{\partial^{1,0}f}\rangle$ denotes the 1-form on $T^{1,0}\mathbb{H}^m_{\mathbb{C}}$ given by $\langle\nabla'\partial^{1,0}f,\overline{\partial^{1,0}f}\rangle(z) = \langle\nabla_z\partial^{1,0}f,\overline{\partial^{1,0}f}\rangle = 2\sum_\beta g_n\big((\nabla_z\partial^{1,0}f)(z_\beta),\overline{\partial^{1,0}f}(z_\beta)\big)$. □

Finally, easy computations show that:

Lemma 2.10.

$$R' = \frac{1}{2}\sum_{\alpha,\beta}\Big[\big|g_n(\partial^{1,0}f(z_\alpha),\overline{\partial^{1,0}f(z_\beta)})\big|^2 - \big|g_n(\partial^{1,0}f(z_\beta),\overline{\partial^{1,0}f(\bar z_\alpha)})\big|^2\Big] + \frac{1}{8}e'(e'-e'')\ ,$$

$$R'' = \frac{1}{2}\sum_{\alpha,\beta}\Big[\big|g_n(\partial^{1,0}f(\bar z_\alpha),\overline{\partial^{1,0}f(\bar z_\beta)})\big|^2 - \big|g_n(\partial^{1,0}f(z_\alpha),\overline{\partial^{1,0}f(\bar z_\beta)})\big|^2\Big] + \frac{1}{8}e''(e''-e')\ .$$

Remark. In the sequel, we shall use the fact that all the functions involved in those three lemmas are well defined on $M = \Gamma\backslash\mathbb{H}^m_{\mathbb{C}}$.

3. Rigidity of representations of lattices of PU$(m,1)$ into PU$(n,1)$

3.1. Burger-Iozzi invariant.

We again assume that $m \geq 2$.

Let Γ be a torsion-free lattice in PU$(m,1)$, and let $\rho : \Gamma \longrightarrow$ PU$(n,1)$ be a homomorphism. M. Burger and A. Iozzi assign to ρ an invariant which can be defined as follows (see [BI01]).

Take any ρ-equivariant map $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ and consider the pull-back $f^\star\omega_n$ of the Kähler form ω_n of $\mathbb{H}^n_{\mathbb{C}}$. Note that we can consider $f^\star\omega_n$ as a 2-form on $M = \Gamma\backslash\mathbb{H}^m_{\mathbb{C}}$. The de Rham cohomology class $[f^\star\omega_n] \in H^2_{DR}(M)$ defined by $f^\star\omega_n$ is independent of the choice of the equivariant map f since all such maps are homotopic, and therefore we call it $[\rho^\star\omega_n]$.

Now, Burger and Iozzi remark that the class $[\rho^\star\omega_n]$ is in the image of the natural comparison map from the L^2-cohomology group $H^2_{(2)}(M)$ of M to the de Rham cohomology group $H^2_{DR}(M)$. Since $m \geq 2$, the comparison map is injective (see [Zu82]; the arithmeticity of the lattice Γ is not necessary for the result in the present case), this yields a well-defined L^2-cohomology class, denoted by $[\rho^\star\omega_n]_{(2)}$, and they define (in a slightly different form)

$$\tau(\rho) := \frac{1}{2m}\int_M \langle\rho^\star\omega_n,\omega_m\rangle dV_m\ ,$$

where $\rho^\star\omega_n$ is any L^2-form representing $[\rho^\star\omega_n]_{(2)}$ (observe that, because ω_m is parallel, $\tau(\rho)$ depends only on $[\rho^\star\omega_n]_{(2)}$, hence on ρ).

Remark. In complex dimension 1 and for a *uniform* lattice Γ, $\tau(\rho)$ is also well-defined and coincides with the classical Toledo invariant (cf. [To89]).

The main result of [BI01] then reads:

Theorem 3.1. *Under the above assumptions,*

$$\left|\tau(\rho)\right| \leq \mathrm{Vol}(M)\ .$$

Moreover, equality holds if and only if there exists a ρ-equivariant totally geodesic isometric embedding $\mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$.

Let Γ be a torsion-free lattice in $\mathrm{SU}(m,1)$. Via the natural inclusion of $\mathrm{SU}(m,1)$ into $\mathrm{PU}(n,1)$ for $n > m$, we obtain the so-called standard representation of Γ into $\mathrm{PU}(n,1)$ (which is of course $\mathbb{C}$-Fuchsian). Theorem 3.1 then implies:

Corollary 3.2. *Let Γ be a torsion-free lattice in $\mathrm{SU}(m,1)$, $m \geq 2$. Then any deformation of the standard representation of Γ into $\mathrm{PU}(n,1)$ $(n > m)$ is also $\mathbb{C}$-Fuchsian.*

Proof. Since Γ is torsion-free, it projects isomorphically into $\mathrm{PU}(m,1)$. We can therefore consider the standard representation of Γ as a representation of a lattice of $\mathrm{PU}(m,1)$ and apply Theorem 3.1. Now, when seen as a cohomology class in $H^2_{DR}(M)$, $[\rho^\star\omega_n]$ is a characteristic class of the principal $\mathrm{PU}(n,1)$-bundle over M associated to ρ and so, it is constant on connected components of $\mathrm{Hom}(\Gamma,\mathrm{PU}(n,1))$. On the other hand, $\left|\tau(\rho)\right| = \mathrm{Vol}(M)$ holds when ρ is the standard representation of Γ into $\mathrm{PU}(n,1)$, hence the result. $\square$

The main tool in [To89] and [BI01] is bounded cohomology. Corlette in [Co88] was the first to obtain Corollary 3.2 for $m \geq 2$ and Γ cocompact. He worked with an invariant similar to $\tau(\rho)$, the *volume* of ρ (see the introduction and the remark at the end of Section 3.2), and he used harmonic maps techniques to obtain a result equivalent to Theorem 3.1.

We will now show that Theorem 3.1 is a consequence of our results on harmonic maps.

3.2. **Proof of Theorem 3.1.**

The next two lemmas have important consequences. They show that the invariant is well-defined, that is, that the de Rham cohomology class $[\rho^\star\omega_n]$ can be represented by an L^2-form. Moreover, they imply that we don't have to worry about non reductive representations and that we can use the finite energy harmonic map we constructed to compute the invariant in the case of reductive ones.

Lemma 3.3. *Suppose the representation ρ is not reductive. Then $[\rho^\star\omega_n] = 0$ and hence $\tau(\rho) = 0$.*

Proof. Let f be any ρ-equivariant map from $\mathbb{H}^m_{\mathbb{C}}$ to $\mathbb{H}^n_{\mathbb{C}}$. The image $\rho(\Gamma)$ fixes a point at infinity in $\mathbb{H}^n_{\mathbb{C}}$ which we can take to be ∞ in the Siegel model. We then have $\omega_n = -\mathrm{dd}^c t'$ where $t' = \log(2\mathrm{Re}(w') - \langle\langle z', z'\rangle\rangle)$ (see section 1). As can be easily checked, the 1-form $\varsigma := -\mathrm{d}^c t'$ is invariant by the stabilizer in $\mathrm{PU}(n,1)$ of the fixed point. Therefore, the form $f^\star\varsigma$ on $\mathbb{H}^m_{\mathbb{C}}$ goes down to a form defined on the whole quotient M such that $f^\star\omega_n = \mathrm{d}f^\star\varsigma$. Hence the de Rham cohomology class of $f^\star\omega_n$ is zero and the invariant $\tau(\rho)$ vanishes. $\square$

We may now assume that ρ is reductive. Theorems 1.1 and 1.2 then guarantee the existence of a finite energy ρ-equivariant harmonic map $f : \mathbb{H}^m_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$.

We first prove that we can use f to compute $\tau(\rho)$. This follows from the

Lemma 3.4. *The 2-form $f^\star\omega_n$ is L^2.*

Proof. If f is (anti)holomorphic, it follows from the generalization of the Schwarz-Pick lemma (see [Ko70]) that in an obvious sense, $f^\star g_n \leq g_m$. An easy computation then shows that $\|f^\star\omega_n\| \leq 2m$ everywhere on M: $f^\star\omega_n$ is a L^2-form.

Assume now that f is not (anti)holomorphic. Then Proposition 2.7 implies that $\mathrm{rk}_{\mathbb{R}} f \leq 2$. When $\mathrm{rk}_{\mathbb{R}}\, f \leq 1$, $f^\star\omega_n = 0$, whereas $\mathrm{rk}_{\mathbb{R}}\, \mathrm{d}_x f = 2$ (or $\mathrm{rk}_{\mathbb{C}}\, \mathrm{d}^{\mathbb{C}}_x f = 2$) is equivalent to

$$\dim_{\mathbb{C}} \mathrm{d}^{\mathbb{C}}_x f(T^{1,0}\mathbb{H}^m_{\mathbb{C}}) = 1 \text{ and } \mathrm{d}^{\mathbb{C}}_x f(T^{1,0}\mathbb{H}^m_{\mathbb{C}}) \cap \overline{\mathrm{d}^{\mathbb{C}}_x f(T^{1,0}\mathbb{H}^m_{\mathbb{C}})} = \{0\} \ .$$

But, if $\mathrm{d}^{\mathbb{C}}_x f(T^{1,0}\mathbb{H}^m_{\mathbb{C}})$ contains no real vectors, it contains no purely imaginary vectors, thus $\mathrm{d}_x f(X) = 0$ if and only if $\mathrm{d}^{\mathbb{C}}_x f(X - iJX) = 0$. This means that $\mathrm{Ker}\, \mathrm{d}_x f$ is J-invariant, hence, we may choose an orthonormal basis $(e_i)_{1\leq i\leq 2m} = (e_i, Je_i)_{1\leq i\leq m}$ of $T_x\mathbb{H}^m_{\mathbb{C}}$ in which computations give

$$\langle f^\star\omega_n, f^\star\omega_n\rangle = \sum_{i,j=1}^{2m} f^\star\omega_n(e_i, e_j)^2 = 2f^\star\omega_n(e_1, Je_1)^2 = 2(e' - e'')^2 \ .$$

Thus, if $\mathrm{rk}_{\mathbb{R}} f = 2$, we have $\|f^\star\omega_n\|^2 = 2(e' - e'')^2$ on $\mathbb{H}^m_{\mathbb{C}}$. Using the fact that $\mathrm{rk}_{\mathbb{R}} f = 2$ in the formulae of Lemma 2.10, we find that $R' \geq \frac{1}{4}e'(e' - e'')$ and $R'' \geq \frac{1}{4}e''(e'' - e')$ so that $(e' - e'')^2 \leq 4(R' + R'')$. Now, adding the two equalities in Lemma 2.8, we get

$$2(e' - e'')^2 \leq 8(R' + R'') = -\Delta e - 2\|\nabla' \mathrm{d}^{\mathbb{C}} f\|^2 + (m + 1)\, e \ .$$

Since f has finite energy, we conclude that $f^\star\omega_n$ is L^2 (see the proof of Lemma 2.3 and the beginning of section 2.3). □

We now prove the inequality $|\tau(\rho)| \leq \mathrm{Vol}(M)$.

The above lemma implies that the Burger-Iozzi invariant of ρ is given by

$$\tau(\rho) = \frac{1}{2m}\int_M \langle f^\star\omega_n, \omega_m\rangle dV_m = \frac{1}{m}\int_M (e' - e'')dV_m \ ,$$

since $\langle f^\star\omega_n, \omega_m\rangle = 2(e' - e'')$.

Therefore, if f is a complex rank r holomorphic map, the Schwarz-Pick lemma implies that $0 \leq \langle f^\star\omega_n, \omega_m\rangle \leq 2r$ at each point of M, whereas if f is a rank r antiholomorphic map, then $-2r \leq \langle f^\star\omega_n, \omega_m\rangle \leq 0$. Integrating these inequalities yields the result.

If f is neither holomophic nor antiholomorphic, then we know from Proposition 2.7 that $\mathrm{rk}_{\mathbb{R}} f \leq 2$. We will prove that in this case,

$$\Big| \int_M \langle f^\star\omega_n, \omega_m\rangle dV_m \Big| \ \leq\ (m + 1)\mathrm{Vol}(M) \ .$$

As in the proof of Lemma 3.4, we have $R' \geq \frac{1}{4}e'(e' - e'')$ and $R'' \geq \frac{1}{4}e''(e'' - e')$. Lemma 2.9 then implies that

$$\Delta \log e' \leq 2(e'' - e') + m + 1 \ ,$$

resp.

$$\Delta \log e'' \leq 2(e' - e'') + m + 1 \ ,$$

at each point where $e' \neq 0$, resp. $e'' \neq 0$.

