Discrete and Continuous Dynamical Systems
doi:10.3934/dcds.2026024

CORRIGENDUM TO "SPECTRAL ESTIMATES FOR RUELLE OPERATORS WITH TWO PARAMETERS AND SHARP LARGE DEVIATIONS" (VOLUME 39, NUMBER 11, 2019, 6391-6417)

Vesselin Petkov[1] and Luchezar Stoyanov[2]

[1]Institut de Mathématiques de Bordeaux, University of Bordeaux, 351, Cours de la Libération, 33405 Talence, France

[2]University of Western Australia, 35 Stirling Hwy, Perth WA 6009, Australia

(Communicated by Enrique Pujals)

Abstract. We correct an estimate in the proof of Lemma 7 in [10], and this leads to a modified condition for the parameter w in Theorems 1 and 3. We propose corrected statements of Theorems 1 and 3. We also correct Theorem 2 and establish a weaker version.

A certain incorrect estimate in the proof of Lemma 7 in the paper [10] led to a rather relaxed condition about the parameter w in Theorems 1 and 3, which made the corresponding statements incorrect. Here, we propose corrected statements of Theorems 1 and 3 and provide a corrected proof of Lemma 7 under a modified condition for the parameter w. For completeness, we present all details in the proof of that lemma. The proofs of the other lemmas remain the same as well as the derivations of Theorems 1 and 3 from Lemma 7 and the other lemmas. The application of the corrected statement of Theorem 1 with $0 < \delta_1 < 1$ is not sufficient to obtain Theorem 2 in [10]. We prove a weaker version of Theorem 2.

Theorem 1. *Let $\varphi_t : M \longrightarrow M$ satisfy the standing assumptions (see Section 4) over the basic set Λ, and let $0 < \beta < \alpha$. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family for φ_t over Λ as in Section 2. Then, for any real-valued functions $f, g \in C^\alpha(\widehat{U})$ and any constant $\epsilon > 0$, there exist constants $0 < \hat{\rho} < 1$, $a_0 > 0$, $b_0 \geq 1$, $0 < \delta_1 < 1$, and $C = C(\epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f-(a+\mathbf{i}b)\tau+(c+\mathbf{i}w)g} h\|_{\beta,b} \leq C\, e^{Pm}\, \hat{\rho}^m\, |b|^\epsilon\, \|h\|_{\beta,b} \tag{1}$$

for all $h \in C^\beta(U)$, all integers $m \geq 1$, and all $b, w \in \mathbb{R}$ with $|b| \geq b_0$ and $|w| \leq \delta_1 |b|$.

As in the original paper [10], the main step in proving Theorem 1 is the following theorem.

Theorem 3. *Under the assumptions in Theorem 1, there exist constants $0 < \hat{\rho} < 1$, $a_0 > 0$, $b_0 \geq 1$, $0 < \delta_1 < 1$, $A_0 > 0$, and $C = C(\epsilon) > 0$ such that if $a, c \in \mathbb{R}$ satisfy $|a|, |c| \leq a_0$, then*

$$\|L^m_{f_{atc} - \mathbf{i}b\tau + (c+\mathbf{i}w)g_t} h\|_{\mathrm{Lip},b} \leq C\, \hat{\rho}^m\, \|h\|_{\mathrm{Lip},b}$$

for all $h \in C^{\mathrm{Lip}}(\widehat{U})$, all integers $m \geq 1$, and all $b, w, t \in \mathbb{R}$ with $|b| \geq b_0$, $te^{A_0 t} \leq |b|$, and $|w| \leq \delta_1 |b|$.

As in [10], we will now assume that $h, H \in C_D^{\mathrm{Lip}}(\widehat{U})$ are **fixed functions such that**

$$H \in K_{E|b|}(\widehat{U}) \quad , \quad |h(u)| \leq H(u) \quad , \quad u \in \widehat{U} \,, \tag{2}$$

and

$$|h(u) - h(u')| \leq E\, t\, |b| H(u')\, D(u, u') \quad \text{whenever } u, u' \in \widehat{U}_i \,,\ i = 1, \ldots, k \,. \tag{3}$$

2020 *Mathematics Subject Classification.* Primary: 37C30; Secondary: 37D20, 37C35.

Again, let $z = c + \mathbf{i}w$. Define the functions $\chi_\ell^{(i)} : \widehat{U} \longrightarrow \mathbb{C}$ ($\ell = 1, \dots, j_0$, $i = 1, 2$) as in [10] by

$$\chi_\ell^{(1)}(u) = \frac{\left| e^{(f^N_{atc} - \mathbf{i}b\tau^N + \mathbf{i}wg^N_t)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f^N_{atc} - \mathbf{i}b\tau^N + \mathbf{i}wg^N_t)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{(1-\mu) e^{f^N_{atc}(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + e^{f^N_{atc}(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

$$\chi_\ell^{(2)}(u) = \frac{\left| e^{(f^N_{atc} - \mathbf{i}b\tau^N + \mathbf{i}wg^N_t)(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) + e^{(f^N_{atc} - \mathbf{i}b\tau^N + \mathbf{i}wg^N_t)(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \right|}{e^{f^N_{atc}(v_1^{(\ell)}(u))} H(v_1^{(\ell)}(u)) + (1-\mu) e^{f^N_{atc}(v_2^{(\ell)}(u))} H(v_2^{(\ell)}(u))},$$

and set

$$\gamma_\ell(b, u) = |b| \, [\tau^N(v_2^{(\ell)}(u)) - \tau^N(v_1^{(\ell)}(u))]$$

for all $u \in \widehat{U}$. Notice that in [10] this was denoted simply by $\gamma_\ell(u)$.

We use the notations of Section 5 in [10]. Recall that $0 < \rho < 1$ depends of the Markov partition, $c_0 > 0, \gamma_1 > \gamma_0 > 1$ are introduced in (2.1) in [10], a_0, C_0, T satisfy (4.2) - (4.5) in [10] and $\hat{\delta} > 0$ is defined in Lemma 3 in [10]. Then, as in [10], fix a sufficiently large constant $E > 1$, a large integer $N \geq N_0$, and the parameter $t = t(a_0, N) > 1$ such that (4.6) and (4.7) in [10] hold. As in Section 4, fix an arbitrary constant $\hat{\gamma}$ with $1 < \hat{\gamma} < \gamma_0$. Fix integers $1 \leq n_1 \leq N_0, p_0 \geq 1$ and $\ell_0 \geq 1$ and assume $Et \leq \frac{c_0 \hat{\delta} \rho \hat{\gamma}^N}{512 \gamma_1^{n_1}}$. We choose the constants $0 < \epsilon_1 < 1, \epsilon_2 = \sqrt{\epsilon_1}$ so that

$$\sqrt{\epsilon_1} = \min\Big\{ \frac{1}{32 C_0}, c, \frac{1}{4E}, \hat{\delta} \rho^{p_0+1}, \frac{c_0 r_0}{\gamma_1^{n_1}}, \frac{c_0^2(\gamma_0 - 1)}{16 T \gamma_1^{n_1}} \Big\}.$$

The choice of ϵ_1 is different from that in page 6407 in [10]. The following lemma remains the same as in the original paper.

Lemma 6. *Let $j, j' \in \{1, 2, \dots, q\}$ be such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are contained in $\mathcal{C}_m$ for some $m = 1, \dots, p$ and are η_ℓ-separable in $\mathcal{C}_m$ for some $\ell = 1, \dots, \ell_0$. Then,*

$$|\gamma_\ell(b, u) - \gamma_\ell(b, u')| \geq c_2 \, \epsilon_1$$

for all $u \in \widehat{Z}_j$ and $u' \in \widehat{Z}_{j'}$, where $c_2 = \frac{\hat{\delta} \rho}{16} < 1$.

In what follows, we assume that

$$\mathbf{0} < \delta_\mathbf{1} < \frac{\mathbf{c_0^2 \, c_2 \, (\gamma_0 - 1)}}{\mathbf{16}\mathrm{Lip}(\mathbf{g_t}) \gamma_\mathbf{1}^{\mathbf{n_1}}}. \tag{4}$$

Here $g_t(x)$ is defined in (4.1) in [10] and $\mathrm{Lip}(g_t)$ denotes the Lipschitz constant of g_t. The following lemma remains the same as in the original paper.

Lemma 7. *Assume $|b| \geq b_0$ for some sufficiently large $b_0 > 0$, $|a|, |c| \leq a_0$, and let $|w| \leq \delta_1 |b|$, where δ_1 satisfies (4). Then, for any $j = 1, \dots, q$, there exist $i \in \{1, 2\}$, $j' \in \{1, \dots, q\}$, and $\ell \in \{1, \dots, \ell_0\}$ such that $\mathcal{D}_j$ and $\mathcal{D}_{j'}$ are adjacent and $\chi_\ell^{(i)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$.*

The next lemma also remains the same as in the original paper.

Lemma 8. *If h and H satisfy (2)-(3), then for any $j = 1, \dots, q$, $i = 1, 2$, and $\ell = 1, \dots, \ell_0$, we have*

(a) $\frac{1}{2} \leq \frac{H(v_i^{(\ell)}(u'))}{H(v_i^{(\ell)}(u''))} \leq 2$ *for all $u', u'' \in \widehat{Z}_j$;*

(b) *Either for all* $u \in \widehat{Z}_j$ *we have* $|h(v_i^{(\ell)}(u))| \leq \frac{3}{4} H(v_i^{(\ell)}(u))$, *or* $|h(v_i^{(\ell)}(u))| \geq \frac{1}{4} H(v_i^{(\ell)}(u))$ *for all* $u \in \widehat{Z}_j$.

Proof of Lemma 7. We use a modification of the proof of Lemma 5.10 in [13]. As in [13] (see the Remark before Lemma 5.9 there), it is easy to see that for any m and any ℓ, the set $\{\gamma_\ell(b,u) : u \in \sigma^{n_1}(\mathcal{C}_m)\}$ is contained in an interval of length $< \epsilon_2/8$. More precisely, for any $i = 1,2$, the set $\{|b|\, \tau_N(v_i^{(\ell)}(u)) : u \in \sigma^{n_1}(\mathcal{C}_m)\}$ is contained in an interval of length $< \frac{\epsilon_2}{16}$.