Let $\varepsilon > 0$ be a regular value of e'. We set $M_\varepsilon = \{x \in M,\ e'(x) > \varepsilon\}$, and we introduce the cut-off function η_R defined in the proof of Proposition 2.2.

By Green's formula we have

$$\int_{M_\varepsilon} \Big[\eta_R \Delta \log e' - \langle \mathrm{grad}\, \eta_R, \mathrm{grad}\, \log e'\rangle\Big] dV_m = \int_{\partial M_\varepsilon} \eta_R \langle \nu, \mathrm{grad}\, \log e'\rangle\, dA \geq 0 \ .$$

The latter is nonnegative because ν is the inward unit vector field along ∂M_ε which is pointwise orthogonal to ∂M_ε. From

$$|\langle \mathrm{grad}\, \eta_R, \mathrm{grad}\, \log e'\rangle| \leq \sqrt{2}\|\mathrm{d}\eta_R\| \frac{\|\nabla' \partial^{1,0} f\|}{\sqrt{e'}} \leq \sqrt{2}\|\mathrm{d}\eta_R\| \frac{\|\nabla' \partial^{1,0} f\|}{\sqrt{\varepsilon}} \quad \text{on } M_\varepsilon \ ,$$

we get

$$\int_{M_\varepsilon} \eta_R\, \Delta \log e'\, dV_m \geq -\sqrt{\frac{2}{\varepsilon}}\frac{C}{R}\int_M \|\nabla'\partial^{1,0} f\| dV_m \geq -\sqrt{\frac{2}{\varepsilon}}\frac{C}{R}\Big[\mathrm{Vol}(M)\int_M \|\nabla'\partial^{1,0} f\|^2 dV_m\Big]^{1/2} .$$

Using the fact that $\|\nabla'\partial^{1,0} f\| \in L^2(M)$, we obtain that, for some constant C' and for all $R > 1$,

$$\int_{M_\varepsilon} \eta_R \Delta \log e'\, dV_m \geq -\sqrt{\frac{2}{\varepsilon}}\frac{C'}{R}$$

and so

$$\int_{M_\varepsilon} \Big(2(e'' - e') + m + 1\Big) dV_m = \lim_{R\to+\infty} \int_{M_\varepsilon} \eta_R \Big(2(e'' - e') + m + 1\Big) dV_m \geq 0 .$$

The subset $\{x \in M,\, e'(x) = 0\}$ has zero measure since $\partial^{1,0} f$ is a holomorphic section of $\mathrm{Hom}(T^{1,0}\mathbb{H}^m_{\mathbb{C}}, f^\star T^{1,0}\mathbb{H}^n_{\mathbb{C}})$. Hence, letting $\varepsilon \to 0$, we conclude that

$$\int_M \langle f^\star\omega_n, \omega_m\rangle dV_m = \int_M 2(e' - e'') dV_m \leq (m+1)\,\mathrm{Vol}(M) .$$

Integrating $\Delta \log e''$ in the same way, we get the required inequality.

Assume now that equality holds in the theorem: $|\int_M \langle f^\star\omega_n, \omega_m\rangle dV_m| = 2m\mathrm{Vol}(M)$.
Since $m \geq 2$, it follows from the proof above that f is a complex rank m holomorphic or antiholomorphic map. Since the inequality $|\langle f^\star\omega_n, \omega_m\rangle| \leq 2m$ is therefore true pointwise, the global equality implies that $|\langle f^\star\omega_n, \omega_m\rangle| = 2m$ everywhere on M: f is an isometry. On the other hand, the Bochner-type formula ($\diamondsuit$) with $Q = I$ reads:

$$\int_M \|\nabla \mathrm{d} f\|^2 dV_m = \frac{1}{2}\int_M \Big(\mathrm{Scal}(f^\star R^n) - \frac{1}{2m}\|\mathrm{d} f\|^2 \mathrm{Scal}(R^m)\Big) dV_m .$$

Since f is an (anti)holomorphic isometry, $\|\mathrm{d} f\|^2 = 2m$ and $f^\star R^n = R^m$. Therefore $\nabla \mathrm{d} f = 0$, namely, f is totally geodesic and we are done.

Remark. It should be noted that if one is interested only in proving Corollary 3.2, it is actually possible to define another invariant, that one could call the L^2-volume of the representation, in the following way: just take the m-th exterior power of any L^2-form representing the L^2-cohomology class $[\rho^\star\omega_n]_{(2)}$ and integrate it over M. The so-obtained number is independent of the choice of the L^2-representative and therefore depends only on ρ. One can then prove Theorem 3.1 with $\tau(\rho)$ replaced by this (suitably normalized) L^2-volume. The proof is in fact easier since this invariant will be zero if the real rank of the harmonic map is less than $2m$. Therefore one does not need to deal with non (anti)holomorphic maps.

Nevertheless, there exist representations ρ with zero L^2-volume but $\tau(\rho) \neq 0$ and one can hope to be able to get informations on these representations from the Burger-Iozzi invariant that the volume would not give. An example of such a representation was given by Livné in his thesis ([Liv81], see also [Ka98]). He constructed a (closed) complex hyperbolic manifold M of complex dimension 2 and a surjective holomorphic map f from M to a (closed) Riemann surface Σ such that the induced map on the fundamental groups is surjective. This gives a representation of the lattice $\pi_1(M) \subset \mathrm{PU}(2,1)$ into $\mathrm{PU}(1,1)$. Of course the volume of this representation is zero. But its Burger-Iozzi invariant does not vanish since the pull-back $f^\star\omega_1$ is a semi-positive $(1,1)$-form on M which is not identically zero.

4. The case of non-uniform lattices of PU$(1,1)$

In this section we want to extend the previous results to the 1-dimensional case, namely the case of non-uniform lattices of $\mathrm{PU}(1,1)$. We remark that if ρ is a representation of such a lattice into $\mathrm{PU}(n,1)$, the Burger-Iozzi invariant of ρ is not defined since $H^2_{DR}(M)=0$ and the comparison map $H^2_{(2)}(M)\longrightarrow H^2_{DR}(M)$ is of course not injective anymore. As we mentionned in the introduction, Corollary 3.2 fails in this case. Indeed, Gusevskii and Parker prove in [GP00] that there exist lattices in $\mathrm{PU}(1,1)$ admitting quasi-Fuchsian deformations into $\mathrm{PU}(2,1)$.

Geometrically, a torsion-free lattice $\Gamma<\mathrm{PU}(1,1)$ is the fundamental group of the complete hyperbolic surface of finite volume $M=\Gamma\backslash\mathbb{H}^1_{\mathbb{C}}$. It turns out that we don't need a Riemannian structure on M to define an invariant associated to representations of its fundamental group into $\mathrm{PU}(1,1)$. We will therefore work in the more general setting of fundamental groups of orientable surfaces of finite topological type.

Let M be the open surface obtained by removing a finite number of points $m_1,\dots,m_p$, called punctures, from a closed orientable surface $\overline{M}$ of genus g. We will call such an M a p-times punctured closed orientable surface of genus g. We assume that the Euler characteristic $\chi(M)=2-2g-p$ of M is negative. Let $\Gamma=\pi_1(M)$ be the fundamental group of M and $\pi:\widetilde{M}\longrightarrow M$ be the universal cover of M.

Loops going once around the puncture m_i in the direction prescribed by the orientation of $\overline{M}$ correspond to a conjugacy class c_i of elements of Γ. The elements of the conjugacy classes c_i are called *peripheral*. For each i, choose $\gamma_i\in c_i$ and denote by $\langle\gamma_i\rangle$ the cyclic subgroup generated by γ_i. There exist small disjoint open topological discs $D_i\subset\overline{M}$ around each m_i and disjoint open simply-connected sets U_i in $\widetilde{M}$, precisely invariant under $\langle\gamma_i\rangle$ (meaning that $\gamma_iU_i=U_i$ and $\gamma U_i\cap U_i=\emptyset$ if $\gamma\notin\langle\gamma_i\rangle$), such that the punctured discs $D_i^\star:=D_i\backslash\{m_i\}\subset M$ are given by $D_i^\star=\langle\gamma_i\rangle\backslash U_i$ (This can for example be seen by uniformizing M as a finite volume hyperbolic surface, and then choosing precisely invariant horospherical neighbourhoods of the parabolic fixed points of Γ).

Let ρ be a homomorphism from Γ to $\mathrm{PU}(n,1)$. At the beginning of section 3.2 (there, Γ was a lattice in $\mathrm{PU}(m,1)$ for $m\geq 2$), we saw how to find a compactly supported 2-form in the de Rham cohomology class $[\rho^\star\omega_n]$. Here, as we said, this class is zero but we shall in the same way associate to ρ a class in the cohomology with compact support.

For each i, choose a fixed point ξ_i of $\rho(\gamma_i)$, in $\mathbb{H}^n_{\mathbb{C}}$ if $\rho(\gamma_i)$ is elliptic, else in $\partial_\infty\mathbb{H}^n_{\mathbb{C}}$, and then a Kähler potential ψ_i of ω_n, invariant by the stabilizer in $\mathrm{PU}(n,1)$ of ξ_i if $\xi_i\in\mathbb{H}^n_{\mathbb{C}}$ or by the stabilizer of the horospheres centered at ξ_i if $\xi_i\in\partial_\infty\mathbb{H}^n_{\mathbb{C}}$. If $\xi_i\in\mathbb{H}^n_{\mathbb{C}}$, we can assume that $\xi_i=0$ in the ball model of $\mathbb{H}^n_{\mathbb{C}}$ and take $\psi_i=\log(1-\langle\langle z,z\rangle\rangle)$ (here $\langle\langle\,,\,\rangle\rangle$ denotes the standard Hermitian form on $\mathbb{C}^n$). If $\xi_i\in\partial_\infty\mathbb{H}^n_{\mathbb{C}}$, we can take $\psi_i=\log(2\mathrm{Re}(w')-\langle\langle z',z'\rangle\rangle)$, where $(w',z')\in\mathbb{C}\times\mathbb{C}^{n-1}$ are horospherical coordinates centered at ξ_i. Note that, up to an additive constant, these potentials are unique. The potential ψ_i is invariant by $\rho(\gamma_i)$ only if $\rho(\gamma_i)$ is elliptic or parabolic but the 1-form $\varsigma_i:=-\mathrm{d}^c\psi_i$, which satisfies $\mathrm{d}\varsigma_i=\omega_n$, is always invariant by $\rho(\gamma_i)$.

Given a ρ-equivariant map $f:\widetilde{M}\longrightarrow\mathbb{H}^n_{\mathbb{C}}$, we can pull-back the Kähler form ω_n and the forms ς_i to get a 2-form $f^\star\omega_n$ invariant by Γ and 1-forms $f^\star\varsigma_i$ invariant by $\langle\gamma_i\rangle$. We can therefore consider $f^\star\omega_n$ as a 2-form on M and each $f^\star\varsigma_i$ as a 1-form on the punctured disc $D_i^\star$ (by restricting it first to U_i). If now χ is a function identically equal to 0 outside the $D_i^\star$'s and to 1 in small neighbourhoods of the punctures, we get a compactly supported 2-form $f^\star\omega_n-\mathrm{d}(\sum_i\chi f^\star\varsigma_i)$ on M. This yields a class $[f^\star\omega_n-\mathrm{d}(\sum_i\chi f^\star\varsigma_i)]_c$ in the second cohomology group with compact support $H^2_c(M)$.

Proposition-definition 4.1. *The class $[f^\star\omega_n - \mathrm{d}(\sum_i \chi f^\star\varsigma_i)]_c$ depends only on the representation ρ, and will therefore be denoted by $[\rho^\star\omega_n]_c$. Moreover, we set*

$$\tau(\rho) = \int_M [\rho^\star\omega_n]_c\ .$$

Proof. This class is clearly independent of the cut-off function χ, and therefore also of the choice of the punctured discs $D_i^\star$ where the $f^\star\varsigma_i$'s are defined.