Indeed, given m, ℓ, $i = 1,2$ and $u, u' \in \sigma^{n_1}(\mathcal{C}_m)$, set $x = v_i^{(\ell)}(u), x' = v_i^{(\ell)}(u')$. Since $d(\sigma^N(x), \sigma^N(x')) = d(u,u') \leq \mathrm{diam}(\sigma^{n_1}(\mathcal{C}_m)) \leq \frac{\gamma_1^{n_1}\epsilon_1}{c_0\,|b|}$, it follows that

$$d(\sigma^j(x), \sigma^j(x')) \leq \frac{1}{c_0\, \gamma_0^{N-j}}\, d(u,u') \leq \frac{\gamma_1^{n_1}\epsilon_1}{c_0^2 \gamma_0^{N-j} |b|}\ .$$

This yields

$$|\tau_N(x) - \tau_N(x')| \leq \sum_{j=0}^{N-1} |\tau(\sigma^j(x)) - \tau(\sigma^j(x'))| \leq \frac{T\, \gamma_1^{n_1}\, \epsilon_1}{c_0^2\, |b|\, (\gamma_0 - 1)} \leq \frac{\epsilon_2}{16\, |b|}\ .$$

Thus, the set $\{\tau_N(v_i^{(\ell)}(u)) : u \in \sigma^{n_1}(\mathcal{C}_m)\}$ is contained in an interval of length $\leq \frac{\epsilon_2}{16|b|}$. Therefore, $\{\gamma_\ell(b,u) : u \in \sigma^{n_1}(\mathcal{C}_m)\}$ is contained in an interval of length $\leq \epsilon_2/8$. Since any Z_j is contained in $\sigma^{n_1}(\mathcal{C}_m)$ for some m, it follows that for any j and ℓ, the set $\{\gamma_\ell(b,u) : u \in Z_j\}$ is contained in an interval of length $\epsilon_2/8$.

In a similar way to the above estimate for σ^N, again for $x = v_i^{(\ell)}(u), x' = v_i^{(\ell)}(u')$, and $u, u' \in \sigma^{n_1}(\mathcal{C}_m)$, we get

$$\begin{aligned} |g_t^N(x) - g_t^N(x')| &\leq \sum_{j=0}^{N-1} |g_t(\sigma^j(x)) - g_t(\sigma^j(x'))| \\ &\leq \mathrm{Lip}(g_t) \sum_{j=0}^{N-1} d(\sigma^j(x), \sigma^j(x')) \\ &\leq \mathrm{Lip}(g_t) \sum_{j=0}^{N-1} \frac{\gamma_1^{n_1}\epsilon_1}{c_0^2 \gamma_0^{N-j} |b|} \\ &\leq \frac{\mathrm{Lip}(g_t)\gamma_1^{n_1}}{c_0^2\, (\gamma_0 - 1)} \frac{\epsilon_1}{|b|}\ . \end{aligned} \tag{5}$$

Thus, if $|w| \leq \delta_1\, |b|$ and δ_1 satisfies (4), then the set

$$\{|w||g_t^N(v_i^{(\ell)}(u))| : u \in \sigma^{n_1}(\mathcal{C}_m)\}$$

is contained in an interval of length $\leq \dfrac{\delta_1 \mathrm{Lip}(g_t)\epsilon_1\gamma_1^{n_1}}{c_0^2\, (\gamma_0 - 1)} \leq \dfrac{c_2\epsilon_1}{16}$.

For later use, notice that with x, x' as above, as in the Appendix in [10], we have

$$|f_{atc}^N(x) - f_{atc}^N(x')| \leq \frac{Tt}{c_0(\gamma_0 - 1)}\, D(u,u') \leq \frac{Tt}{c_0(\gamma_0 - 1)}\, \frac{\epsilon_1}{|b|}.$$

Given $j = 1, \ldots, q$, let $m = 1, \ldots, p$ be such that $\mathcal{D}_j \subset \mathcal{C}_m$. As in [13], we find $j', j'' = 1, \ldots, q$ such that $\mathcal{D}_{j'}, \mathcal{D}_{j''} \subset \mathcal{C}_m$ and $\mathcal{D}_{j'}$ and $\mathcal{D}_{j''}$ are η_ℓ-separable in $\mathcal{C}_m$. Then, Lemma 6 applies for any $u \in \widehat{Z}_{j'}$ and $u' \in \widehat{Z}_{j''}$.

Fix ℓ, j' and j'' with the above properties, and set $\widehat{Z} = \widehat{Z}_j \cup \widehat{Z}_{j'} \cup \widehat{Z}_{j''}$. If there exist $t \in \{j, j', j''\}$ and $i = 1, 2$ such that the first alternative in Lemma 8(b) holds for $\widehat{Z}_t$, ℓ and i, then $\mu \leq 1/4$ implies $\chi_\ell^{(i)}(u) \leq 1$ for any $u \in \widehat{Z}_t$.

Now assume that for every $t \in \{j, j', j''\}$ and every $i = 1, 2$ the second alternative in Lemma 8(b) holds for $\widehat{Z}_t$, ℓ, and i, i.e.

$$|h(v_i^{(\ell)}(u))| \geq \frac{1}{4}\, H(v_i^{(\ell)}(u)), \quad u \in \widehat{Z}_t. \tag{6}$$

Since $\psi(\widehat{Z}) = \widehat{\mathcal{D}}_j \cup \widehat{\mathcal{D}}_{j'} \cup \widehat{\mathcal{D}}_{j''} \subset \mathcal{C}_m$, given $u, u' \in \widehat{Z}$, both $\sigma^{N-n_1}(v_i^{(\ell)}(u))$ and $\sigma^{N-n_1}(v_i^{(\ell)}(u')) \in \mathcal{C}_m$. Moreover, $\mathcal{C}' = v_i^{(\ell)}(\sigma^{n_1}(\mathcal{C}_m))$ is a cylinder with

$$\mathrm{diam}(\mathcal{C}') \leq \frac{\epsilon_1}{c_0\, \gamma_0^{N-n_1}\, |b|}. \tag{7}$$

It follows from (5) above that

$$|g_t^N(v_i^{(\ell)}(u)) - g_t^N(v_i^{(\ell)}(u'))| \leq \frac{\mathrm{Lip}(g_t)\gamma_1^{n_1}}{c_0^2\,(\gamma_0 - 1)} \frac{\epsilon_1}{|b|}\,. \tag{8}$$

Recall that $z = c + \mathbf{i}\, w$, where $|w| \leq \delta_1\, |b|$.

We will now compare the lengths and the arguments of the complex numbers

$$\begin{aligned} \rho_1(u) &= e^{(f_{atc}^N - \mathbf{i} b\tau_N)(v_1^{(\ell)}(u))} e^{z g_t^N(v_1^{(\ell)}(u))} h(v_1^{(\ell)}(u)) \\ &= e^{(f_{atc}^N + c g_t^N)(v_1^{(\ell)}(u))}\, e^{-\mathbf{i} b\tau_N(v_1^{(\ell)}(u))}\, e^{\mathbf{i} w g_t^N(v_1^{(\ell)}(u))}\, h(v_1^{(\ell)}(u)), \end{aligned}$$

and

$$\begin{aligned} \rho_2(u) &= e^{(f_{atc}^N - \mathbf{i} b\tau_N)(v_2^{(\ell)}(u))} e^{z g_t^N(v_2^{(\ell)}(u))} h(v_2^{(\ell)}(u)) \\ &= e^{(f_{atc}^N + c g_t^N)(v_2^{(\ell)}(u))}\, e^{-\mathbf{i} b\tau_N(v_2^{(\ell)}(u))}\, e^{\mathbf{i} w g_t^N(v_2^{(\ell)}(u))}\, h(v_2^{(\ell)}(u)). \end{aligned}$$

First, given $i = 1, 2$ and $u, u' \in \widehat{Z}$, we look at

$$e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u))}\, h(v_i^{(\ell)}(u)) \quad \text{and} \quad e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u'))}\, h(v_i^{(\ell)}(u')).$$

Assume for example that

$$|h(v_i^{(\ell)}(u))| \geq |h(v_i^{(\ell)}(u'))|.$$

Then, using (3), (7), and (5), we get

$$\begin{aligned} &\frac{|e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u))}\, h(v_i^{(\ell)}(u)) - e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u'))}\, h(v_i^{(\ell)}(u'))|}{\min\{|h(v_i^{(\ell)}(u))|, |h(v_i^{(\ell)}(u'))|\}} \\ &= \frac{|e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u))}\, h(v_i^{(\ell)}(u)) - e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u'))}\, h(v_i^{(\ell)}(u'))|}{|h(v_i^{(\ell)}(u'))|} \\ &\leq \frac{|e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u))}\, h(v_i^{(\ell)}(u)) - e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u))}\, h(v_i^{(\ell)}(u'))|}{|h(v_i^{(\ell)}(u'))|} \\ &\quad + \frac{|e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u))}\, h(v_i^{(\ell)}(u')) - e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u'))}\, h(v_i^{(\ell)}(u'))|}{|h(v_i^{(\ell)}(u'))|} \\ &\leq \frac{|h(v_i^{(\ell)}(u)) - h(v_i^{(\ell)}(u'))|}{|h(v_i^{(\ell)}(u'))|} + |e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u))} - e^{\mathbf{i} w g_t^N(v_i^{(\ell)}(u'))}| \end{aligned}$$

$$\begin{aligned}
&\le \frac{Et|b|\,H(v_i^{(\ell)}(u'))}{|h(v_i^{(\ell)}(u'))|} D(v_i^{(\ell)}(u), v_i^{(\ell)}(u')) + |w|\,\|g_t^N(v_i^{(\ell)}(u)) - g_t^N(v_i^{(\ell)}(u'))\| \\
&\le 4Et|b|\,\mathrm{diam}(\mathcal{C}') + \delta_1\,|b|\,\frac{\mathrm{Lip}(g_t)\gamma_1^{n_1}}{c_0^2\,(\gamma_0-1)}\frac{\epsilon_1}{|b|} \\
&\le \frac{4Et|b|}{c_0\gamma_0^{N-n_1}} \cdot \frac{\epsilon_1}{|b|} + \delta_1\,\frac{\mathrm{Lip}(g_t)\epsilon_1\gamma_1^{n_1}}{c_0^2\,(\gamma_0-1)} \\
&\le \frac{c_2\epsilon_1}{8} + \frac{c_2\epsilon_1}{16} < \frac{c_2\epsilon_1}{4} < \frac{\pi}{3}\,,
\end{aligned}$$