Now, let f_1 and f_2 be two ρ-equivariant maps $\widetilde{M} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$. The map f_2 is homotopic to a ρ-equivariant map f_3 such that, when seen as sections of the $\mathbb{H}^n_{\mathbb{C}}$-bundle on M associated to ρ, $f_3 = f_1$ on the set $\{\chi < 1\}$ and $f_3 = f_2$ close enough to the punctures. Then there exists a compactly supported 1-form α such that $f_2^\star\omega_n = f_3^\star\omega_n + \mathrm{d}\alpha$. Hence we have

$$f_2^\star\omega_n - \mathrm{d}\Big(\sum_i \chi f_2^\star\varsigma_i\Big) = f_3^\star\omega_n - \mathrm{d}\Big(\sum_i \chi f_3^\star\varsigma_i\Big) + \mathrm{d}\Big(\sum_i \chi(f_3^\star\varsigma_i - f_2^\star\varsigma_i)\Big) + \mathrm{d}\alpha\ .$$

But

$$f_3^\star\omega_n - \mathrm{d}\Big(\sum_i \chi f_3^\star\varsigma_i\Big) = \left\{\begin{array}{l} f_1^\star\omega_n - \mathrm{d}(\sum_i \chi f_1^\star\varsigma_i) \text{ on } \{\chi<1\} \\ 0 \text{ on } \{\chi = 1\}\end{array}\right. = f_1^\star\omega_n - \mathrm{d}\Big(\sum_i \chi f_1^\star\varsigma_i\Big) \text{ on } M\ .$$

Therefore

$$f_2^\star\omega_n - \mathrm{d}\Big(\sum_i \chi f_2^\star\varsigma_i\Big) = f_1^\star\omega_n - \mathrm{d}\Big(\sum_i \chi f_1^\star\varsigma_i\Big) + \mathrm{d}\Big[\sum_i \chi(f_3^\star\varsigma_i - f_2^\star\varsigma_i) + \alpha\Big]\ .$$

From the definition of f_3, the 1-form inside the brackets is compactly supported. Hence $[f^\star\omega_n - \mathrm{d}(\sum_i \chi f^\star\varsigma_i)]_c$ does not depend on the ρ-equivariant map f.

If $\rho(\gamma_j)$ is elliptic for some j, it might fix more than one point in $\mathbb{H}^n_{\mathbb{C}}$. We therefore have to check that choosing another fixed point, say ξ'_j, to define the Kähler potential does not change the class. Let ψ'_j be the Kähler potential associated to ξ'_j and $\varsigma'_j = -\mathrm{d}^c\psi'_j$ the corresponding 1-form. To compute our class, we can choose the ρ-equivariant map $f : \widetilde{M} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ to be constant equal to ξ_j in U_j so that $f^\star\varsigma_j = f^\star\varsigma'_j = 0$ on $D_j^\star$. Therefore

$$f^\star\omega_n - \mathrm{d}\Big(\sum_i \chi f^\star\varsigma_i\Big) = f^\star\omega_n - \mathrm{d}\Big(\sum_{i\neq j} \chi f^\star\varsigma_i\Big)\ ,$$

and the cohomology class is not affected.

In the same way, if $\rho(\gamma_j)$ is hyperbolic for some j, we must show that we can choose any of the two fixed points ξ_j and ξ'_j of $\rho(\gamma_j)$ indifferently. In this case, we can arrange that the equivariant map f maps U_j to the axis of $\rho(\gamma_j)$. If we take the potential ψ_j associated to ξ_j (resp. ξ'_j) to define ς_j then $\varsigma_j = -\mathrm{d}^c t = \mathrm{d}t \circ J$ in horospherical coordinates (z, v, t) chosen so that $\xi_j = \infty$ and $\xi'_j = 0$ (resp. $\xi'_j = \infty$ and $\xi_j = 0$). Since in these coordinates the axis of $\rho(\gamma_j)$ is the set $\{z = 0, v = 0\}$, $f^\star\varsigma_j = 0$ on $D_j^\star$. Again,

$$f^\star\omega_n - \mathrm{d}\Big(\sum_i \chi f^\star\varsigma_i\Big) = f^\star\omega_n - \mathrm{d}\Big(\sum_{i\neq j} \chi f^\star\varsigma_i\Big)\ .$$

Finally, it is easy to check that a different choice of the peripheral elements γ_i (and hence of the U_i's) gives the same cohomology class. Indeed, let γ_j and $\gamma'_j = \gamma\gamma_j\gamma^{-1}$ be two elements of the conjugacy class c_i. We denote with primes the objects associated to γ'_j (for example, ψ'_j is the Kähler potential associated to a fixed point ξ'_j of $\rho(\gamma'_j)$). If $\rho(\gamma_j)$, and hence $\rho(\gamma'_j)$, is elliptic or hyperbolic, we can choose as above the equivariant map f so that $f^\star\varsigma_i = 0$ on U_i and $f^\star\varsigma'_i = 0$ on $U'_i = \gamma U_i$. Thus we can assume that $\rho(\gamma_j)$ and $\rho(\gamma'_j)$ are both parabolic. But then $\xi'_j = \rho(\gamma)\xi_j$, $\psi'_j = \psi_j \circ \rho(\gamma^{-1})$ and $f^\star\varsigma'_j = (\gamma^{-1})^\star f^\star\varsigma_j$ on $\mathbb{H}^1_{\mathbb{C}}$. Therefore the restrictions of $f^\star\varsigma'_j$ to $U'_j = \gamma U_j$ and of $f^\star\varsigma_j$ to U_j induce the same form on $D^\star_j = \langle\gamma_j\rangle\backslash U_j = \langle\gamma'_j\rangle\backslash U'_j$. $\square$

Before stating the main theorem of this section, we need the following definitions.

Definition 4.2. *Let Γ be the fundamental group of a p-times punctured closed orientable surface M.*

A homomorphism ρ from Γ in $\mathrm{PU}(n,1)$ *is called tame if it maps no peripheral element of Γ to a hyperbolic isometry of $\mathbb{H}^n_{\mathbb{C}}$.*

A homomorphism ρ from Γ in $\mathrm{PU}(1,1)$ *is called a uniformization representation if it is an isomorphism onto a torsion-free discrete subgroup $\rho(\Gamma) < \mathrm{PU}(1,1)$ such that M and $\rho(\Gamma)\backslash\mathbb{H}^1_{\mathbb{C}}$ are diffeomorphic.*

Theorem 4.3. *Let M be a p-times punctured closed orientable surface of genus g. Assume that $\chi(M) = 2 - 2g - p < 0$. Let Γ be the fundamental group of M and $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$ be a homomorphism. Then $|\tau(\rho)| \leq -2\pi\chi(M)$ and equality holds if and only if the image $\rho(\Gamma)$ of Γ stabilizes a totally geodesic copy of $\mathbb{H}^1_{\mathbb{C}}$ in $\mathbb{H}^n_{\mathbb{C}}$, hence ρ can be seen as a homomorphism from Γ into* $\mathrm{PU}(1,1)$ *and moreover, as such, ρ is a uniformization representation. If in addition ρ is tame (and equality holds) then $\rho(\Gamma) < \mathrm{PU}(1,1)$ is a lattice.*

Remark. (i) The number $\tau(\rho)$, up to sign, depends only on the diffeomorphism type of the surface M in the following sense. If $\phi : M' \longrightarrow M$ is a diffeomorphism and if we consider the homomorphism $\rho' = \rho \circ \phi_\star$ of the fundamental group Γ' of M' into $\mathrm{PU}(n,1)$, then it is easily checked that $\tau(\rho') = \pm\tau(\rho)$ depending on whether ϕ is orientation preserving or reversing.

(ii) In particular, if $\rho : \Gamma \longrightarrow \Gamma' < \mathrm{PU}(1,1)$ is a uniformization representation, so that M is diffeomorphic to $M' := \Gamma'\backslash\mathbb{H}^1_{\mathbb{C}}$, then $\tau(\rho) = \pm\int_{M'}[\omega_1 - \mathrm{d}(\sum_i \chi\varsigma_i)]$. Let M'_0 be the convex core of M'. M'_0 is a finite volume complete hyperbolic surface whose boundary consists of finitely many disjoint closed simple geodesics c_k (corresponding to the conjugacy classes of peripheral elements of Γ sent by ρ to hyperbolic isometries of $\mathbb{H}^1_{\mathbb{C}}$). We can assume that $\chi = 0$ on the boundary of M'_0. It is easy to see (cf. the proof of Proposition 4.5 below) that the 1-forms ς_j corresponding to punctures of M'_0 are L^1 forms and therefore that

$$\int_{M'_0}[\omega_1 - \mathrm{d}(\sum_j \chi\varsigma_j)] = \int_{M'_0}\omega_1 = \mathrm{Vol}(M'_0) = -2\pi\chi(M'_0) = -2\pi\chi(M)\ .$$

Moreover,

$$\int_{M'\backslash M'_0}[\omega_1 - \mathrm{d}(\sum_k \chi\varsigma_k)] = \sum_k \int_{c_k}\mathrm{d}\varsigma_k = 0\ ,$$

as can be seen from the proof of Proposition 4.1. Hence $\tau(\rho) = \pm 2\pi\chi(M)$ as Theorem 4.3 says.

(iii) Our definition of the invariant τ makes no reference to a Riemannian structure on the surface M. This means that we can equip M with any Riemannian metric we want and that our results will be independent of this particular choice. Now, the uniformization theorem implies that there exist complete hyperbolic metrics of finite volume on M. If we choose such a metric g_1 on M, we obtain a uniformization representation $u : \Gamma \longrightarrow \mathrm{PU}(1,1)$ and M is diffeomorphic to $u(\Gamma)\backslash\mathbb{H}^1_{\mathbb{C}}$. The composition $\rho \circ u^{-1}$ is a homomorphism from the torsion-free lattice $u(\Gamma)$ of $\mathrm{PU}(1,1)$ to $\mathrm{PU}(n,1)$. These identifications are completely transparent for the invariant τ and we can therefore assume that Γ is a torsion-free lattice in $\mathrm{PU}(1,1)$ and that $M = \Gamma\backslash\mathbb{H}^1_{\mathbb{C}}$ with its complete hyperbolic metric g_1 of finite volume. The punctured neighbourhoods $D^\star_i$ of the punctures m_i will then be seen as cusps of M and will often be denoted by C_i.

The metric on M allows to talk about L^2-cohomology groups and if we call $[\rho^\star\omega_n]_{(2)}$ the image in $H^2_{(2)}(M)$ of $[\rho^\star\omega_n]_c$ under the comparison map $H^2_c(M) \longrightarrow H^2_{(2)}(M)$, we have

$$\tau(\rho) = \frac{1}{2}\int_M \langle [\rho^\star\omega_n]_{(2)}, \omega_1\rangle dV_1 \ ,$$

where ω_1 is the Kähler form of g_1.

(iv) If ρ is not reductive, that is, if $\rho(\Gamma)$ fixes a point in $\partial_\infty\mathbb{H}^n_{\mathbb{C}}$, then $\tau(\rho) = 0$. Indeed, we can use this fixed point to define as before a 1-form ς invariant by $\rho(\Gamma)$. Then $f^\star\omega_n = \mathrm{d}f^\star\varsigma$ on M. Hence $f^\star\omega_n - \mathrm{d}(\chi f^\star\varsigma) = \mathrm{d}((1-\chi)f^\star\varsigma)$, that is $[f^\star\omega_n - \mathrm{d}(\chi f^\star\varsigma)]_c = 0$ and $\tau(\rho) = 0$.

Hence we assume from now on that all considered representations are reductive.

The proof of Theorem 4.3 is easier if one deals only with *tame* representations. Since almost all the ideas are needed in this case, we present it separately in Section 4.1 and we explain in Section 4.2 how to adapt the arguments to handle the general case.

4.1. **Tame representations.**

Note that a representation ρ of a torsion-free lattice Γ of $\mathrm{PU}(1,1)$ into $\mathrm{PU}(n,1)$ is tame if and only if ρ maps no parabolic elements of Γ to hyperbolic elements of $\mathrm{PU}(n,1)$. The name "tame" is motivated by the following proposition:

Proposition 4.4. *Let $\Gamma < \mathrm{PU}(1,1)$ be a torsion-free lattice, and let $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$ be a homomorphism. There exists a ρ-equivariant map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ of finite energy if and only if ρ is tame.*

Proof. When ρ is tame, we may easily construct a ρ-equivariant map with finite energy. In fact, we use the same method as in Theorem 1.2; since the fundamental group of each cusp of M is generated by a single parabolic element, the construction is much simpler. Namely, it is sufficient to define the ρ-equivariant map on each cusp of M. Let C be a cusp of M and let γ be a parabolic element of Γ generating $\pi_1(C)$ (via the usual identification). We can choose (horospherical) coordinates (v,t) on $\mathbb{H}^1_{\mathbb{C}}$ such that $\gamma(v,t) = (v+a,t)$ and such that C is isometric to the quotient by $\langle\gamma\rangle$ of the subset $D := [0,a]\times[0,+\infty)$ of $\mathbb{H}^1_{\mathbb{C}}$ endowed with the metric $g_1 = e^{-2t}dv^2 + dt^2$.

If $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ is any ρ-equivariant map, the energy of f in the cusp is given by

$$E_C(f) = \frac{1}{2}\int_0^{+\infty}\int_0^a \|\mathrm{d}f\|^2_{(v,t)} e^{-t} dv\, dt \ .$$

When $\rho(\gamma)$ is elliptic, we map D to a fixed point of $\rho(\gamma)$ and the energy of f in the cusp is zero. If $\rho(\gamma)$ is parabolic, let (z', v', t') be adapted horospherical coordinates on $\mathbb{H}^n_{\mathbb{C}}$. We define

$$\begin{array}{rcl} f : [0,a]\times[0,+\infty) & \longrightarrow & \mathbb{H}^n_{\mathbb{C}} \\ (v,t) & \longmapsto & (\varphi(v), 2t) \end{array}$$

where φ is a map from $[0,a]$ into a horosphere $HS' \subset \mathbb{H}^n_{\mathbb{C}}$ such that $\varphi(a) = \rho(\gamma)\varphi(0)$. Computing as in the proof of Theorem 1.2 (*case 1: $\rho(\gamma_0)$ is parabolic*), we get the following estimation of the energy of f in the cusp:

$$E_C(f) \leq \frac{1}{2}\int_0^{+\infty}\int_0^a (4 + \|\mathrm{d}\varphi\|^2)e^{-t} dv\, dt \leq 2a + E(\varphi) < +\infty \ .$$

Suppose now that $\rho(\gamma)$ is hyperbolic and let $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ be any ρ-equivariant map. For every t, we denote by $c_t : [0,a] \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ the curve $v \longmapsto f(v,t)$. Since $\rho(\gamma)$ is hyperbolic, there exists $\delta > 0$ such that the distance between $f(0,t)$ and $f(a,t) = \rho(\gamma)f(0,t)$ is at least δ, hence the length $l(c_t)$ of c_t is at least δ. This implies that

$$\frac{1}{2}\int_0^a \|\mathrm{d}f\|^2_{(v,t)}dv \geq e^{2t}E(c_t) = \frac{e^{2t}}{2}\int_0^a \|\mathrm{d}c_t\|^2 dv \geq \frac{e^{2t}}{2a}\big(l(c_t)\big)^2 \geq \frac{e^{2t}}{2a}\delta^2$$

and

$$E_C(f) \geq \frac{\delta^2}{2a}\int_0^{+\infty} e^t dt = +\infty \ .$$

□

The next proposition shows that for a tame representation ρ, the invariant $\tau(\rho)$ can be computed with any finite energy ρ-equivariant map $\mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$.

Proposition 4.5. *Let $\Gamma < \mathrm{PU}(1,1)$ be a torsion-free lattice and let $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$ be a tame homomorphism. Then,*

$$\tau(\rho) = \frac{1}{2}\int_M \langle f^\star\omega_n, \omega_1\rangle \, dV_1$$

for any finite energy ρ-equivariant map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$.

Remark. One could therefore take this as a definition of the invariant for tame representations. This gives a formulation very similar to the classical one for the Toledo invariant of a uniform lattice.

Note that the energy finiteness assumption is necessary as the following simple example shows. Let Γ be a torsion-free lattice in $\mathrm{SU}(1,1)$ generated by two hyperbolic elements γ_1 and γ_2 such that $\gamma_0 = [\gamma_1, \gamma_2]$ is parabolic. Then $M = \Gamma\backslash\mathbb{H}^1_{\mathbb{C}}$ is diffeomorphic to a once-punctured torus. Let ρ be the inclusion $\Gamma \longrightarrow \mathrm{PU}(1,1)$. Keeping the same notations as in the proof of Proposition 4.4, we can define C^∞ maps $f_\mu : M \longrightarrow M$ which equal identity outside the cusp, and given by $f_\mu : [0,a]\times[0,+\infty) \longrightarrow [0,a]\times\mathbb{R}$, $(v,t) \longmapsto (v,\mu(t))$ in D, where $\mu \in C^\infty([0,+\infty),\mathbb{R})$. The energy of f_μ in the cusp is $\frac{a}{2}\int_0^{+\infty}(e^{t-2\mu(t)} + \mu'(t)^2e^{-t})\,dt$. Moreover, a formal computation gives: $\frac{1}{2}\int_C\langle f_\mu^\star\omega_1, \omega_1\rangle\,dV_1 = \int_C f_\mu^\star\omega_1 = a\int_0^{+\infty}\mu'(t)e^{-\mu(t)}\,dt = a(1 - \lim_{t\to+\infty} e^{-\mu(t)})$. Note that this limit needs not exist and that for every $c \in [-\infty, \tau(\rho)]$, we may choose μ such that $\int_M f_\mu^\star\omega_1 = c$. But if the energy of f_μ is finite, then $\mu(t) \to +\infty$ as $t \to +\infty$, and then, $\int_M f_\mu^\star\omega_1 = \int_M \omega_1 = \tau(\rho)$.

Proof. Let $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ be a finite energy ρ-equivariant map. Since f has finite energy, $\langle f^\star\omega_n, \omega_1\rangle = 2(e'(f) - e''(f))$ is integrable on M and we can write

$$\tau(\rho) = \frac{1}{2}\int_M \langle f^\star\omega_n, \omega_1\rangle\,dV_1 - \frac{1}{2}\int_M \langle \mathrm{d}(\sum_i \chi f^\star\varsigma_i), \omega_1\rangle\,dV_1 \ .$$

The function $\langle \mathrm{d}(\sum_i \chi f^\star\varsigma_i), \omega_1\rangle$ is integrable on M (because so is $\langle f^\star\omega_n, \omega_1\rangle$ in the cusps of M). Therefore, if we prove that the 1-form $\sum_i \chi f^\star\varsigma_i$ is an L^1-form, the Stokes formula of [Ga54] will apply and $\int_M\langle \mathrm{d}(\sum_i \chi f^\star\varsigma_i), \omega_1\rangle\,dV_1$ will vanish as wanted.

Now, if the generator γ_i of the fundamental group of the cusp C_i is sent by ρ to a parabolic element of $\mathrm{PU}(n,1)$, we know that in horocyclic coordinates relative to the fixed point of $\rho(\gamma_i)$ we have $g_n = dt'^2 + \varsigma_i^2 + 4e^{-t'}\langle\langle dz', dz'\rangle\rangle$. Therefore,

$$\|\mathrm{d}f\|^2 = \mathrm{tr}_{g_1}(f^\star g_n) \geq \|f^\star\varsigma_i\|^2 \ .$$

If $\rho(\gamma_i)$ is elliptic, we can assume that one of its fixed points is 0 in the ball model of $\mathbb{H}^n_{\mathbb{C}}$. In this case, we have $\varsigma_i = -\mathrm{d}^c\psi_i$ with $\psi_i = \log(1 - \langle\langle z, z\rangle\rangle)$. The metric g_n is then given by

$$g_n = 4\frac{\langle\langle dz, dz\rangle\rangle}{1 - \langle\langle z, z\rangle\rangle} + (\mathrm{d}\psi_i)^2 + (\mathrm{d}^c\psi_i)^2 \ .$$

Again, we have $\|f^\star\varsigma_i\| \leq \|\mathrm{d}f\|$ on C_i.

The form $f^\star\varsigma_i$ is therefore L^1 on C_i for each i. The proposition follows. □

We are now in position to prove Theorem 4.3 in the case of reductive and tame representations.

Proof of Theorem 4.3. Since ρ is assumed to be tame and reductive, Proposition 4.4 and Theorem 1.1 imply the existence of a ρ-equivariant harmonic map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ of finite energy.

Proceeding exactly as in section 3.2 yields the inequality $|\tau(\rho)| \leq \mathrm{Vol}(M)$.

Before treating the equality case, we remark that there are particular representations for which it is immediate that equality can not hold:

Claim 4.6. *Suppose that ρ maps the conjugacy classes $c_{q+1}, \ldots, c_p$ of peripheral elements of Γ to $\mathrm{id} \in \mathrm{PU}(n,1)$ and denote by M' the surface obtained by removing only the points $m_1, \ldots, m_q$ from the closed orientable surface $\overline{M}$ of genus g. Then, $|\tau(\rho)| \leq \max\big(0, -2\pi\chi(M')\big)$.*

Proof. The representation ρ admits a factorisation $\rho = \rho' \circ i_\star$ where $i_\star : \Gamma \longrightarrow \Gamma' := \pi_1(M')$ is the representation induced by the inclusion $i : M \longrightarrow M'$ and $\rho' : \Gamma' \longrightarrow \mathrm{PU}(n,1)$ is a homomorphism. We may equip M' with a complex structure such that the inclusion i is holomorphic.

If $\chi(M') < 0$, we endow M' with a complete hyperbolic metric of finite volume. Then, $|\tau(\rho')| \leq -2\pi\chi(M')$ by the above argument. If f is any section of the bundle $M' \times_{\rho'} \mathbb{H}^n_{\mathbb{C}}$, the restriction of f to M can be seen as a section of the bundle $M \times_\rho \mathbb{H}^n_{\mathbb{C}}$ whose energy is finite on neighbourhoods of the punctures $m_{q+1}, \ldots, m_p$ (since energy finiteness only depends on the conformal structure of M). Then, by Proposition 4.5, $\tau(\rho) = \int_M [f^\star\omega_n - \mathrm{d}(\sum_{i=1}^q \chi f^\star\varsigma_i)] = \int_{M'} [f^\star\omega_n - \mathrm{d}(\sum_{i=1}^q \chi f^\star\varsigma_i)] = \tau(\rho')$.

If $\chi(M') = 0$, then M' is holomorphically equivalent to either $\mathbb{C}^\star$ or an elliptic curve. In the first case, $\Gamma' \simeq \mathbb{Z}$ and so $\rho(\Gamma)$ is generated by a single element. If this element is parabolic or hyperbolic, then ρ is not reductive and if it is elliptic, there exists a constant ρ-equivariant map. Thus, $\tau(\rho) = 0$. In the second case, Γ' is abelian, generated by two elements γ_1 and γ_2. If $\rho'(\gamma_1)$ is parabolic or hyperbolic, by commutation, $\rho'(\gamma_2)$ must fix the fixed point(s) of $\rho'(\gamma_1)$ and ρ is not reductive. If $\rho'(\gamma_1)$ and $\rho'(\gamma_2)$ are elliptic, they must have a common fixed point in $\mathbb{H}^n_{\mathbb{C}}$, as they commute. In either case, $\tau(\rho) = 0$.

Finally, if $\chi(M') > 0$, M' is simply connected and $\tau(\rho)$ is of course zero. □

We suppose now that the equality $|\tau(\rho)| = -2\pi\chi(M)$ holds.

The harmonic map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ needs not be (anti)holomorphic as in the higher dimensional case, but we know that its real rank is 2 at some point, hence on a dense open subset $\mathcal{U}$ of $\mathbb{H}^1_{\mathbb{C}}$ by a result of [Sa78], and that in the proof of Theorem 3.1, one of the inequalities concerning R' and R'' is necessarily an equality on M.

Suppose for example that $R' = \frac{1}{4}e'(e' - e'')$ (the case where the inequality becoming an equality is $R'' \geq \frac{1}{4}e''(e'' - e')$ is handled similarly), that is

$$\Big|g_n\Big(\partial^{1,0}_x f(z_1), \overline{\partial^{1,0}_x f(\bar{z}_1)}\Big)\Big|^2 = g_n\Big(\partial^{1,0}_x f(z_1), \overline{\partial^{1,0}_x f(z_1)}\Big)\, g_n\Big(\partial^{1,0}_x f(\bar{z}_1), \overline{\partial^{1,0}_x f(\bar{z}_1)}\Big)$$

for every $x \in \mathbb{H}^1_{\mathbb{C}}$.

This has the following simple but very important consequence: for all $x \in \mathbb{H}^1_{\mathbb{C}}$, $\mathrm{d}_x f(T_x\mathbb{H}^1_{\mathbb{C}})$ is contained in a complex one-dimensional subspace of $T_{f(x)}\mathbb{H}^n_{\mathbb{C}}$. To prove this fact, we only need to consider points where $\mathrm{rk}_{\mathbb{R}}\,\mathrm{d}_x f = 2$. We remark that a vector subspace $V \subset T_{f(x)}\mathbb{H}^n_{\mathbb{C}}$ is J-invariant if and only if

$$V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1} = V^{1,0} \oplus \overline{V^{1,0}}$$

where $V^{\mathbb{C}} \subset T^{\mathbb{C}}_{f(x)}\mathbb{H}^n_{\mathbb{C}}$ is the complexification of V and $V^{1,0}$ resp. $V^{0,1}$ is the projection of $V^{\mathbb{C}}$ on $T^{1,0}_{f(x)}\mathbb{H}^n_{\mathbb{C}}$ resp. $T^{0,1}_{f(x)}\mathbb{H}^n_{\mathbb{C}}$. So, if $V = \mathrm{d}_x f(T_x\mathbb{H}^1_{\mathbb{C}})$ is a 2-dimensional real subspace of $T_{f(x)}\mathbb{H}^n_{\mathbb{C}}$, V is J-invariant if and only if $\dim_{\mathbb{C}} V^{1,0} = 1$. Since $V^{1,0}$ is spanned by $\partial^{1,0}_x f(z_1)$ and $\partial^{1,0}_x f(\bar{z}_1)$, the above equality holds if and only if $\mathrm{d}_x f(T_x\mathbb{H}^1_{\mathbb{C}})$ is contained in a complex one-dimensional subspace of $T_{f(x)}\mathbb{H}^n_{\mathbb{C}}$, according to the equality part of Cauchy-Schwarz's inequality.