where we have used the estimate for Et above. Thus, the angle between the complex numbers

$$e^{\mathbf{i}wg_t^N(v_i^{(\ell)}(u))}\, h(v_i^{(\ell)}(u)) \quad \text{and} \quad e^{\mathbf{i}wg_t^N(v_i^{(\ell)}(u'))}\, h(v_i^{(\ell)}(u'))$$

(regarded as vectors in $\mathbb{R}^2$) is less than $\pi/3$. In particular, for each $i = 1, 2$, we can choose a real continuous function $\theta_i(u)$, $u \in \widehat{Z}$, with values in $[0, \pi/3]$ and a constant λ_i such that

$$e^{\mathbf{i}wg_t^N(v_i^{(\ell)}(u))}\, h(v_i^{(\ell)}(u)) = e^{\mathbf{i}(\lambda_i+\theta_i(u))}|h(v_i^{(\ell)}(u))|, \quad u \in \widehat{Z}\,.$$

Fix an arbitrary $u_0 \in \widehat{Z}$ and set $\lambda = \gamma_\ell(b, u_0)$. Replacing e.g. λ_2 by $\lambda_2 + 2m\pi$ for some integer m, we may assume that $|\lambda_2 - \lambda_1 + \lambda| \le \pi$.

The difference between the arguments of the complex numbers

$$\begin{aligned}
&e^{-\mathbf{i}\,b\,\tau^N(v_1^{(\ell)}(u))}\, e^{\mathbf{i}\,w\,g_t^N(v_1^{(\ell)}(u))}\, h(v_1^{(\ell)}(u)) \\
\text{and}\quad &e^{-\mathbf{i}\,b\,\tau^N(v_2^{(\ell)}(u))}\, e^{\mathbf{i}\,w\,g_t^N(v_2^{(\ell)}(u))}\, h(v_2^{(\ell)}(u))
\end{aligned}$$

is given by the function

$$\begin{aligned}
\Gamma^{(\ell)}(b,u) &= [-b\,\tau^N(v_1^{(\ell)}(u)) + \theta_1(u) + \lambda_1] - [-b\,\tau^N(v_2^{(\ell)}(u)) + \theta_2(u) + \lambda_2] \\
&= b\left[\tau^N(v_2^{(\ell)}(u)) - \tau_N(v_1^{(\ell)}(u))\right] + (\lambda_1 - \lambda_2) + (\theta_1(u) - \theta_2(u)).
\end{aligned}$$

It follows from the observation in the beginning of the proof of the lemma that $|\gamma_\ell(b,u) - \gamma_\ell(b,u_0)| \le 2\,\frac{\epsilon_2}{8}$ for any $u \in \widehat{Z}$. Thus, for any such u, using $\lambda = \gamma_\ell(b, u_0)$, we get

$$\begin{aligned}
|\Gamma^{(\ell)}(b,u)| &\le |\gamma_\ell(b,u) - \gamma_\ell(b,u_0)| + |\lambda_2 - \lambda_1 + \lambda| + |\theta_2(u) - \theta_1(u)| \\
&\le 2\,\frac{\epsilon_2}{8} + \pi + \frac{\pi}{3} < \frac{3\pi}{2}.
\end{aligned}$$

Thus, $|\Gamma^{(\ell)}(b,u)| < \frac{3\pi}{2}$ for all $u \in \widehat{Z}$.

Given $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$, since $\widehat{\mathcal{D}}_{j'}$ and $\widehat{\mathcal{D}}_{j''}$ are contained in $\mathcal{C}_m$ and are η_ℓ-separable in $\mathcal{C}_m$, it follows from Lemma 6 and the above that

$$\begin{aligned}
&|\Gamma^{(\ell)}(b,u') - \Gamma^{(\ell)}(b,u'')| \\
&\ge |\gamma_\ell(b,u') - \gamma_\ell(b,u'')| - |\theta_1(u') - \theta_1(u'')| - |\theta_2(u') - \theta_2(u'')| \\
&\ge c_2\epsilon_1 - 2\frac{c_2\epsilon_1}{4} \ge \frac{c_2\epsilon_1}{2}.
\end{aligned}$$

Thus, $|\Gamma^{(\ell)}(b,u') - \Gamma^{(\ell)}(b,u'')| \ge \frac{c_2}{2}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ and $u'' \in \widehat{Z}_{j''}$. Hence, either $|\Gamma^{(\ell)}(b,u')| \ge \frac{c_2}{4}\epsilon_1$ for all $u' \in \widehat{Z}_{j'}$ or $|\Gamma^{(\ell)}(b,u'')| \ge \frac{c_2}{4}\epsilon_1$ for all $u'' \in \widehat{Z}_{j''}$.