Now, we will use the same trick as A. G. Reznikov in his paper [Re93]. Namely, we equip the product $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^n_{\mathbb{C}}$ with the metric $h_\varepsilon := \varepsilon g_1 + g_n$ (for a given $\varepsilon > 0$) and we consider the map $\phi : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^n_{\mathbb{C}}$ given by $\phi(x) = (x, f(x))$. ϕ is then a harmonic embedding of $\mathbb{H}^1_{\mathbb{C}}$ into $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^n_{\mathbb{C}}$. Let $\kappa_{\phi^\star h_\varepsilon}$ be the Gaussian curvature of the induced metric $\phi^\star h_\varepsilon$ on $\mathbb{H}^1_{\mathbb{C}}$. Since ϕ is harmonic, Lemma C.4 of [Re93] implies that for all $x \in \mathbb{H}^1_{\mathbb{C}}$, $\kappa_{\phi^\star h_\varepsilon}(x)$ is less than or equal to the value of the sectional curvature of h_ε on the 2-plane $T_{\phi(x)}\phi(\mathbb{H}^1_{\mathbb{C}}) \subset T_{\phi(x)}(\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^n_{\mathbb{C}})$ (which we will call $\kappa_{h_\varepsilon}(T_{\phi(x)}\phi(\mathbb{H}^1_{\mathbb{C}}))$).

Since ϕ is equivariant w.r.t. the isometric diagonal action of Γ on $\mathbb{H}^1_{\mathbb{C}} \times \mathbb{H}^n_{\mathbb{C}}$, these quantities are well defined over the Riemann surface $M = \Gamma\backslash\mathbb{H}^1_{\mathbb{C}}$. Moreover, we have

Claim 4.7. *For each $\varepsilon > 0$,*

$$\mathrm{Area}_{g_1}(M) = -2\pi\chi(M) = -\int_M \kappa_{\phi^\star h_\varepsilon}\, dV_{\phi^\star h_\varepsilon}$$

where $dV_{\phi^\star h_\varepsilon}$ denotes the volume element of the pull-back metric $\phi^\star h_\varepsilon$ on M.

Proof. We shall apply the main theorem of [Li97] to $\phi^\star h_\varepsilon$. This metric is clearly complete (because g_1 is), and has everywhere non-positive curvature. We have to prove that

$$\lim_{r\to+\infty} \frac{\mathrm{Area}_{\phi^\star h_\varepsilon}\big(B_{\phi^\star h_\varepsilon}(r)\big)}{r^2} = 0$$

where $B_{\phi^\star h_\varepsilon}(r)$ is the geodesic ball of radius r (at a fixed point that belongs, for example, to the compact part of M) w.r.t. the metric $\phi^\star h_\varepsilon$. Now, simple computations show that

$$dV_{\phi^\star h_\varepsilon} = \sqrt{\varepsilon^2 + 2\varepsilon\, e(f) + \frac{1}{2}\|f^\star\omega_n\|^2}\, dV_1\ .$$

Since f has finite energy and $f^\star\omega_n$ is L^2, the metric $\phi^\star h_\varepsilon$ has in fact finite volume and we obtain the expected limit. Then, it follows from [Li97] that $\phi^\star h_\varepsilon$ has finite total curvature and this implies the result (as it is explained in the same paper). □

On $\mathcal{U}$, the pull-back $f^\star g_n$ is a Riemannian metric. Let $\kappa_{f^\star g_n}$ denote its Gaussian curvature. Again, since f is harmonic, for all $x \in \mathcal{U}$, $\kappa_{f^\star g_n}(x)$ is less than or equal to $\kappa_{g_n}(T_{f(x)}f(\mathbb{H}^1_{\mathbb{C}}))$, the sectional curvature of the 2-plane $T_{f(x)}f(\mathbb{H}^1_{\mathbb{C}}) \subset T_{f(x)}\mathbb{H}^n_{\mathbb{C}}$ w.r.t. the metric g_n. Since $T_{f(x)}f(\mathbb{H}^1_{\mathbb{C}})$ is a complex line in $T_{f(x)}\mathbb{H}^n_{\mathbb{C}}$, its sectional curvature is -1. Hence, for all $x \in \mathcal{U}$, $\kappa_{f^\star g_n}(x) \leq -1$.

Assume that at a certain point p of $\mathcal{U}$, $\kappa_{f^\star g_n}(p)$ is stricly less than -1. We are going to prove that this can not happen. For, if it does, there exist $\alpha > 0$ and a small disc D around p whose closure $\overline{D}$ is contained in $\mathcal{U}$ such that for all $x \in D$, $\kappa_{f^\star g_n}(x) \leq -1 - \alpha$. We can assume that the image of D in M is diffeomorphic to D, and we still call it D. Now, if K is any compact subset of $\mathcal{V} := \Gamma\backslash\mathcal{U}$ containing D, we have

$$\begin{aligned}\mathrm{Area}_{g_1}(M) &\geq -\int_K \kappa_{\phi^\star h_\varepsilon}(x)\, dV_{\phi^\star h_\varepsilon} \\ &\geq -\int_D \kappa_{\phi^\star h_\varepsilon}(x)\, dV_{\phi^\star h_\varepsilon} - \int_{K\backslash D} \kappa_{h_\varepsilon}(T_{\phi(x)}\phi(\mathbb{H}^1_{\mathbb{C}}))\, dV_{\phi^\star h_\varepsilon}\ ,\end{aligned}$$

since on K, $\kappa_{\phi^\star h_\varepsilon} \leq \kappa_{h_\varepsilon} \leq 0$.

When ε goes to 0, the induced metric $\phi^\star h_\varepsilon$ clearly goes to $f^\star g_n$ on D and therefore, on D, the curvature and the volume form of $\phi^\star h_\varepsilon$ respectively go to the curvature and volume form of $f^\star g_n$. In the same manner, on the compact set K, $\kappa_{h_\varepsilon}(T_{\phi(x)}\phi(\mathbb{H}^1_{\mathbb{C}}))$ goes to $\kappa_{g_n}(T_{f(x)}f(\mathbb{H}^1_{\mathbb{C}}))$. Again, all involved quantities are well defined on $\mathcal{V}$. Hence

$$\mathrm{Area}_{g_1}(M) \geq -\int_D \kappa_{f^\star g_n}(x)\, dV_{f^\star g_n} - \int_{K\backslash D} \kappa_{g_n}(T_{f(x)}f(\mathbb{H}^1_{\mathbb{C}}))\, dV_{f^\star g_n}\ .$$

Using $\kappa_{g_n}(T_{f(x)}f(\mathbb{H}^1_{\mathbb{C}})) = -1$ and remembering our assumption on D, we obtain

$$\mathrm{Area}_{g_1}(M) \geq (1+\alpha)\,\mathrm{Area}_{f^\star g_n}(D) + \mathrm{Area}_{f^\star g_n}(K\backslash D) = \alpha\,\mathrm{Area}_{f^\star g_n}(D) + \mathrm{Area}_{f^\star g_n}(K)\ .$$

This is true for any compact subset K of $\mathcal{V}$, hence

$$\mathrm{Area}_{g_1}(M) \geq \alpha\,\mathrm{Area}_{f^\star g_n}(D) + \mathrm{Area}_{f^\star g_n}(\mathcal{V})\ .$$

Since $f(\mathcal{U})$ is (locally) a 1-dimensional complex submanifold of $\mathbb{H}^n_{\mathbb{C}}$, the volume form of the metric induced on it by g_n is simply the restriction of ω_n. Therefore the volume form of the metric $f^\star g_n$ on $\mathcal{V}$ is $|f^\star\omega_n| = |e' - e''|\,\omega_1$. Hence

$$\begin{aligned}\mathrm{Area}_{g_1}(M) &\geq \alpha\,\mathrm{Area}_{f^\star g_n}(D) + \textstyle\int_{\mathcal{V}} |e'-e''|\,\omega_1 \\ &= \alpha\,\mathrm{Area}_{f^\star g_n}(D) + \textstyle\int_M |e'-e''|\,\omega_1 \\ &\geq \alpha\,\mathrm{Area}_{f^\star g_n}(D) + \left|\textstyle\int_M (e'-e'')\,\omega_1\right|\ .\end{aligned}$$

But we assumed that $\int_M (e'-e'')\,\omega_1 = \mathrm{Area}_{g_1}(M)$, so this gives $e'-e'' \geq 0$ on M and the desired contradiction: we conclude that the curvature of the metric induced by g_n on $f(\mathcal{U})$ is everywhere -1.

If we now take a small open set U in $\mathcal{U}$ on which f is an embedding, $f(U)$ is a complex, hence minimal, submanifold of $\mathbb{H}^n_{\mathbb{C}}$ whose sectional curvature equals the restriction of the sectional curvature of the ambient manifold $\mathbb{H}^n_{\mathbb{C}}$: $f(U)$ must be totally geodesic and therefore contained in a complex geodesic. By a result of Sampson ([Sa78]), this implies that f maps $\mathbb{H}^1_{\mathbb{C}}$ entirely into this complex geodesic. In particular, $\rho(\Gamma)$ stabilizes this totally geodesic copy of $\mathbb{H}^1_{\mathbb{C}}$ and ρ thus induces a tame representation of Γ into $\mathrm{PU}(1,1)$ that we shall still denote by ρ in the sequel. In the same way, f will be seen as a ρ-equivariant map from $\mathbb{H}^1_{\mathbb{C}}$ into $\mathbb{H}^1_{\mathbb{C}}$.

Claim 4.8. *The map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^1_{\mathbb{C}}$ is a local diffeomorphism.*

Proof. We apply the method of Toledo in [To79] Theorem 4.2, to prove that $e' - e'' > 0$ everywhere on M. We first go back to the proof of Theorem 3.1, keeping the same notations. We know that $\Delta \log e' = 2(e'' - e') + 2$, wherever $e' > 0$. The zeros of e' (if any) are known to be isolated and of finite order. Suppose that there exists $x \in M$ such that $e'(x) = 0$, and let r_0, $\varepsilon_0 > 0$ such that $e' \geq \varepsilon_0$ on $\partial\mathrm{D}(x, r_0)$ (where $\mathrm{D}(x, r_0)$ is the disc of radius r_0 about x). For any $0 < r < r_0$ and $0 < \varepsilon < \varepsilon_0$, we set $M_{r,\varepsilon} = \big(M_\varepsilon\backslash\mathrm{D}(x,r_0)\big) \cup \big(\mathrm{D}(x,r_0)\backslash\mathrm{D}(x,r)\big)$ and $M'_\varepsilon = M_\varepsilon \cup \mathrm{D}(x, r_0)$.

By Green's formula, we have

$$\int_{M_{r,\varepsilon}} \Big[\eta_R \Delta \log e' - \langle \operatorname{grad} \eta_R, \operatorname{grad} \log e' \rangle\Big] dV_1 = \int_{\partial M_{r,\varepsilon}} \eta_R \langle \nu, \operatorname{grad} \log e' \rangle \, ds \ ,$$

and letting $r \to 0$, then $R \to +\infty$, we get

$$\int_{M'_\varepsilon} \Big(2(e'' - e') + 2\Big) dV_1 \geq 4\pi q \ ,$$

where $2q > 0$ is the order of the zero of e' at x (see [To79] and the proof of Theorem 3.1). But, letting $\varepsilon \to 0$, we get a contradiction with $\tau(\rho) = -2\pi\chi(M)$. Finally, we thus have $e' > 0$ and $e' - e'' \geq 0$ everywhere on M.

Now, if $e'(x) = e''(x)$ at some point $x \in M$, there exists a positive constant A such that

$$-\Delta \log \frac{e'}{e''} \leq 4(e' - e'') \leq A \log \frac{e'}{e''}$$

in a neighbourhood of x. This implies (see [SY78] and references therein) that $\log(e'/e'')$ is identically zero on a non empty open subset of M, which is impossible, as the rank of $\mathrm{d}f$ is generically 2. □

We may therefore pull-back the complex structure of the target to obtain a complex structure on $\mathbb{H}^1_{\mathbb{C}}$ with respect to which f is holomorphic. In the following discussion, M will be endowed with the new induced complex structure.