Assume for example that $|\Gamma^{(\ell)}(b,u)| \geq \frac{c_2}{4}\epsilon_1$ for all $u \in \widehat{Z}_{j'}$. As observed above, for any $u \in \widehat{Z}$, we have $|\Gamma^{(\ell)}(b,u)| < \frac{3\pi}{2}$. Thus,

$$\frac{c_2}{4}\epsilon_1 \leq |\Gamma^{(\ell)}(b,u)| < \frac{3\pi}{2}$$

for all $u \in \widehat{Z}_{j'}$.

Now, as in [4] (see also [13]), one derives that $\chi_\ell^{(1)}(u) \leq 1$ and $\chi_\ell^{(2)}(u) \leq 1$ for all $u \in \widehat{Z}_{j'}$. This proves the lemma. □

Remark 1. The statement of Theorem 5.1(c) in [9] must be corrected, changing the condition $|w| \leq B|b|$ by $|w| \leq \delta_1|b|$ with $0 < \delta_1 < 1$ as we have done in Theorem 1 above. Notice that Theorem 5.1(c) is a particular case of Theorem 1 since one assumes that the functions f, g are in $C^{Lip}(\hat{U})$. Similar corrections must be done in Theorem 6.1 in [9], replacing the condition $|b| \leq B|w|$ by $|b| \leq \delta_2|w|$ with a suitable small constant $0 < \delta_2 < 1$ and in Theorem 7.3 in [9] replacing $\eta_0 \leq |w|$ by $\eta_0 \leq |w| \leq \delta_1 |\operatorname{Im} s|$.

We pass to a corrected version of Theorem 2 in [10] and we use the notations of [10]. Recall that $F : \Lambda \to \mathbb{R}$, $G : \Lambda \to (0, \infty)$ are Hölder continuous functions. By adding a constant, we may assume that $\Pr_{\sigma_\tau}(F) = 0$, and this does not change the equilibrium state m_{F+tG}. Therefore, $\Pr_\sigma(f - \Pr_{\sigma_\tau}(F)\tau) = 0$ yields $\Pr_\sigma(f) = 0$. Given $p \in \mathcal{I}_G = \big\{ \int_{\mathcal{R}^\tau} G dm_{F+tG} : t \in \mathbb{R}\big\}$, we have

$$\frac{dPr_{\sigma^\tau}(F + tG)}{dt}\Big|_{t=\xi(p)} = \int_{\mathcal{R}^\tau} G dm_{F+\xi(p)G} = p$$

and $\xi(p) \in \mathbb{R}$ is uniquely determined by the above equality. Set $g_p(x) := g(x) - p\tau(x)$ and

$$g_p^n(x) = g^n(x) - p\tau^n(x) = \int_0^{\tau^n(x)} (G(\pi(x,t)) - p)dt.$$

We have

$$\begin{aligned} 0 &= \int_{\mathcal{R}^\tau} (G-p) dm_{F+\xi(p)G} \\ &= \int_{\mathcal{R}^\tau} (G-p) dm_{F+\xi(p)(G-p)} \\ &= \Big(\int_{\mathcal{R}} \tau d\mu_{f+\xi(p)g_p}\Big)^{-1} \int_{\mathcal{R}} \Big(\int_0^{\tau(x)} \Big(G(\pi(x,t) - p\Big) dt\Big) d\mu_{f+\xi(p)g_p}, \end{aligned}$$

which implies

$$\frac{d\Pr_\sigma(f + tg_p)}{dt}\Big|_{t=\xi(p)} = \int_{\mathcal{R}} g_p d\mu_{f+\xi(p)g_p} = 0. \tag{9}$$

Following [11, 12], we say that the function $g_p(x)$ satisfies the *Diophantine condition* if there exist three periodic orbits $\tau^{n_k}(x_k) = x_k$, $n_k \in \mathbb{N}$, $k = 1, 2, 3$, issued from x_k, so that

$$\alpha = \frac{g_p^{n_3}(x_3) - g_p^{n_1}(x_1)}{g_p^{n_2}(x_2) - g_p^{n_1}(x_1)}$$

is a Diophantine number, that is, there exist $d > 0$ and $\nu > 1$ such that

$$\big|\alpha - \frac{m}{q}\big| \geq dq^{-1-\nu},\ \forall q \in \mathbb{N},\ \forall m \in \mathbb{Z}.$$

Under this condition, we have the following Dolgopyat type estimate for the Ruelle operator $L_{f+(\xi(p)+\mathbf{i}u)g_p}$.

Proposition 1 (Prop. 2.2 in [12])**.** Assume that $g_p(x)$ satisfies the Diophantine condition. Then, there exist $b_0 > 1$, $\gamma > 0$, $c_2 > 0$, $C > 0$, and $D > 0$ such that for $|u| \geq b_0$, we have

$$\|L^{2Nm}_{f+(\xi(p)+\mathbf{i}u)g_p} 1\|_\infty \leq Ce^{2NmPr_\sigma(f+\xi(p)g_p)}\Big(1 - \frac{c_2}{|u|^\gamma}\Big)^m, \ \forall m \in \mathbb{N}, \tag{10}$$

where $N = [D \log |u|]$.

Below we assume that G is not cohomologous to a constant. Then, g_p is not cohomologous to 0, and the set $\mathcal{I}_G$ has a non-empty interior Int $\mathcal{I}_G$. From (9), it follows that $g_p(x)$ is not cohomologous to a constant $b \neq 0$. Obviously, the same holds if $g_p(x)$ satisfies the Diophantine condition. Thus, g_p is not cohomologous to a constant and according to the results in [6], one obtains

$$\frac{d^2 Pr_\sigma(f + tg_p)}{dt^2}\Big|_{t=\xi(p)} = \omega(p) > 0.$$

Moreover, equality (9) shows that $g_p(x)$ cannot be everywhere on $\mathcal{R}$ a non-negative or a non-positive function. The change of sign of $g_p(x)$ leads to difficulties when we study the Dolgopyat type estimate of $L_{f+(\xi(p)+\mathbf{i}u)g_g}$.

The function $g_p(x)$ is called non-lattice if $tg_p(x)$ cohomologous to $b + F(x)$ with $t \in \mathbb{R}$, $b \in \mathbb{R}$, $F \in C(\mathcal{R}; \mathbb{Z})$, implies $t = b = 0$. The sharp deviation results concern the estimate of the measure

$$G_n = \mu\big\{x \in \mathcal{R} : \ \frac{1}{\tau^n(x)} \int_0^{\tau^n(x)} G(\pi(x,t))dt - p \in (-\delta_n, \delta_n)\big\}, \ n \to \infty,$$

where $\mu = \mu_f$ is the equilibrium state of f. Let

$$0 < \tau_0 = \min_{x \in R} \tau(x), \ \tau_1 = \max_{x \in R} \tau(x).$$

Then, $\tau_0 n \leq \tau^n(x) \leq \tau_1 n$, and we set

$$\nu_n = \tau_0 n\delta_n, \ \epsilon_n = \tau_1 n \delta_n, \ n \in \mathbb{N}.$$

Clearly, if $\delta_n = \frac{1}{n^{1+\kappa}}$, $\kappa > 0$, then $\nu_n = \mathcal{O}(n^{-\kappa})$, $\epsilon_n = \mathcal{O}(n^{-\kappa})$, and

$$\mu\{x : \ g^n(x) - p\tau^n(x) \in (-\nu_n, \nu_n)\} \leq G_n \leq \mu\{x : \ g^n(x) - p\tau^n(x) \in (-\epsilon_n, \epsilon_n)\}.$$

We prove a weaker version of Theorem 2 in [10], where the sequence $\delta_n = e^{-\delta n}$, $\delta > 0$, is replaced by $\epsilon_n = \mathcal{O}(n^{-\kappa})$. $\kappa > 0$.

Theorem 2. *Assume that $G : \Lambda \to (0, \infty)$ is a Hölder continuous function which is not cohomologous to a constant function, and there exists a Markov family $\mathcal{R} = \{R_i\}_{i=1}^k$ of the flow φ_t on Λ such that G is constant on the stable leaves of all rectangular boxes*

$$B_i = \{\varphi_t(x) : \ x \in R_i, 0 \leq t \leq \tau(x)\}, \ i = 1, ..., k.$$

Assume that the suspended flow σ_τ^t is topologically weak mixing. Let $p \in \mathrm{Int}\mathcal{I}_G$, $g_p(x)$ be non-lattice and let $g_p(x)$ satisfy the Diophantine condition. Then, there exists $\kappa > 0$ such that if $\epsilon_n = \mathcal{O}(n^{-\kappa})$, we have

$$\mu\{x \in \mathcal{R} : \ g^n(x) - p\tau^n(x) \in (-\epsilon_n, \epsilon_n)\} \sim \frac{2\epsilon_n}{\sqrt{2\pi\omega(p)n}} e^{nJ(p)}, \ n \to \infty. \tag{11}$$

Here, $J(p) = \mathrm{Pr}_\sigma(f + \xi(p)g_p) \leq 0$ and $J(p) = 0$ if and only if $p = \int G dm_F$ and $\xi(p) = 0$.

Remark 2. To satisfy some of the assumptions in the above theorem, we may assume that the functions τ and g are $\sigma-$independent. This means that if $t_1\tau + t_2 g$ with $t_1 \in \mathbb{R}, t_2 \in \mathbb{R}$, is cohomologous to a function in $C(\mathcal{R}; \mathbb{Z})$, then $t_1 = t_2 = 0$. The $\sigma-$independence of τ and g implies that the flow σ_τ^t is topologically weak mixing and G is not cohomologous to a constant.

If $\mu_{f+\xi(p)g_p}(R_i) \in \mathbb{Q}$ for $i = 1, ..., k$ and $g_p(x)$ satisfies the Diophantine condition, then $g_p(x)$ is non-lattice. Indeed, assuming $tg_p(x) \sim b + F(x)$ with $t \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$, $F(x) \in C(\mathcal{R}; \mathbb{Z})$, and applying (9), one deduces

$$b + \sum_{i=1}^k m_i \mu_{f+\xi(p)g_p}(R_i) = 0,$$

where $m_i = F(x)|_{R_i} \in \mathbb{Z}$, $i = 1, ..., k$. This implies $b \in \mathbb{Q}$, and one obtains a contradiction with the Diophantine condition for g_p. In general, the Diophantine condition for g_p does not imply that g_p is non-lattice.