As a consequence of the uniformization theorem, there exists a properly discontinuous subgroup $\Gamma' < \mathrm{PU}(1,1)$ such that M is holomorphically equivalent to $\Gamma' \backslash \mathbb{H}^1_{\mathbb{C}}$.

Claim 4.9. *The discrete subgroup* $\Gamma' < \mathrm{PU}(1,1)$ *is a lattice.*

Proof. Each puncture of M has a neighbourhood which is holomorphically equivalent to either a punctured disc or an annulus. We have to show that the latter cannot happen.

Let m be a puncture of M and suppose that m has a neighbourhood U which is holomorphically equivalent to an annulus

$$A_b = \{z \in \mathbb{C} \ , \ e^{-\pi/b} < |z| < 1\} \ , \ b > 0 \ .$$

In the former complex structure, we have horocyclic coordinates $(v,t) \in [0,a] \times [0,+\infty)$ on a fundamental domain of the cusp C corresponding to m. Let us denote as before by $c_t : [0,a] \longrightarrow \mathbb{H}^1_{\mathbb{C}}$ the curve $v \longmapsto f(v,t)$. Since the harmonic ρ-equivariant map f has finite energy, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ going to infinity such that $c_k := c_{t_k}$ verifies

$$\lim_{k\to+\infty} \big(l(c_k)\big)^2 \leq \lim_{k\to+\infty} a \int_0^a \|\mathrm{d}c_k\|^2 dv \leq \lim_{k\to+\infty} a\, e^{-t_k} \int_0^a \|\mathrm{d}f\|^2_{(v,t_k)} e^{-t_k} dv = 0 \ ,$$

where $l(c_k)$ is the length of c_k.

If γ is a peripheral element of Γ generating $\pi_1(C)$, $\rho(\gamma)$ is either parabolic or elliptic, because ρ is tame. In the first case, f induces a holomorphic function f_U from A_b into $\langle\rho(\gamma)\rangle\backslash\mathbb{H}^1_{\mathbb{C}}$ (which is holomorphically equivalent to a punctured disc). The loops $\alpha_k : [0,a] \longrightarrow C$, $v \longmapsto (v,t_k)$ define a sequence (β_k) of simple loops in A_b (with index 1 with respect to the origin), and whose supports converge to the circle $C_b = \{z \in \mathbb{C} \ , \ |z| = e^{-\pi/b}\}$. As $\lim_{k\to+\infty} l(c_k) = 0$, for every $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that $k \geq k_\varepsilon$ implies $|f_U \circ \beta_k(t)| \leq \varepsilon$ for any $t \in [0,a]$. By the maximum principle, $|f_U(z)| \leq \varepsilon$ for any z in the connected component of A_b lying between the support of β_{k_ε} and C_b. But the holomorphic function f_U admits a Laurent series expansion $f_U(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ on A_b and since, for every $e^{-\pi/b} < r < 1$,

$$|a_n| = \left| \frac{1}{2i\pi} \int_{|z|=r} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta \right| \leq e^{n\pi/b} \sup_{|z|=r} |f(z)| \ ,$$

f_U should be identically zero, which is a contradiction.

If $\rho(\gamma)$ is elliptic, we know from Claim 4.6 that $\rho(\gamma) \neq \mathrm{id}$ and thus we may assume that it is a rotation with center $0 \in \mathbb{H}^1_{\mathbb{C}}$ and angle $\theta \in (0, 2\pi)$. We now use the cover map $S \longrightarrow A_b$, $w \longmapsto e^{iz/b}$ where S is the strip $\{w \in \mathbb{C}\ ,\ 0 < \operatorname{Im} w < \pi\}$. The ρ-equivariant map f induces a holomorphic map $f_S : S \longrightarrow \mathbb{H}^1_{\mathbb{C}}$ such that $f(w + 2\pi b) = e^{i\theta} f(w)$ for every $w \in S$. But the holomorphic map $g_S : S \longrightarrow \mathbb{C}$, $w \longmapsto e^{-i\frac{\theta}{2\pi b}w} f_S(w)$ descends to A_b and, since $|g_S| \leq e^{\frac{\theta}{2b}} |f_S|$, the previous arguments applied to g_S instead of f_U do imply the same contradiction. $\square$

The conclusion of Theorem 4.3 now easily follows. Indeed, denoting by ω_1' the Kähler form of the induced complete hyperbolic metric of finite volume on $M \simeq \Gamma'\backslash\mathbb{H}^1_{\mathbb{C}}$, we may apply the Schwarz-Pick lemma to obtain $0 \leq \langle f^\star\omega_n, \omega_1'\rangle \leq 2$ pointwise. Moreover,

$$\tau(\rho) = \int_M f^\star\omega_n = \frac{1}{2}\int_M \langle f^\star\omega_n, \omega_1'\rangle\, dV_1' \leq \int_M dV_1' = -2\pi\chi(M) \ .$$

Therefore $\langle f^\star\omega_n, \omega_1'\rangle = 2$ everywhere and hence, if $\rho' : \Gamma' \longrightarrow \mathrm{PU}(1,1)$ is the representation induced by ρ, f defines a ρ'-equivariant isometry from $\mathbb{H}^1_{\mathbb{C}}$ onto $\mathbb{H}^1_{\mathbb{C}}$. In particular, ρ' (and so ρ) is injective, and $\rho'(\Gamma') = \rho(\Gamma)$ must be discrete. Finally, since f descends to an isometry between $\Gamma'\backslash\mathbb{H}^1_{\mathbb{C}}$ and $\rho(\Gamma)\backslash\mathbb{H}^1_{\mathbb{C}}$, ρ is a uniformization representation and $\rho(\Gamma)$ is a lattice in $\mathrm{PU}(1,1)$. $\square$

4.2. **The general case.**

Let now Γ be the fundamental group of a p-times punctured closed orientable surface of negative Euler characteristic M and $\rho : \Gamma \longrightarrow \mathrm{PU}(n,1)$ be a reductive representation. In the following, we shall prove Theorem 4.3 in this general setting.

As usual, we identify Γ with a non-uniform torsion-free lattice in $\mathrm{PU}(1,1)$ and M with $\Gamma\backslash\mathbb{H}^1_{\mathbb{C}}$, and we call $C_1, \ldots, C_p$ the cusps of M. We can assume that ρ is not tame, namely that for some $1 \leq k \leq p$, ρ maps the peripheral elements corresponding to the punctures $m_1, \ldots, m_k$ to hyperbolic isometries of $\mathrm{PU}(n,1)$.

We begin by showing that the proof given in the previous section, until Claim 4.8 included, still holds here.

If one looks carefully at the preceding arguments, one sees that what is really needed is the existence of a ρ-equivariant harmonic map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ such that:

(a) the norm $\|\nabla \mathrm{d}f\|$ is square integrable and hence $f^\star\omega_n$ is an L^2 form;
(b) we have $\tau(\rho) = \frac{1}{2}\int_M \langle f^\star\omega_n, \omega_1\rangle\, dV_1$, which together with (a) implies as before the inequality $|\tau(\rho)| \leq \mathrm{Vol}(M)$;
(c) Claim 4.7 holds and hence the equality $|\tau(\rho)| = \mathrm{Vol}(M)$ implies as for tame representations that f is a ρ-equivariant immersion from $\mathbb{H}^1_{\mathbb{C}}$ into a totally geodesic copy of $\mathbb{H}^1_{\mathbb{C}}$ in $\mathbb{H}^n_{\mathbb{C}}$.

According to Proposition 4.4, there is no finite energy ρ-equivariant map $\mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$, so that we can not apply Theorem 1.1. Nevertheless, we shall prove that:

(i) there exists a (infinite energy) ρ-equivariant harmonic map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$;
(ii) we have a control on the energy density of f at infinity;
(iii) this control implies (a), (b) and (c).

We shall use the notations of the proofs of the tame case.

(i) Our method is the same as in [JZ97]. We first fix a particular ρ-equivariant map $\Phi : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^n_{\mathbb{C}}$. As usual, we only define it in each cusp C_i of M and extend it on the compact part of M. We call γ_i a generator of the fundamental group of C_i (when working on a single cusp, we shall drop the subscript i). The notations are as in the proof of Proposition 4.4. Let C be a cusp of M. If $\rho(\gamma)$ is parabolic or elliptic, then we define Φ as we defined f in Proposition 4.4. If $\rho(\gamma)$ is hyperbolic, let $\varphi : [0,a] \longrightarrow \mathbb{H}^n_{\mathbb{C}}$ be a map sending $[0,a]$ proportionally to arclength into the axis of $\rho(\gamma)$ (which is a geodesic in $\mathbb{H}^n_{\mathbb{C}}$), and such that $\varphi(a) = \rho(\gamma)\varphi(0)$. Then, define Φ by $\Phi(v,t) = \varphi(v)$ (we have $\|\mathrm{d}\Phi\|_{(v,t)} = e^t\frac{\delta}{a}$ where δ is the translation length of $\rho(\gamma)$).

For each $s \in \mathbb{R}_+$, let M_s be the compact Riemann surface with boundary, obtained from M by deleting the end $\{t > s\}$ in each cusp of M, and let F_s be the restriction of the fiber bundle $F := \mathbb{H}^1_{\mathbb{C}} \times_\rho \mathbb{H}^n_{\mathbb{C}}$ to M_s. From [Co92], we get the existence of a harmonic section f_s of F_s which agrees with Φ on ∂M_s and whose energy satisfies $E(f_s) \leq E(\Phi_{|M_s})$.

We want to prove that there exists a strictly increasing sequence $(s_n)_{n\in\mathbb{N}}$ going to infinity, such that f_{s_n} converges uniformly on every compact subset of M. The limiting function f will be the (infinite energy) harmonic section of F we are looking for. Using the same argument as in [Co92], it is sufficient to prove that for each $0 \leq t \leq s$, the energy $E(f_{s|M_t})$ is bounded independently of s.

Let C be a cusp of M such that $\rho(\gamma)$ is hyperbolic. If we set $C_s = C \cap M_s$, from the proof of Proposition 4.4, it is immediate that

$$E(f_{s|C_s\setminus C_t}) \geq E(\Phi_{|C_s\setminus C_t}) = \frac{\delta^2}{2a}(e^s - e^t)\ .$$

Moreover, denoting by M' the union of the compact part of M with the cusps $C_{k+1}, \ldots, C_p$, and letting $M'' = M\setminus M'$ ($M'_s = M' \cap M_s$, $M''_s = M'' \cap M_s$), we have

$$E(\Phi_{|M'_s}) \leq E(\Phi_{|M'}) < +\infty$$

and so, because of the energy minimizing property of f_s,

$$E(f_{s|M_t}) \leq E(f_s) - E(\Phi_{|M''_s\setminus M''_t}) \leq E(\Phi_{|M'\cup M''_t})$$

which is independent of s.

(ii) For each $i \in \{1, \ldots, k\}$, we now define

$$\begin{array}{rcl} \alpha_i : [0,+\infty) & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \dfrac{1}{2}\displaystyle\int_0^{a_i} \big(\|\mathrm{d}f\|^2_{(v,t)} - \|\mathrm{d}\Phi\|^2_{(v,t)}\big)e^{-t}dv\ , \end{array}$$

where $(v,t) \in [0,a_i] \times [0,+\infty)$ are the usual horospherical coordinates in the cusp C_i. The α_i's are non-negative functions (see the proof of Proposition 4.4).

We claim that the energy density $e(f)$ is controlled by $e(\Phi)$, namely:

$$\int_0^{+\infty} \alpha_i(t)dt < +\infty \text{ for any } i \in \{1, \ldots, k\},\ \text{and } E(f_{|M'}) < +\infty\ .$$

Otherwise, there exists $r > 0$ such that

$$E\big(f_{|M_r}\big) \geq 2 + E\big(\Phi_{|M'\cup M''_r}\big) < +\infty\ .$$

We may choose $s > r$ such that

$$\big|e(f_s) - e(f)\big| \leq \frac{1}{\mathrm{Vol}(M_r)}$$

on M_r and so,

$$E\big(f_{s|M_r}\big) \geq 1 + E\big(\Phi_{|M' \cup M''_r}\big) \ .$$

Now, we have

$$\begin{aligned} E(f_s) = \int_{M_s} e(f_s)\, dV_1 \ &\geq \ 1 + E\big(\Phi_{|M' \cup M''_r}\big) + E\big(\Phi_{|M''_s \setminus M''_r}\big) \\ &\geq \ 1 + E\big(\Phi_{|M_s}\big) \ . \end{aligned}$$

But f_s and Φ coincide on ∂M_s and this contradicts the energy minimizing property of the harmonic map f_s.

(iii) *Proof of (a).* First, we remark that the map Φ used to construct f is a rank one harmonic map on M'', and in fact, it is totally geodesic. From the formula of Eells-Sampson (see Lemma 2.3), we thus get $\Delta e(\Phi) = 2e(\Phi)$ on M''. Let now be $e_\Phi : M \longrightarrow \mathbb{R}$ be a (C^2) function such that $e_{\Phi|M'} = 0$ and $e_{\Phi|M''\setminus M''_1} = e(\Phi)_{|M''\setminus M''_1}$. If K is the compact subset $\overline{M''_1 \setminus M''_0}$, we have

$$\Delta\big(e(f) - e_\Phi\big) = -\|\nabla \mathrm{d} f\|^2 + \mathrm{Scal}(f^\star R^n) + 2\big(e(f) - e_\Phi\big)$$

on $M\setminus K$. Denoting by η_R ($R>1$) the usual cut-off functions, and using Green's formula, we obtain

$$\begin{aligned} \int_M \eta_R \Delta\big(e(f) - e_\Phi\big)\, dV_1 \ &= \ \int_M \big(e(f) - e_\Phi\big)(\Delta\eta_R)\, dV_1 \\ &= \ \int_{M'} e(f)(\Delta\eta_R)\, dV_1 + \sum_{i=1}^k \int_0^{+\infty} (\Delta\eta_R)\alpha_i(t)dt + A \ , \end{aligned}$$

because η_R only depends on t in the cusps (here A is a constant independent of R). Like in Lemma 2.3, the inequality $\|\nabla \mathrm{d} f\|^2 \leq -\Delta\big(e(f) - e_\Phi\big) + 2\big(e(f) - e_\Phi\big)$, which is valid on $M\setminus K$, now implies that $\|\nabla \mathrm{d} f\| \in L^2(M)$ and, like in Lemma 3.4, we also obtain that $f^\star\omega_n$ is L^2. □

Proof of (b). Since $f^\star\omega_n$ is L^2, we see from the proof of Proposition 4.5 that we only need to show that each $f^\star\varsigma_i$ is an L^2 form to get $\tau(\rho) = \frac{1}{2}\int_M \langle f^\star\omega_n, \omega_1\rangle\, dV_1$.

Let $i \in \{1, \ldots, k\}$. We take horospherical coordinates $(z', v', t') \in \mathbb{H}^n_{\mathbb{C}}$ such that the axis of $\rho(\gamma_i)$ is given by $z' = v' = 0$. Then $\Phi^\star\varsigma_i = \Phi^\star dz' = 0$, that is $\|\mathrm{d}\Phi\| = \|\Phi^\star dt'\|$. Since $\int_0^{a_i}\big(\|f^\star dt'\|^2_{(v,t)} - \|\Phi^\star dt'\|^2_{(v,t)}\big)dv \geq 0$ for each t, the fact that $\|\mathrm{d} f\|^2 \geq \|f^\star dt'\|^2 + \|f^\star\varsigma_i\|^2$ together with (ii) implies that $f^\star\varsigma_i$ is L^2.

The proof of the inequality $|\tau(\rho)| \leq \mathrm{Vol}(M)$ is like in section 3.2, the key point being that $\|\nabla \mathrm{d} f\| \in L^2(M)$. □

Proof of (c). We only have to show that for each $\varepsilon > 0$,

$$\lim_{r\to+\infty} \frac{\mathrm{Area}_{\phi^\star h_\varepsilon}\big(B_{\phi^\star h_\varepsilon}(r)\big)}{r^2} = 0 \ .$$

In a cusp C_i for which $\rho(\gamma_i)$ is hyperbolic, we first compare the volume element $dV_{\phi^\star h_\varepsilon}$ to $dV_{\varepsilon g_1 + \Phi^\star g_n} = \sqrt{\varepsilon^2 + 2\varepsilon\, e(\Phi)}\, dV_1$ induced by the model map Φ. For any $s > 0$, we immediately get

$$\mathrm{Area}_{\phi^\star h_\varepsilon}(C_{i,s}) \leq \mathrm{Area}_{\varepsilon g_1 + \Phi^\star g_n}(C_{i,s}) + 2\int_0^s \alpha_i(t)dt + \frac{1}{2\varepsilon}\int_{C_{i,s}} \|f^\star\omega_n\|^2 dV_1 \ .$$

Recall that $f^\star\omega_n$ is L^2 and, since $E(f_{|M'}) < +\infty$, the other cusps have finite area w.r.t. $\phi^\star h_\varepsilon$. So

$$\lim_{r\to+\infty} \frac{\mathrm{Area}_{\phi^\star h_\varepsilon}\big(B_{\phi^\star h_\varepsilon}(r)\big)}{r^2} \leq \lim_{r\to+\infty} \frac{\mathrm{Area}_{\varepsilon g_1+\Phi^\star g_n}\big(B_{\phi^\star h_\varepsilon}(r)\cap M''\big)}{r^2}$$
$$\leq \lim_{r\to+\infty} \frac{\mathrm{Area}_{\varepsilon g_1+\Phi^\star g_n}\Big(B_{g_1}\big(\frac{r}{\sqrt{\varepsilon}}\big)\cap M''\Big)}{r^2}$$

where $B_{g_1}(r)$ is the geodesic ball of radius r with respect to the metric g_1.

But $B_{g_1}\big(\frac{r}{\sqrt{\varepsilon}}\big)\cap C_i \subset C_{i,\frac{r}{\sqrt{\varepsilon}}}$, and $dV_{\varepsilon g_1+\Phi^\star g_n} = \sqrt{\varepsilon^2+\varepsilon\,\frac{\delta_i^2}{a_i^2}e^{2t}}\,e^{-t}dv\,dt$ on C_i. From this, we conclude that there exists a positive constant A_ε such that

$$\mathrm{Area}_{\varepsilon g_1+\Phi^\star g_n}\Big(B_{g_1}\big(\frac{r}{\sqrt{\varepsilon}}\big)\cap M''\Big) \leq A_\varepsilon\, r\ .$$

Thus the area of balls for the metric $\phi^\star h_\varepsilon$ grows at most linearly and the result follows. $\square$

As in the tame case, we therefore know that $\rho(\Gamma)$ stabilizes a totally geodesic copy of $\mathbb{H}^1_{\mathbb{C}}$ in $\mathbb{H}^n_{\mathbb{C}}$, and hence we can consider ρ as a homomorphism from Γ to PU$(1,1)$. Moreover, the harmonic map $f:\mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^1_{\mathbb{C}}$ is a local diffeomorphism and therefore we can pull-back the complex structure of the target to M. The uniformization theorem implies that there exists a complete hyperbolic metric g_1' on M compatible with this new complex structure and a discrete torsion-free subgroup $\Gamma' = u(\Gamma)$ of PU$(1,1)$, isomorphic to Γ, such that (M, g_1') is isometric to $\Gamma'\backslash\mathbb{H}^1_{\mathbb{C}}$.

However, contrary to the case of tame representations, Claim 4.9 does not a priori hold and we will assume that Γ' is not a lattice, namely that the volume of M with respect to g_1' is infinite (in fact, Claim 4.9 and the proof of Claim 4.10 below imply that Γ' is a lattice if and only if ρ is tame). This means that some punctures, say $m_1,\dots,m_q$, have neighbourhoods of infinite g_1'-volume. More precisely, peripheral elements of Γ corresponding to these punctures are sent by the new uniformization representation u to hyperbolic isometries of PU$(1,1)$ and for each $i\in\{1,\dots,q\}$, m_i has a neighbourhood g_1'-isometric to the annulus

$$A_i := \{z\in\mathbb{C}\,,\, e^{-\pi/b_i} < |z| < a_i\},\ b_i>0,\ 1>a_i>e^{-\pi/2b_i},$$

endowed with the metric

$$\Big(\frac{2b_i}{|z|\sin(b_i\log|z|)}\Big)^2\,dzd\overline{z}\ .$$

In particular, there exist disjoint simple closed g_1'-geodesics $c_1,\dots,c_q$ in M, corresponding to the circles $\{|z| = e^{-\pi/2b_i}\}$ in A_i, such that c_i is null-homotopic in $M\cup\{m_i\}$ for each $i\in\{1,\dots,q\}$.

If we cut M along these geodesics and double the remaining finite volume part M_0 (whose interior is diffeomorphic to M) along its (possibly disconnected) g_1'-geodesic boundary, we obtain a surface $2M$, on which g_1' extends by symmetry to a complete hyperbolic metric of finite volume. We call σ the g_1'-isometric (antiholomorphic) involution of $2M$. Note that the Euler characteristic $\chi(2M)$ of $2M$ equals $2\chi(M)$.

We want to extend the representation $\rho:\Gamma\longrightarrow$ PU$(1,1)$ to a representation 2ρ of the fundamental group 2Γ of $2M$. For this we need an adapted presentation of 2Γ.

The hyperbolic metric g_1' on $2M$ allows us to identify 2Γ with a lattice in PU$(1,1)$ and $2M$ with $2\Gamma\backslash\mathbb{H}^1_{\mathbb{C}}$. Call again $\pi:\mathbb{H}^1_{\mathbb{C}}\longrightarrow 2M$ the covering projection and choose a connected component X' of $\mathbb{H}^1_{\mathbb{C}}\backslash\bigcup_{i=1}^q\pi^{-1}(c_i)$. Since $\bigcup_{i=1}^q\pi^{-1}(c_i)$ consists of disjoint geodesics of $\mathbb{H}^1_{\mathbb{C}}$, X' is a convex set whose stabilizer in 2Γ can be identified with Γ'. Moreover, $\pi(X') = \Gamma'\backslash X'$

is just the interior of M_0, seen as a subset of $2M$. For each i choose a lift $\tilde{c}_i$ of c_i in the boundary of X', a generator γ'_i of the cyclic stabilizer of $\tilde{c}_i$ in Γ', and call σ_i the symmetry w.r.t. $\tilde{c}_i$. It is easy to see that the symmetries σ_i are lifts of σ and therefore for all i there exists $h_i \in 2\Gamma$ such that $\sigma_i = h_i\sigma_1$.

Let $X'' = \sigma_1 X'$ be the other component of $\mathbb{H}^1_{\mathbb{C}} \backslash \bigcup_{i=1}^q \pi^{-1}(c_i)$ adjacent to $\tilde{c}_1$, and call Γ'' $(= \sigma_1\Gamma'\sigma_1)$ its stabilizer in 2Γ. Note that, for all i, $h_i^{-1}\gamma'_i h_i = \sigma_1\sigma_i\gamma'_i\sigma_i\sigma_1 = \sigma_1\gamma'_i\sigma_1$ belongs to Γ''.

The fundamental group 2Γ of $2M$ has the following abstract presentation (see for example [Se77]). It is generated by the sets Γ' and Γ'', together with the h_i's, subject to the relations:
- the relations of the groups Γ' and Γ'';
- for all $i \in \{1, \ldots, q\}$, the element γ'_i of Γ' is identified with the element $\gamma''_i := h_i^{-1}\gamma'_i h_i$ of Γ''.

Now, it follows from the discussion in the proof of Claim 4.9 that for all $i \in \{1, \ldots, q\}$, $g_i := \rho \circ u^{-1}(\gamma'_i)$ is an hyperbolic isometry of $\mathbb{H}^1_{\mathbb{C}}$. Call s_i the symmetry w.r.t. the axis of g_i and set

$$\begin{cases} 2\rho(\gamma') = \rho \circ u^{-1}(\gamma') \text{ for } \gamma' \in \Gamma' \ ; \\ 2\rho(\gamma'') = s_1(\rho \circ u^{-1}(\sigma_1\gamma''\sigma_1))s_1 \text{ for } \gamma'' \in \Gamma'' \ ; \\ 2\rho(h_i) = s_i s_1 \text{ for } i \in \{1, \ldots, q\} \ . \end{cases}$$

This is clearly compatible with the relations of 2Γ and hence 2ρ is a well-defined representation of 2Γ into $\mathrm{PU}(1,1)$.

Claim 4.10. *The representation $2\rho : 2\Gamma \longrightarrow \mathrm{PU}(1,1)$ is tame.*

Proof. It is enough to show that, for any $i \in \{q+1, \ldots, p\}$, the peripheral elements γ_i corresponding to the punctures m_i are not mapped by ρ to hyperbolic isometries of $\mathbb{H}^1_{\mathbb{C}}$.