The only point in the proof of Theorem 2 in [10] that must change is the analysis of the term

$$I_3(n) = \frac{\epsilon_n}{2\pi} \int_{|u|>a} \Big(\int L^n_{f+(\xi(p)+\mathbf{i}ug_p)} \mathbf{1}(x) d\mu(x)\Big) \hat{\chi}(\epsilon_n(u - \mathbf{i}\xi(p)) du$$

where $a > 0$ is large. By using estimate (10), we obtain

$$I_3(n) = e^{nJ(p)} \mathcal{O}\Big(\epsilon_n^{1-k} \int_a^\infty \Big(1 - \frac{c_2}{u^\gamma}\Big)^{\frac{n}{2[D \log u]}} u^{-k} du\Big),\ k \gg 1,$$

The purpose is to show that the term $\mathcal{O}(...)$ goes to 0 as $n \to \infty$ faster than $\epsilon_n n^{-1/2}$. To prove this, we repeat the argument of subsection 4.2 in [12]. For reader's convenience we present some details.

Write

$$\int_a^\infty \epsilon_n^{1-k} \Big(1 - \frac{c_2}{u^\gamma}\Big)^{\frac{n}{2[D \log u]}} u^{-k} du = \int_a^{n^{\delta'}} (...) + \int_{n^{\delta'}}^\infty (...) = I_1 + I_2$$

with $0 < \kappa < \delta' < 1/\gamma$. Since $\delta'\gamma < 1$, it is easy to see that for every $r \geq 1$ and $n \geq N_0(r, k)$ sufficiently large, we have

$$I_1 \leq a^{-k} n^{\frac{1}{\gamma} + (k-1)\kappa} \Big(1 - \frac{c_2}{n^{\delta'\gamma}}\Big)^{\frac{n}{D\delta' \log n}} \leq \frac{1}{n^r}.$$

Next,

$$I_2 \leq n^{(1-k)\delta' + (k-1)\kappa} = n^{(k-1)(\kappa - \delta')}.$$

Since $\kappa - \delta' < 0$, choosing k large, we obtain $I_2 = \mathcal{O}(n^{-r})$, $n \geq N_1(r)$. This completes the proof. □

Acknowledgments. We would like to thank Thibault Lefeuvre who informed us that the statement (c) of Theorem 5.1 in [9] with $|w| \leq B|b|$ and any $B > 0$ is false. This led to the corrections of the corresponding results in [9] and [10].

REFERENCES

[1] R. Bowen, Equilibrium states and the ergodic theory of anosov diffeomorphisms, *Lect. Notes in Math.*, Springer-Verlag, Berlin, **470** (1975).

[2] R. Bowen, Symbolic dynamics for hyperbolic flows, *Amer. J. Math.*, **95** (1973), 429-460.

[3] R. Bowen and D. Ruelle, The ergodic theory of Axiom A flows, *Invent. Math.*, **29** (1975), 181-202.

[4] D. Dolgopyat, Decay of correlations in Anosov flows, *Ann. Math.*, **147** (1998), 357-390.

[5] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, Cambridge, 1995.

[6] S. P. Lalley, Distribution of period orbits of symbolic and Axiom A flows, *Adv. Appl. Math.*, **8** (1987), 154-193.

[7] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Astérisque*, (1990), No. 187-188, 268 pp.

[8] V. Petkov and L. Stoyanov, Sharp large deviations for some hyperbolic systems, *Ergodic Theory Dynam. Systems*, **35** (2015), 249-273.

[9] V. Petkov and L. Stoyanov, Ruelle transfer operators with two complex parameters and applications, *Discr. Cont. Dyn. Sys.*, **36** (2016), 6413-6451.

[10] V. Petkov and L. Stoyanov, Spectral estimates for Ruelle transfer operators with two parameters and sharp large deviations, *Discr. Cont. Dyn. Sys.*, **39** (2019), 6391-6417.

[11] M. Pollicott and R. Sharp, Error terms for closed orbits of hyperbolic flows, *Ergodic Theory Dynamical Systems*, **21** (2001), 545-562.

[12] M. Pollicott and R. Sharp, Large deviations, fluctuations and shrinking intervals, *Comm. Math. Phys.*, **290** (2009), 321-334.

[13] L. Stoyanov, Spectra of Ruelle transfer operators for Axiom A flows, *Nonlinearity*, **24** (2011), 1089-1120.

[14] L. Stoyanov, Pinching conditions, linearization and regularity of Axiom A flows, *Discr. Cont. Dyn. Sys.*, **33** (2013), 391-412.

[15] S. Waddington, Large deviations for Anosov flows, *Ann. Inst. H. Poincaré, Analyse non-Linéaire*, **13** (1996), 445-484.

Received for publication August 13, 2025; early access January 2026.