Suppose that one of them is. We drop the subscript i. The quotient $\langle u(\gamma)\rangle\backslash\mathbb{H}^1_{\mathbb{C}}$ is holomorphically equivalent to a punctured disc $D^\star = \{z \in \mathbb{C}\,,\, 0 < |z| < 1\}$, whereas the quotient $\langle\rho(\gamma)\rangle\backslash\mathbb{H}^1_{\mathbb{C}}$ is holomorphically equivalent to an annulus $A = \{z \in \mathbb{C}\,,\, a < |z| < 1\}$. The ρ-equivariant holomorphic map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^1_{\mathbb{C}}$ induces a holomorphic map f from $D^\star$ to A. This map is bounded and hence extends to a holomorphic map from the whole disc D to $\overline{A}$. This is impossible since the ρ-equivariance of f implies that for all $0 < r < 1$, the loops $f(\{z \in D^\star,\, |z| = r\})$ are homotopically non trivial. □

Claim 4.11. *The invariant $\tau(2\rho)$ is maximal: $\tau(2\rho) = -2\pi\chi(2M)$.*

Proof. Since 2ρ is tame, there exists a finite energy 2ρ-equivariant map $f : \mathbb{H}^1_{\mathbb{C}} \longrightarrow \mathbb{H}^1_{\mathbb{C}}$ and $\tau(2\rho) = \int_{2M} f^\star\omega_1$.

For each $i \in \{1, \ldots, q\}$, we may choose a symmetric tubular neighbourhood N_i of the geodesic c_i and a lift $\widetilde{N}_i$ in $\mathbb{H}^1_{\mathbb{C}}$ such that $N_i = \langle\gamma'_i\rangle\backslash\widetilde{N}_i$. Let ψ_i be a Kähler potential associated as before to a fixed point ξ_i of $2\rho(\gamma'_i)$ and let ς_i be the corresponding 1-form. Then $f^\star\varsigma_i$ is a 1-form on N_i such that $f^\star\omega_1 = \mathrm{d}f^\star\varsigma_i$. If now η is a symmetric cut-off function on $2M$ identically equal to 0 outside the N_i's and to 1 close to the geodesics c_i, we get:

$$\begin{aligned} \tau(2\rho) &= \int_{2M} \Big[f^\star\omega_1 - \mathrm{d}\Big(\sum_{i=1}^q \eta f^\star\varsigma_i\Big)\Big] \\ &= \int_{M_0} \Big[f^\star\omega_1 - \mathrm{d}\Big(\sum_{i=1}^q \eta f^\star\varsigma_i\Big)\Big] + \int_{\sigma M_0} \Big[f^\star\omega_1 - \mathrm{d}\Big(\sum_{i=1}^q \eta f^\star\varsigma_i\Big)\Big] \\ &= \int_{M_0} \Big[f^\star\omega_1 - \mathrm{d}\Big(\sum_{i=1}^q \eta f^\star\varsigma_i\Big)\Big] - \int_{M_0} \Big[\sigma^\star f^\star\omega_1 - \mathrm{d}\Big(\sum_{i=1}^q \eta \sigma^\star f^\star\varsigma_i\Big)\Big] \ . \end{aligned}$$

Extending the form $f^\star\omega_1 - \mathrm{d}(\sum_{i=1}^q \eta f^\star\varsigma_i)$ on M_0 by zero to a form on M, one sees that $\int_M [f^\star\omega_1 - \mathrm{d}(\sum_{i=1}^q \eta f^\star\varsigma_i)] = \tau(\rho)$.

The forms $\sigma^\star f^\star\omega_1$ on M and $\sigma^\star f^\star\varsigma_i$ on N_i are respectively induced by $\sigma_1^\star f^\star\omega_1$ on $\mathbb{H}^1_\mathbb{C}$ and $\sigma_1^\star f^\star\varsigma_i$ on $\sigma_1\widetilde{N}_i$. Since $s_1^\star\omega_1 = -\omega_1$, one has $\sigma_1^\star f^\star\omega_1 = -(s_1\circ f\circ\sigma_1)^\star\omega_1$. Moreover, $\varsigma_i = s_1^\star s_1^\star(-\mathrm{d}^c\psi_i) = s_1^\star\mathrm{d}^c(\psi_i\circ s_1)$, and $\psi_i\circ s_1 = \psi_i\circ s_i\circ s_1 = 2\rho(h_i)^\star\psi_i$. Hence $\sigma_1^\star f^\star\varsigma_i = -(s_1\circ f\circ\sigma_1)^\star 2\rho(h_i)^\star\varsigma_i$. Since $s_1\circ f\circ\sigma_1$ is also ρ-equivariant, the 1-form $(s_1\circ f\circ\sigma_1)^\star 2\rho(h_i)^\star\varsigma_i = h_i^\star(s_1\circ f\circ\sigma_1)^\star\varsigma_i$ on $h_i^{-1}\widetilde{N}_i = \sigma_1\widetilde{N}_i$ induces $(s_1\circ f\circ\sigma_1)^\star\varsigma_i$ on N_i. Finally,

$$-\int_{M_0}\Big[\sigma^\star f^\star\omega_1 - \mathrm{d}\Big(\sum_{i=1}^q \eta\sigma^\star f^\star\varsigma_i\Big)\Big] = \int_{M_0}\Big[(s_1\circ f\circ\sigma_1)^\star\omega_1 - \mathrm{d}\Big(\sum_{i=1}^q \eta(s_1\circ f\circ\sigma_1)^\star\varsigma_i\Big)\Big]$$

and the r.h.s. again equals $\tau(\rho)$ since $s_1\circ f\circ\sigma_1$ is a ρ-equivariant map. Hence $\tau(2\rho) = 2\tau(\rho) = -4\pi\chi(M) = -2\pi\chi(2M)$ and the lemma is proved. $\square$

The representation $2\rho : 2\Gamma \longrightarrow \mathrm{PU}(1,1)$ is therefore a tame representation of maximal invariant. It follows from the results of section 4.1 that there exists a 2ρ-equivariant isometry f from $\mathbb{H}^1_\mathbb{C}$ onto $\mathbb{H}^1_\mathbb{C}$. Since f is a fortiori $(\rho\circ u^{-1})$-equivariant, ρ is a uniformization representation and we are done.

References

[Au60] L. Auslander, *Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups,* Ann. of Math. **71**, 1960, 579-590 6

[Bi97] O. Biquard, *Métriques d'Einstein à cusps et équations de Seiberg-Witten,* J. Reine Angew. Math. **490**, 1997, 129-154 5

[BI01] M. Burger and A. Iozzi, *Bounded cohomology and representation varieties of lattices in* PSU$(1,n)$, Preprint, 2001 2, 17, 18

[BI03] M. Burger and A. Iozzi, Letter, 2003 4

[BM02] M. Burger and N. Monod, *Continuous bounded cohomology and applications to rigidity theory,* Geom. Funct. Anal. **12**, 2002, 219-280 2

[CT89] J. A. Carlson and D. Toledo, *Harmonic mappings of Kähler manifolds to locally symmetric spaces,* Publ. Math. I.H.E.S. **69**, 1989, 173-201 14, 15

[Co88] K. Corlette, *Flat G-bundles with canonical metrics,* J. Diff. Geom. **28**, 1988, 361-382 2, 18

[Co92] K. Corlette, *Archimedean superrigidity and hyperbolic geometry,* Ann. of Math. **135**, 1992, 164-182 1, 3, 4, 6, 11, 31

[ES64] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. **86**, 1964, 109-160 2, 12

[Ga54] M. P. Gaffney, *A special Stokes's theorem for complete Riemannian manifolds,* Ann. of Math. **60**, 1954, 140-145 25

[Go85] W. M. Goldman, *Representations of fundamental groups of surfaces,* Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math. **1167**, 95-117, Springer, Berlin, 1985 1

[Go99] W. M. Goldman, *Complex hyperbolic geometry,* The Clarendon Press, Oxford University Press, New York, 1999 5

[GM87] W. M. Goldman and J. J. Millson, *Local rigidity of discrete groups acting on complex hyperbolic space,* Invent. Math. **88**, 1987, 495-520 1

[GS92] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one,* Publ. Math. I.H.E.S. **76**, 1992, 165-246 1

[GP00] N. Gusevskii and J. R. Parker, *Representations of free Fuchsian groups in complex hyperbolic space,* Topology **39**, 2000, 33-60 3, 21

[GP03] N. Gusevskii and J. R. Parker, *Complex hyperbolic quasi-Fuchsian groups and Toledo's invariant,* Geom. Dedicata **97**, 2003, 151-185 3

[Hel84] S. Helgason, *Groups and geometric analysis,* Pure and Applied Math. **113**, Academic Press, New-York, 1984 12

[Her91] L. Hernandez, *Kähler manifolds and $\frac{1}{4}$-pinching,* Duke Math. J. **62**, 1991, 601-611 14

[HS96] C. Hummel and V. Schroeder, *Cusp closing in rank one symmetric spaces,* Invent. Math. **123**, 1996, 283-307 5

[Io02] A. Iozzi, *Bounded cohomology, boundary maps and rigidity of representations into* $\mathrm{Homeo}_+(\mathrm{S}^1)$ *and* $\mathrm{SU}(1,n)$*,* Rigidity in dynamics and geometry (Cambridge, 2000), Springer, Berlin, 2002, 237-260 2

[JM87] D. Johnson and J. J. Millson, *Deformation spaces associated to compact hyperbolic manifolds,* Discrete groups in geometry and analysis (New Haven, Conn., 1984), Progr. Math. **67**, Birkhäuser, Boston, MA, 1987, 48-106 1

[JZ97] J. Jost and K. Zuo, *Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasiprojective varieties,* J. Diff. Geom. **47**, 1997, 469-503 3, 31

[Ka98] M. Kapovich, *On normal subgroups in the fundamental groups of complex surfaces,* arXiv:math.GT/9808085 20

[Ko70] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings,* Marcel Dekker, New York, 1970 18

[Li97] P. Li, *Complete surfaces of at most quadratic area growth,* Comm. Math. Helv. **72**, 1997, 67-71 27

[Lic70] A. Lichnerowicz, *Applications harmoniques et variétés kählériennes,* Symp. Math III Bologna, 1970, 341-402 16

[Liv81] R. Livné, *On certain covers of the universal elliptic curve,* Ph.D. Thesis, Harvard University, 1981 20

[Ma91] G. A. Margulis, *Discrete subgroups of semisimple Lie groups,* Ergeb. der Math. **17**, Springer-Verlag, Berlin, 1991 1

[MSY93] N. Mok, Y.-T. Siu and S.-K. Yeung, *Geometric superrigidity,* Invent. Math. **113**, 1993, 57-83 3, 10

[Mo73] G. D. Mostow, *Strong rigidity of locally symmetric spaces,* Annals of Math. Studies **78**, Princeton Univ. Press, Princeton N.J., 1973 1

[Pa95] P. Pansu, *Sous-groupes discrets des groupes de Lie : rigidité, arithméticité,* Séminaire Bourbaki, Vol. 1993/94, Astérisque **227**, Société Mathématique de France, Paris, 1995, 69-105 10

[Re93] A. G. Reznikov, *Harmonic maps, hyperbolic cohomology and higher Milnor inequalities,* Topology **32**, 1993, 899-907 27

[Sa78] J. H. Sampson, *Some properties and applications of harmonic mappings,* Ann. Sci. École Norm. Sup. **11**, 1978, 211-228 26, 28

[Se77] J.-P. Serre, *Arbres, amalgames,* SL_2*,* Astérisque **46**, Société Mathématique de France, Paris, 1977 34

[Si80] Y.-T. Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds,* Ann. of Math. **112**, 1980, 73-111 2

[SY78] R. Schoen and S.-T. Yau, *On univalent harmonic maps between surfaces,* Invent. Math. **44**, 1978, 265-278 29

[To79] D. Toledo, *Harmonic mappings of surfaces to certain Kähler manifolds,* Math. Scand. **45**, 1979, 13-26 16, 28, 29

[To89] D. Toledo, *Representations of surface groups in complex hyperbolic space,* J. Diff. Geom. **29**, 1989, 125-133 2, 3, 17, 18

[Wo79] J. C. Wood, *Holomorphicity of certain harmonic maps from a surface to complex projective n-space,* J. London Math. Soc. **20**, 1979, 137-142 16

[Zu82] S. Zucker, *L_2-cohomology of warped products and arithmetic groups,* Invent. Math. **70**, 1982, 169-218 17

Institut Elie Cartan, Université Henri Poincaré, B. P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France

E-mail address: `koziarz@iecn.u-nancy.fr`

E-mail address: `maubon@iecn.u-nancy.fr`